GEOMETRIC RIGIDITY OF A CLASS OF FRACTAL SETS

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ABSTRACT. We study geometric rigidity of a class of fractals, which is slightly larger than the collection of self-conformal sets. Namely, we shall prove that a set of this class is contained in a smooth submanifold or is totally spread out.

1. Introduction

We study limit sets of certain iterated function systems on $\mathbb{R}^d$. A self-conformal set is a limit set of an iterated function system in which the mappings are conformal on a neighborhood of the limit set. To define the class of limit sets we are interested in, we use mappings that are required to be conformal only on the limit set. With the conformality here, we mean that the derivative of the mapping is an orthogonal transformation. This class is larger than the collection of self-conformal sets.

To illustrate the type of results we are interested in, we recall the following known theorems dealing with self-conformal sets. The latter one is a generalization of Mattila's rigidity theorem for self-similar sets ([5, Corollary 4.3]). The method we use in this paper delivers a new proof and generalization of these theorems. To find other rigidity results of similar kind, the reader is referred to [6] and [10]. Let E be a self-conformal set, $\mathcal{H}^t$ denote the t-dimensional Hausdorff measure, and $\dim_{\mathrm{T}}$ and $\dim_{\mathrm{H}}$ be the topological dimension and the Hausdorff dimension, respectively.

Theorem 1.1 (Mayer and Urbański [8, Corollary 1.3]). *Suppose $l = \dim_{\mathrm{T}}(E)$. Then either*

(1) $\dim_{\mathrm{H}}(E) > l$ or

(2) E is contained in an l-dimensional affine subspace or an l-dimensional geometric sphere whenever d exceeds 2 and if d equals 2, E is contained in an analytic curve.

Theorem 1.2 ([3, Theorem 2.1]). *Suppose $t = \dim_{\mathrm{H}}(E)$ and $0 < l < d$. Then either*

(1) $\mathcal{H}^t(E \cap M) = 0$ for every l-dimensional C^1-submanifold $M \subset \mathbb{R}^d$ or

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(2) E is contained in an l-dimensional affine subspace or an l-dimensional geometric sphere whenever d exceeds 2 and if d equals 2, E is contained in an analytic curve.

Our aim is to prove results of similar kind for the previously mentioned class of limit sets. We define the class rigorously in the next chapter.

2. Class of fractal sets

We consider the sets obtained as geometric projections of the symbol space I^∞: Take a finite set I with at least two elements and set $I^* = \bigcup_{n=1}^\infty I^n$ and $I^\infty = I^{\mathbb{N}}$. If $\mathtt{i} \in I^*$ and $\mathtt{j} \in I^* \cup I^\infty$, then with the notation $\mathtt{i}, \mathtt{j}$ we mean the element obtained by juxtaposing the terms of $\mathtt{i}$ and $\mathtt{j}$. The *length* of $\mathtt{i}$, that is, the number of terms in $\mathtt{i}$, is denoted by $|\mathtt{i}|$. Let $X \subset \mathbb{R}^d$ be a compact set and choose a collection $\{X_\mathtt{i} : \mathtt{i} \in I^*\}$ of nonempty closed subsets of X satisfying

(L1) $X_{\mathtt{i},i} \subset X_\mathtt{i}$ for every $\mathtt{i} \in I^*$ and $i \in I$,
(L2) $\operatorname{diam}(X_\mathtt{i}) \to 0$ as $|\mathtt{i}| \to \infty$.

Now the *projection mapping* is the function $\pi \colon I^\infty \to X$ for which

$$\{\pi(\mathtt{i})\} = \bigcap_{n=1}^\infty X_{\mathtt{i}|_n}$$

when $\mathtt{i} \in I^\infty$. The compact set $E = \pi(I^\infty)$ is called a *limit set*.

Since this setting is too general to study the geometry, we assume the limit set is constructed by using the sets of the form $X_\mathtt{i} = \varphi_\mathtt{i}(X)$, where $\varphi_\mathtt{i} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_{|\mathtt{i}|}}$ for $\mathtt{i} = (i_1, \ldots, i_{|\mathtt{i}|}) \in I^*$ and the mappings φ_i belong into the following category: Suppose $\Omega' \subset \mathbb{R}^d$ is open and Ω is open and bounded such that $\overline{\Omega} \subset \Omega'$ and $X \subset \Omega$. We consider mappings $\varphi \in C^2(\Omega')$ for which $\varphi(X) \subset X$ and

(F1) there exist constants $0 < \underline{s}, \overline{s} < 1$ for which $\overline{s}^2 \le \underline{s}$ and
$$\underline{s} \le |(\varphi'(x))^{-1}|^{-1} \le |\varphi'(x)| \le \overline{s}$$
when $x \in \Omega$,

(F2) the derivative of φ is an orthogonal transformation on E, that is,
$$|(\varphi'(x))^{-1}|^{-1} = |\varphi'(x)|$$
when $x \in E$.

Here $|\cdot|$ denotes the usual operator norm for linear mappings. Furthermore, we set $||\varphi_\mathtt{i}'|| = \sup_{x\in\Omega} |\varphi_\mathtt{i}'(x)|$.

For example, each contractive conformal mapping satisfies both assumptions (F1) and (F2). At first glance, it might feel that requiring mappings that define the limit set to be conformal on the limit set, to be very restrictive assumption for nonconformal mappings. In the following, we shall give an example of a nonconformal setting.

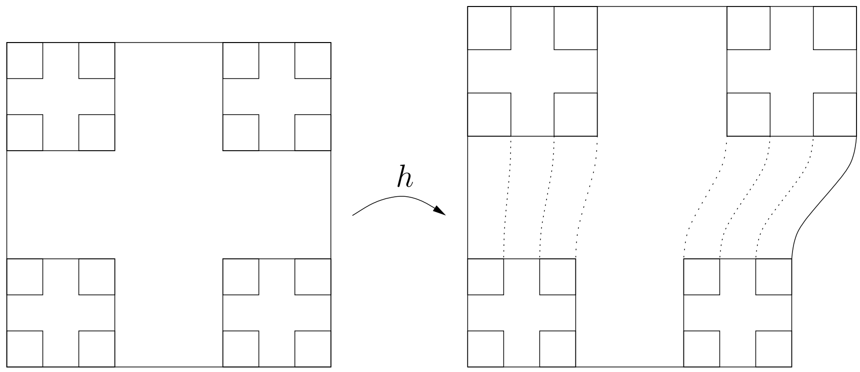

Figure 1. A nonconformal example.

Example 2.1. Suppose the mappings $\varphi_1, \dots, \varphi_k$ defined on an open set $\Omega' \subset \mathbb{R}^d$ are conformal and contractive on an open and bounded set Ω for which $\overline{\Omega} \subset \Omega'$. Assume also that there is a compact set $X \subset \Omega$ such that $\varphi_i(X) \subset X$ for each $i \in \{1, \dots, k\}$. The limit set E associated to this setting is called a *self-conformal set*. Furthermore, we require that $\max_i ||\varphi_i'||^2 \max_i ||(\varphi_i^{-1})'|| < 1$.

Next choose $\max_i ||\varphi_i'|| < \overline{s} < 1$ and $0 < \underline{s} < (\max_i ||(\varphi_i^{-1})'||)^{-1}$ such that $\overline{s}^2 < \underline{s}$. Suppose $h \colon \mathbb{R}^d \to \mathbb{R}^d$ is a C^2 diffeomorphism such that it is conformal on E. We assume also that

$$1 \le ||h'||\,||(h^{-1})'|| \le \min\left\{ \frac{\overline{s}}{\max_i ||\varphi_i'||}, \frac{1}{\underline{s}\max_i ||(\varphi_i^{-1})'||} \right\}. \tag{2.1}$$

Define $\tilde{\varphi}_i = h \circ \varphi_i \circ h^{-1}$ for every $i \in \{1, \dots, k\}$ and set $\tilde{\Omega}' = h(\Omega')$, $\tilde{\Omega} = h(\Omega)$, and $\tilde{X} = h(X)$. Since $\tilde{\varphi}_i(\tilde{X}) \subset \tilde{X}$ for every i, the assumption (L1) is satisfied for the collection $\{\tilde{\varphi}_{\mathtt{i}}(\tilde{X}) : \mathtt{i} \in I^* \}$. We claim that also the assumption (L2) is satisfied and the mappings $\tilde{\varphi}_i$ satisfy the assumptions (F1) and (F2). To see this, notice that

$$\begin{aligned} |\tilde{\varphi}_i'(x)| &\le |(h'(h^{-1}(x)))^{-1}||h'(\varphi_i \circ h^{-1}(x))||\varphi_i'(h^{-1}(x))|, \\ |(\tilde{\varphi}_i'(x))^{-1}| &\le |h'(h^{-1}(x))||(h'(\varphi_i \circ h^{-1}(x)))^{-1}||(\varphi_i'(h^{-1}(x)))^{-1}| \end{aligned} \tag{2.2}$$

for every $x \in \tilde{\Omega}$. The condition (F1), and hence also the condition (L2), can now be verified by using (2.1). Denoting the limit set associated to this setting with $\tilde{E}$, it is straightforward to see that $\tilde{E} = h(E)$. Assumptions on h guarantee that equations in (2.2) hold with an equality provided that $x \in \tilde{E}$. Therefore also (F2) holds.

The class of limit sets obtained by this method clearly includes all the self-conformal sets. Since the collection of mappings that generate the limit set is not necessarily unique, we shall next give an example of a self-conformal set E and a mapping h such that there are no conformal mappings having $h(E)$ as the limit set. Let E be the usual Cantor dust on $\mathbb{R}^3$, that is, $E = C^3$, where C is the middle third Cantor set on the unit interval. Define $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(x,y,z) = g(z)(x,y,z)$, where g is an increasing C^2 function with the following properties: $g' < c_1$, $g \equiv 1$ on $[0,\frac{1}{3}]$ and $g \equiv c_2$ on $[\frac{2}{3},1]$, see Figure 1. Now, with suitable choices of $0 < \underline{s} < \frac{1}{3}$, $\frac{1}{3} < \overline{s} < 1$, $c_1 > 0$, and $c_2 > 1$, the mapping h satisfies the condition (2.1). If the set $h(E)$ were a limit set of a collection of conformal mappings, it would be invariant with respect to these mappings. Hence there exists a conformal mapping taking a cylinder set small enough (if Ω is connected, then a first level cylinder would suffice) to the whole set $h(E)$ such that the image of a 2-dimensional affine subspace containing one side of the small cylinder set includes sides of two first level cylinder sets located in two distinct 2-dimensional affine subspaces (the sides on the right in Figure 1). According to Liouville's Theorem this is not possible. Therefore, the class of limit sets obtained by this method is strictly larger than the collection of all self-conformal sets.

To avoid too much overlapping among the sets $\varphi_{\mathtt{i}}(X)$, we assume the open set condition, that is, $\varphi_i\big(\mathrm{int}(X)\big) \cap \varphi_j\big(\mathrm{int}(X)\big) = \emptyset$ for $i \neq j$, and the existence of $\varrho_0 > 0$ for which

$$\inf_{x\in\partial X}\inf_{0<r<\varrho_0} \frac{\mathcal{H}^d\big(B(x,r)\cap \mathrm{int}(X)\big)}{\mathcal{H}^d\big(B(x,r)\big)} > 0, \tag{2.3}$$

where ∂X denotes the boundary of X. These assumptions are crucial in determining the so called conformal measure, see, for example, [2], [7] and [4] . From now on, without mentioning it explicitly, this is the setting we are working with.

As a consequence of the assumption (F1), we have the following proposition. Observe that the assumption (F2) is not needed here.

Proposition 2.2 (Falconer [1, Proposition 4.3]). *There exists a constant $c > 0$ such that*

$$|\varphi_{\mathtt{i}}'(x) - \varphi_{\mathtt{i}}'(y)| \le c|\varphi_{\mathtt{i}}'(x)||x-y|$$

for every $\mathtt{i} \in I^*$ and $x, y \in \Omega$.

As a corollary, Falconer [1, Corollary 4.4] shows that there exists a bounded function $1 \le K(t) \le K_0$, $K(t) \to 1$ as $t \to 0$, such that

$$\begin{aligned} |\varphi_{\mathtt{i}}'(x)| &\le K(|x-y|)\,|\varphi_{\mathtt{i}}'(y)|, \\ |(\varphi_{\mathtt{i}}'(x))^{-1}|^{-1} &\le K(|x-y|)\,|(\varphi_{\mathtt{i}}'(y))^{-1}|^{-1} \end{aligned} \tag{2.4}$$

for every $\mathtt{i} \in I^*$ and $x, y \in \Omega$. In the following, $B(a,r)$ denotes the open ball centered at $a \in \mathbb{R}^d$ with radius $r > 0$. The closed ball is denoted by $\overline{B}(a,r)$ whereas the closure of a given set A is denoted with $\overline{A}$. The boundary of A is denoted by ∂A. Finally, we set $[x,y] = \{\lambda x + (1-\lambda)y : 0 \le \lambda \le 1\}$.

Lemma 2.3. *(1) If $x \in E$, then*

$$B\big(\varphi_{\mathtt{i}}(x), K_0^{-1}|\varphi_{\mathtt{i}}'(x)|r\big) \subset \varphi_{\mathtt{i}}\big(B(x,r)\big)$$

for every $\mathtt{i} \in I^*$ and $0 < r < \operatorname{dist}(E, \partial\Omega)$.

(2) If $x \in X$, then

$$\varphi_{\mathtt{i}}\big(B(x,r)\big) \subset B\big(\varphi_{\mathtt{i}}(x), ||\varphi_{\mathtt{i}}'||r\big)$$

for every $\mathtt{i} \in I^*$ and $0 < r < \operatorname{dist}(X, \partial\Omega)$.

(3) There exists a constant $D \geq 1$ such that

$$\operatorname{diam}\big(\varphi_{\mathtt{i}}(X)\big) \leq D||\varphi_{\mathtt{i}}'||$$

for every $\mathtt{i} \in I^*$.

Proof. Take $x \in E$, $\mathtt{i} \in I^*$, and $0 < r < \operatorname{dist}(E, \partial\Omega)$. Iterating (F2) and using (2.4), we have

$$|\varphi_{\mathtt{i}}'(x)| \leq K(|x-y|)\,|(\varphi_{\mathtt{i}}'(y))^{-1}|^{-1} \tag{2.5}$$

when $y \in \Omega$. Let $r_1 > 0$ be the supremum of all radii for which $B\big(\varphi_{\mathtt{i}}(x), r_1\big) \subset \varphi_{\mathtt{i}}\big(B(x,r)\big)$. Using now the Mean Value Theorem, we find, for each $z, w \in \overline{B}\big(\varphi_{\mathtt{i}}(x), r_1\big)$ and $\theta \in \mathbb{R}^d$, a point $\xi \in [z,w]$ such that

$$\theta \cdot \big(\varphi_{\mathtt{i}}^{-1}(z) - \varphi_{\mathtt{i}}^{-1}(w)\big) = \theta \cdot \big((\varphi_{\mathtt{i}}^{-1})'(\xi)(z-w)\big).$$

Thus, choosing $\theta = (x-y)/|x-y|$, where $y \in \partial B(x,r)$ is such that $\varphi_{\mathtt{i}}(y) \in \partial B\big(\varphi_{\mathtt{i}}(x), r_1\big)$, we get, using (2.5),

$$\begin{aligned} r = |x-y| &= \big|\varphi_{\mathtt{i}}^{-1}\big(\varphi_{\mathtt{i}}(x)\big) - \varphi_{\mathtt{i}}^{-1}\big(\varphi_{\mathtt{i}}(y)\big)\big| \\ &\leq |(\varphi_{\mathtt{i}}^{-1})'(\xi)||\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \\ &= \big|\big(\varphi_{\mathtt{i}}'(\varphi_{\mathtt{i}}^{-1}(\xi))\big)^{-1}\big||\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \\ &\leq K(|\varphi_{\mathtt{i}}^{-1}(\xi) - x|)\,|\varphi_{\mathtt{i}}'(x)|^{-1}|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)|, \end{aligned} \tag{2.6}$$

where $\xi \in [\varphi_{\mathtt{i}}(x), \varphi_{\mathtt{i}}(y)]$. Hence $K_0^{-1}|\varphi_{\mathtt{i}}'(x)|r \leq r_1$, which proves the first claim.

Assume now that $x \in X$ and $0 < r < \operatorname{dist}(X, \partial\Omega)$. Take $z \in \varphi_{\mathtt{i}}\big(B(x,r)\big)$ and $y \in B(x,r)$ for which $z = \varphi_{\mathtt{i}}(y)$. Applying the Mean Value Theorem, we obtain

$$|\varphi_{\mathtt{i}}(x) - z| \leq ||\varphi_{\mathtt{i}}'||\,|x-y| \tag{2.7}$$

from which the second claim follows immediately. Since X is compact and Ω is open and connected, we can cover X by a finite chain of balls in Ω. We assume that there are balls $B_1, \ldots, B_p \subset \Omega$ with $X \subset \bigcup_{i=1}^p B_i$ and $B_i \cap B_{i+1} \neq \emptyset$ for every $i \in \{1,\dots,p-1\}$. Now

$$\begin{aligned} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(X)\bigr) &\le \sum_{i=1}^{p} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(B_i)\bigr) \\ &\le ||\varphi_{\mathtt{i}}'|| \sum_{i=1}^{p} \operatorname{diam}(B_i) \\ &\le p \operatorname{diam}(\Omega) ||\varphi_{\mathtt{i}}'|| \end{aligned}$$

using (2.7). The proof is finished. $\square$

Let $0 < l < d$ be an integer and $G(d,l)$ the collection of all l-dimensional linear subspaces of $\mathbb{R}^d$. The orthogonal projection onto $V \in G(d,l)$ is denoted by P_V. We denote the orthogonal complement of V with $V^\perp \in G(d,d-l)$ and the projection onto that by $Q_V = P_{V^\perp}$. We can metricize $G(d,l)$ by identifying $V \in G(d,l)$ with the projection Q_V and defining for $V, W \in G(d,l)$

$$d(V,W) = |Q_V - Q_W|,$$

where $|\cdot|$ is the usual operator norm for linear mappings. With this metric, $G(d,l)$ is compact. Furthermore, we denote $V + \{x\} = \{v + x : v \in V\}$ for $x \in \mathbb{R}^d$ and $AV = \{Av : v \in V\}$ for a nonsingular linear mapping $A \colon \mathbb{R}^d \to \mathbb{R}^d$.

If $a \in \mathbb{R}^d$, $V \in G(d,l)$, $0 < \delta < 1$, and $r > 0$, we set

$$\begin{aligned} X(a,V,\delta) &= \{x \in \mathbb{R}^d : |Q_V(x-a)| < \delta^{1/2}|x-a|\}, \\ X(a,r,V,\delta) &= X(a,V,\delta) \cap B(a,r), \\ V_a(\delta) &= \{x \in \mathbb{R}^d : |Q_V(x-a)| < \delta\}. \end{aligned}$$

Notice that the closure of $X(a,V,\delta)$ is the complement of $X(a,V^\perp,1-\delta)$. Salli [9] has shown that $d(V,W) = \sup_{x \in V \cap S^{d-1}} \operatorname{dist}(x,W)$. Hence the set $X(0,V,\delta)$ is an open ball in $G(d,l)$ centered at V with radius $\delta^{1/2}$.

3. Geometric rigidity

For the purpose of verifying our main result, we need the following lemma. In the lemma we study images of small angles. We work in the setting described in the previous chapter.

Lemma 3.1. *Suppose $a \in E$, $\mathtt{i} \in I^*$, $0 < l < d$, $0 < \delta < 1$, $\frac{1}{2} \le \varrho < 1$, and $V \in G(d,l)$. Then there exists $r_0 > 0$ depending only on δ and ϱ such that*

$$\varphi_{\mathtt{i}}\bigl(X(a,r,V,\varrho\delta)\bigr) \subset X\bigl(\varphi_{\mathtt{i}}(a), ||\varphi_{\mathtt{i}}'||r, \varphi_{\mathtt{i}}(a)V, \delta\bigr)$$

whenever $0 < r < r_0$.

Proof. First of all, choose $r_0 > 0$ small enough such that $r_0 < \operatorname{dist}(E,\partial\Omega)$. Then by Lemma 2.3(2) we have $\varphi_{\mathtt{i}}\bigl(B(a,r)\bigr) \subset B\bigl(\varphi_{\mathtt{i}}(a), ||\varphi_{\mathtt{i}}'||r\bigr) \subset \Omega$ for every $0 < r < r_0$. Take $0 < r < r_0$ and $x \in X(a,r,V,\varrho\delta)$. Denote $V' = \varphi_{\mathtt{i}}'(a)V$,

$y = P_V(x-a)+a$, and $\theta = Q_{V'}\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)\big)/\big|Q_{V'}\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)\big)\big|$. Using the Mean Value Theorem we choose $\xi \in [x,a]$ such that

$$\begin{aligned}\big|Q_{V'}\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)\big)\big| &= \theta\cdot\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)\big)\\ &= \theta\cdot\big(\varphi_{\mathtt{i}}'(\xi)(x-a)\big).\end{aligned}\tag{3.1}$$

Since $\varphi_{\mathtt{i}}'(a)(y-a)\in V'$, we have

$$\begin{aligned}\big|Q_{V'}\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)\big)\big| &= \big|\theta\cdot\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)-\varphi_{\mathtt{i}}'(a)(x-a)\\ &\qquad -\varphi_{\mathtt{i}}'(a)(y-a)+\varphi_{\mathtt{i}}'(a)(x-a)\big)\big|\\ &\le \big|\theta\cdot\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)-\varphi_{\mathtt{i}}'(a)(x-a)\big)\big|\\ &\qquad +\big|\theta\cdot\big(\varphi_{\mathtt{i}}'(a)(y-a)-\varphi_{\mathtt{i}}'(a)(x-a)\big)\big|\\ &\le |\varphi_{\mathtt{i}}'(\xi)(x-a)-\varphi_{\mathtt{i}}'(a)(x-a)|+|\varphi_{\mathtt{i}}'(a)(x-y)|\end{aligned}\tag{3.2}$$

using (3.1) and the Cauchy-Schwartz inequality. Calculating as in (2.6), we notice that

$$|\varphi_{\mathtt{i}}'(a)||x-a| \le K(|\varphi_{\mathtt{i}}^{-1}(\xi')-a|)\,|\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)|,\tag{3.3}$$

where $\xi'\in[\varphi_{\mathtt{i}}(x),\varphi_{\mathtt{i}}(a)]$. Observe that $|\varphi_{\mathtt{i}}^{-1}(\xi')-a| \le K_0|\varphi_{\mathtt{i}}'(a)|^{-1}|\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)| \le K_0^2|x-a|$ by (2.4). Therefore, when $|x-a|$ is small, also $|\varphi_{\mathtt{i}}^{-1}(\xi')-a|$ is small, and hence, to simplify the notation, we may replace in the following $K(|\varphi_{\mathtt{i}}^{-1}(\xi')-a|)$ with $K(|x-a|)$. Using Proposition 2.2 and (3.3), we obtain

$$\begin{aligned}|\varphi_{\mathtt{i}}'(\xi)(x-a)-\varphi_{\mathtt{i}}'(a)(x-a)| &\le |\varphi_{\mathtt{i}}'(\xi)-\varphi_{\mathtt{i}}'(a)||x-a|\\ &\le c|\varphi_{\mathtt{i}}'(a)||\xi-a||x-a|\\ &\le cK(|x-a|)\,|\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)||x-a|.\end{aligned}\tag{3.4}$$

Using (3.3) we also have

$$\frac{|\varphi_{\mathtt{i}}'(a)(x-y)|}{|\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)|} \le K(|x-a|)\frac{|\varphi_{\mathtt{i}}'(a)||x-y|}{|\varphi_{\mathtt{i}}'(a)||x-a|} \le K(|x-a|)(\varrho\delta)^{1/2}\tag{3.5}$$

and hence, combining (3.2), (3.4), and (3.5), we conclude

$$\frac{\big|Q_{V'}\big(\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)\big)\big|}{|\varphi_{\mathtt{i}}(x)-\varphi_{\mathtt{i}}(a)|} \le K(|x-a|)\big(c|x-a|+(\varrho\delta)^{1/2}\big).$$

Finally, choosing $r_0 \le \delta^{1/2}c^{-1}\big(((\varrho+1)/2)^{1/2}-\varrho^{1/2}\big)$ so small such that $K(t)\le(2/(\varrho+1))^{1/2}$ for all $0<t\le r_0$, we have finished the proof. $\square$

With this geometrical lemma we are able to study tangents of the limit set E. Let m be a Borel measure on $\mathbb{R}^d$, $0<l<d$, and $t>0$. Take $a\in E$ and $V\in G(d,l)$. We say that V is a *weak (t,l)-tangent plane for E at a* if

$$\liminf_{r\downarrow 0}\frac{m\big(B(a,r)\setminus V_a(\delta r)\big)}{r^t}=0$$

for all $0 < \delta < 1$. We also say that V is an *l-tangent plane for E at a* if for every $0 < \delta < 1$ there exists $r_\delta > 0$ such that

$$E \cap B(a,r) \subset X(a,V,\delta)$$

whenever $0 < r < r_\delta$. Furthermore, the set E is said to be *uniformly l-tangential* if every point $a \in E$ has an l-tangent plane and the previously mentioned $r_\delta > 0$ does not depend on a. An application of Whitney's Extension Theorem shows that a uniformly l-tangential set is a subset of an l-dimensional C^1-submanifold, see Proposition 3.3.

For each $\mathtt{i} \in I^*$ and $t \geq 0$ the function $\mathtt{h} \mapsto \big|\varphi'_{\mathtt{i}}\big(\pi(\mathtt{h})\big)\big|^t$ defined on I^∞ is a cylinder function satisfying the chain rule, see [4, Chapter 2], and hence, by the open set condition, (2.3), and [4, Theorems 2.5, 3.7, and 3.8], there exists a Borel probability measure m on E such that for each $\mathtt{i} \in I^*$

$$m\big(\varphi_{\mathtt{i}}(E)\big) = \int_E |\varphi'_{\mathtt{i}}(x)|^t dm(x),$$

where $t = \dim_{\mathrm{H}}(E)$. The measure m is called a *conformal measure*. It can be easily shown that there exists a constant $C > 0$ such that

$$m\big(B(x,r)\big) \geq Cr^t \tag{3.6}$$

for all $x \in E$ and $0 < r < r_0$. Namely, take $\mathtt{i} = (i_1, i_2, \ldots) \in I^\infty$ such that $\pi(\mathtt{i}) = x$ and n to be the smallest integer for which $\varphi_{\mathtt{i}|_n}(E) \subset B(x,r)$. Now, using (F2), (2.4), and Lemma 2.3(3), we obtain

$$\begin{aligned} m\big(B(x,r)\big) \geq m\big(\varphi_{\mathtt{i}|_n}(E)\big) &= \int_E |\varphi'_{\mathtt{i}|_n}(x)|^t dm(x) \\ &= \int_E \big|\varphi'_{\mathtt{i}|_{n-1}}\big(\varphi_{i_n}(x)\big)\big|^t |\varphi'_{i_n}(x)|^t dm(x) \\ &\geq K_0^{-2t} \min_{i \in I} ||\varphi'_i||^t ||\varphi'_{\mathtt{i}|_{n-1}}||^t \\ &\geq D^{-t} K_0^{-2t} \min_{i \in I} ||\varphi'_i||^t \operatorname{diam}\big(\varphi_{\mathtt{i}|_{n-1}}(X)\big)^t, \end{aligned}$$

where $t = \dim_{\mathrm{H}}(E)$. The claim follows since the set $\varphi_{\mathtt{i}|_{n-1}}(X)$ is not included in $B(x,r)$.

We are now ready to prove the main theorem.

Theorem 3.2. *Suppose $t = \dim_{\mathrm{H}}(E)$ and $0 < l < d$. If a point of E has a weak (t,l)-tangent plane, then E is uniformly l-tangential.*

Proof. Let us first sketch the main idea of the proof: Assuming there is a point $x \in E$ with no tangent, we find for each plane W a point $y \in E$ close to x such that the angle between $y - x$ and W is large. Since the set $\{\varphi_{\mathtt{i}}(x) : \mathtt{i} \in I^*\}$ is dense in E, we are able to, using Lemma 3.1, map this setting arbitrary close to any given point in E. Hence, if $a \in E$ has a weak tangent plane V, we obtain an immediate contradiction, since either the image of x or the image of y is not included in a small neighborhood of $V + \{a\}$ provided that W is chosen in the beginning such that the image of W is close to V.

Suppose $a \in E$ has a weak (t,l)-tangent plane V. Assume on the contrary that there exists $x \in E$ such that for every $W \in G(d,l)$ there is $0 < \delta < 1$ such that for each $r_0 > 0$ there exists $0 < r' < r_0$ for which

$$E \cap B(x,r') \setminus X(x,W,\delta) \neq \emptyset. \tag{3.7}$$

Take $\mathtt{i} \in I^\infty$ such that $\pi(\mathtt{i}) = a$. Then clearly $\varphi_{\mathtt{i}|_k}(x) \to a$ as $k \to \infty$. Setting $A_k = \varphi'_{\mathtt{i}|_k}(x)/|\varphi'_{\mathtt{i}|_k}(x)|$ for all $k \in \mathbb{N}$ and using the compactness of $G(d,l)$, we notice $(A_k^{-1}V)_k$ has a subsequence converging to some $W \in G(d,l)$. Denoting the subsequence as the original sequence and setting $W_k = A_k W$, we have $W_k \to V$ as $k \to \infty$. Choose also $0 < \delta < 1$ such that (3.7) holds for this W.

Put $1/(\delta+1) < \varrho < 1$ and let $r_0 = r_0(\varrho, 1-\varrho\delta) < \operatorname{dist}(E, \partial\Omega)$ be as in Lemma 3.1. Then fix $0 < r'' < r_0/2$ such that (3.7) remains satisfied. Choosing $y \in E \cap B(x,r'') \setminus X(x,W,\delta)$ we notice that there exists $0 < \eta < 1$ depending only on ϱ and δ such that

$$B(y,\eta r') \subset B(x,r') \setminus X(x,W,\varrho\delta), \tag{3.8}$$

where $r' = 2|x-y|$. Applying Lemma 3.1 we obtain

$$\begin{aligned}\varphi_{\mathtt{i}|_k}\big(B(x,r') \setminus X(x,W,\varrho\delta)\big) &= \varphi_{\mathtt{i}|_k}\big(\overline{X(x,r',W^\perp,1-\varrho\delta)}\big) \\ &\subset \overline{X\big(\varphi_{\mathtt{i}|_k}(x), ||\varphi'_{\mathtt{i}|_k}||r', W_k^\perp, (1-\varrho\delta)/\varrho\big)} \\ &= B\big(\varphi_{\mathtt{i}|_k}(x), ||\varphi'_{\mathtt{i}|_k}||r'\big) \setminus X\big(\varphi_{\mathtt{i}|_k}(x), W_k, \delta-(1/\varrho-1)\big)\end{aligned} \tag{3.9}$$

whenever $k \in \mathbb{N}$. Hence, using Lemma 2.3(1), (3.8), and (3.9), we have

$$\begin{aligned}B\big(\varphi_{\mathtt{i}|_k}(y), K_0^{-1}|\varphi'_{\mathtt{i}|_k}(y)|\eta r'\big) &\subset \varphi_{\mathtt{i}|_k}\big(B(y,\eta r')\big) \\ &\subset B\big(\varphi_{\mathtt{i}|_k}(x), ||\varphi'_{\mathtt{i}|_k}||r'\big) \setminus X\big(\varphi_{\mathtt{i}|_k}(x), W_k, \delta-(1/\varrho-1)\big)\end{aligned} \tag{3.10}$$

whenever $k \in \mathbb{N}$. Since $W_k \to V$ as $k \to \infty$, we may take k_0 large enough such that $|Q_{W_k} - Q_V| < 2^{-1}(\delta-(1/\varrho-1))^{1/2}$ whenever $k \geq k_0$. Recalling that the set $X(0,V,\delta)$ is an open ball in $G(d,l)$ centered at V with radius $\delta^{1/2}$, we notice, using the triangle inequality, that

$$X\big(\varphi_{\mathtt{i}|_k}(x), V, (\delta-(1/\varrho-1))/4\big) \subset X\big(\varphi_{\mathtt{i}|_k}(x), W_k, \delta-(1/\varrho-1)\big) \tag{3.11}$$

whenever $k \geq k_0$.

Let $r > 0$ and choose n to be the smallest integer for which

$$||\varphi'_{\mathtt{i}|_n}|| < D^{-1}r/2.$$

By choosing $r > 0$ small enough we may assume that $n \geq k_0$. Since by (3.10) and (3.11)

$$\begin{aligned}B\big(\varphi_{\mathtt{i}|_n}(y), K_0^{-1}|\varphi'_{\mathtt{i}|_n}(y)|\eta r'\big) \subset B\big(\varphi_{\mathtt{i}|_n}(x), ||\varphi'_{\mathtt{i}|_n}||r'\big)\setminus \\ X\big(\varphi_{\mathtt{i}|_n}(x), V, (\delta-(1/\varrho-1))/4\big),\end{aligned}$$

this choice gives, using (F2) and (2.4),

$$\begin{aligned}\big|Q_V\big(\varphi_{\mathtt{i}|_n}(x)-\varphi_{\mathtt{i}|_n}(y)\big)\big| &\geq 2^{-1}(\delta-(1/\varrho-1))^{1/2}|\varphi_{\mathtt{i}|_n}(x)-\varphi_{\mathtt{i}|_n}(y)| \\ &\geq 2^{-1}(\delta-(1/\varrho-1))^{1/2}K_0^{-1}|\varphi'_{\mathtt{i}|_n}(y)|\eta r' \\ &\geq 2^{-1}(\delta-(1/\varrho-1))^{1/2}K_0^{-2}\eta r'||\varphi'_{\mathtt{i}|_{n-1}}||\min_{i\in I}|\varphi'_i(y)| \\ &\geq 2^{-1}(\delta-(1/\varrho-1))^{1/2}K_0^{-2}\eta r'\min_{i\in I}|\varphi'_i(y)|D^{-1}r/2 \\ &=: \lambda r,\end{aligned}$$

where $\lambda > 0$ does not depend on r. Assuming now $\operatorname{dist}\big(\varphi_{\mathtt{i}|_n}(x)-a, V\big) \leq \lambda r/2$, we have

$$\begin{aligned}\operatorname{dist}\big(\varphi_{\mathtt{i}|_n}(y)-a, V\big) &\geq \big|Q_V\big(\varphi_{\mathtt{i}|_n}(x)-\varphi_{\mathtt{i}|_n}(y)\big)\big| - \big|Q_V\big(\varphi_{\mathtt{i}|_n}(x)-a\big)\big| \\ &\geq \lambda r - \lambda r/2 = \lambda r/2.\end{aligned}$$

Changing the roles of x and y above, we observe that there exists $z \in \{x,y\}$ such that

$$\operatorname{dist}\big(\varphi_{\mathtt{i}|_n}(z)-a, V\big) \geq \lambda r/2.$$

Since by Lemma 2.3(3)

$$\begin{aligned}\operatorname{dist}\big(\varphi_{\mathtt{i}|_n}(z)-a, V\big) &\leq |\varphi_{\mathtt{i}|_n}(z)-a| \leq \operatorname{diam}\big(\varphi_{\mathtt{i}|_n}(X)\big) \\ &\leq D||\varphi'_{\mathtt{i}|_n}|| < r/2,\end{aligned}$$

we have

$$B\big(\varphi_{\mathtt{i}|_n}(z), \lambda r/8\big) \subset B(a,r) \setminus V_a(\lambda r/8).$$

Therefore, using (3.6),

$$m\big(B(a,r) \setminus V_a(\lambda r/8)\big) \geq C(\lambda/8)^t r^t$$

for all $r > 0$. This contradicts the assumption that V is a weak (t,l)-tangent plane of E at a. □

Let us next discuss applications of this theorem. At first, we study uniformly l-tangential sets of $\mathbb{R}^d$. Our aim is to embed each such a set into a C^1-submanifold.

Proposition 3.3. *If $0 < l < d$ and a closed set $A \subset \mathbb{R}^d$ is uniformly l-tangential, then A is a subset of an l-dimensional C^1-submanifold.*

Proof. Take $a \in A$ and denote the l-tangent plane associated to a point $x \in A$ with V_x. We shall prove that there exists $r_0 > 0$ not depending on a such that

$$A \cap B(a,r_0) \subset X(x, V_a, 1/2) \tag{3.12}$$

whenever $x \in A \cap B(a,r_0)$. From this the claim follows by applying Whitney's Extension Theorem to the bi-Lipschitz mapping $P_{V_a}^{-1}\colon P_{V_a}\big(A \cap \overline{B}(a,r_0)\big) \to V_a^{\perp}$ (we identify $\mathbb{R}^d$ with the direct sum $V_a + V_a^{\perp}$). To prove (3.12), we shall first show that there exists $r_1 > 0$ such that

$$d(V_x, V_a) < 1/8^{1/2} \tag{3.13}$$

for every $x \in A \cap B(a, r_1)$. Suppose this is not true. Then with any choice of $r > 0$ there is $x \in A \cap B(a, r)$ for which $d(V_x, V_a) \geq 1/8^{1/2}$. Recalling that the set $X(0, V, \delta)$ is an open ball in $G(d, l)$ centered at V with radius $\delta^{1/2}$, we infer

$$X(0, V_x, 1/32) \cap X(0, V_a, 1/32) = \emptyset.$$

Hence $x \notin X(a, V_a, 1/32)$ or $a \notin X(x, V_x, 1/32)$. According to the assumptions, both cases are clearly impossible provided that $r > 0$ is chosen small enough.

Observe that (3.13) implies immediately that

$$X(x, V_x, 1/8) \subset X(x, V_a, 1/2)$$

whenever $x \in A \cap B(a, r_1)$. Using the assumptions, we choose $r_2 > 0$ such that

$$A \cap B(x, r_2) \subset X(x, V_x, 1/8).$$

Now, defining $r_0 = \min\{r_1, r_2/2\}$, we have shown (3.12) and therefore finished the proof.

□

The generalizations for Theorems 1.1 and 1.2 are now straightforward.

Corollary 3.4. *Suppose* $l = \dim_{\mathrm{T}}(E)$. *Then either*

(1) $\dim_{\mathrm{H}}(E) > l$ *or*

(2) E is contained in an l-dimensional C^1*-submanifold.*

Proof. The claim follows from [8, Lemma 2.1], Theorem 3.2, Proposition 3.3, and the fact that $\mathcal{H}^t(E) > 0$ as $t = \dim_{\mathrm{H}}(E)$ (see [4, Theorem 3.8]). □

Corollary 3.5. *Suppose* $t = \dim_{\mathrm{H}}(E)$ *and* $0 < l < d$. *Then either*

(1) $\mathcal{H}^t(E \cap M) = 0$ *for every l-dimensional* C^1*-submanifold* $M \subset \mathbb{R}^d$ *or*

(2) E is contained in an l-dimensional C^1*-submanifold.*

Proof. The claim follows from [3, Lemma 2.2], Theorem 3.2, and Proposition 3.3. □

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