Surface families and boundary behavior of quasiregular mappings

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. We call a mapping $f : \Omega \to \mathbb{R}^n$ quasiregular, if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, and if there exists $1 \leq K_O < \infty$ so that

$$||Df(x)||^n \leq K_O J_f(x)$$

for almost all $x \in \Omega$. By the results of Reshetnyak, non-constant quasiregular mappings are discrete, open and locally Hölder continuous, and map sets of measure zero to sets of measure zero. For the theory of quasiregular mappings, see the monographs [10] and [11].

One of the most interesting open problems on quasiregular mappings is to find out to what extend one can generalize Fatou's theorem on the boundary behavior of analytic functions. Recall that Fatou's theorem says that a bounded analytic function on the unit disc has radial limits at almost every boudary point, cf. [8], page 5. This result is not true for planar quasiregular mappings in this generality; for any $\epsilon > 0$ there exists a bounded quasiregular mapping $f : D(0,1) \to \mathbb{C}$ and a set $E_\epsilon \subset S^1(0,1)$ whose Hausdorff dimension is smaller ϵ, so that f fails to have radial limits in $S^1(0,1) \setminus E_\epsilon$, see [8], pages 119–120. The basic reason for this failure is the fact that the boundary extension of a quasiconformal homeomorphism of the unit disc onto itself may carry sets of arbitrarily small Hausdorff dimension to sets of full linear measure, see [1]. On the other hand, the Stoilow factorization theorem implies that radial limits do exist in a set of positive Hausdorff dimension.

In dimensions higher than two, it is not even known if there exists a bounded quasiregular mapping of the unit ball without any radial limits. One can, though, prove, by using normal families, that if a radial limit exists at $y \in S^{n-1}(0,1)$, then it is also a non-tangential limit, see [6]. In [6]

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it is also proved that, if an additional assumption that there exist $C > 0$ and $0 < s < n$ so that

$$\int_{B(0,r)} J_f(x)\, dx \le C(1-r)^{1-s} \quad \forall\, 0 < r < 1, \tag{1.1}$$

on a bounded quasiregular mapping $f : B(0,1) \to \mathbb{R}^n$ is put, then f has radial limits almost everywhere. This result is sharpened in [5] in the effect of having an upper bound on the Hausdorff dimension on the exceptional set. For bounded quasiregular mappings, (1.1) with $s = n$ always holds true. In [11], VII Theorem 2.7, Assumption (1.1) is weakened to

$$\int_0^1 \Big(\int_{B(0,r)} J_f(x)\, dx \Big) (1-r)^{n-2} \Big(\log \frac{1}{1-r} \Big)^{n+\delta} dr < \infty \quad \text{for some } \delta > 0.$$

In the other direction, in [7] and [4] examples of bounded quasiregular mappings $f : B(0,1) \to \mathbb{R}^n$, $n \ge 3$, are constructed, so that these mappings fail to have radial limits in subsets of the $(n-1)$-sphere of Hausdorff dimension arbitrarily close to $n-1$. For other results on the boundary behavior of quasiregular mappings, see [13] and the references therein.

In this note we take a different viewpoint to the boundary behavior. Namely, the existence of radial limits at points y on the unit sphere is related to the stronger property that the curves fL_y are rectifiable, where $L_y = \{ty : t \in [0,1)\}$. For results on the latter property for analytic functions, see [12], [2]. In fact, the proofs of the results in [6] and [11], VII Theorem 2.7 mentioned above conclude the latter property. We study, for $n \ge 3$, the behavior of the images of certain $(n-1)$-dimensional sets that are symmetric with respect to L_y:s. We prove that, under a condition that requires the sets to be slightly cusplike (compared to cones with vertices at the points y), one finds images of finite $\mathcal{H}^{n-1}$-measure for almost all points $y \in S^{n-1}(0,1)$. The main advantage of this result is that it holds true for all bounded quasiregular mappings in dimensions three or higher, without any multiplicity assumptions.

Theorem 1.1. *Let $f : B(0,1) \to \mathbb{R}^n$, $n \ge 3$, be a bounded quasiregular mapping. Moreover, let $\mathcal{E} : (1/2,1) \to (0,1)$ be a smooth function satisfying*

$$\mathcal{E}(t) \le C(1-t) \log^{-\beta} \frac{1}{1-t}$$

for some

$$\beta > \frac{3n-1}{n(n-2)}.$$

Then, for $\mathcal{H}^{n-1}$- almost all $x \in S^{n-1}$, there exists a set V_x so that $\mathcal{H}^1(V_x) = 0$ and so that, for each $\lambda \in [1,2] \setminus V_x$,

$$\mathcal{H}^{n-1}(fS_{x,\lambda}) < \infty,$$

where

$$S_{x,\lambda} = \{t\varphi_t : t \in (1/2,1), \varphi_t \in S^{n-1}, |x-\varphi_t| = \lambda\mathcal{E}(t)\}.$$

In the notation of the theorem, for cones one has $\mathcal{E}(t) \approx C\alpha(1-t)$, where α is the opening angle of the cone. The discussion at the beginning of the introduction shows that results similar to Theorem 1.1 do not in general hold true for planar quasiregular mappings. Theorem 1.1 is proved by using the conformal modulus of families of $(n-1)$-dimensional sets. For a different application of this method, see [9].

Notation

We will denote the euclidean norm by $|\cdot|$, while the operator norm of a matrix is denoted by $||\cdot||$. Moreover, a k-dimensional ball with center x and radius r in $\mathbb{R}^n$ is denoted by $B^k(x,r)$. When $k=n$, the notation $B(x,r)$ is used. Similarly, corresponding k-dimensional spheres are denoted by $S^k(x,r)$. We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure.

For notational convenience we denote, for $i \in \mathbb{N}$, $r_i = 1-2^{-i}$. Also, we use the notation $A_i = B(0,r_{i+1}) \setminus B(0,r_i)$. For a Sobolev mapping $f : \Omega \to \mathbb{R}^n$, $Df(x)$ denotes the differential matrix of f at $x \in \Omega$. Then, $J_f(x)$ is the Jacobian determinant and $D^{\#}f(x)$ the adjoint matrix of $Df(x)$, respectively. For a quasiregular mapping there exists, in addition to K_O, a constant $1 \leq K_I \leq K_O^{n-1}$, so that

$$||D^{\#}f(x)||^{\frac{n}{n-1}} \leq K_I^{\frac{1}{n-1}} J_f(x)$$

for almost every $x \in \Omega$. Finally, we use the notation

$$N(y,f,U) = \operatorname{card}\{f^{-1}(y) \cap U\}.$$

2 Proof of Theorem 1.1

We will consider a quasiregular mapping satisfying the assumptions of Theorem 1.1. We first define the conformal modulus for $(n-1)$-dimensional sets (surfaces). For a family Λ of Borel measurable subsets of $\mathbb{R}^n$, set

$$\begin{aligned} M_S\Lambda \quad = \quad \inf \Big\{ & \int_{\mathbb{R}^n} \rho(x)^{\frac{n}{n-1}}\, dx \,|\, \rho : \mathbb{R}^n \to [0,\infty] \text{ is Borel measurable,} \\ & \int_S \rho(x)\, d\mathcal{H}^{n-1}(x) \geq 1 \quad \forall\, S \in \Lambda \Big\}. \end{aligned}$$

Next we describe the families of surfaces that we will be concerned with. Throughout this paper, we will assume that $n \geq 3$. Fix a smooth function

$$\mathcal{E} : (1/2,1) \to (0,1),$$

and require that $\mathcal{E}_t := \mathcal{E}(t) < \sqrt{1-t^2}$ for all t, so that the sets defined below will become subsets of the unit ball $B(0,1)$. Let $E \subset S^{n-1}$ be a measurable set, and $i \in \mathbb{N}$. Assume that there exists a constant $A > 0$ and a map F from E to the measurable subsets of $[1,2]$, so that (denote $F(x) = F_x$)

$$\mathcal{H}^1(F_x) \geq A \text{ for all } x \in E. \tag{2.1}$$

Next, for each $x \in E$ and $\lambda \in F_x$, define a surface $S^i_{x,\lambda} \subset B(0,1)$ by

$$S^i_{x,\lambda} = \{t\varphi_t : t \in [r_i, r_{i+1}), \varphi_t \in S^{n-1}, |x - \varphi_t| = \lambda \mathcal{E}_t\}.$$

Now, we define $\Lambda^i_{E,F}$ to be the family of $S^i_{x,\lambda}$:s;

$$\Lambda^i_{E,F} = \bigcup_{x \in E} \bigcup_{\lambda \in F_x} S^i_{x,\lambda}.$$

Of course, $\Lambda^i_{E,F}$ depends also on $\mathcal{E}$, but we will consider it to be fixed throughout.

We now have the following lower bound for the conformal modulus of $\Lambda^i_{E,F}$.

Proposition 2.1. *There exists a constant $C_n > 0$, only depending on n, so that*

$$M_S \Lambda^i_{E,F} \geq C_n A^{\frac{n}{n-1}} \Big(\int_{r_i}^{r_{i+1}} \mathcal{E}_t^{n(n-2)}\, dt \Big)^{\frac{-1}{n-1}} \mathcal{H}^{n-1}(E).$$

Proof. Let ρ be a test function for $M_S \Lambda^i_{E,F}$. Then we have, for each $x \in E$ and $\lambda \in F_x$,

$$1 \leq \int_{S^i_{x,\lambda}} \rho(y)\, d\mathcal{H}^{n-1}(y) \leq C_n \int_{r_i}^{r_{i+1}} \int_{S^{n-1}(0,t) \cap S^i_{x,\lambda}} \rho(z)\, d\mathcal{H}^{n-2}(z)\, dt.$$

Hence, if we denote $G_{x,t,\lambda} = S^{n-1}(0,t) \cap S^i_{x,\lambda}$, and $H_{x,t} = S^{n-1}(0,t) \cap B^{n-1}(xt, 4\mathcal{E}_t)$, integrating over F_x and E yields (in what follows, C_n may vary from line to line)

$$\begin{aligned}
A\mathcal{H}^{n-1}(E) &\leq C_n \int_E \int_{F_x} \int_{r_i}^{r_{i+1}} \int_{G_{x,t,\lambda}} \rho(z)\, d\mathcal{H}^{n-2}(z)\, dt\, d\lambda\, d\mathcal{H}^{n-1}(x) \\
&= C_n \int_{r_i}^{r_{i+1}} \int_E \int_{F_x} \int_{G_{x,t,\lambda}} \rho(z)\, d\mathcal{H}^{n-2}(z)\, d\lambda\, d\mathcal{H}^{n-1}(x)\, dt \\
&\leq C_n \int_{r_i}^{r_{i+1}} \mathcal{E}_t^{-1} \int_E \int_{H_{x,t}} \rho(y)\, d\mathcal{H}^{n-1}(y) \mathcal{H}^{n-1}(x)\, dt \\
&\leq C_n \int_{r_i}^{r_{i+1}} \mathcal{E}_t^{n-2} \int_{tE} M_t\rho(x) \mathcal{H}^{n-1}(x)\, dt,
\end{aligned}$$

where $tE = \{tx : x \in E\}$, and M_t is the Hardy-Littlewood maximal function in $(S^{n-1}(0,t), |\cdot|)$. By applying Hölder's inequality, the right hand integral can be estimated from above by

$$C_n \Big(\int_{r_i}^{r_{i+1}} \mathcal{E}_t{}^{n(n-2)}\, dt \Big)^{\frac{1}{n}} \Big(\int_{r_i}^{r_{i+1}} \Big(\int_{tE} M_t \rho(x)\, d\mathcal{H}^{n-1}(x) \Big)^{\frac{n}{n-1}} dt \Big)^{\frac{n-1}{n}}.$$

Furthermore, by applying Hölder's inequality and the L^p-boundedness of the Hardy-Littlewood maximal function for $p > 1$, the last term can be estimated from above by

$$\begin{aligned} & C_n \mathcal{H}^{n-1}(E)^{\frac{1}{n}} \Big(\int_{r_i}^{r_{i+1}} \int_{S^{n-1}(0,t)} (M_t\rho(x))^{\frac{n}{n-1}}\, d\mathcal{H}^{n-1}(x)\, dt \Big)^{\frac{n-1}{n}} \\ \le\ & C_n \mathcal{H}^{n-1}(E)^{\frac{1}{n}} \Big(\int_{r_i}^{r_{i+1}} \int_{S^{n-1}(0,t)} \rho(x)^{\frac{n}{n-1}}\, d\mathcal{H}^{n-1}(x)\, dt \Big)^{\frac{n-1}{n}} \\ \le\ & C_n \mathcal{H}^{n-1}(E)^{\frac{1}{n}} \Big(\int_{B(0,1)} \rho(x)^{\frac{n}{n-1}}\, dx \Big)^{\frac{n-1}{n}}. \end{aligned}$$

Combining the estimates yields

$$\int_{B(0,1)} \rho(x)^{\frac{n}{n-1}}\, dx \ge C_n A^{\frac{n}{n-1}} \Big(\int_{r_i}^{r_{i+1}} \mathcal{E}_t^{n(n-2)}\, dt \Big)^{\frac{-1}{n-1}} \mathcal{H}^{n-1}(E).$$

Since this holds true for every test function ρ, the proof is complete. $\square$

Next we show that the natural generalization of the K_O-inequality for path families (see [11], II Theorem 2.4) holds true in our case.

Lemma 2.2. *Let $\rho : \mathbb{R}^n \to [0,\infty]$ be a test function for $M_S f\Lambda^i_{E,F}$. Then*

$$M_S\Lambda^i_{E,F} \le K_I^{\frac{1}{n-1}} \int_{\mathbb{R}^n} \rho(y)^{\frac{n}{n-1}}\, N(y,f,A_i)\, dy.$$

Proof. Denote by Λ' the family of all $S^i_{x,\lambda} \in \Lambda^i_{x,\lambda}$ on which there exists a subset $V \subset S^i_{x,\lambda}$ so that $\mathcal{H}^{n-1}(V) = 0$ but $\mathcal{H}^{n-1}(fV) > 0$. Then, by Fuglede's lemma [3], Theorem 3, $M_S\Lambda' = 0$. Let ρ be a test function for $M_S f\Lambda^i_{E,F}$. Define a Borel function $\rho' : \mathbb{R}^n \to [0,\infty]$ by setting

$$\rho'(x) = \rho(f(x))||D^\# f(x)||\chi_{A_i}$$

when f is differentiable at x, and $\rho'(x) = 0$ otherwise. By [11], I Theorem 2.4, f is differentiable outside a set Y of Lebesgue measure zero. Hence, if

$$\Lambda'' = \{S \in \Lambda^i_{E,F} : \mathcal{H}^{n-1}(S \cap Y) > 0\},$$

we have $M_S\Lambda'' = 0$. Then, for all $S \in \Lambda^i_{x,\lambda} \setminus (\Lambda' \cup \Lambda'')$,

$$||D^\# f(x)|| \ge \lim_{r\to 0} \frac{\mathcal{H}^{n-1}(f(S \cap B(x,r)))}{\mathcal{H}^{n-1}(S \cap B(x,r))}$$

for $\mathcal{H}^{n-1}$-almost every $x \in S$. Hence, for such S,

$$\int_S \rho'(x)\, d\mathcal{H}^{n-1}(x) = \int_S \rho(f(x))||D^{\#} f(x)||\, d\mathcal{H}^{n-1}(x) \geq \int_{fS} \rho(y)\, d\mathcal{H}^{n-1}(y) \geq 1,$$

where the last inequality holds true since ρ is assumed to be a test function for $M_S f \Lambda^i_{E,F}$. Moreover, by the quasiregularity of f, and by the change of variable formula, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho'(x)^{\frac{n}{n-1}}\, dx = \int_S \rho(f(x))^{\frac{n}{n-1}} ||D^{\#} f(x)||^{\frac{n}{n-1}}\, dx \\ \leq \; & K_I^{\frac{1}{n-1}} \int_S \rho(f(x))^{\frac{n}{n-1}} J_f(x)\, dx = K_I^{\frac{1}{n-1}} \int_{\mathbb{R}^n} \rho(y)^{\frac{n}{n-1}}\, N(y, f, A_i)\, dy, \end{aligned}$$

as desired. □

Now we are ready to prove our main result.

Proof of Theorem 1.1. With the notation as above, fix r_i and set, for $x \in S^{n-1}$,

$$F^i_x = \Big\{\lambda \in [1,2] : \mathcal{H}^{n-1}(fS^i_{x,\lambda}) > i^{-1-\alpha}\Big\},$$

where

$$\alpha = \frac{1}{2}\Big(\frac{\beta(n-2)n}{3n-1} - 1\Big).$$

By our assumption on β, $\alpha > 0$. Moreover, set

$$E_i = \Big\{x \in S^{n-1} : \mathcal{H}^1(F^i_x) > i^{-1-\alpha}\Big\}.$$

By Proposition 2.1, we have, for $E = E_i$ and $F_x = F^i_x$,

$$\begin{aligned} M\Lambda^i_{E,F} \;\; & \geq \;\; C_n \mathcal{H}^{n-1}(E_i) i^{\frac{-(1+\alpha)n}{n-1}} \Big(\int_{r_i}^{r_{i+1}} \Big((1-t)\log^{-\beta}\frac{1}{1-t}\Big)^{n(n-2)} dt\Big)^{\frac{-1}{n-1}} \\ (2.2) \qquad & \geq \;\; C_n \mathcal{H}^{n-1}(E_i) i^{\frac{-(1+\alpha)n}{n-1}} (1-r_i)^{1-n} \Big(\log\frac{1}{1-r_i}\Big)^{\frac{\beta n(n-2)}{n-1}}. \end{aligned}$$

On the other hand we have, by the definition of E_i and F^i_x,

$$\mathcal{H}^{n-1}(fS) > i^{-1-\alpha}$$

for all $S \in \Lambda^i_{E,F}$, and hence the constant function $\rho = i^{1+\alpha}$ is a test function for $Mf\Lambda^i_{E,F}$. Moreover, by the boundedness of f, there exists a constant $C > 0$ so that, for all $1/2 < r < 1$,

$$\int_{B(0,r)} J_f(x)\, dx \leq C(1-r)^{1-n},$$

cf. [11], page 172. Hence, by Lemma 2.2,

$$\begin{aligned} M\Lambda^i_{E,F} &\le K_I^{\frac{1}{n-1}} i^{\frac{(1+\alpha)n}{n-1}} \int_{\mathbb{R}^n} N(y,f,B(0,r_{i+1}))\,dy \\ &= K_I^{\frac{1}{n-1}} i^{\frac{(1+\alpha)n}{n-1}} \int_{B(0,r_{i+1})} J_f(x)\,dx \le C K_I^{\frac{1}{n-1}} i^{\frac{(1+\alpha)n}{n-1}} (1-r_i)^{1-n}. \end{aligned} \tag{2.3}$$

By combining (2.2) and (2.3), and by using $1-r_i = 2^{-i}$, we have

$$\begin{aligned} \mathcal{H}^{n-1}(E_i) &\le C(n,K) i^{\frac{2(1+\alpha)n}{n-1}} \Big(\log \frac{1}{1-r_i}\Big)^{\frac{-\beta(n-2)n}{n-1}} \\ &= C(n,K) i^{\frac{2(1+\alpha)n}{n-1}} i^{\frac{-\beta(n-2)n}{n-1}} \le C(n,K) i^{-1-\alpha}, \end{aligned}$$

where the last inequality follows from our assumption on β and our choice of α.

In conclusion, we have, for the set

$$E := \bigcap_{N=1}^{\infty} \bigcup_{i\ge N} E_i,$$

$$\mathcal{H}^{n-1}(E) \le \lim_{N\to\infty} \sum_{i\ge N} \mathcal{H}^{n-1}(E_i) \le C \lim_{N\to\infty} \sum_{i\ge N} i^{-1-\alpha} = 0.$$

Thus, for almost every $x \in S^{n-1}$, there exists a constant $N_x \in \mathbb{N}$, so that

$$x \in S^{n-1} \setminus \bigcup_{i\ge N_x} E_i.$$

Fix such x. Then we have, for the set

$$Q_x := \{\lambda \in [1,2] : \text{ there exists } i \ge N_x \text{ so that } \mathcal{H}^{n-1}(fS^i_{x,\lambda}) > i^{-1-\alpha}\},$$

$$\mathcal{H}^1(Q_x) \le \sum_{i\ge N_x} \mathcal{H}^1(F^i_x) \le \sum_{i\ge N_x} i^{-1-\alpha}.$$

By choosing, as we may, N_x to be arbitrarily large, we see that, for all $\lambda \in [1,2]\setminus W_x$, $\mathcal{H}^1(W_x)=0$, there exists $N_{x,\lambda}$, so that

$$\mathcal{H}^{n-1}(f(S_{x,\lambda} \setminus \cup_{i=1}^{N_{x,\lambda}} S^i_{x,\lambda})) \le \sum_{i\ge N_{x,\lambda}} \mathcal{H}^{n-1}(fS^i_{x,\lambda}) \le \sum_{i\ge N_{x,\lambda}} i^{-1-\alpha} < \infty.$$

In order to finish the proof, we will show that, for $\mathcal{H}^1$- almost all $[1,2]\setminus W_x$,

$$\mathcal{H}^{n-1}(f(\cup_{i=1}^{N_{x,\lambda}} S^i_{x,\lambda})) < \infty.$$

Clearly,

$$\begin{aligned} &\{\lambda \in [1,2]\setminus W_x : f(\cup_{i=1}^{N_{x,\lambda}} S^i_{x,\lambda}) = \infty\} \\ \subset\ &\{\lambda \in [1,2]\setminus W_x : f(S^i_{x,\lambda}) = \infty \quad \text{for some } i \in \mathbb{N}\} =: Z_x. \end{aligned}$$

Assume that $\mathcal{H}^1(Z_x) > 0$. Then $\mathcal{H}^1(Z_x^i) > 0$ for some i, where

$$Z_x^i = \{\lambda \in [1,2] \setminus W_x : f(S^i_{x,\lambda}) = \infty\}.$$

Fix such i. If

$$\Lambda^i_{x,Z} = \{S^i_{x,\lambda} : \lambda \in Z^i_x\},$$

then, by Lemma 2.2, $M_S \Lambda^i_{x,Z} = 0$. We show that this cannot be the case.

Let ρ be a test function for $M_S \Lambda^i_{x,Z}$. Then, for all $\lambda \in Z^i_x$,

$$1 \le \int_{S^i_{x,\lambda}} \rho(y)\, d\mathcal{H}^{n-1}(y) \le C_n \int_{r_i}^{r_{i+1}} \int_{S^{n-1}(0,t)\cap S^i_{x,\lambda}} \rho(z)\, d\mathcal{H}^{n-2}(z)\, dt,$$

and hence Hölder's inequality yields

$$\begin{aligned}
\mathcal{H}^1(Z^i_x) &\le \int_{Z^i_x} \int_{S^i_{x,\lambda}} \rho(y)\, d\mathcal{H}^{n-1}(y)\, d\lambda \\
&\le C_n \int_{Z^i_x} \int_{r_i}^{r_{i+1}} \int_{S^{n-1}(0,t)\cap S^i_{x,\lambda}} \rho(z)\, d\mathcal{H}^{n-2}(z)\, dt\, d\lambda \\
&\le C\Big(\int_{A_i} \rho(x)^{\frac{n}{n-1}}\, dx\Big)^{\frac{n-1}{n}}
\end{aligned}$$

for some positive constant C (compare the proof of Proposition 2.1). Hence $M_S \Lambda^i_{x,Z} > 0$, which is a contradiction. The proof is complete.

□

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