Radial limits of quasiregular local homeomorphisms

Kai Rajala*

Mathematics Subject Classification (2000): 30C65.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. We call a mapping $f : \Omega \to \mathbb{R}^n$ K-quasiregular, if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, and if there exists $1 \leq K < \infty$ so that

$$||Df(x)||^n \leq K J_f(x)$$

for almost every $x \in \Omega$. Here $||Df(x)||$ and $J_f(x)$ are the operator norm and the Jacobian determinant of the differential matrix of f at x, respectively. We say that f is quasiregular, or of bounded distortion, if it is K-quasiregular for some $1 \leq K < \infty$. When $n = 2$ and $K = 1$, we recover the class of analytic functions. If the target of f is the Riemann sphere $\overline{\mathbb{R}}^n$ instead of $\mathbb{R}^n$, we call f (K-) quasimeromorphic.

The theory of quasiregular mappings, initiated by the works of Reshetnyak and Martio, Rickman and Väisälä, shows that they form, from the geometric function theoretic point of view, the correct generalization of the class of analytic functions to higher dimensions. In particular, Reshetnyak proved that non-constant quasiregular mappings are continuous, discrete and open, and that they preserve sets of measure zero. For the theory of quasiregular mappings, see the monographs [9], [18] and [19].

Fatou's theorem [4] says that a bounded analytic function of the unit disc has radial limits at almost every point of the unit circle. One of the most important open questions in the theory of quasiregular mappings is to find out the correct analogue of Fatou's theorem for higher-dimensional mappings. Indeed, while Fatou's theorem is a central result in classical function theory, with several important developments based on it, the topic is poorly understood in higher dimensions; it is not even known if there exists a bounded n-dimensional quasiregular mapping of the unit ball without any radial limits when $n \geq 3$, cf. [11], [24]. In this paper we give the first step towards the solution of this problem. Namely, we prove that radial

*Research supported by the Academy of Finland, and by NSF grant DMS 0244421. This research was done when the author was visiting at the University of Michigan. He wishes to thank the department for hospitality.

limits do exist for spatial quasiregular local homeomorphisms. In fact, no boundedness or growth assumptions are needed here.

Theorem 1.1. *Let $f : B(0,1) \to \mathbb{R}^n$, $n \geq 3$, be a quasiregular local homeomorphism. Then there exists a point $x \in S^{n-1}(0,1)$, so that the radial limit*

$$\lim_{t \to 1} f(tx) = b_x \in \mathbb{R}^n$$

exists.

Theorem 1.1 is a special case of a more general result, Theorem 3.1 below. In particular, radial limits exist for infinitely many points of the unit sphere, see the discussion after Theorem 3.1. We do not know any better estimates for the size of the set on which radial limits exist.

Since $\mathcal{C}^{\frac{n}{n-2}-\delta(n,K)}$-smooth spatial quasiregular mappings are local homeomorphisms by [3] (also see [19], p. 12), the local homeomorphism assumption may be replaced by this smoothness assumption in Theorem 1.1. Also, since spatial K-quasiregular mappings are local homeomorphisms when K is close to one, see [12], [5], [17], the local homeomorphism assumption may be replaced by the assumption $K < 1 + \epsilon(n)$ in Theorem 1.1. This corollary also follows from the rigidity result of Sarvas [20]. However, using [20] does not give a quantitative bound for the constant $\epsilon(n)$ above, while Theorem 1.1 combined with [17] gives such a bound.

We next recall some earlier results concerning radial limits of mappings of the unit ball. First, by using Bojarski's factorization theorem, one can prove that bounded planar quasiregular mappings have radial limits in a set E of positive Hausdorff dimension. On the other hand, by the Beurling-Ahlfors theorem [2], E can be of arbitrarily small Hausdorff dimension. The first results on radial limits in higher dimensions are due to Martio and Rickman [11]. They proved the existence of radial limits almost everywhere in $S^{n-1}(0,1)$ for bounded quasiregular mappings of the unit ball under a growth condition on the integral of the Jacobian determinant. This result has been generalized by Koskela, Manfredi and Villamor [10], Rickman [19], VII Theorem 2.7, and Martio and Srebro [15]. In [15] the mappings are assumed to be local homeomorphisms. In these generalizations the boundedness assumption is dropped. Moreover, in Rickman's result the growth assumption is milder than the one in [11], while in [10] and [15] the size of the exceptional set is shown to be small in terms of Hausdorff dimension. Also, Vuorinen has several related results on the boundary behavior, see [28], [19], Chapter VII and the references therein. In particular, in [27] it is shown that for quasiregular local homeomorphisms the existence of a radial limit at a point implies the existence of a conical limit.

On the other direction, Martio and Srebro [15], and Heinonen and Rickman [7] have constructed examples to show that bounded quasiregular mappings of the unit ball may fail to have radial limits in a set of Hausdorff dimension arbitrarily close to $n-1$. The example given in [15] is a local homeomorphism, while the one in [7] has wild branching behavior. Martio and Srebro have also constructed, for each $n \geq 3$, quasimeromorphic mappings $f : B(0,1) \to \overline{\mathbb{R}}^n$ without any radial limits, see [13], [14]. We believe that radial limits do exist for spatial quasimeromorphic local homeomorphisms of the unit ball. On the other hand, we do not know if there are spatial quasiregular mappings $f : B(0,1) \to \mathbb{R}^n$ without radial limits.

The assumption $n \geq 3$ is needed in Theorem 1.1, unless it is replaced by some other assumptions; there exists a locally univalent analytic function f of the unit disc without any radial limits. Such a function f has been constructed by Barth and Rippon [1]. This construction uses Arakelian's approximation theorem. By using conformal sewing, one can give another, more constructive proof for the existence of such f[1]: Consider the locally univalent function

$$g(z) = \int_0^z \exp(\exp w)\, dw$$

on the closed upper half plane $\overline{\mathbb{H}}^+$, and the identity map h on the closed lower half plane $\overline{\mathbb{H}}^-$. Then the restrictions of g and h to $\mathbb{R}$ are homeomorphisms mapping onto $\mathbb{R}$. By glueing $\overline{\mathbb{H}}^+$ and $\overline{\mathbb{H}}^-$ so that each point $x \in \partial\overline{\mathbb{H}}^+$ is identified with the point $g(x) \in \partial\overline{\mathbb{H}}^-$, one has an open, simply connected Riemann surface X. From results of Volkovyskii [26], IV 12.52, it follows that X is of hyperbolic type. Hence there exist disjoint domains $D_1, D_2 \subset \mathbb{D}$, with common boundary, so that $\overline{D_1} \cup \overline{D_2} = \overline{\mathbb{D}}$, and conformal maps $G : \overline{\mathbb{H}}^+ \to \overline{D_1} \setminus S^1(0,1)$, $H : \overline{\mathbb{H}}^- \to \overline{D_2} \setminus S^1(0,1)$, so that

$$H^{-1}(G(x)) = g(x) \quad \text{for all } x \in \mathbb{R}. \tag{1.1}$$

Now, define

$$f(x) = \begin{cases} g(G^{-1}(z)), & z \in D_1, \\ h(H^{-1}(z)), & z \in D_2. \end{cases}$$

Then, by (1.1), f extends to a locally univalent function of the unit disc. On the other hand, the hyperbolicity of X implies that the domains D_1 and D_2 spiral around each other while approaching $S^1(0,1)$. It follows that f does not have radial limits at any point $x \in S^1(0,1)$.

We next give a brief outline of the proof of Theorem 1.1. We choose a suitable line segment γ on the image of f, so that γ terminates at a point z inside the cluster set of $S^{n-1}(0,1)$, and so that it has a lift γ' approaching a point x on $S^{n-1}(0,1)$. Then we try to relate the properties of γ' to the behavior of the image of the radial line segment I starting at 0 and terminating at x. In general, if γ' behaves in a non-tangential way, then $f(|I|)$ and γ are closely related, and it follows that the radial limit at x

[1]This example was communicated to the author by Alexandre Eremenko.

exists and equals z. On the other hand, if γ' behaves tangentially, then such a good relation does not have to exist. However, the tangentiality implies a strong continuity estimate for the restriction of f to γ'. It turns out that this estimate compensates the absence of the good relation, so that radial convergence to z occurs also in this case.

Acknowledgements
We thank Juha Heinonen for inspiring discussions and several helpful comments on this work. Also, we thank Karl F. Barth, Alexandre Eremenko and Sergiy Merenkov for valuable information on the boundary behavior of analytic functions, and Jani Onninen and Jang-Mei Wu for useful discussions on the topic.

2 Preliminaries

We will mostly use the notation of [19]. The euclidean norm is denoted by $|\cdot|$. Euclidean n-balls with center x and radius r are denoted by $B(x,r)$, while the notation $S^{n-1}(x,r)$ for the corresponding $(n-1)$-spheres is used. The distance of two sets $E, F \subset \mathbb{R}^n$ is denoted by $\mathrm{d}(E,F)$. Similar notation is used for the distance of a set E and a point x. The diameter of a set E is denoted by $\operatorname{diam} E$. The topological boundary of a set $E \subset \mathbb{R}^n$ is denoted by ∂E. For a topological sphere $S \subset \mathbb{R}^n$, we denote the bounded component of $\mathbb{R}^n \setminus S$ by $\operatorname{int} S$. For a path $\gamma : I \to \mathbb{R}^n$, where I is an interval, we will use the notation $|\gamma| := \gamma(I)$. We define the spherical cap $C(z,\alpha,w)$ by

$$C(z,\alpha,w) = \{x \in \mathbb{R}^n : |x-z| = |w-z|, (w-z)\cdot(x-z) > |x-z|^2 \cos\alpha\}.$$

We will use the notion of a quasisymmetric map, see [21], [25]. Let (X,d) and (Y,d') be metric spaces. We call a homeomorphism $f : X \to f(X) \subset Y$ (η-) quasisymmetric, or a quasisymmetric embedding into Y, if there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ so that $d'(f(x),f(y)) \le \eta(t) d'(f(x),f(z))$ whenever $t \ge 0$ and $d(x,y) \le t d(x,z)$.

Now suppose that Γ is a family of paths γ in $\Omega \subset \mathbb{R}^n$. The (conformal) n-modulus $M\Gamma$ of Γ is defined by

$$M\Gamma = \inf_{\rho \in X(\Gamma)} \int_{\mathbb{R}^n} \rho(x)^n \, dx,$$

where $X(\Gamma)$ is the family of all Borel functions $\rho : \mathbb{R}^n \to [0,\infty]$ so that

$$\int_\gamma \rho \, ds \ge 1 \quad \text{for all } \gamma \in \Gamma.$$

Then, Poletsky's inequality (see [19], II Theorem 8.1) says that, for a K-quasiregular mapping $f : \Omega \to \mathbb{R}^n$, and a family Γ of paths in Ω,

$$Mf(\Gamma) \le K^{n-1} M\Gamma.$$

When Γ is a family of paths in $S^{n-1}(x,r)$ equipped with the induced metric and the Hausdorff $(n-1)$-measure, the n-modulus of Γ (in $S^{n-1}(x,r)$) is denoted by $M_n^S\Gamma$.

We will also consider the conformal modulus of $(n-1)$-dimensional subsets of $\mathbb{R}^n$. For a family Λ of Borel sets $E \subset \mathbb{R}^n$, the conformal modulus $M_S\Lambda$ is defined by

$$M_S\Lambda = \inf_{\rho \in Y(\Lambda)} \int_{\mathbb{R}^n} \rho(x)^{\frac{n}{n-1}}\, dx,$$

where $Y(\Lambda)$ is the family of all Borel functions $\rho : \mathbb{R}^n \to [0,\infty]$ so that

$$\int_E \rho(y)\, d\mathcal{H}^{n-1}(x) \geq 1 \quad \text{for all } E \in \Lambda \text{ with } \mathcal{H}^{n-1}(E) > 0.$$

Here $\mathcal{H}^{n-1}$ is the Hausdorff $(n-1)$-measure. We will use the following result. Suppose that $g : \Omega \to \mathbb{R}^n$ is a K-quasiconformal map, i.e. an injective K-quasiregular mapping, and $a \in \mathbb{R}^n$. Furthermore, suppose that $\Lambda = \{E_r : r \in (s,t)\}$, where the sets E_r satisfy $E_r \subset S^{n-1}(a,r) \cap \Omega$. Then

$$M_S\Lambda \leq K M_S g(\Lambda).$$

This result is an easy consequence of the Sobolev embedding theorem, applied to the $(n-1)$-dimensional sets $S^{n-1}(a,r) \cap \Omega$, and basic properties of quasiconformal maps.

3 Basic setting

In this section we define the concepts needed to formulate Theorem 3.1 below. Also, we give an auxiliary result that guarantees the existence of certain paths whose lifts have desirable properties. For information on lifts of paths, see [19], II Section 3. The setting in this section is similar to the one in the proof of the Zorich Global Homeomorphism Theorem, see [29], [8], [19], III Theorem 3.4. In particular, the proof of Lemma 3.2 essentially uses the ideas used to prove the Global Homeomorphism Theorem and its generalizations.

From now on we assume that f satisfies the assumptions of Theorem 1.1. Also, we may assume that $f(0) = 0$. Choose a small enough radius $\delta > 0$ so that $U(0,f,\delta)$, the 0-component of $f^{-1}(B(0,\delta))$, is a normal neighborhood of 0, i.e. $f(\partial U(0,f,\delta)) = S^{n-1}(0,\delta)$ and $U(0,f,\delta) \cap f^{-1}(0) = \{0\}$. Such radius exists by [19], I Lemma 4.9.

Next, for every $y \in S^{n-1}(0,1)$, define

$$\gamma_y : [0,\infty) \to \mathbb{R}^n, \quad \gamma_y(t) = ty,$$

and let $\tilde{\gamma}_y$ be the maximal f-lifting of γ_y starting at 0. For each $y \in S^{n-1}(0,1)$ there exist $\lambda(y) \in [\delta, \infty]$, and a point $x_y \in S^{n-1}(0,1)$, so that there exists an increasing sequence (t_i), $t_i \to \lambda(y)$, so that

$$\lim_{i\to\infty} \tilde{\gamma}_y(t_i) = x_y. \tag{3.1}$$

Denote

$$z_y = \begin{cases} \gamma_y(\lambda(y)), & \delta \le \lambda(y) < \infty, \\ \infty, & \lambda(y) = \infty, \end{cases}$$

and

$$G = \{ty : y \in S^{n-1}(0,1), t \in [0, \lambda(y))\}.$$

Then, by construction, and by the local homeomorphism property of f, G is a domain that is starlike with respect to the origin. We denote the 0-component of $f^{-1}(G)$ by G'. Moreover, we denote

$$F := \{z_y : y \in S^{n-1}(0,1)\} \setminus \{\infty\}.$$

Furthermore, the restriction of f to G' will be denoted by g;

$$g : G' \to G, \quad g(x) = f(x) \quad \text{for all } x \in G'.$$

Then g is a homeomorphism: If we set $g^{-1}(z) = \tilde{\gamma}_y(t)$ for $z = ty$, then g^{-1} is a well-defined, continuous one-to-one mapping by the local homeomorphism property of f. It follows in particular that $F \neq \emptyset$, since otherwise we would have a quasiconformal homeomorphism g^{-1} from $\mathbb{R}^n$ into $B(0,1)$ (recall that the inverse of a quasiconformal homeomorphism is also quasiconformal). This cannot be true since the image of $\mathbb{R}^n$ under a quasiconformal map is $\mathbb{R}^n$.

For every $z_y \in F$, define cones $K(y, \varphi)$ by

$$K(y,\varphi) := \{x \in \mathbb{R}^n : z_y \cdot (z_y - x) > |z_y||z_y - x| \cos\varphi\}. \tag{3.2}$$

That is, $K(y,\varphi)$ is a cone with vertex z_y and opening angle φ, and it is symmetric with respect to γ_y. We denote by H the set of all points $z_y \in F$ with the following property: there exist constants $r_y, \varphi_y > 0$ so that

$$K(y, \varphi_y) \cap B(z_y, r_y) \subset G.$$

By the local homeomorphism property of f, we see that there exists a point $z_y \in F$ minimizing $|z|$ over all $z \in F$. Then $z_y \in H$, so that H in non-empty. Hence Theorem 1.1 is a consequence of the following result.

Theorem 3.1. *For each $z_y \in H$,*

$$\lim_{t\to 1} f(tx_y) = z_y. \tag{3.3}$$

In fact, H contains infinitely many elements. To see this, first notice that if H were finite, then also F would be finite. But if F were finite, the set $\mathbb{R}^n \setminus G$ would be of zero Hausdorff $(n-1)$-measure (recall that $n \geq 3$). By a theorem of Väisälä [22], such sets are removable for quasiconformal maps. Hence, g^{-1} would again extend to a quasiconformal map of $\mathbb{R}^n$ mapping into a ball, which is impossible.

Cone conditions on the image of a map, similar to the one used here, appear in the study of the boundary behavior of conformal maps, cf. [16], Chapter 6. Also, such a condition is used in [6] to study the boundary absolute continuity properties of quasiconformal maps of the unit ball. It would be interesting to know that if one assumes that the set H in Theorem 3.1 has positive Hausdorff dimension, can one then deduce measure estimates for the size of the set at which radial limits exist.

The rest of the paper is concerned with the proof of Theorem 3.1. We first prove an auxiliary result that will be used later in order to estimate the conformal moduli of certain path families. As indicated above, the ideas used in the proof are from the proof of the Zorich Global Homeomorphism Theorem. We point out that the assumption $n \geq 3$ is necessary here.

Lemma 3.2. *Suppose that $z_y \in F$, and that $\gamma : [0,1] \to B(0,1)$ is a path so that $\gamma(0) \in |\tilde{\gamma}_y|$. Moreover, assume that for $r > 0$,*

$$f(\gamma(0)) \in B(z_y, r),$$

and that

$$T := \{t : f(\gamma(t)) \in S^{n-1}(z_y, r)\} \neq \emptyset.$$

Denote $t^r = \min_{t \in T} t$. Then there exist $0 < \alpha \leq \pi$ and a point

$$k_r \in \partial C(z_y, \alpha, f(\gamma(t^r))),$$

so that every path η joining k_r and $f(\gamma(t^r))$ in $\overline{C}(z_y, \alpha, f(\gamma(t^r)))$ has the property that all maximal lifts η' of η starting at $\gamma(t^r)$ satisfy

$$\overline{|\eta'|} \cap S^{n-1}(0,1) \neq \emptyset.$$

Proof. For $\varphi > 0$, consider the $\gamma(t^r)$-component C'_φ of $f^{-1}(C(z_y, \varphi, f(\gamma(t^r))))$. Denote by $\alpha \in (0, \pi]$ the maximal angle for which

$$h := f_{|C'_\alpha} : C'_\alpha \to C(z_y, \alpha, f(\gamma(t^r)))$$

is a homeomorphism. We first consider the case $\alpha < \pi$. Then, if the conclusion of the lemma does not hold true, there exists, for each $p \in \partial C(z_y, \alpha, f(\gamma(t^r)))$, a path η joining p to $f(\gamma(t^r))$ in $C(z_y, \alpha, f(\gamma(t^r)))$ so that the lift η' of η starting at $\gamma(t^r)$ has the property $|\eta'| \subset B(0, 1-\epsilon)$ for some $\epsilon > 0$. It follows from [19], III Lemma 3.3 that h^{-1} extends to a map of a neighborhood of p. Since this is true for all $p \in \partial C(z_y, \alpha, f(\gamma(t^r)))$,

h^{-1} extends to a homeomorphism of $\overline{C}(z_y, \alpha, f(\gamma(t^r)))$ onto $\overline{C'_\alpha}$. Furthermore, [19], III Lemma 3.2 says that h^{-1} extends to a homeomorphism of a neighborhood of $\overline{C}(z_y, \alpha, f(\gamma(t^r)))$ onto its image. This contradicts the maximality of α, and hence the proof is complete in the case $\alpha < \pi$.

Next assume that $\alpha = \pi$. Denote by p the unique boundary point of $C(z_y, \pi, f(\gamma(t^r)))$. If the claim of the lemma does not hold true, there exists a path η joining $f(\gamma(t^r))$ and p so that the lift η' of η starting at $\gamma(t^r)$ has the property

$$|\eta'| \subset B(0, 1 - \epsilon) \tag{3.4}$$

for some $\epsilon > 0$. Now define a nested sequence of compact sets $\overline{W_i}$ by setting

$$W_i = C'_\pi \cap h^{-1}(B(p, i^{-1}) \cap C(z_y, \pi, f(\gamma(t^r)))).$$

Since each W_i is connected (notice that we use the assumption $n \geq 3$ here), also each $\overline{W_i}$ is connected, and hence

$$K = \bigcap_{i=1}^{\infty} \overline{W_i}$$

is a compact connected set. Consequently, (3.4) implies that

$$K \cap B(0, 1) \neq \emptyset.$$

But then K has to be a point by the discreteness of f and by the inclusion

$$K \cap B(0, 1) \subset f^{-1}(p).$$

We conclude that if we denote

$$S = S^{n-1}(z_y, |f(\gamma(t^r)) - z_y|),$$

and the $\gamma(t^r)$-component of $f^{-1}(S)$ by S', then

$$h = f_{|S'} : S' \to S$$

is a homeomorphism. In particular, S' is a topological sphere.

Denote $B' = \operatorname{int} S'$, and $B = \operatorname{int} S$. We next show that h extends to a homeomorphism $\overline{B'} \to \overline{B}$. First, if there exists a point $y \notin \overline{B}$ so that $y = f(x)$ for some $x \in B'$, then there exists a path η joining y and ∞ outside $\overline{B}$. But then the lift η' of η starting at x has to leave $\overline{B'}$ by the openness of f, and has to intersect S' in particular. This is a contradiction, and hence $fB' \subset B$. Moreover, since $f_{|S'} = S$, $f\overline{B'} = \overline{B}$ by the openness of f. Assume then that $f(x_1) = f(x_2) = y$ for some $x_1, x_2 \in B'$. Then, if we choose a path η joining y to a point in S, we have two lifts, η'_1 and η'_2, of η starting at x_1 and x_2, respectively, and terminating at S'. Since $h : S' \to S$

is a homeomorphism, η_1' and η_2' have to terminate at the same point $x \in S'$. This is a contradiction, however, since f is a local homeomorphism. We conclude that

$$h = f_{|\overline{B'}} : \overline{B'} \to \overline{B}$$

is a homeomorphism.

Finally, define $\beta : [0, t^r + 1) \to \overline{B}$ by setting

$$\beta(t) = \begin{cases} f(\gamma(t^r - t)), & t \in [0, t^r] \\ f(\gamma(0)) + (t - t^r)(z_y - f(\gamma(0))), & t \in [t^r, t^r + 1). \end{cases}$$

Now, since $B' \cap S^{n-1}(0,1) = \emptyset$, we have $x_y \notin B'$. Hence β must have two lifts, β_1' and β_2', starting at $\gamma(t^r)$, one lift going near x_y and another one joining $\gamma(t^r)$ to $h^{-1}(z_y) \in B'$. This contradicts the local homeomorphism property of f, and thus the proof is complete. $\square$

4 Continuity estimate

From now on we consider $z_y \in H$ to be fixed, and denote $\varphi_y = \varphi_0$, $r_y = r_0$. We may assume that $r_0 \leq 1$. Our goal is to prove that the radial limit at x_y exists and equals z_y.

For $0 < r < 1$, choose a point

$$x_r \in |\tilde{\gamma}_y| \cap S^{n-1}(x_y, r) \tag{4.1}$$

as follows: $x_r = \tilde{\gamma}_y(t_r)$, where

$$t_r = \min\{t \in [0, \lambda(y)) : \tilde{\gamma}_y(t) \in S^{n-1}(x_y, r)\}. \tag{4.2}$$

Then we have

$$|g(x_{r_1}) - z_y| < |g(x_{r_2}) - z_y|$$

whenever $r_1 < r_2$. Define a function $\theta : (0,1] \to (0,1]$ by setting

$$\theta(r) = \mathrm{d}(x_r, S^{n-1}(0,1)). \tag{4.3}$$

We now have the following continuity estimate.

Proposition 4.1. *If $f(x_s) \in B(z_y, r_0/2)$ and $0 < t < s/2$, then*

$$\int_t^s \frac{dr}{\theta(r)} \leq C_0 \log \frac{1}{|f(x_t) - z_y|}, \tag{4.4}$$

where $C_0 > 0$ only depends on φ_0, K and n.

Proof. Fix $t < r < s$. Then, since $f(x_r) \in B(z_y, r_0/2)$, [25], Theorem 2.4, implies that we have the following quasisymmetry condition:

$$\frac{\max_{z\in\partial K_r} |g^{-1}(z) - x_r|}{\theta(r)} \le \frac{\max_{z\in\partial K_r} |g^{-1}(z) - x_r|}{\min_{z\in\partial K_r} |g^{-1}(z) - x_r|} \le C(\varphi_0, K, n),$$

where $C(\varphi_0, K, n) > 0$ only depends on φ_0, K and n, and

$$K_r = K(y, \varphi_0/2) \cap (B(z_y, 2|g(x_r) - z_y|) \setminus B(z_y, 2^{-1}|g(x_r) - z_y|)). \tag{4.5}$$

Hence, if we denote

$$V_r = S^{n-1}(x_y, r) \cap g^{-1}(K_r), \tag{4.6}$$

then

$$\operatorname{diam} V_r \le C_1(\varphi_0, K, n)\theta(r),$$

and so

$$\mathcal{H}^{n-1}(V_r) \le C_2(\varphi_0, K, n)\theta(r)^{n-1}. \tag{4.7}$$

If we now define

$$\Lambda = \{V_r : t < r < s\},$$

then we have the following lower bound for $M_S\Lambda$. Take an arbitrary test function ρ of $M_S\Lambda$, and fix r. Then, by Hölder's inequality,

$$1 \le \int_{V_r} \rho(x)\, d\mathcal{H}^{n-1}(x) \le \mathcal{H}^{n-1}(V_r)^{\frac{1}{n}} \Big(\int_{V_r} \rho(x)^{\frac{n}{n-1}}\, d\mathcal{H}^{n-1}(x) \Big)^{\frac{n-1}{n}}. \tag{4.8}$$

Estimates (4.7) and (4.8) yield

$$C_3(\varphi_0, K, n)\theta(r)^{-1} \le \int_{V_r} \rho(x)^{\frac{n}{n-1}}\, d\mathcal{H}^{n-1}(x),$$

and so, by integrating over r, we have

$$M_S\Lambda \ge C(\varphi_0, K, n) \int_t^s \frac{dr}{\theta(r)}. \tag{4.9}$$

We will next give an upper bound for $M_S g(\Lambda)$.

For $t < r < s$, choose

$$s_r = \min_{z\in\partial K_r} \frac{|g^{-1}(z) - x_r|}{2}.$$

Then, if we denote

$$W_r = B(x_r, s_r) \cap S^{n-1}(x_y, r),$$

we have

$$\mathrm{d}(\partial g^{-1}(K_r), W_r) \geq s_r,$$

and [25], Theorem 2.4 implies that the restriction

$$g_r : W_r \to g(W_r)$$

of g is an η-quasisymmetric embedding, where the homeomorphism η only depends on K and n. Furthermore, [25], Theorem 5.2 implies that

$$\mathcal{H}^{n-1}(g(V_r)) \geq \mathcal{H}^{n-1}(g(W_r)) \geq C\,\mathrm{d}(g(x_r), g(\partial W_r))^{n-1}, \tag{4.10}$$

where $C > 0$ only depends on K and n.

We want to show that

$$\mathrm{d}(g(x_r), g(\partial W_r)) \geq C|g(x_r) - z_y|, \tag{4.11}$$

where $C > 0$ only depends on φ_0, K and n. First, notice that g^{-1} restricted to the set K_r is an η_2-quasisymmetric map since

$$\mathrm{d}(\partial K, \partial G) \geq C\,\mathrm{diam}(K_r),$$

where $C > 0$ only depends on φ_0 and n. Here the homeomorphism η_2 only depends on φ_0, K and n. Choose points $v \in \partial W_r$ and $w \in g^{-1}(\partial K_r)$ so that

$$\mathrm{d}(g(x_r), g(\partial W_r)) = |g(x_r) - g(v)|$$

and

$$|x_r - w| = 2s_r.$$

Then, by quasisymmetry,

$$\frac{1}{2} = \frac{|x_r - v|}{|x_r - w|} \leq \eta_2^{-1}\Big(\frac{|g(x_r) - g(v)|}{|g(x_r) - g(w)|}\Big),$$

i.e.

$$|g(x_r) - g(v)| \geq \eta_2(2^{-1})|g(x_r) - g(w)| \geq C|g(x_r) - z_y|,$$

where $C > 0$ only depends on φ_0, K and n. This proves (4.11).

Recall that we have the inequality

$$M_S\Lambda \leq K M_S g(\Lambda). \tag{4.12}$$

Also, notice that, by our definition of the sets V_r,

$$g(V_r) \subset B(z_y, r_0) \setminus B(z_y, 2^{-1}|f(x_t) - z_y|). \tag{4.13}$$

By (4.10), (4.11) and (4.13), there exists a constant $C > 0$ only depending on φ_0, K and n, so that the function $\rho : \mathbb{R}^n \to [0, \infty]$,

$$\rho(x) = \begin{cases} C|x - z_y|^{1-n}, & x \in B(z_y, r_0) \setminus \overline{B}(z_y, |f(x_t) - z_y|/2) \\ 0 & \text{elsewhere,} \end{cases}$$

is a test function for $M_S g(\Lambda)$. Thus we have, by integrating $\rho^{\frac{n}{n-1}}$ in polar coordinates,

$$M_S g(\Lambda) \le C \log \frac{1}{|f(x_t) - z_y|}. \tag{4.14}$$

By combining (4.9), (4.12) and (4.14), we have (4.4). The proof is complete.
□

5 Proof of Theorem 3.1

Denote

$$I_y : [0,1) \to B(0,1), \quad I_y(t) = tx_y.$$

Suppose that (3.3) does not hold true. Then there exist $m > 0$ and a sequence $(a_j) = (I_y(t_j))$, where t_j increases to 1 as $j \to \infty$, so that

$$|f(a_j) - z_y| \ge m \quad \text{for all } j \in \mathbb{N}. \tag{5.1}$$

We fix a constant s as in Proposition 4.1, and a large enough integer j_s depending on s. Moreover, we choose, for each $j \ge j_s \ge 1$, a point $b_j = x_{\alpha(j)} = \tilde{\gamma}_y(t_{\alpha(j)})$ for some $0 < \alpha(j) < s$, where x_r and t_r are as in (4.1) and (4.2), respectively. We will make precise choices of the points b_j later. For now, we only require that

$$|b_j - x_y| < |a_j - x_y|. \tag{5.2}$$

Define

$$\beta_0 : [0,1/2] \to B(0,1), \quad \beta_0(t) = b_j + 2|b_j|^{-1} t b_j(|a_j| - |b_j|),$$

and, similarly, let $I_0 : [1/2,1] \to B(0,1)$ be a line segment connecting $b_j|a_j|/|b_j|$ and a_j. Furthermore, define $\beta : [0,1] \to B(0,1)$ by setting

$$\beta(t) = \begin{cases} \beta_0(t), & 0 \le t \le 1/2 \\ I_0(t), & 1/2 \le t \le 1. \end{cases}$$

Then β is a path satisfying the assumptions of Lemma 3.2.

We denote $r_j = |f(b_j) - z_y|$ and $s_j = |f(a_j) - z_y|$. By passing to a subsequence, if necessary, we may assume that $s_j > 10r_j$ for all $j \in \mathbb{N}$. We find that, for all $r \in (r_j, s_j)$, $f(b_j) \in B(z_y, r)$ but $f(a_j) \in \mathbb{R}^n \setminus B(z_y, r)$. For each $r \in (r_j, s_j)$, choose t^r and a point

$$f(\beta(t^r)) = p_r \in f(|\beta|) \cap S^{n-1}(z_y, r)$$

as in Lemma 3.2. Then, Lemma 3.2 provides a point k_r and a spherical cap C_r so that, if Γ_r is the family of all paths joining p_r and k_r in $\overline{C}_r$, then each lift γ' of $\gamma \in \Gamma_r$ starting at $\beta(t^r)$ has the property

$$\overline{|\gamma'|} \cap S^{n-1}(0,1) \ne \emptyset.$$

By [23], Theorem 10.2, there exists a constant $C > 0$, only depending on n, so that

$$M_n^S \Gamma_r \geq \frac{C}{r}$$

for all $r \in (r_j, s_j)$. Hence, if we denote

$$\Gamma = \{\gamma : \gamma \in \Gamma_r,\, r \in (r_j, s_j)\},$$

then integration over r yields

$$M\Gamma \geq C \log \frac{s_j}{r_j}. \tag{5.3}$$

Moreover, denote

$$\Gamma' = \{\gamma' : \gamma' \text{ is a lift of some } \gamma \in \Gamma_r \subset \Gamma \text{ starting at } \beta(t^r)\}.$$

In order to estimate $M\Gamma'$, we use the subadditivity of the conformal modulus. Denote by N_j the smallest positive integer larger than $\log_2 \frac{|x_y - a_j|}{\theta(\alpha(j))}$ (recall the definition of the function θ, (4.3)). Then

$$M\Gamma' \leq \sum_{i=0}^{N_j+1} M\Gamma_j, \tag{5.4}$$

where Γ_0 is the family of all paths joining I_0 and $S^{n-1}(0,1)$, and Γ_i, $i = 1, \ldots N_j + 1$ is the family of all paths joining

$$|\beta_i| = \{tb_j/|b_j| : t \in [1 - 2^i\theta(\alpha(j)), 1 - 2^{i-1}\theta(\alpha(j))]\}$$

and $S^{n-1}(0,1)$. By (5.2), and since $\operatorname{diam} |\beta_i| \leq C\, \mathrm{d}(|\beta_i|, S^{n-1}(0,1))$ for each $i = 1, \ldots, N_j + 1$, we have

$$M\Gamma_i \leq C \tag{5.5}$$

for each $i = 0, 1, \ldots, N_j + 1$, where $C > 0$ only depends on n. Hence, combining (5.4) and (5.5) gives

$$M\Gamma' \leq C(N_j + 2) \leq C\Big(\log \frac{|x_y - a_j|}{\theta(\alpha(j))} + 2\Big). \tag{5.6}$$

By combining (5.3), (5.6) and Poletsky's inequality, we have

$$\log s_j = \log \frac{s_j}{r_j} - \log \frac{1}{r_j} \leq C_1\Big(\log \frac{|x_y - a_j|}{\theta(\alpha(j))} + 2\Big) - \log \frac{1}{r_j} \tag{5.7}$$

where $C_1 > 0$ only depends on φ_0, K and n.

We will use the following auxiliary result. Its proof is postponed until Section 6.

Lemma 5.1. *Suppose that $\theta : (0,1] \to (0,1]$ is a function so that if $\theta(t_i) \to 0$, then $t_i \to 0$. Furthermore, fix $\epsilon > 0$. Then either*

1. *there exists a constant $\theta > 0$ and a decreasing sequence (t_i), $t_i \to 0$, so that*
$$\theta(t_i) \geq \theta t_i$$
for all $i \in \mathbb{N}$, or

2. *there exists a constant $M = M(\epsilon) > 0$ and a decreasing sequence (T_i), $T_i \to 0$, so that*
$$\log \frac{1}{\theta(T_i)} \leq \epsilon \int_{T_i}^{1} \frac{dr}{\theta(r)} + M \tag{5.8}$$
for all $i \in \mathbb{N}$.

Now, the function θ defined in (4.3) satisfies the assumptions of Lemma 5.1. We split the rest of the proof into two cases, depending on whether 1. or 2. are satisfied when Lemma 5.1 is applied to this θ. In particular, we will make precise choices of the points b_j.

We first assume that 1. is satisfied with constant $\theta > 0$. Notice that we may assume that θ, although fixed, can be chosen to be as small as we wish. Suppose that
$$|x_y - a_j| = m_j < s.$$
Denote by $\epsilon_j m_j$ the supremum of all radii $r \leq m_j$ for which
$$\theta(r) \geq \theta r. \tag{5.9}$$
By our assumption, $\epsilon_j m_j > 0$ for all $j \in \mathbb{N}$. Hence, for each j, there exists a point
$$b_j = \tilde{\gamma}_y(t_{\alpha(j)}),$$
so that $\alpha(j) \in [\epsilon_j m_j/2, \epsilon_j m_j]$, and so that (5.9) is satisfied. Then, if $\epsilon_j > \epsilon > 0$ for all $j \in \mathbb{N}$, (5.7) implies
$$\log s_j \leq C_1 \Big(\log \frac{2m_j}{\theta \epsilon m_j} + 2 \Big) - \log \frac{1}{r_j} \leq C(\theta, \epsilon) - \log \frac{1}{r_j} \to -\infty$$
as $j \to \infty$. This contradicts (5.1). Hence we may, by passing to a subsequence if necessary, assume that (ϵ_j) is decreasing and $\epsilon_j \to 0$.

Now we apply Proposition 4.1 with $t = \epsilon_j m_j$. We have, by (4.4), and since $\theta(r) < \theta r$ for all $r \in [\epsilon_j m_j, m_j]$,
$$\log \frac{1}{r_j} \geq C_0 \int_{\epsilon_j m_j}^{s} \frac{dr}{\theta(r)} \geq \frac{C_0 \log \frac{1}{\epsilon_j}}{\theta}. \tag{5.10}$$

Hence, (5.7) and (5.10) yield

$$\log s_j \le C_1\Big(\log\frac{2}{\theta}+2\Big)+(C_1-\theta^{-1}C_0)\log\frac{1}{\epsilon_j}. \tag{5.11}$$

If we now choose $\theta < C_0/C_1$, then the right hand side of (5.11) tends to $-\infty$ as $j \to \infty$. Then, however, (5.11) contradicts (5.1). This proves the case that 1. is satisfied for the function θ in Lemma 5.1.

Now we assume that 2. is satisfied in Lemma 5.1. We choose, for each $j \in \mathbb{N}$, a point $b_j = x_{T_i} = \tilde{\gamma}_y(t_{T_i})$ for some T_i as in Lemma 5.1, so that (5.2) is satisfied. Again, x_r and t_r are as in (4.1) and (4.2), respectively. But now, by Proposition 4.1 and (5.2),

$$\begin{aligned} C_1\log\frac{1}{\theta(T_i)}-\log\frac{1}{r_j} &\le C_1\log\frac{1}{\theta(T_i)}-C_0\int_{T_i}^{s}\frac{dr}{\theta(r)} \\ &\le M(C_0/C_1)C_1+C_0\int_s^1\frac{dr}{\theta(r)}, \end{aligned} \tag{5.12}$$

where $M(\cdot)$ is as in (5.8). Thus, combining (5.12) and (5.7) yields

$$\log s_j \le C_1(\log|x_y-a_j|+2+M(C_0/C_1))+C_0\int_s^1\frac{dr}{\theta(r)} \to -\infty$$

as $j \to \infty$. This contradicts (5.1). Hence, Theorem 3.1 is proved, except for the proof of Lemma 5.1.

6 Proof of Lemma 5.1

We assume that 1. does not hold true. Without loss of generality, we may assume that $\theta(t) < t$ for all $0 < t < 1$. First let $h : (0,1] \to (0,2]$ be a convex, strictly increasing absolutely continuous function for which 1. does not hold true. We have, for $r \in (0,1)$,

$$\int_r^1\frac{\epsilon\,dt}{h(t)}-\log\frac{1}{h(r)}=\int_r^1\frac{\epsilon-h'(t)}{h(t)}\,dt+\log h(1). \tag{6.1}$$

By the properties of h, there exists a constant $\kappa > 0$ so that $h'(t) \le \epsilon$ for almost every $t < \kappa$. Hence, when $r < \kappa$,

$$\int_r^1\frac{\epsilon-h'(t)}{h(t)}\,dt \ge -\int_\kappa^1\frac{h'(t)}{h(t)}\,dt=-\log\frac{h(1)}{h(\kappa)}. \tag{6.2}$$

By combining (6.1) and (6.2), we have

$$\int_r^1\frac{\epsilon\,dt}{h(t)} \ge \log\frac{1}{h(r)}-\log\frac{1}{h(\kappa)}. \tag{6.3}$$

Now consider the original function θ. We will construct a function h as above, so that

$$h(t) \geq \theta(t) \quad \text{for all } t \in (0,1], \tag{6.4}$$

and so that, for a decreasing sequence (T_i), $T_i \to 0$,

$$h(T_i) \leq 3\theta(T_i) \quad \text{for all } i \in \mathbb{N}. \tag{6.5}$$

Denote $t_1 = 1$, and for $j \geq 2$,

$$t_j = \sup\{t : \theta(s) < j^{-1}s \text{ for all } s < t\}. \tag{6.6}$$

Then (t_j) is a nonincreasing sequence, and $t_j > 0$ for all $j \in \mathbb{N}$. Moreover, $t_j \to 0$. Denote

$$A_1 = 2, \quad A_j = (j-1)^{-1}t_j, \text{ for } j \geq 2,$$

and define $h_1 : \mathbb{R} \to \mathbb{R}$,

$$h_1(t) = A_2 + \frac{A_1 - A_2}{t_1 - t_2}(t - t_2),$$

and $h(t) = h_1(t)$ for $t \in [t_2, t_1]$. Then $h(t) \geq t > \theta(t)$ for all $t \in [t_2, t_1]$, and, since

$$\limsup_{t \to t_2} \theta(t) \geq 2^{-1}t_2 = 2^{-1}h(t_2),$$

there exists a point $T_1 \in [t_2, t_1]$ so that

$$h(T_1) \leq 3\theta(T_1).$$

Now set $j_1 = 1$, $j_2 = 2$ and, for $i \geq 2$,

$$j_{i+1} = \min\left\{ j \in \mathbb{N} : j > j_i,\ \frac{A_{j_i} - A_j}{t_{j_i} - t_j} \leq \frac{A_{j_{i-1}} - A_{j_i}}{t_{j_{i-1}} - t_{j_i}} \right\}. \tag{6.7}$$

Such j_{i+1} exists for all i, since

$$\frac{A_{j_i} - A_j}{t_{j_i} - t_j} \to \frac{1}{j_i - 1} \quad \text{as } j \to \infty,$$

while

$$\frac{A_{j_{i-1}} - A_{j_i}}{t_{j_{i-1}} - t_{j_i}} > \frac{1}{j_i - 1}.$$

Denote, for $i \in \mathbb{N}$,

$$K_i = \frac{A_{j_i} - A_{j_{i+1}}}{t_{j_i} - t_{j_{i+1}}},$$

and define $h_i : \mathbb{R} \to \mathbb{R}$,

$$h_i(t) = A_{j_{i+1}} + K_i(t - t_{j_{i+1}}), \tag{6.8}$$

and $h : (0,1] \to (0,2]$,

$$h(t) = h_i(t), \quad t \in [t_{j_{i+1}}, t_{j_i}],\ i \in \mathbb{N}.$$

Then h is continuous, increasing and piecewise linear in $[a,1]$ for all $a > 0$. Moreover, h is convex by (6.7). We next show that (6.4) and (6.5) hold true.

First, by (6.6),

$$\limsup_{t \to t_{j_i+1}} \theta(t) \geq j_{i+1}^{-1} t_{j_{i+1}} = \frac{j_{i+1}-1}{j_{i+1}} (j_{i+1}-1)^{-1} t_{j_{i+1}} \geq 2^{-1} h(t_{j_{i+1}})$$

for each $i \geq 2$. Hence, by (6.8), there exists a point T_i near $t_{j_{i+1}}$ so that (6.5) holds true.

Assume then that $h(t) < \theta(t)$ for some $t \in [t_{j_{k+1}}, t_{j_k}]$, and choose $N \in \mathbb{N}$ so that

$$\frac{t}{N} \leq \theta(t) \leq \frac{t}{N-1}.$$

Then $N < j_{k+1}$ and $t_N \leq t$. Since $h(t_N)/t_N \leq h(t)/t$, we have

$$h(t_N) \leq \frac{h(t)t_N}{t} < \frac{\theta(t)t_N}{t} \leq \frac{t_N}{N-1}.$$

But then

$$\frac{A_{j_k} - A_N}{t_{j_k} - t_N} < K_{k-1}.$$

This contradicts the minimality of j_{k+1} in (6.7). Hence (6.4) holds true.

By using (6.3), (6.4) and (6.5), we finally have, for $i \in \mathbb{N}$,

$$\begin{aligned} \log \frac{1}{\theta(T_i)} &\leq \log 3 + \log \frac{1}{h(T_i)} \leq \log \frac{3}{h(\kappa)} + \epsilon \int_{T_i}^1 \frac{dt}{h(t)} \\ &\leq \log \frac{3}{h(\kappa)} + \epsilon \int_{T_i}^1 \frac{dt}{\theta(t)}. \end{aligned}$$

The proof is complete.

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University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35
FIN-40014 University of Jyväskylä
Finland
e-mail: kirajala@maths.jyu.fi