Global integrability of the derivative of a quasiconformal mapping

Tomi Nieminen *

Abstract

We establish an essentially sharp growth condition on the quasihyperbolic metric of a domain $\Omega \subset \mathbf{R}^n$ sufficient for the global higher integrability of the derivative of a quasiconformal mapping $f : \Omega' \to \Omega$.

1 Introduction

Recall that a homeomorphism $f : \Omega' \to \Omega$ with $\Omega, \Omega' \subset \mathbf{R}^n$ is K-quasiconformal if $f \in W^{1,n}_{loc}(\Omega')$ and $|Df|^n \leq K J_f(x)$ holds for almost every $x \in \Omega'$. Suppose that we are given two domains $\Omega', \Omega \subset \mathbf{R}^n$, $n \geq 2$, and a quasiconformal mapping $f : \Omega' \to \Omega$. What can be said about the global higher integrability of Df in this case? In other words, under which conditions does the inequality

$$\int_{\Omega'} |Df|^n \varphi(|Df|) dx < \infty \tag{1.1}$$

hold with an unbounded increasing function φ? This problem is well-known and it was considered already, for example, in [AK] where the following important result was established. If Ω satisfies the growth condition

$$k_\Omega(y, y_0) \leq \phi\Big(\frac{d(y, \partial\Omega)}{d(y_0, \partial\Omega)}\Big) + C_0 \tag{1.2}$$

on the quasihyperbolic metric k_Ω with the function $\phi(t) = C \log \frac{1}{t}$, then

$$\int_{\Omega'} |Df|^p dx < \infty$$

for some $p > n$. Here the exponent p depends only on n, C and the dilatation $K = K(f)$. In this paper we prove an extension of this theorem. We show that (1.2) implies (1.1) provided that ϕ satisfies certain conditions formulated in the following. We also give a sharp estimate for φ in this case.

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Definition 1.1. *We say that a decreasing function $\phi:]0,1]\to]0,\infty[$ is of logarithmic type, if there exist positive constants t_0 and β such that ϕ satisfies the following conditions for all $t\le t_0$:*

$$\phi(t) \text{ is twice differentiable and } -\phi'(t)t \text{ is a decreasing function;} \tag{1.3}$$

$$\phi(t)\le\beta\phi(\sqrt{t}). \tag{1.4}$$

Note that, for example, a function of the form

$$\phi(t)=\begin{cases} C(\log\frac{1}{t})^{s_1}(\log\log\frac{1}{t})^{s_2}...(\log^{(m)}\frac{1}{t})^{s_m}+C,\ t<a_m; \\ C,\ t\ge a_m, \end{cases}$$

where $C>0$, $m\in\mathbf{Z}^+$, $a_m=1/\exp^{(m-1)}(e)$, $s_1\ge 1$, $s_2,...,s_m\ge 0$, is of logarithmic type.

Theorem 1.2. *Let $\Omega\subset\mathbf{R}^n$ be a bounded domain satisfying* (1.2) *for some fixed point $y_0\in\Omega$ and for all $y\in\Omega$, where ϕ is of logarithmic type satisfying*

$$\int_0\frac{dt}{(-\phi'(t)t)^{n-1}t}=\infty. \tag{1.5}$$

If $\Omega'\subset\mathbf{R}^n$ and $f:\Omega'\to\Omega$ is K-quasiconformal, then

$$\int_{\Omega'}|Df|^n\varphi(\log(e+|Df|))dx<\infty \tag{1.6}$$

where

$$\varphi(r)=\exp\Big(C_1\int_{[\phi^{-1}(C_2r),\ t_1]}\frac{dt}{(-\phi'(t)t)^{n-1}t}\Big) \tag{1.7}$$

for all sufficiently large r. Here $t_1<1$ and the constants C_1 and C_2 depend only on β, K and n.

We will show in Section 3 that this result is essentially sharp at least in $\mathbf{R}^2$. Indeed, for a given ϕ there exists a domain Ω satisfying (1.2) and a (quasi)conformal mapping $f:B^2\to\Omega$ such that inequality (1.6) fails with large constants C_1, C_2.

If, for example, $\phi(t)=C(\log\frac{1}{t})^{\frac{n}{n-1}}$, then Theorem 1.2 implies the global integrability condition

$$\int_{\Omega'}|Df|^n(\log(e+|Df|))^p dx<\infty$$

with some $p>0$ depending only on C, K and n. Note that if $\phi(t)=C(\log\frac{1}{t})^s$ with some $s>\frac{n}{n-1}$, then the integral $\int_{\Omega'}|Df|^n\varphi(\log(e+|Df|))dx$ can be made to diverge with any unbounded increasing function φ by choosing Ω,Ω' and f suitably, see Section 3. Indeed, the exponent $\frac{n}{n-1}$ is the limit for the divergence condition (1.5) to hold for functions of this scale.

As another example, let us consider the classical situation when Ω is a Hölder domain, i.e. it satisfies (1.2) with $\phi(t) = \frac{1}{\varepsilon}\log\frac{1}{t}$. Now, by Theorem 1.2, $\int_{\Omega'} |Df|^{n+C\varepsilon^n} < \infty$. This result is essentially equivalent with [AK, Theorem 1.2]. In particular, the dependency on the constant ε is sharp at least in $\mathbf{R}^2$, see Section 3.

In [AK] the problem at hand was considered also from the viewpoint of uniform continuity of the (quasi)conformal mapping f. Astala and Koskela concluded that, for univalent functions in the unit disk, the higher integrability of the derivative is equivalent to Hölder continuity. More precisely, $\int_\Delta |Df|^p dx < \infty$ for some $p > 2$ if and only if the inequality $|f(x) - f(x')| \leq M|x - x'|^\alpha$ holds for all $x, x' \in \Delta$ with some constants $M < \infty$, $\alpha \leq 1$. Here Δ denotes the unit disk. The "if" part of this result has a counterpart which holds also in the general case. It is formulated in the following.

Corollary 1.3. *Let* $\psi :]0,1[\to]0,1[$ *be an increasing bijection and let* $u := \psi^{-1}$. *Suppose that* $\log(\frac{1}{u(t)})$ *is of logarithmic type and that*

$$\int_0 \Big(\frac{u(t)}{u'(t)}\Big)^{n-1} \frac{dt}{t^n} = \infty. \tag{1.8}$$

If $f : B^n \to \mathbf{R}^n$ *is a* K*-quasiconformal map such that the inequality*

$$|f(x) - f(x')| \leq \psi(|x - x'|) \tag{1.9}$$

holds for all $x, x' \in B^n$ *sufficiently close to each other, then*

$$\int_{B^n} |Df|^n \varphi(\log(e + |Df|))dx < \infty$$

where

$$\varphi(r) = \exp\Big(C_1 \int_{[\psi(\exp(-C_2 r)),\ t_1]} \Big(\frac{u(t)}{u'(t)}\Big)^{n-1} \frac{dt}{t^n}\Big)$$

for all sufficiently large r. *Here* $t_1 < 1$ *and the constants* C_1 *and* C_2 *depend only on* β, K *and* n.

Note that this result is optimal at least in $\mathbf{R}^2$ by the example given in Section 3. Considering the special case of a Hölder continuous f, Corollary 1.3 implies a result which is essentially equivalent with [AK, Corollary 1.4].

2 Proofs of the results

For the proof of Theorem 1.2 we need some preliminary results. As in [AK], we will use the *average derivative* (introduced in [AG]) to estimate the distortion properties of a quasiconformal mapping in terms of the Jacobian.

Definition 2.1. *Let f be a quasiconformal mapping in a proper subdomain $\Omega \subset \mathbf{R}^n$ and set $B(x) = B(x, d(x, \partial\Omega)/2)$. Then the average derivative is defined by*

$$a_f(x) = \exp\Big(\frac{1}{n|B(x)|}\int_{B(x)} \log J_f(y)dy\Big), \quad x \in \Omega.$$

Recall that the counterpart of the Koebe distortion theorem holds for a_f :

Lemma 2.2 ([AG]). *If f is K-quasiconformal in a domain $\Omega \subset \mathbf{R}^n$, then*

$$c_1 \frac{d(f(x), \partial f\Omega)}{d(x, \partial\Omega)} \leq a_f(x) \leq c_2 \frac{d(f(x), \partial f\Omega)}{d(x, \partial\Omega)},$$

where c_1, c_2 depend only on n and K.

We need also the following result proven in [AK, Theorem 3.4].

Lemma 2.3 ([AK]). *There exists $\varepsilon = \varepsilon(n, K) > 0$ such that whenever f is K-quasiconformal in a domain $\Omega \subset \mathbf{R}^n$, the estimate*

$$c_1 \int_Q a_f(x)^p dx \leq \int_Q |Df|^p dx \leq c_2 \int_Q a_f(x)^p dx$$

holds for all cubes Q in any Whitney decomposition of Ω and for all $-\varepsilon < p < n + \varepsilon$. Here the constants c_1, c_2 depend only on n, K and p.

Here a Whitney decomposition refers to a decomposition of Ω into cubes so that the interiors of the cubes are pairwise disjoint and the inequality $\mathrm{diam}(Q) \leq \mathrm{dist}(Q, \partial\Omega) \leq 4\ \mathrm{diam}(Q)$ holds for each cube Q in the decomposition. See [S] for the construction and properties of the Whitney decomposition.

Since we will use Jensen's inequality in the proof of Theorem 1.2, we show in the next lemma that a certain function is convex.

Lemma 2.4. *Let ϕ be of logarithmic type and let $p > n$. Then there are constants C_1, C_2, depending on p, and a function $\varphi :]0, \infty[\to]0, \infty[$ so that*

$$\varphi(r) = \exp\Big(C_1 \int_{[\phi^{-1}(C_2 r),\ t_1]} \frac{dt}{(-\phi'(t)t)^{n-1}t}\Big)$$

for all sufficiently large r, and the inverse function of

$$\theta(t) := t^{n/p}\varphi(\log(e + t^{1/p}))$$

is convex.

Proof. Denote $\vartheta := \theta^{-1}$. Since $\vartheta''(\theta(t)) = -\frac{1}{\theta'(t)^3}\theta''(t)$ and $\theta(t)$ is an increasing function, it suffices to show that $\theta''(t) < 0$ for all $t > 0$. This is trivially true for $t < t_0$ when we choose $\varphi(r)$ to be constant for all $r < r_0$. On the other hand, notice that $\varphi'(r) = \varphi(r)\frac{C_1C_2}{(-\phi'(\phi^{-1}(C_2r))\phi^{-1}(C_2r))^n}$ for all sufficiently large r, where $\frac{C_1C_2}{(-\phi'(\phi^{-1}(C_2r))\phi^{-1}(C_2r))^n}$ is a decreasing function by (1.3). Hence $\varphi''(r) \le \varphi'(r) \le \alpha\varphi(r)$ with any $\alpha > 0$ provided that C_1C_2 is chosen small enough depending on α. This estimate combined with a straightforward calculation implies $\theta''(t) < 0$ for sufficiently large t, and hence the claim follows. $\square$

In [N] the generalized dimension of the boundary of a domain Ω satisfying (1.2) is estimated. These estimates imply the next lemma which is crucial for the final arguments in this paper. We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbf{R}^n$.

Lemma 2.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain satisfying the conditions of Theorem 1.2 and set*

$$\Omega_j := \{z \in \Omega : d(z, \partial\Omega) < 2^{-j}\}.$$

Then there is an integer j_0 and a positive constant $C(\beta, n)$ such that

$$\sum_{j=j_0}^{\infty} |\Omega_j \setminus \Omega_{j+1}| \exp\Big(C(\beta, n)\int_{[2^{-j}, 2^{-j_0}]} \frac{dt}{(-\phi'(t)t)^{n-1}t}\Big) < \infty.$$

Proof. Given a sequence (a_j) of positive numbers with $\sum_{j=1}^{\infty} a_j < \infty$, set $r_m = \sum_{j=m}^{\infty} a_j$. Then

$$\sum_{j=1}^{\infty} \frac{a_j}{\sqrt{r_j}} < \infty. \tag{2.1}$$

To see this notice that

$$\sqrt{r_j} + \sqrt{r_{j+1}} < 2\sqrt{r_j}$$

and hence

$$\frac{a_j}{\sqrt{r_j}} = \frac{(\sqrt{r_j} - \sqrt{r_{j+1}})(\sqrt{r_j} + \sqrt{r_{j+1}})}{\sqrt{r_j}} < 2(\sqrt{r_j} - \sqrt{r_{j+1}}).$$

We set $a_j = |\Omega_j \setminus \Omega_{j+1}|$. By [N] we know that there is an integer j_0 and a positive constant $\tilde{C}(\beta, n)$ such that

$$|\Omega_j| \le C\exp\Big(-\tilde{C}(\beta, n)\int_{[2^{-j}, 2^{-j_0}]} \frac{dt}{(-\phi'(t)t)^{n-1}t}\Big) \tag{2.2}$$

for all $j > j_0$. Combining (2.1) and (2.2) we obtain that

$$\begin{aligned}\infty &> \sum_{j=j_0}^{\infty} \frac{|\Omega_j \setminus \Omega_{j+1}|}{\sqrt{|\Omega_j|}} \\ &\geq C \sum_{j=j_0}^{\infty} |\Omega_j \setminus \Omega_{j+1}| \exp\Big(\frac{1}{2}\tilde{C}(\beta,n)\int_{[2^{-j},2^{-j_0}]} \frac{dt}{(-\phi'(t)t)^{n-1}t}\Big).\end{aligned}$$

□

Proof of Theorem 1.2.

Let $\mathcal{W}$ be a Whitney decomposition of Ω' and let $Q \in \mathcal{W}$. Denote by x_0 the center of Q. Let $\varepsilon = \varepsilon(n,K)$ be as in Lemma 2.3 and set $p = n + \frac{\varepsilon}{2}$. Define the function θ as in Lemma 2.4 and choose the constants C_1, C_2 so small that θ^{-1} is convex. Then we have by Jensen's inequality that

$$\begin{aligned}\frac{1}{|Q|}\int_Q |Df|^n \varphi(\log(e+|Df|))dx &= \theta(\theta^{-1}(\frac{1}{|Q|}\int_Q |Df|^n \varphi(\log(e+|Df|))dx)) \\ &\leq \theta(\frac{1}{|Q|}\int_Q |Df|^p dx). \qquad (2.3)\end{aligned}$$

By Lemma 2.3 we obtain

$$\theta(\frac{1}{|Q|}\int_Q |Df|^p dx) \leq \theta(\frac{c_1}{|Q|}\int_Q a_f(x)^p dx) \qquad (2.4)$$

with a constant c_1 depending only on n and K. As in [AK, 3.4] we have that $a_f(x) \leq C a_f(y)$ for all $x, y \in Q$, and hence

$$\theta(\frac{c_1}{|Q|}\int_Q a_f(x)^p dx) \leq \theta(c_2 a_f(x_0)^p) \leq c_3 a_f(x_0)^n \varphi(\log(e + c_4 a_f(x_0))). \quad (2.5)$$

Applying Lemma 2.3 again, we obtain that

$$c_3 a_f(x_0)^n \varphi(\log(e + c_4 a_f(x_0))) \leq \frac{c_5}{|Q|}\int_Q |Df|^n \varphi(\log(e + c_6 a_f(x)))dx. \quad (2.6)$$

Since the inequalities (2.3), (2.4), (2.5) and (2.6) hold for all cubes $Q \in \mathcal{W}$, we arrive at

$$\int_{\Omega'} |Df|^n \varphi(\log(e+|Df|))dx \leq c_5 \int_{\Omega'} |Df|^n \varphi(\log(e + c_6 a_f(x)))dx. \quad (2.7)$$

Combining (2.7) and Lemma 2.2 we write

$$\begin{aligned}\int_{\Omega'} |Df|^n \varphi(\log(e+|Df|))dx &\leq c_5 \int_{\Omega'} |Df|^n \varphi\Big(\log\Big(e + c_7 \frac{d(f(x),\partial\Omega)}{d(x,\partial\Omega')}\Big)\Big)dx \\ &\leq c_5 \int_{\Omega'} K J_f \varphi\Big(\log\Big(e + c_7 \frac{d(f(x),\partial\Omega)}{d(x,\partial\Omega')}\Big)\Big)dx \\ &\leq c_8 \int_{\Omega} \varphi\Big(\log\Big(e + c_7 d(y,\partial\Omega)\frac{1}{d(f^{-1}(y),\partial\Omega')}\Big)\Big)dy\end{aligned}$$

where $c_8 = c_8(n, K)$ and $c_7 = c_7(n, K)$. Furthermore, since Ω is bounded and $k_{\Omega'}(x, x_0) \geq \log \frac{d(x_0,\partial\Omega')}{d(x,\partial\Omega')}$ and quasiconformal maps are quasi-isometries for large distances in the quasihyperbolic metrics (see [GO, p. 62]), we deduce that

$$c_8 \int_\Omega \varphi\Big(\log \Big(e + c_7 d(y, \partial\Omega) \frac{1}{d(f^{-1}(y), \partial\Omega')} \Big) \Big) dy$$

$$\leq c_8 \int_\Omega \varphi\Big(\log \Big(e + c_7 \frac{d(y, \partial\Omega)}{d(f^{-1}(y_0), \partial\Omega')} \exp(k_{\Omega'}(f^{-1}(y), f^{-1}(y_0))) \Big) \Big) dy$$

$$\leq c_8 \sum_{j=j_0}^{\infty} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi\Big(C k_\Omega(y, y_0) \Big) dy + \tilde{C}$$

for sufficiently large j_0. Here Ω_j is defined as in Lemma 2.5 and the constant C depends only on n and K. Finally, we choose j_0 so large that $2^{j_0} \geq d(y_0, \partial\Omega)$ and $\phi(2^{-j_0}) \geq C_0$ and thus by combining (1.2), (1.4) and (1.7) with the previous calculations, we obtain the following chain of inequalities:

$$\int_{\Omega'} |Df|^n \varphi(\log(e + |Df|)) dx \leq c_8 \sum_{j=j_0}^{\infty} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi\Big(C k_\Omega(y, y_0) \Big) dy + \tilde{C}$$

$$\leq c_8 \sum_{j=j_0}^{\infty} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi\Big(C\phi\Big(\frac{d(y, \partial\Omega)}{d(y_0, \partial\Omega)} \Big) + CC_0 \Big) dy + \tilde{C}$$

$$\leq c_8 \sum_{j=j_0}^{\infty} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi\Big(2C\phi(2^{-4j}) \Big) dy + \tilde{C}$$

$$\leq c_8 \sum_{j=j_0}^{\infty} \int_{\Omega_j \setminus \Omega_{j+1}} \varphi\Big(2C\beta^2\phi(2^{-j}) \Big) dy + \tilde{C}$$

$$\leq c_8 \sum_{j=j_0}^{\infty} |\Omega_j \setminus \Omega_{j+1}| \exp\Big(C_1 \int_{[\phi^{-1}(C_2 2C\beta^2\phi(2^{-j})),\ t_1]} \frac{dt}{(-\phi'(t)t)^{n-1}t} \Big) + \tilde{C}.$$

The last sum above converges by Lemma 2.5 if $C_2 \leq \frac{1}{2C\beta^2}$ and $C_1 \leq C(\beta, n)$ and $t_1 = 2^{-j_0}$. This completes the proof. □

Note that as a by-product we establish an essentially sharp integrability condition for the quasihyperbolic metric k_Ω in the domain Ω satisfying (1.2) with some ϕ. Indeed, suppose that the assumptions of Theorem 1.2 hold. Then the estimates above imply

$$\int_\Omega \varphi(k_\Omega(y, y_0)) dy < \infty$$

with the function

$$\varphi(r) = \exp\Big(C_1 \int_{[\phi^{-1}(C_2 r), t_1]} \frac{dt}{(-\phi'(t)t)^{n-1}t} \Big).$$

This improves on [KN], where the case $\phi(t) = C(\log \frac{1}{t})^{\frac{n}{n-1}}$ was considered.

Proof of Corollary 1.3. It follows from (1.9) that

$$d(f(x), \partial\Omega) \leq \psi(1 - |x|). \tag{2.8}$$

for all $x \in B^n$ sufficiently close to the boundary. Combining (2.8) with the quasi-isometry property of f we obtain

$$\begin{aligned} k_\Omega(f(x), f(0)) &\leq C(K,n) k_{B^n}(x, 0) \\ &= C(K,n) \log \frac{1}{1 - |x|} \\ &\leq C(K,n) \log \frac{1}{\psi^{-1}(d(f(x), \partial\Omega))} \\ &\leq C(K,n)\beta \log \frac{1}{\psi^{-1}(\frac{d(f(x),\partial\Omega)}{d(f(0),\partial\Omega)})} \end{aligned}$$

for all x sufficiently close to the boundary ∂B^n. Thus we conclude that $\Omega = f(B^n)$ is a bounded domain which satisfies the growth condition (1.2) with the function

$$\phi(t) = C(\beta, K, n) \log \frac{1}{\psi^{-1}(t)}. \tag{2.9}$$

The claim now follows by Theorem 1.2. The assumption (1.8) guarantees that also the divergence condition (1.5) holds. $\square$

3 Sharpness of the results

To show the essential sharpness of Theorem 1.2, we construct a simply connected domain $\Omega \subset \mathbf{R}^2$ such that (1.2) holds, but the integral

$$\int_{B^2} |Df|^2 \varphi(\log(e + |Df|)) dx \tag{3.1}$$

diverges for some (quasi)conformal $f : B^2 \to \Omega$ with

$$\varphi(r) = \exp\Big(\tilde{C}_1 \int_{[\phi^{-1}(\tilde{C}_2 r),\ t_1]} \frac{dt}{-\phi'(t) t^2}\Big). \tag{3.2}$$

Let ϕ be a function satisfying the conditions of Theorem 1.2 and let $c = c(\beta) \geq 1$. By condition (1.3) we can take j_0 to be the smallest integer such that $\frac{c}{-\phi'(r) r} \leq \frac{1}{16}$ for all $r \leq 2^{-j_0}$. Let $\alpha :]0,1[\to]0, \infty[$,

$$\alpha(t) = \frac{c}{-\phi'(t)}.$$

Let $Q_{j_0} = \{x \in \mathbf{R}^2 : |x_i| < 2^{-j_0-1} \text{ for } i = 1, 2\}$, and denote the side length of Q_{j_0} by $r_{j_0} = 2^{-j_0}$. Let Ω^{j_0} be the $\alpha(r_{j_0})$-neighborhood of the coordinate axes in Q_{j_0}. Let $Q_{j_0+1} = Q_{j_0} \setminus \Omega^{j_0}$. Now Q_{j_0+1} consists of 4 squares with side lengths $r_{j_0+1} = \frac{1}{2} r_{j_0}(1 - \frac{2c}{-\phi'(r_{j_0})r_{j_0}})$. Denote the components of Q_{j_0+1} by $Q^l_{j_0+1}$. Let Ω^{j_0+1} be the union of the $\alpha(r_{j_0+1})$-neighborhoods of the centered coordinate axes in the squares $Q^l_{j_0+1}$. Then let $Q_{j_0+2} = Q_{j_0+1} \setminus \Omega^{j_0+1}$. Now Q_{j_0+2} consists of 4^2 squares with side lengths $r_{j_0+2} = (\frac{1}{2})^2 r_{j_0}(1 - \frac{2c}{-\phi'(r_{j_0})r_{j_0}})(1 - \frac{2c}{-\phi'(r_{j_0+1})r_{j_0+1}})$. Define for every $k \geq j_0 + 2$ the sets Ω^k and Q_k accordingly. Now the set Ω^k consists of the $\alpha(r_k)$-neighborhoods of the centered coordinate axes in the squares $Q^l_k, l = 1, 2, ..., 4^{k-j_0}$, with side lengths

$$r_k = r_{j_0}(\frac{1}{2})^{k-j_0} \prod_{i=j_0}^{k-1} (1 - \frac{2c}{-\phi'(r_i)r_i}).$$

A trivial estimation $r_i \leq 2^{-i}$ implies

$$r_k \geq 2^{-k} \prod_{i=j_0}^{k-1} (1 - \frac{2c}{-\phi'(2^{-i})2^{-i}}) \geq 2^{-k} \exp\Big(- 4c \int_{[2^{-k}, 2^{-j_0}]} \frac{dt}{-\phi'(t)t^2}\Big) \quad (3.3)$$

since $-\phi'(t)t$ is a decreasing function and $\prod(1 - a_i) \geq \exp(-2 \sum a_i)$ for $0 < a_i \leq \frac{1}{8}$.

Define a domain $\tilde{\Omega}$ by setting

$$\tilde{\Omega} = \bigcup_{k=1}^{\infty} \Omega^k.$$

Now, the domain $\tilde{\Omega}$ must be modified slightly to make it simply connected. This can be done easily by closing certain gates in the construction. We leave this task to the reader. The resulting domain Ω is simply connected and hence there exists a conformal mapping $f : B^2 \to \Omega$. Furthermore, a straightforward calculation shows that the growth condition (1.2) holds in Ω provided that c is chosen large enough depending only on β.

Next we show that the integral (3.1) diverges. Notice first that $a_f(x) = |f'(x)|$ for the conformal mapping f and hence Lemma 2.2 implies

$$\begin{aligned} \int_{B^2} |Df|^2 \varphi(\log(e + |Df|))dx &\geq \int_{B^2} |Df|^2 \varphi\Big(\log \Big(e + C_0 \frac{d(f(x), \partial\Omega)}{d(x, \partial B^2)}\Big)\Big)dx \\ &= \int_{\Omega} \varphi\Big(\log \Big(e + C_0 d(y, \partial\Omega) \frac{1}{d(f^{-1}(y), \partial B^2)}\Big)\Big)dy \\ &= \int_{\Omega} \varphi\Big(\log \Big(e + C_0 d(y, \partial\Omega) \exp(k_{B^2}(f^{-1}(y), 0))\Big)\Big)dy. \end{aligned} \quad (3.4)$$

By the quasi-isometry property of f we deduce that

$$\begin{aligned}
&\int_\Omega \varphi\Big(\log\Big(e+C_0 d(y,\partial\Omega)\exp(k_{B^2}(f^{-1}(y),0))\Big)\Big)dy \\
&\quad\geq \int_\Omega \varphi\Big(\log\Big(e+C_0 d(y,\partial\Omega)\exp(Ck_\Omega(y,y_0))\Big)\Big)dy \\
&\quad\geq \sum_{j=j_0+1}^{\infty} 4^{j-j_0} r_j\alpha(r_j)\varphi\Big(\log\Big(e+C_0\frac{\alpha(r_j)}{2}\exp(C\sum_{i=j_0}^{j-1}\frac{r_i}{\alpha(r_i)})\Big)\Big). \qquad (3.5)
\end{aligned}$$

Notice that properties (1.3) and (1.4) imply $\alpha(r)\geq r^2$ for all small r. Combining this estimate with (1.3), (3.3) and the assumption $\frac{c}{-\phi'(r)r}\leq\frac{1}{16}$ we obtain that

$$\begin{aligned}
&\sum_{j=j_0+1}^{\infty} 4^{j-j_0} r_j\alpha(r_j)\varphi\Big(\log\Big(e+C_0\frac{\alpha(r_j)}{2}\exp(C\sum_{i=j_0}^{j-1}\frac{r_i}{\alpha(r_i)})\Big)\Big) \\
&\quad\geq \sum_{j=j_0+1}^{\infty}\frac{c4^{j-j_0}r_j^2}{-\phi'(r_j)r_j}\varphi\Big(\log\Big(e+\tilde{C}_0 r_j^2\exp(C\sum_{i=j_0}^{j-1}-\phi'(2^{-i})2^{-i})\Big)\Big) \\
&\quad\geq \sum_{j=2j_0}^{\infty}\frac{c4^{-j_0}}{-\phi'(r_j)r_j}\exp\Big(-8c\int_{[2^{-j},2^{-j_0}]}\frac{dt}{-\phi'(t)t^2}\Big)\varphi\Big(\tilde{C}\phi(2^{-j})\Big). \qquad (3.6)
\end{aligned}$$

The last sum diverges by (1.3) and (1.5) when the constants $\tilde{C}_1,\tilde{C}_2$ in (3.2) are chosen large enough. The desired conclusion then follows by combining (3.4), (3.5) and (3.6).

Note that if the divergence condition (1.5) fails for the function ϕ, then for any given increasing, unbounded function g we find a domain Ω, by the construction above, satisfying (1.2) and a (quasi)conformal mapping $f: B^2\to\Omega$ such that

$$\int_{B^2}|Df|^2 g(|Df|)dx=\infty.$$

This example shows also the essential sharpness of Corollary 1.3 in $\mathbf{R}^2$. Indeed, since the domain Ω satisfies the growth condition (1.2), the quasiconformal mapping $f: B^2\to\Omega$ has the modulus of continuity $\psi(t)=C\phi^{-1}(C\log\frac{1}{t})$ (see [HK] for a detailed discussion), and in this case Theorem 1.2 and Corollary 1.3 give us essentially equivalent results.

Let us point out that this example also shows the essential sharpness of the integrability condition of the quasihyperbolic metric in the domain Ω discussed at the end of Section 2.

We promised to discuss further the special case of a Hölder domain Ω satisfying (1.2) with the function $\phi(t)=\frac{1}{\varepsilon}\log\frac{1}{t}$. Indeed, the construction above with the choice $\alpha(t) = c\varepsilon t$, where c is large enough but independent of ε, gives us exactly such a domain. The calculations above imply that

$$\int_{B^2} |Df|^2 \varphi(\log(e + |Df|))dx = \infty$$

for the function $\varphi(t) = \exp(\tilde{C}\varepsilon^2 t)$, when the constant $\tilde{C}$ is chosen large enough independently of ε. Thus we see that the dependency on the constant ε implied by Theorem 1.2 is sharp.

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Tomi Nieminen, University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35, FIN-40014 Jyväskylä, Finland.

E-mail address: `tominiem@maths.jyu.fi`