NONSYMMETRIC CONICAL UPPER DENSITY THEOREM FOR MEASURES WITH FINITE LOWER DENSITY

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ABSTRACT. We study how measures with finite lower density are distributed around $(n-m)$-planes in small balls in $\mathbb{R}^n$. Our result may be applied to a large collection of Hausdorff and packing type measures.

1. Introduction

Conical density theorems are used in geometric measure theory to derive geometric information from given metric information. Classically, they deal with the distribution of the s-dimensional Hausdorff measure, $\mathcal{H}^s$. The main applications of conical density theorems deal with rectifiability, see [9], but they have been applied also elsewhere in geometric measure theory, for example, in the study of porous sets, see [8] and [6]. The upper conical density results, going back to Besicovitch [1] and Marstrand [7], show that under certain conditions there is a lot of A near k-dimensional linear subspaces of $\mathbb{R}^n$. Besides Besicovitch and Marstrand, the theory of upper conical density theorems has been developed by Morse and Randolph [10], Federer [5], and Salli [11]. A sample result is the following (Salli [11, Theorem 3.1]): If $V \in G(n, n-m)$, $G(n, n-m)$ denoting the space of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$, $0 < \alpha < 1$, $A \subset \mathbb{R}^n$, $0 < \mathcal{H}^s(A) < \infty$, and $s > m$, then

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^s\big(A \cap X(x, r, V, \alpha)\big)}{r^s} \geq c \tag{1.1}$$

for $\mathcal{H}^s$-almost all $x \in A$, where $c > 0$ is a constant depending only on n, m, s, and α. Here

$$\begin{aligned} X(x, V, \alpha) &= \{y \in \mathbb{R}^n : \operatorname{dist}(y - x, V) < \alpha |y - x|\}, \\ X(x, r, V, \alpha) &= B(x, r) \cap X(x, V, \alpha). \end{aligned}$$

Here $B(x, r) \subset \mathbb{R}^n$ is the closed ball with centre at x and radius $r > 0$. Generalisations of (1.1) for measures other than $\mathcal{H}^s$ are proved in [12, §3].

In 1988, Mattila [8] improved the above result by showing that it is not necessary to fix V in (1.1). More precisely, he proved that if $A \subset \mathbb{R}^n$, $0 < \mathcal{H}^s(A) < \infty$,

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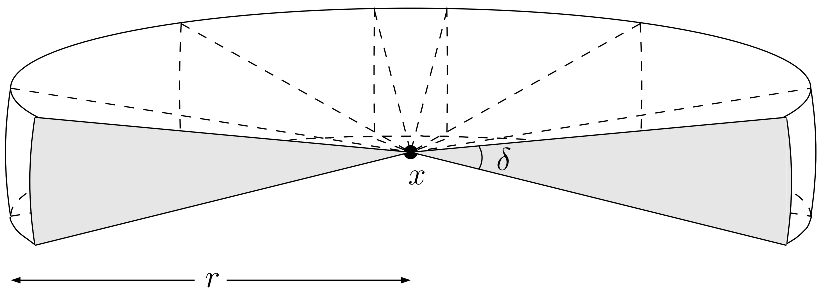

FIGURE 1. The set $X(x,r,V,\alpha) \setminus H(x,\theta,\eta)$ when $n = 3$, $m = 1$, and $\alpha = \sin(\delta/2)$.

$s > m$, and $0 < \alpha < 1$, then for a constant $c > 0$ depending only on n, m, s, and α,

$$\limsup_{r\downarrow 0} \inf_{C} \frac{\mathcal{H}^s\big(A \cap B(x,r) \cap C_x\big)}{r^s} \geq c, \tag{1.2}$$

where the infimum is taken over all Borel sets $C \subset G(n, n-m)$ for which $\gamma_{n,n-m}(C) > \alpha$, where $\gamma_{n,n-m}$ denotes the unique Borel regular probability measure on $G(n, n-m)$ which is invariant under the orthogonal group $O(n)$, and $C_x = \{x\} + \bigcup C$. As an immediate corollary to Mattila's result, under the same assumptions as in (1.1), we have

$$\limsup_{r\downarrow 0} \inf_{V \in G(n,n-m)} \frac{\mathcal{H}^s\big(A \cap X(x,r,V,\alpha)\big)}{r^s} \geq c \tag{1.3}$$

for $\mathcal{H}^s$-almost all $x \in A$, where $c > 0$ depends only on n, m, s, and α, see [9, §11]. Although the constant in (1.1) is much better than that of (1.3), still (1.3) is a significant improvement of (1.1): It shows that in the sense of the measure $\mathcal{H}^s$, there are arbitrary small scales such that almost all points of A are quite effectively surrounded by A.

The proof of (1.2) is nontrivial and it is based on Fubini-type arguments and an elegant use of the so-called sliced measures. Since the geometry of the cones $X(x,V,\alpha)$ is simpler than that of the cones C_x in (1.2), it is natural to ask for an elementary proof of (1.3). In [6], such a proof was given and the technique used there does not require the cones to be symmetric. Namely, given $s > m$, $0 < \alpha < 1$, $0 < \eta \leq 1$, and $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$, it was shown that there is a constant $c > 0$ depending only on n, m, s, α, and η so that

$$\limsup_{r\downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mathcal{H}^s\big(A \cap X(x,r,V,\alpha) \setminus H(x,\theta,\eta)\big)}{r^s} \geq c \tag{1.4}$$

for $\mathcal{H}^s$-almost all $x \in A$, see [6, Theorem 2.5]. Here $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and

$$H(x, \theta, \eta) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \eta |y - x|\},$$

see Figure 1.

In Theorem 2.3, we generalise the result (1.4) for measures with finite lower density. The main application of this generalisation, Corollary 2.4, is a conical density theorem for the s-dimensional packing measure, $\mathcal{P}^s$. Our result may also be applied to a large collection of Hausdorff and packing type measures which are determined using gauges other than power functions. As far as we know, there have been previously no conical density theorems of a similar type for other measures than the Hausdorff measure.

We finish the introduction by setting down some notation. If μ is a measure on $\mathbb{R}^n$, $r_0 > 0$, $h\colon (0, r_0) \to (0, \infty)$, and $x \in \mathbb{R}^n$, the upper and lower μ-densities at x with respect to h are given by

$$\underline{D}_h(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu\big(B(x, r)\big)}{h(r)},$$
$$\overline{D}_h(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu\big(B(x, r)\big)}{h(r)}.$$

If $V \in G(n, m)$, $x \in \mathbb{R}^n$, and $\lambda > 0$, we define

$$V_x(\lambda) = \{y \in \mathbb{R}^n : \operatorname{dist}(y - x, V) \le \lambda\}.$$

Open balls are denoted by $U(x, r)$. If μ is a measure on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$, we use the notation $\mu|_A$ for the restriction measure, that is $\mu|_A(B) = \mu(A \cap B)$ for $B \subset \mathbb{R}^n$.

2. The results

To verify our main result, Theorem 2.3, we need the following two geometrical lemmas. The first one is due to Erdős and Füredi [3], see also [6, Lemma 2.1].

Lemma 2.1. *For given $0 < \beta < \pi$, there is $q = q(n, \beta) \in \mathbb{N}$ such that in any set of q points in $\mathbb{R}^n$, there are always three points which determine an angle between β and π.*

For $0 < \eta \le 1$ we define $t(\eta) = (\eta^2 + 4)^{1/2}/\eta$ and $\gamma(\eta) = 1/t(\eta)$. Notice that $t(\eta) \ge 2$ and $\eta/5^{1/2} \le \gamma(\eta) \le \eta/2$. An easy calculation yields the following, see [6, Lemma 2.3].

Lemma 2.2. *Suppose $y \in \mathbb{R}^n$, $\theta \in S^{n-1}$, $0 < \eta \le 1$, $t \ge t(\eta)$, and $\gamma = \gamma(\eta)$. If $z \in \mathbb{R}^n \setminus \big(B(y, tr) \cup H(y, \theta, \gamma)\big)$, then $B(z, r) \cap H(y, \theta, \eta) = \emptyset$.*

We have now the necessary tools to prove our main result concerning the distribution of measures with finite lower density.

Theorem 2.3. *Let $0 \le m < s < n$ and $0 < \alpha, \eta \le 1$. Then there is a constant $c = c(n, m, s, \eta, \alpha) > 0$ such that if $h\colon (0, r_0) \to (0, \infty)$ is a function, and μ is a measure on $\mathbb{R}^n$ with $\underline{D}_h(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$, and for all $0 < r < r_0$ and $0 < \varepsilon < 1$ one has*

$$\varepsilon^s h(r) \ge h(\varepsilon r), \tag{2.1}$$

then

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \eta)\big)}{h(r)} \ge c\overline{D}_h(\mu, x)$$

for μ almost all $x \in \mathbb{R}^n$.

Proof. Let us first sketch the main idea of the proof: Suppose our theorem is false. Then there is a closed exceptional set $F \subset \mathbb{R}^n$ with positive μ-measure so that for all small scales r and for all points x of F, there are θ and V so that $\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \eta)\big)$ is small compared to $h(r)$. A simple covering argument on $G(n, n-m)$ implies that at each small ball $B = B(z, r)$ centred at F, we may fix $V \in G(n, n-m)$ so that the measure $\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \eta)$ is small for some θ for a set of points $x \in F \cap B$ whose measure is comparable to $h(r)$. This implies that for $\lambda > 0$, we may find $y \in F \cap B$ so that the measure in $V_y(\lambda r)$ is comparable to $\lambda^m h(r)$. But our antithesis implies that if λ is small, then this measure is essentially contained in at most $q - 1$ balls of radius λr, the number q being determined by Lemma 2.1. Thus, there is a ball $B(w, \lambda r) \subset B$ so that $\mu\big(F \cap B(w, \lambda r)\big) \approx \lambda^m h(r)$. Iterating this, we find a sequence of balls $B_1 \supset B_2 \supset \ldots$ so that $\operatorname{diam}(B_k) \approx \lambda^k$ and $\mu(F \cap B_k) \approx \lambda^{mk}$. By (2.1), this implies $\underline{D}_h(\mu, x) = \infty$ for the point x being determined by $\{x\} = \bigcap_k B_k$. This gives a contradiction since we may choose F in the beginning so that the lower density $\underline{D}_h(\mu, x)$ is finite for all points of F.

We shall now verify in detail the steps described heuristically above. Since all the sets used in the formulation are Borel sets and there is a Borel measure ν which equals μ for Borel sets, we may assume that μ is a Borel measure. We may also assume that μ is finite since μ-almost all of $\mathbb{R}^n$ is contained in a countable union of open balls, each of finite μ-measure. We shall prove that for any finite collection, $\{V^1, \ldots, V^l\} \subset G(n, n-m)$,

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ i \in \{1, \ldots, l\}}} \frac{\mu\big(X(x, r, V^i, \alpha) \setminus H(x, \theta, \eta)\big)}{h(r)} \ge c(n, m, s, \eta, \alpha, l)\overline{D}_h(\mu, x)$$

for μ-almost all $x \in \mathbb{R}^n$ from which the claim follows by the compactness of $G(n, n-m)$, see [6, proof of Theorem 2.5].

Set $t = \max\{t(\eta), 1 + 3/\alpha\}$, $\gamma = \gamma(\eta)$, and take $\beta < \pi$ so that the opening angle of $H(x, \theta, \gamma)$ is smaller than β. Let $q = q(n, \beta)$ be as in Lemma 2.1. Moreover, define $c_1 = 2^m m^{m/2}$, $c_2 = 2^{n-m} n^{n/2}$, $d = \big(3c_1 l(q-1)\big)^{-1}$, $\lambda = \min\{2^{-1} t^{s/(m-s)} d^{1/(s-m)}, 3^{-1} t^{-1}\}$, and $c = c(n, m, s, \eta, \alpha, l) = \lambda^n / (6 c_1 c_2 \ell 3^s)$.

These definitions together with (2.1) guarantee the following three facts: If $0 < r < r_0$, $k \in \mathbb{N}$, $V \in G(n, n-m)$, $z \in \mathbb{R}^n$, and $x, y \in V_z(\lambda r)$ with $|x-y| \geq t\lambda r$, then

$$B(y, \lambda r) \subset X(x, V, \alpha), \tag{2.2}$$

$$h\big(3(t\lambda)^k r\big) \leq 3^s d^k \lambda^{km} h(r), \tag{2.3}$$

$$d\lambda^{m-s} t^{-s} \geq 2^{s-m}. \tag{2.4}$$

Let $0 < M < \infty$ and define

$$A = \{x \in \mathbb{R}^n : \overline{D}_h(\mu, x) > M \text{ and } \underline{D}_h(\mu, x) < \infty\}.$$

It suffices to show that

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ i \in \{1,\dots,l\}}} \frac{\mu\big(X(x, r, V^i, \alpha) \setminus H(x, \theta, \eta)\big)}{h(r)} \geq cM$$

for almost all $x \in A$. Suppose on the contrary that there exists a set $F \subset A$ with $\mu(F) > 0$ and $0 < r_1 < r_0$ such that for every $x \in F$ and $0 < r < r_1$, there are $i \in \{1, \dots, l\}$ and $\theta \in S^{n-1}$ with

$$\mu\big(X(x, r, V^i, \alpha) \setminus H(x, \theta, \eta)\big) < cMh(r). \tag{2.5}$$

Going into a subset, if necessary, we may assume that F is closed.

Choose $x \in F$ such that $\lim_{r \downarrow 0} \mu\big(F \cap B(x, r)\big) / \mu\big(B(x, r)\big) = 1$ and $0 < r < r_1/3$ such that $\mu\big(F \cap B(x, r)\big) \geq Mh(r)$. To simplify the notation, assume that $r = 1$ and $h(1) = 1$. Let $B_0 = B(x, 1)$. Suppose that $B_k = B\big(x_k, (t\lambda)^k\big)$ has been defined so that $\mu(F \cap B_k) \geq Md^k \lambda^{mk}$. Take $x_{k+1} \in F \cap B_k$ which maximises the function $y \mapsto \mu\big(F \cap B(y, (t\lambda)^{k+1})\big)$ in $F \cap B_k$. There is such a point because $F \cap B_k$ is compact and the function $y \mapsto \mu\big(F \cap B(y, (t\lambda)^{k+1})\big)$ is upper semicontinuous on $F \cap B_k$. Define $B_{k+1} = B\big(x_{k+1}, (t\lambda)^{k+1}\big)$. Our aim is to estimate the measure $\mu(F \cap B_{k+1})$ from below. Define, for $i \in \{1, \dots, l\}$,

$$\begin{aligned} \widetilde{C}_i = \big\{x \in F \cap B_k : \mu\big(X(x, 3(t\lambda)^k, V^i, \alpha) \setminus H(x, \theta, \eta)\big) \\ < cMh\big(3(t\lambda)^k\big) \text{ for some } \theta \in S^{n-1}\big\}. \end{aligned}$$

Fix $i \in \{1, \dots, l\}$ for which $\mu(\widetilde{C}_i) \geq \mu(F \cap B_k)/l \geq Md^k \lambda^{mk}/l$ and take a compact $C_i \subset \widetilde{C}_i$ with $\mu(C_i) > \mu(\widetilde{C}_i)/2$. We may cover the set ${V^i}^\perp \cap B_k$ with $c_1 \lambda^{-m}$ balls of radius $t^k \lambda^{k+1}$ and hence there exists $y \in {V^i}^\perp \cap B_k$ for which

$$\mu\big(C_i \cap V^i_y(t^k \lambda^{k+1})\big) \geq 2^{-1} c_1^{-1} \ell^{-1} Md^k \lambda^{m(k+1)}. \tag{2.6}$$

Next we shall choose q points as follows: Choose a point $y_1 \in C_i \cap V^i_y(t^k \lambda^{k+1})$ such that the ball $B(y_1, t^k \lambda^{k+1})$ has largest $\mu|_F$ measure among the balls centred at $C_i \cap V^i_y(t^k \lambda^{k+1})$ with radius $t^k \lambda^{k+1}$. If $y_1, \dots, y_p$, $p \in \{1, \dots, q-1\}$, have already been chosen, we choose $y_{p+1} \in C_i \cap V^i_y(t^k \lambda^{k+1}) \setminus \bigcup_{j=1}^p U(y_j, (t\lambda)^{k+1})$ so that the

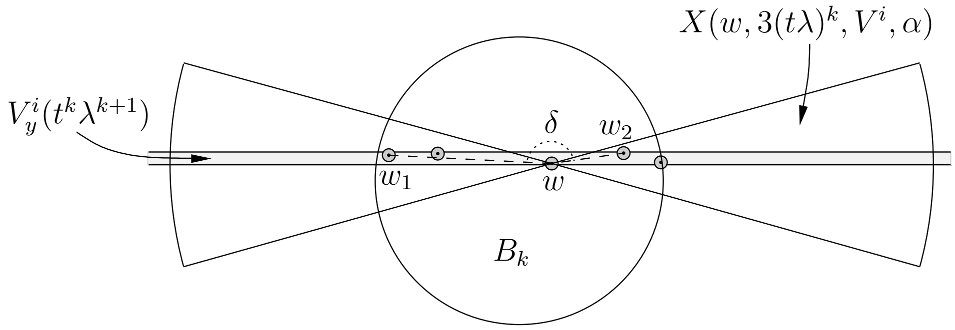

FIGURE 2. Illustration for the proof of Theorem 2.3. The angle δ formed by the points w_1, w, and w_2 is greater than β.

ball $B(y_{p+1}, t^k\lambda^{k+1})$ has maximal $\mu|_F$ measure among the balls centred at $C_i \cap V_y^i(t^k\lambda^{k+1}) \setminus \bigcup_{j=1}^p U\big(y_j,(t\lambda)^{k+1}\big)$ with radius $t^k\lambda^{k+1}$. Since the set $V_y^i(t^k\lambda^{k+1}) \cap B_k$ may be covered by $c_2\lambda^{m-n}$ balls of radius $t^k\lambda^{k+1}$, using (2.6), we estimate

$$\begin{aligned}\mu\big(F \cap B(y_q, t^k\lambda^{k+1})\big) \geq c_2^{-1}\lambda^{n-m}\bigg(&2^{-1}c_1^{-1}\ell^{-1}Md^k\lambda^{m(k+1)} \\ &- \sum_{j=1}^{q-1}\mu\big(F \cap B(y_j,(t\lambda)^{k+1})\big)\bigg).\end{aligned} \tag{2.7}$$

According to Lemma 2.1, we may choose three points w, w_1, w_2 from the set $\{y_1, \ldots, y_q\}$ such that for each $\theta \in S^{n-1}$ there is $j \in \{1,2\}$ for which $w_j \in \mathbb{R}^n \setminus \big(B(w,(t\lambda)^{k+1}) \cup H(w,\theta,\gamma)\big)$. We obtain, using Lemma 2.2, that for each $\theta \in S^{n-1}$ there is $j \in \{1,2\}$ such that

$$B(w_j, t^k\lambda^{k+1}) \subset B\big(w, 3(t\lambda)^k\big) \setminus H(w,\theta,\eta)$$

and hence (2.2) implies that also

$$B(w_j, t^k\lambda^{k+1}) \subset X\big(w, 3(t\lambda)^k, V^i, \alpha\big) \setminus H(w,\theta,\eta), \tag{2.8}$$

see Figure 2. Since $w \in C_i$ there is $\theta \in S^{n-1}$ so that $\mu\big(X(w,3(t\lambda)^k,V^i,\alpha) \setminus H(w,\theta,\eta)\big) < cMh\big(3(t\lambda)^k\big)$. Choosing $j \in \{1,2\}$ for which (2.8) holds, we get

$$\begin{aligned}\mu\big(F \cap B(y_q, t^k\lambda^{k+1})\big) &\leq \mu\big(F \cap B(w_j, t^k\lambda^{k+1})\big) \\ &\leq \mu\big(X(w,3(t\lambda)^k,V^i,\alpha) \setminus H(w,\theta,\eta)\big) \\ &< cMh\big(3(t\lambda)^k\big)\end{aligned} \tag{2.9}$$

Consequently, using (2.7), (2.9), (2.3), and the definitions of c and d, we get

$$\begin{aligned}\sum_{j=1}^{q-1}\mu\big(F\cap B(y_j,(t\lambda)^{k+1})\big) &> 2^{-1}c_1^{-1}\ell^{-1}Md^k\lambda^{m(k+1)} - c_2cMh\big(3(t\lambda)^k\big)\lambda^{m-n}\\ &\geq 2^{-1}c_1^{-1}\ell^{-1}Md^k\lambda^{m(k+1)} - c_2cM3^sd^k\lambda^{m(k+1)}\lambda^{-n}\\ &= 3^{-1}c_1^{-1}\ell^{-1}Md^k\lambda^{m(k+1)}\\ &= (q-1)Md^{k+1}\lambda^{m(k+1)}.\end{aligned}$$

It follows that there is $y \in \{y_1,\dots,y_{q-1}\}$ for which $\mu\big(F\cap B(y,(t\lambda)^{k+1})\big) \geq M(d\lambda^m)^{k+1}$. This implies also that

$$\mu(F\cap B_{k+1}) \geq M(d\lambda^m)^{k+1}. \tag{2.10}$$

Let $z = \lim_{k\to\infty} x_k$. Since $t\lambda \leq 1/3$, we have $|z-x_k| \leq \sum_{i=k}^{\infty}(t\lambda)^i < 2(t\lambda)^k$. Thus $B_k \subset B\big(z,3(t\lambda)^k\big)$ for all $k\in\mathbb{N}$. If $(t\lambda)^{k+1} \leq r' < (t\lambda)^k$, then $3r' < (t\lambda)^{k-1}$, and hence, using (2.1), (2.10), and (2.4), we get

$$\begin{aligned}\frac{\mu\big(B(z,3r')\big)}{h(3r')} &\geq \frac{(t\lambda)^{s(k-1)}\mu(B_{k+1})}{(3r')^s h\big((t\lambda)^{k-1}\big)} > \frac{Md^{k+1}\lambda^{m(k+1)}}{h\big((t\lambda)^{k-1}\big)}\\ &= Md^2\lambda^{2m}\big(d\lambda^{m-s}t^{-s}\big)^{k-1}\frac{(t\lambda)^{s(k-1)}}{h\big((t\lambda)^{k-1}\big)}\\ &\geq Md^2\lambda^{2m}2^{(s-m)(k-1)} \longrightarrow \infty\end{aligned}$$

as $r' \downarrow 0$. This implies $\underline{D}_h(\mu,z)=\infty$, a contradiction since $z\in F$. □

We remark that the condition (2.1) could be weakened slightly. Namely, it suffices to assume that there is $s>m$ so that

$$\frac{h(\varepsilon r)}{\varepsilon^s h(r)} \xrightarrow{\varepsilon\downarrow 0} 0 \quad \text{uniformly for all } 0<r<r_0. \tag{2.11}$$

We wrote down the proof in the case (2.1) to avoid technicalities, and also since many natural gauge functions satisfy (2.1), see the discussion below.

Let us now consider the applications of our result. We denote $h_s(r)=r^s$ as $r\geq 0$. As noted in the introduction, Theorem 2.3 is a generalisation of (1.4). This follows from the well known fact according to which

$$1 \leq \overline{D}_{h_s}(\mathcal{H}^s|_A,x) \leq 2^s$$

for $\mathcal{H}^s$-almost all $x\in A$ provided $A\subset\mathbb{R}^n$ with $0<\mathcal{H}^s(A)<\infty$. The most important improvement in Theorem 2.3 compared to (1.4) is related to the s-dimensional packing measure, $\mathcal{P}^s$. See [9, §5.10] for the definition. If $A\subset\mathbb{R}^n$ with $0<\mathcal{P}^s(A)<\infty$, then

$$\underline{D}_{h_s}(\mathcal{P}^s|_A,x) = 2^s$$

for $\mathcal{P}^s$-almost all $x \in A$, see [9, Theorem 6.10]. Thus we get the following corollary:

Corollary 2.4. *Suppose $0 \leq m < s \leq n$ and $0 < \alpha, \eta \leq 1$. Then there is a constant $c = c(n, m, s, \alpha, \eta) > 0$ such that*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mathcal{P}^s\big(A \cap X(x,r,V,\alpha) \setminus H(x,\theta,\eta)\big)}{r^s} \geq c\,\overline{D}_{h_s}(\mathcal{P}^s|_A, x) \geq c\,2^s \tag{2.12}$$

for $\mathcal{P}^s$-almost every $x \in A$ whenever $A \subset \mathbb{R}^n$ with $0 < \mathcal{P}^s(A) < \infty$.

It is remarkable to note that it is certainly possible that the upper density $\overline{D}_{h_s}(\mathcal{P}^s|_A, x)$ is infinity almost everywhere on the set A. In this case Corollary 2.4 states that also the upper density (2.12) is infinity for $\mathcal{P}^s$-almost every $x \in A$.

For many fractals some other gauge function than h_s might be more useful in measuring the fractal set in a delicate manner. Let $h\colon [0,\infty) \to [0,\infty)$, $h(0) = 0$, and denote the Hausdorff and packing measures constructed using h as a gauge function by $\mathcal{H}_h$ and $\mathcal{P}_h$, respectively. See [9, §4.9] and [2, Definition 2.2] for the definitions. If $A, B \subset \mathbb{R}^n$, $0 < \mathcal{H}_h(A) < \infty$, $0 < \mathcal{P}_h(B) < \infty$, $\mu = \mathcal{H}_h|_A$, and $\nu = \mathcal{P}_h|_B$, then $\overline{D}_h(\mu, x) \leq \limsup_{r\downarrow 0} h(2r)/h(r)$ for μ-almost every $x \in \mathbb{R}^n$ and $\underline{D}_h(\nu, x) \leq \limsup_{r\downarrow 0} h(2r)/h(r)$ for ν-almost every $x \in \mathbb{R}^n$. Thus Theorem 2.3 may be applied to measures μ and ν provided h satisfies (2.1) and the doubling condition $\limsup_{r\downarrow 0} h(2r)/h(r) < \infty$. The above estimates for $\overline{D}_h(\mu, x)$ and $\underline{D}_h(\nu, x)$ may be proved by imitating the proofs of Theorems 6.2 and 6.10 in [9]. The condition (2.1) holds for functions such as $h(r) = r^s/\log(1/r)$ or $h(r) = r^t \log(1/r)$, $t > s$. However, some interesting (see [13, page 13] for discussion) gauge functions such as $h(r) = r^m/\log(1/r)$ fail to satisfy (2.1) or even the weaker condition (2.11).

In spite of all, most measures are so unevenly distributed that there are no functions that could be used to approximate the measures in small balls. For these measures it is natural to study upper densities such as

$$\limsup_{r\downarrow 0} \frac{\mu\big(X(x,r,V,\alpha)\big)}{\mu\big(B(x,r)\big)}.$$

We conclude the article with the following open problem. It is stated here in it's simplest form though natural generalisations arise comparing (1.1)–(1.4): Suppose that μ is a measure on $\mathbb{R}^n$ whose packing dimension, $\dim_{\mathrm{P}}(\mu)$, equals s (see [4, §10]). If $0 < \alpha < 1$, $m \in \mathbb{N}$ with $m < s$, and $V \in G(n, n-m)$, is it true that

$$\limsup_{r\downarrow 0} \frac{\mu\big(X(x,r,V,\alpha)\big)}{\mu\big(B(x,r)\big)} \geq c$$

for μ-almost every $x \in \mathbb{R}^n$, where $c > 0$ depends only on n, m, s, and α?

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