Absolute continuity of quasiconformal mappings on curves

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Abstract

We show that a quasiconformal mapping between two proper, locally Ahlfors Q-regular metric spaces, $Q > 1$, is absolutely continuous on almost every curve. We further relax the limes superior in the definition of quasiconformality to a limes inferior and verify that exceptional sets analogous to the Euclidean setting can be allowed.

1 Introduction

The history of various definitions of quasiconformality is long. Grötzsch and Teichmüller considered smooth mappings in the 1920's and 1930's and used an analytic definition. In 1954, Ahlfors [1] initiated the study of non-smooth quasiconformal mappings in the plane using a geometric definition. We take as our starting point the following metric definition.

A homeomorphism $f\colon X \to X'$ between metric spaces (X, d) and (X', d') is said to be quasiconformal if it satisfies

$$H_f(x) := \limsup_{r\to 0} H_f(x,r) \le H < \infty \tag{1}$$

for all $x \in X$ for some H independent of x, where

$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)},$$

$$L_f(x,r) = \sup\{d'(f(x), f(y)) : \ d(x,y) \le r\}$$

and

$$l_f(x,r) = \inf\{d'(f(x), f(y)) : \ d(x,y) \ge r\}.$$

The equivalence of the metric (and geometric) definition in $\mathbb{R}^n$, $n \ge 2$, with the analytic definition which requires that $f \in W^{1,n}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ and that $|Df(x)|^n \le KJ_f(x)$ almost everywhere, was established by Gehring in the 1960's [8],[9]. Especially, Gehring proved that a quasiconformal mapping is absolutely continuous on almost

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all lines by a technique that goes back to Menchoff [25]. This method relies on the foliation of $\mathbb{R}^n$ by parallel lines.

The first to consider quasiconformal mappings in a non-Riemannian setting was Mostow [26], [27]. Various definitions of quasiconformality on the Heisenberg and Carnot groups were then used by Pansu [29], Korányi and Reimann [21], Bourdon and Pajot [6] and Vodopyanov and Greshnov [32]. Regarding the metric definition, the associated foliation of the space is more complicated in the group setting and the approach used by Gehring faces formidable difficulties. Nevertheless, Mostow [28] and Margulis and Mostow [23] were able to prove an analog of the absolute continuity on almost all lines by a variant of this technique. It then followed that a version of the analytic definition was also satisfied. An alternate approach was given by Heinonen and Koskela [13]. This was based on first proving that, for a Carnot group $X = X'$, quasiconformality guarantees the global condition of quasisymmetry, i.e. that $H_f(x,r) \leq H' < \infty$ for all x, r, which allowed one to invoke the regularity results by Pansu [29]. This method was shown in [14] to be robust enough to extend to a class of Ahlfors Q-regular metric spaces that support a suitable Poincaré inequality. In fact, (localizing) this general setting covers all the above absolute continuity results.

A problem of general interest is then to give minimal assumptions on metric spaces X and X' and on a homeomorphism f between these spaces so as to guarantee absolute continuity on almost all curves or even quasisymmetry. The goal would be to relax the Poincaré inequality or the foliation properties of X by geodesics as well as to relax (1).

In the Euclidean setting, it was shown by Heinonen and Koskela in 1995 that, surprisingly, $H_f(x)$ can be replaced with

$$h_f(x) := \liminf_{r\to 0} H_f(x,r)$$

in the definition of quasiconformality, see [13]. It should come as no surprise that the resulting new definition is easier to verify in practical situations and of importance in complex dynamics [11], [10], [30]. This improvement was obtained by invoking the powerful Besicovitch covering theorem that is in nature Euclidean and already fails in Heisenberg groups [21], [31]. The question whether this improvement could also hold in non-Euclidean settings was then raised. This has remained an open problem until now. The technique from [13] was however shown by Balogh and Koskela [2] to yield an intermediate result where the limit superior can be replaced with a less stringent limit.

Furthermore, in the Euclidean setting, one can allow for an exceptional set in the definition. Already Gehring [8],[9] showed that the uniform boundedness of $H_f(x)$ in the definition of quasiconformality can be relaxed to the assumption that $H_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-dimensional measure and that $H_f(x) \leq H$ almost everywhere. Kallunki and Koskela have recently showed that this also works for h_f [17], [18].

Our first result gives a striking generalization of the above results.

Theorem 1.1 *Let X, X' be proper, locally Ahlfors Q-regular metric spaces, $Q > 1$. Suppose that a homeomorphism $f\colon X \to X'$ satisfies $h_f(x) < \infty$ for all $x \in X \setminus E$,* *where E has σ-finite $(Q-1)$-dimensional Hausdorff measure and that $h_f(x) \leq H < \infty$ almost everywhere. Then $f \in W^{1,1}_{loc}(X; X')$.*

Because each homeomorphism in the Sobolev class $W^{1,1}_{loc}(X; X')$ is absolutely continuous on 1-modulus almost every curve, Theorem 1.1 encompasses all previous results. For the definition of $W^{1,1}_{loc}(X; X')$ see Section 3 below.

The local Ahlfors Q-regularity of a space X requires that X be equipped with a Borel regular measure with the property that, given a compact set A, there are constants $C > 0$ and $\delta > 0$ so that

$$C^{-1}r^Q \leq \mu(B(x,r)) \leq Cr^Q$$

whenever $x \in A$ and $0 < r < \delta$. The properness requires that each closed ball in X be compact.

As pointed out above, no result in terms of h_f was known outside the Euclidean space. Moreover, the earlier results did not allow for an exceptional set even for H_f.

Our proof of Theorem 1.1 is substantially different from the two approaches described above. Gehring's method, together with its variants, require the space X to have a nice foliation by curves. We assume no such structure for X. On the other hand, the technique used in [13], [14] heavily relies on the Poincaré inequality that now gets entirely disposed of, as we assume only the Ahlfors regularity of X and X'. The h_f-result further bases on the Besicovitch covering theorem that surely fails in our generality. Let us mention here two key points in our proof. We prove a new covering theorem that is tailored for our needs and essentially consider all the curves in X at once. As a consequence, we obtain a new proof even in the Euclidean setting that is simpler than the argument in [17], [18].

In the case of Carnot groups, we obtain a much stronger result than Theorem 1.1.

Theorem 1.2 *Let f be a self-homeomorphism of a Carnot group G with homogeneous dimension $Q > 1$ such that $h_f(x) < \infty$ for all $x \in X \setminus E$, where E has σ-finite $(Q-1)$-dimensional Hausdorff measure and that $h_f(x) \leq H < \infty$ almost everywhere. Then f is quasisymmetric, $f \in W^{1,Q}_{loc}(G; G)$, and $|f_*(x)|^Q \leq KJ_f(x)$ almost everywhere.*

Here f_* is the horizontal differential of f at x and J_f is the determinant of f_*.

Theorem 1.1 and Theorem 1.2 are new even for Heisenberg groups [21], [13], [23], [2], even if we replace h_f with H_f. It seems that such a result cannot be established by a variant of the technique used by Gehring, Margulis and Mostow. To comment on the size of the exceptional set in the above statement, recall that the topological dimension of the Heisenberg group $\mathbb{H}^n$, $n \geq 1$, is $2n+1$ and the homogeneous is $2n+2$. The best conclusion one could hope for, from a Gehring type argument, would then be an exceptional set of Euclidean dimension $2n = Q-2$. Notice, however, that there are examples of sets in $\mathbb{H}^n$ of Heisenberg dimension $Q-1$ whose Euclidean dimension is strictly larger than $Q-2$, see [3], [4], [20].

Theorem 1.2 will be obtained as a corollary to our more general results. Indeed, the statement continues to hold when G is replaced with a proper, Q-regular metric space that supports a 1-Poincaré inequality. Regarding the regularity $f \in W^{1,Q}_{loc}$, one only needs to assume, in addition to the 1-Poincaré inequality for the initial space, that the target space be Q-regular. This appears to be a new conclusion even in the Euclidean setting. Our approach also allows for a version of Theorem 1.2 under a p-Poincaré inequality assumption for $1 < p \leq Q$.

The paper is organized as follows. In Section 2 we introduce a new covering lemma and its consequence for quasiconformal mappings. This will allow us to bypass the lack of the Besicovitch covering theorem. We discuss path families in Section 3 and prove a proposition that will be crucial for allowing an exceptional set. Theorem 1.1 is proven in Section 4. Finally, in Section 5, we prove a quasisymmetry result, and deduce Theorem 1.2 from it.

2 Covering lemmas

Let us begin by recalling the usual Vitali covering theorem, see [24], pages 23–25. In what follows, $A \subset\subset X$ requires that the closure of A be compactly contained in X, and, given a ball $B = B(x, r)$ and $\lambda > 0$, we use the notation λB for $B(x, \lambda r)$.

Lemma 2.1 *Let $\mathcal{B}$ be a family of closed balls in a metric space X so that $\cup_{B\in\mathcal{B}} B \subset\subset X$. Then there is a finite or countable sequence $B_i \in \mathcal{B}$ of pairwise disjoint balls such that*

$$\cup_{B\in\mathcal{B}} B \subset \cup_i 5B_i.$$

The following variant of the above Vitali covering theorem will be crucial for us. The point here is that we can ask for more information on the selected sequence when we confine ourselves with only covering the set of centers of the original balls.

Lemma 2.2 *Let $\mathcal{B}$ be a collection of balls $B(x, r_x)$ (open or closed) with $x \in A$ in a metric space X such that*

$$\cup_{B\in\mathcal{B}} B \subset\subset X.$$

Then there exists a finite or countable sequence $B_i = B(x_i, r_i) \in \mathcal{B}$ with the following properties:

(i) $A \subset \cup_i B(x_i, r_i)$.

(ii) If $i \neq j$, $i, j \in \mathbb{N}$, it follows that

either (a) $x_i \in X \setminus B(x_j, r_j)$ *and* $B(x_j, r_j) \setminus B(x_i, r_i) \neq \emptyset$

or (b) $x_j \in X \setminus B(x_i, r_i)$ *and* $B(x_i, r_i) \setminus B(x_j, r_j) \neq \emptyset$.

(iii) $B(x_i, \frac{1}{3}r_i) \cap B(x_j, \frac{1}{3}r_j) = \emptyset$ *when* $i \neq j$.

Proof. Denote $\mathcal{B} = \{B(x, r_x) : x \in A\}$. Let $M = \sup_{x\in A} r_x < \infty$. Set

$$A_1 = \{x \in A : \frac{3}{4}M < r_x \leq M\}.$$

Choose $x_1 \in A_1$, $x_2 \in A_1 \setminus B(x_1, r_{x_1})$, $x_3 \in A_1 \setminus \cup_{i=1}^2 B(x_i, r_{x_i})$ and so on. For some $J \in \mathbb{N}$

$$A_1 \setminus \bigcup_{j=1}^{J} B(x_j, r_{x_j}) = \emptyset$$

since $\cup_{B\in\mathcal{B}} B \subset\subset X$, $r_x \approx M$ for $x \in A_1$ and the balls $B(x_j, \frac{1}{3} r_{x_j})$ are pairwise disjoint. However, our covering construction is not good enough, because (ii) is not necessarily satisfied. For this purpose, set $\mathcal{F}_1 = \{1\}$ and

$$\mathcal{F}_{j+1} = \{i \in \mathcal{F}_j : B(x_i, r_{x_i}) \not\subset B(x_{j+1}, r_{x_{j+1}})\} \cup \{j+1\}.$$

Now $\cup_{j=1}^J B(x_j, r_{x_j}) = \cup_{j\in\mathcal{F}_J} B(x_j, r_{x_j})$ since we only remove those balls which are already covered. Now set $\mathcal{B}_1 = \{B(x_i, r_{x_i}) : i \in \mathcal{F}_J\}$. This family has the property (ii). Property (iii) clearly also holds.

We continue inductively. Once $\mathcal{B}_k$ is chosen, repeat the above construction for

$$A_{k+1} = \{x \in A \setminus (\cup_{i=1}^k \cup_{B\in\mathcal{B}_i} B) : \left(\frac{3}{4}\right)^{k+1} M < r_x \le \left(\frac{3}{4}\right)^{k} M\}$$

to obtain $\mathcal{B}_{k+1}$.

Set $\mathcal{B}_S = \cup_{j=1}^\infty \mathcal{B}_j = \{B_k = B(x_k, r_k) : k \in \mathbb{N}\}$. Clearly by construction $A \subseteq \cup_{B\in\mathcal{B}_S} B$. So let us check property (ii): Let $j \neq i$. If $B_j, B_i \in \mathcal{B}_k$ for some k then (ii) is automatically valid by construction. If $B_j \in \mathcal{B}_k$ and $B_i \in \mathcal{B}_l$, we can assume that $k < l$, then by construction $x_i \notin B_j$ and

$$d(x_i, x_j) \ge r_j > \left(\frac{3}{4}\right)^{k} M \ge \left(\frac{3}{4}\right)^{k-l+1} r_i \ge r_i.$$

Thus $x_j \notin B_i$ and we conclude that (ii) is valid.

To prove (iii), let $B(x_i, r_i), B(x_j, r_j) \in \mathcal{B}_S$. By symmetry we may assume that they satisfy part (a) of (ii). Notice that (a) yields that $d(x_i, x_j) \ge r_j$ and $r_i < d(x_i, x_j) + r_j$. If there were a point $y \in B(x_i, \frac{1}{3} r_i) \cap B(x_j, \frac{1}{3} r_j)$, then we would conclude that

$$d(x_i, x_j) \le \frac{1}{3} r_i + \frac{1}{3} r_j < \frac{1}{3} d(x_i, x_j) + \frac{2}{3} r_j.$$

Then $d(x_i, x_j) < r_j$, which is a contradiction.

The power of Lemma 2.2 is demonstrated by the following result.

Lemma 2.3 *Let $f\colon X \to Y$ be a homeomorphism between metric spaces X and Y and $\mathcal{B} = \{B(x_i, r_i) : i \in \mathbb{N}\}$ a family of (open or closed) balls. Assume that there exists $H \ge 1$ such that for each $B_i = B(x_i, r_i) \in \mathcal{B}$ we have that*

$$B(f(x_i), \frac{1}{H} \operatorname{diam}(f(B_i))) \subset f(B_i)$$

and $\mathcal{B}$ satisfies the condition (ii) of Lemma 2.2. Then for $i \neq j$ we have

$$B\left(f(x_i), \frac{\operatorname{diam}(f(B_i))}{10H^2}\right) \cap B\left(f(x_j), \frac{\operatorname{diam}(f(B_j))}{10H^2}\right) = \emptyset.$$

Proof. Denote by $B_j = B(x_j, r_j)$ and $B_i = B(x_i, r_i)$. Assume that $x_i \notin B_j$ and let $z \in B_j \setminus B_i$. Then $f(x_i) \notin f(B_j)$ and $f(z) \in f(B_j) \setminus f(B_i)$. We have two cases to consider.

Case 1: $d_Y(f(x_i), f(x_j)) > \frac{\operatorname{diam} f(B_i)}{2H}$.

Since $x_i \notin B_j$ we have $d_Y(f(x_i), f(x_j)) > \frac{\operatorname{diam} f(B_j)}{2H}$. This implies by the triangle inequality that

$$B(f(x_i), \frac{\operatorname{diam} f(B_i)}{5H}) \cap B(f(x_j), \frac{\operatorname{diam} f(B_j)}{5H}) = \emptyset$$

Case 2: $d_Y(f(x_i), f(x_j)) \leq \frac{\operatorname{diam} f(B_i)}{2H}$.

Since $f(z) \notin B(f(x_i), \frac{\operatorname{diam} f(B_i)}{H})$ we see that

$$d_Y(f(z), f(x_j)) + d_Y(f(x_i), f(x_j)) \geq \frac{\operatorname{diam} f(B_i)}{H},$$

which implies

$$\operatorname{diam} f(B_j) + \frac{\operatorname{diam} f(B_i)}{2H} \geq \frac{\operatorname{diam} f(B_i)}{H},$$

and therefore

$$\operatorname{diam} f(B_j) \geq \frac{\operatorname{diam} f(B_i)}{2H}.$$

From the above estimate we infer that

$$d_Y(f(x_i), f(x_j)) \geq \frac{\operatorname{diam} f(B_j)}{H} \geq \frac{\operatorname{diam} f(B_i)}{2H^2}.$$

And by the triangle inequality we obtain

$$B(f(x_i), \frac{\operatorname{diam} f(B_i)}{10H^2}) \cap B(f(x_j), \frac{\operatorname{diam} f(B_j)}{10H^2}) = \emptyset,$$

finishing the proof.

3 Path families and Sobolev spaces

Let Γ be a path family in a metric measure space (X, d, μ). Here path refers to a continuous, non-constant map $\gamma\colon I \to X$, where $I \subset \mathbb{R}$ is a non-degenerate interval. We call a Borel function $\rho\colon X \to [0, \infty]$ *admissible* for the path family Γ, if

$$\int_\gamma \rho \, ds \geq 1$$

for each locally rectifiable $\gamma \in \Gamma$. The p-modulus of Γ is defined by

$$\operatorname{mod}_p(\Gamma) = \inf \left\{ \int_X \rho^p(x) \, d\mu \quad : \quad \rho\colon X \to [0, \infty] \text{ is admissible for } \Gamma \right\}.$$

We say that a condition holds for p-almost every path in Γ if $\operatorname{mod}_p(\hat{\Gamma}) = 0$, where $\hat{\Gamma} \subset \Gamma$ consists of those paths $\gamma \in \Gamma$ for which this condition fails. The p-modulus is an outer measure in the collection of all path families in X. For the basic properties of the p-modulus we refer the reader to [14], [16].

Let (X, d, μ) be a metric measure space and (X', d') be a metric space. Given an open set $U \subset X$ and a continuous mapping $f\colon U \to X'$, we say that a non-negative Borel-function g is an upper gradient of f in U if, for each rectifiable path $\gamma\colon [0,1] \to U$, we have that

$$d'(f(\gamma(1)), f(\gamma(0))) \leq \int_\gamma g\, ds. \tag{2}$$

We recall here that each such γ can be parametrized by a 1-Lipschitz map $\tilde{\gamma}\colon [0, L] \to U$. For $1 \leq p < \infty$, we call a non-negative Borel-function g a p-weak upper gradient of f in U if the above inequality holds for p-almost every rectifiable path in U. It then follows from the properties of the p-modulus that (2) also holds for each subpath of p-almost every γ in U. The existence of a p-integrable p-weak upper gradient always guarantees the existence of a p-integrable upper gradient, see [22]. Let $f\colon X \to X'$ be continuous. Then f is in the Sobolev space $W^{1,p}_{loc}(X; X')$ if, for each relatively compact open set $U \subset X$, f has an upper gradient $g \in L^p(U)$ in U, and if there is $x_0 \in U$ so that $u(x) = d'(f(x), f(x_0)) \in L^p(U)$. In what follows, we will typically have $\mu(U) < \infty$, and thus only a p-integrable p-weak upper gradient is asked for. Notice that for a proper X, each $f \in W^{1,p}_{loc}(X; X')$ is absolutely continuous on p-almost every rectifiable path. For the purposes of this paper it suffices to consider continuous mappings f; for the definition and properties of general Sobolev classes we refer the reader to [16].

Because of the importance of p-weak upper gradients, we now give a sufficient condition for a path family to be of p-modulus zero.

Given a set $E \subset X$ and a path $\gamma\colon I \to X$, we let $\sharp(\gamma \cap E)$ denote the cardinality of $\gamma(I) \cap E$. We also write Γ_{rect} for the collection of all rectifiable paths in X. We denote the λ-dimensional Hausdorff measure by $\mathcal{H}^\lambda$, and say that a set $E \subset X$ has σ-finite $\mathcal{H}^\lambda$-measure if it is contained in a countable union of sets with finite $\mathcal{H}^\lambda$-measure.

Proposition 3.1 *Let (X, d, μ) be a proper, locally Q-regular metric space, $E \subset X$ and $1 < p < Q$. If E has σ-finite $\mathcal{H}^{Q-p}$-measure, then*

$$\operatorname{mod}_p(\{\gamma \in \Gamma_{rect} : \gamma \cap E \textit{ is not countable } \}) = 0. \tag{3}$$

We do not know if Proposition 3.1 could also hold for $p = 1$, but the following version of it will be sufficient for our needs.

Proposition 3.2 *Let $f\colon X \to Y$ be a homeomorphism between metric spaces, where X is proper and locally Q-regular. Let $E \subset X$ have σ-finite $\mathcal{H}^{Q-1}$-measure. Then*

$$\operatorname{mod}_1(\{\gamma \in \Gamma_{rect} : \mathcal{H}^1(f(\gamma \cap E)) > 0\}) = 0. \tag{4}$$

For the proofs of these propositions we need a technical result that will also be applied later on.

Lemma 3.3 *Let (X, d, μ) be a proper, locally Q-regular metric space. Further, let $B_0 = B(x_0, r_0)$ and fix $1 \leq p < \infty$. There are constants δ and C so that given any collection $B_1, B_2, \cdots$ of balls in B_0 with radii at most δ and non-negative numbers a_i, we have the estimate*

$$\int_{B_0} \left(\sum_i a_i \chi_{6B_i}(x) \right)^p d\mu \leq C \int_{B_0} \left(\sum_i a_i \chi_{\frac{1}{6}B_i}(x) \right)^p d\mu.$$

In fact, we may take $\delta = \frac{1}{6}\delta_0$, where δ_0 is the constant for B_0 in the local Q-regularity condition.

For $p = 1$ the claim immediately follows from the local Q-regularity and measure comparison. The general case is obtained using the $L^p - L^{p/(p-1)}$-duality and the boundedness of an appropriate restricted maximal function. In fact, already the doubling of the measure μ for scales up to 6δ would be sufficient; see [5] for a proof that generalizes to our setting.

A crucial step in proving Proposition 3.1 is the following result.

Lemma 3.4 *Let (X, d, μ) be a proper, locally Q-regular metric space, $E \subset X$ bounded and $1 < p < Q$. Denote by Γ the collection of all rectifiable curves γ in X such that $\sharp(\gamma \cap E) = \infty$. If $\mathcal{H}^{Q-p}(E) < \infty$, then $mod_p(\Gamma) = 0$.*

Proof. Fix a ball B_0 so that $E \subset B_0$. Set for $k, l \in \mathbb{N}$

$$\Gamma_{k,l} = \{\gamma \in \Gamma : \exists \{x_1, x_2, \ldots, x_k\} \in \gamma \cap E \text{ s.t. } d(x_i, x_j) > \frac{1}{l} \text{ when } i \neq j\}.$$

Then $\Gamma = \cap_k \cup_l \Gamma_{k,l}$ and $\Gamma_{k,l} \subset \Gamma_{k,l+1}$. Thus

$$\text{mod}_p(\cup_l \Gamma_{k,l}) = \lim_{l \to \infty} \text{mod}_p(\Gamma_{k,l}) \tag{5}$$

(cf. [15]).

Fix k and l and let $\varepsilon > 0$. Since $\mathcal{H}^{Q-p}(E) < \infty$, there is a cover of E by balls $(B_i)_i$ such that $\text{diam}(5B_i) < \frac{1}{2l}$, $E \subset \cup_i B_i$ and $\sum_i r_i^{Q-p} \leq \mathcal{H}^{Q-p}(E) + \varepsilon$. We may assume that $B_i \subset B_0$ and that $\text{diam}(5B_i) < \delta$, where δ is the constant for B_0 in the local Q-regularity condition.

By Lemma 2.1, we find a subfamily of these balls (denoted the same way) that are pairwise disjoint and with $E \subset \cup_i 5B_i$. Set

$$\rho(x) = \frac{1}{k} \sum_i \frac{1}{r_i} \chi_{6B_i}(x).$$

Now ρ is admissible for $\Gamma_{k,l}$, since we have $\int_\gamma \chi_{6B_i}\, ds \geq r_i$ for at least k different indices. By Lemma 3.3 we have that

$$\int_X \rho^p(x)\, d\mu \leq \frac{C}{k^p} \int_X \left(\sum_i \frac{1}{r_i} \chi_{B_i}(x) \right)^p d\mu.$$

Here C depends on Q, p, B_0. Since the balls B_i are pairwise disjoint, by the local Q-regularity of μ we obtain

$$\int_X \rho^p(x)\, d\mu \leq \frac{C}{k^p} \sum_i r_i^{Q-p} \leq \frac{C}{k^p} \left(\mathcal{H}^{Q-p}(E) + \varepsilon\right). \tag{6}$$

Our ε was arbitrary, and thus

$$\operatorname{mod}_p(\cup_l \Gamma_{k,l}) = \lim_{l\to\infty} \operatorname{mod}_p(\Gamma_{k,l}) \leq \frac{C}{k^p}\mathcal{H}^{Q-p}(E).$$

Since $\Gamma \subset \cup_l \Gamma_{k,l}$ for every k, the claim follows.

We continue with the case $p = 1$. The difficulity in extending the above proof to this case lies in inequality 5. It only holds for $p > 1$, the basic issue being reflexivity of L^p.

Lemma 3.5 *Let $f\colon X \to Y$ be a homeomorphism between metric spaces, and assume that X is proper and locally Q-regular. Let $E \subset X$ have finite $\mathcal{H}^{Q-1}$-measure. Fix $\varepsilon > 0$. Then*

$$\operatorname{mod}_1(\{\gamma \in \Gamma_{rect} : \mathcal{H}^1(f(\gamma \cap E)) > \varepsilon\}) = 0. \tag{7}$$

Proof. Denote our family of curves by Γ_ε. By subadditivity, we may without loss of generality assume that $E \subset B_0$ for some ball B_0. Let $k \geq 1$ be an integer. Because $\overline{B}_0$ is compact and f is continuous, we find $\delta_k > 0$ so that, given $x, x' \in \overline{B}_0$ with $d_X(x, x') < \delta_k$, we have that

$$d_Y(f(x), f(x')) < \frac{\varepsilon}{2^{k+3}}. \tag{8}$$

As in the proof of the previous lemma, we find a sequence of balls $(B_i^k)_i$ so that $B_i^k \cap B_j^k = \emptyset$ when $i \neq j$, $\operatorname{diam}(B_i^k) \leq \frac{1}{6}\min\{\delta, \delta_k\}$, $E \subset \cup_i 5B_i^k$ and so that

$$\sum_k (\operatorname{diam}(5B_i^k))^{Q-1} < 5^{Q-1}\mathcal{H}^{Q-1}(E) + \varepsilon. \tag{9}$$

Consider the sequence $(\rho_k)_k$ of Borel-functions, defined by

$$\rho_k(x) = \frac{1}{2^k} \sum_i \frac{1}{\operatorname{diam}(B_i^k)} \chi_{6B_i^k}(x).$$

By (9) we see that

$$\int_X \rho_k\, d\mu \leq \frac{C}{2^k}(\mathcal{H}^{Q-1}(E) + \varepsilon). \tag{10}$$

Set

$$\Gamma_{(k)} = \{\gamma \in \Gamma_\varepsilon : \int_\gamma \rho_k\, ds \geq 1\}.$$

By (10) and subadditivity we conclude that

$$\mathrm{mod}_1(\cup_{j\geq k}\Gamma_{(j)}) \leq \frac{C}{2^k}(\mathcal{H}^{Q-1}(E)+\varepsilon). \tag{11}$$

Let $\gamma \in \Gamma_\varepsilon$. Since $\mathcal{H}^1(f(\gamma\cap E)) > \varepsilon$, for all sufficiently large integers m there exist points $y_1, \ldots, y_{2^m} \in f(\gamma \cap E)$ so that

$$d_Y(y_i, y_j) > \frac{\varepsilon}{2^{m+3}}.$$

Notice that, by (8), there have to be at least 2^m balls $5B_i^m$ in our sum which intersect γ. Thus,

$$\int_\gamma \rho_m \, ds \geq 1.$$

It follows that $\Gamma_\varepsilon \subset \cup_{j\geq k}\Gamma_{(j)}$, for each k. From (11) we then conclude that $\mathrm{mod}_1(\Gamma_\varepsilon) = 0$.

Proof of Proposition 3.1. Suppose that $p > 1$. We may write $E = \cup_i E_i$, where each E_i is bounded with $\mathcal{H}^{Q-p}(E_i) < \infty$. Write

$$\Gamma_i = \{\gamma \in \Gamma_{\text{rect}} : \gamma \cap E_i \text{ is not countable }\}.$$

By the Lemma 3.4, $\mathrm{mod}_p(\Gamma_i) = 0$, and the claim follows by subadditivity of the p-modulus.

Proof of Proposition 3.2. Lemma 3.5 shows that $\mathrm{mod}_p(\Gamma_{1/j}) = 0$ for each $j \geq 1$. Here $\Gamma_{1/j}$ consists of those rectifiable curves for which $\mathcal{H}^1(f(\gamma\cap E)) > 1/j$. The claim follows by subadditivity.

4 Absolute continuity on almost all paths

Theorem 1.1 is an immediate consequence of the following slightly more general result.

Theorem 4.1 *Let (X, d_X, μ) and (Y, d_Y, ν) be locally Q-regular metric spaces, $Q > 1$, and suppose that $E \subset X$ has σ-finite $\mathcal{H}^{Q-1}$-measure. Assume also that X is proper. If $f\colon X \to Y$ is a homeomorphism such that $h_f(x) < \infty$ for every $x \in X \setminus E$ and $h_f(x) \leq H < \infty$ for μ-almost every $x \in X$, then $f \in W^{1,1}_{loc}(X;Y)$.*

Proof. Let $B_0 = B(x_0, r_0)$ be a fixed ball. Without loss of generality assume that $H > 1$. For each $k = 1, 2, \ldots$ write

$$A_k = \{x \in B(x_0, r_0) : H^k < h_f(x) \leq H^{k+1}\}.$$

The set A_k is a Borel set since h_f is a Borel function. Moreover $\mu(\cup_k A_k) = 0$. Fix $1 < p < Q$, and let $0 < \varepsilon < \varepsilon_0$, where $0 < \varepsilon_0 < \delta$ with δ from the local Q-regularity condition for $\overline{B}(x_0, r_0)$. Choose ε_0 so small that $f(B(x, \varepsilon_0)) \subset B(f(x), \delta')$ for each $x \in \overline{B}(x_0, r_0)$ for the constant δ' from the local Q-regularity condition for $f(\overline{B}(x_0, r_0))$.

Since $\mu(A_k)=0$, for every k there exists an open set U_k such that $A_k \subset U_k$ and

$$\mu(U_k) \le \frac{1}{H^{2kpQ/(Q-p)}} \left(\frac{1}{2^k \nu(f(B(x_0, r_0+\varepsilon)))^{\frac{p}{Q}}} \right)^{\frac{Q}{Q-p}}, \tag{12}$$

since we assumed that our measure is Borel regular. Notice here that $\nu(f(B(x_0, r_0+\varepsilon)))$ is finite since X is proper and ν is locally Q-regular.

For points $x \in B(x_0,r_0) \setminus (\cup_k A_k \cup E)$ pick a radius $0<r_x<\varepsilon$ such that $H_f(x,r_x) \le 2H$. For points $x \in A_k$ choose a radius $0<r_x<\varepsilon$ such that $H_f(x,r_x) \le 2H^{k+1}$ and $B(x,r_x) \subset U_k$. Consider now the family $\{B(x,r_x)\}$ of balls. By applying Lemma 2.2 for these balls we find a countable collection $\mathcal{B} = \{B(x_i,r_i)\}$ such that $B(x_0,r_0) \setminus E \subset \cup_{B\in\mathcal{B}} B$ and $\frac{1}{3}B$, $B \in \mathcal{B}$, are pairwise disjoint. Denote by $\mathcal{B}_H$ the subcollection of the balls $B(x,r_x) \in \mathcal{B}$ for which $x \in B(x_0,r_0) \setminus (\cup_k A_k \cup E)$ and by $\mathcal{B}_k$ those for which $x \in A_k$.

Relying on Lemma 2.3 we notice that

(i) $B(f(x_i), \frac{L_f(x_i,r_i)}{250H^2})$, where $B(x_i,r_i) \in \mathcal{B}_H$, are pairwise disjoint and

(ii) $B(f(x_i), \frac{L_f(x_i,r_i)}{250H^{2k+2}})$, where $B(x_i,r_i) \in \mathcal{B}_k$, are pairwise disjoint.

Set

$$\rho_\varepsilon(x) = \sum_i \frac{L_f(x_i,r_i)}{r_i} \chi_{2B_i}(x).$$

The function ρ_ε is clearly Borel measurable. Let Γ_ε denote all rectifiable paths $\gamma\colon [0,1] \to B(x_0,r_0)$ such that $\mathcal{H}^1(f(\gamma \cap E)) = 0$ and $\operatorname{diam}(\gamma) > \varepsilon$.

Let γ be a member of Γ_ε. If $B_i \cap \gamma \neq \emptyset$, then $\mathcal{H}^1(\gamma \cap 2B_i) \ge r_i$. Thus

$$\int_\gamma \rho_\varepsilon \, ds \ge \sum_{B_i\cap\gamma\neq\emptyset} L_f(x_i,r_i) \ge \frac{1}{2} \sum_{B_i\cap\gamma\neq\emptyset} \operatorname{diam}(fB_i) \ge \frac{1}{2} d_Y(f(\gamma(0)), f(\gamma(1))),$$

where the last inequality comes from the fact that the sets $f(B_i)$ cover $f(\gamma)$ up to a set of zero $\mathcal{H}^1$-measure.

By Lemma 3.3 and the pairwise disjointness of the balls $\frac{1}{3}B_i$, we have the estimate

$$\int_X \rho_\varepsilon^p \, d\mu \;\le\; C \int_{B(x_0,r_0+2\varepsilon)} \sum_i \frac{L_f(x_i,r_i)^p}{r_i^p} \chi_{\frac{1}{3}B_i} \, d\mu,$$

where $C = C(B_0,\mu,p)$.

Next we estimate this integral from above in two parts. First we consider the sum over $\mathcal{B}_H$-terms, which we denote by S_H. By Hölder's inequality we have

$$\begin{aligned} S_H &= C \int_{B(x_0,r_0+2\varepsilon)} \sum_{B_i\in\mathcal{B}_H} \frac{L_f(x_i,r_i)^p}{r_i^p} \chi_{\frac{1}{3}B_i} \, d\mu \\ &\le C\mu(B(x_0,r_0+2\varepsilon))^{\frac{Q-p}{Q}} \left(\int_{B(x_0,r_0+2\varepsilon)} \Big(\sum_{B_i\in\mathcal{B}_H} \frac{L_f(x_i,r_i)^p}{r_i^p} \chi_{\frac{1}{3}B_i} \Big)^{\frac{Q}{p}} d\mu \right)^{\frac{p}{Q}} \\ &\le C\mu(B(x_0,r_0+2\varepsilon))^{\frac{Q-p}{Q}} \left(\sum_{B_i\in\mathcal{B}_H} L_f(x_i,r_i)^Q \right)^{\frac{p}{Q}}. \end{aligned}$$

The last inequality comes from the pairwise disjointness of $\frac{1}{3}B_i$'s and local Q-regularity of μ. Since, by (i), the balls $B(f(x_i), \frac{L_f(x_i,r_i)}{250H^2})$ with $B(x_i,r_i) \in \mathcal{B}_H$ are pairwise disjoint and Y is locally Q-regular, we obtain the estimate

$$S_H \leq C\mu(B(x_0, r_0 + 2\varepsilon))^{\frac{Q-p}{Q}} \nu(f(B(x_0, r_0 + \varepsilon)))^{\frac{p}{Q}} < \infty,$$

where $C = C(B_0, \mu, f(B_0), \nu, p, H)$. The fact that the above term is finite comes from the properness of X and local Q-regularity of both measures.

Next we need to estimate from above the sum over $\mathcal{B}_k$'s. We denote the sum corresponding to $\mathcal{B}_k$ by S_k. Using the local Q-regularity and Hölder's inequality we obtain

$$\begin{aligned} S_k &\leq C \sum_{B_i \in \mathcal{B}_k} \frac{L_f(x_i,r_i)^p}{H^{2kp+2p}} H^{2kp} r_i^{Q-p} \\ &\leq C \left(\sum_{B_i \in \mathcal{B}_k} (\frac{L_f(x_i,r_i)}{H^{2k+2}})^Q \right)^{\frac{p}{Q}} \left(\sum_{B_i \in \mathcal{B}_k} H^{2kpQ/(Q-p)} \mu(\frac{1}{3}B_i) \right)^{\frac{Q-p}{Q}}, \end{aligned}$$

where $C = C(B_0, \mu, p, H)$. Now, by (ii), the balls $B(f(x_i), \frac{L_f(x_i,r_i)}{250H^{2k+2}})$ with $B(x_i,r_i) \in \mathcal{B}_k$, are pairwise disjoint, and therefore the first term is no more than $C\nu(f(B(x_0, r_0 + \varepsilon)))^{\frac{p}{Q}}$. For the second term we use (12), so that

$$S_k \leq C\left(\nu(f(B(x_0, r_0 + \varepsilon)))\right)^{\frac{p}{Q}} \frac{1}{2^k \nu(f(B(x_0, r_0 + \varepsilon)))^{\frac{p}{Q}}} \leq \frac{C}{2^k}.$$

Thus

$$\begin{aligned} \int_X \rho_\varepsilon^p \, d\mu &\leq C(S_H + \sum_k S_k) \\ &\leq C\mu(B(x_0, r_0 + 2\varepsilon))^{\frac{Q-p}{Q}} \nu(f(B(x_0, r_0 + \varepsilon)))^{\frac{p}{Q}} + C. \end{aligned}$$

So, for all paths $\gamma \in \Gamma_\varepsilon$, we have the estimate

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma 2\rho_\varepsilon \, ds$$

with $\int (2\rho_\varepsilon)^p \, d\mu \leq M < \infty$ when $0 < \varepsilon < \varepsilon_0$. The weak compactness of L^p guarantees that there is $\rho \in L^p$ and a sequence of ε_i's that decreases to zero such that ρ is a L^p-weak limit of $2\rho_{\varepsilon_i} =: \rho_i$. Here we needed the fact that $p > 1$. Notice that

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma \rho_i \, ds \tag{13}$$

for each $i \geq j$ when $\gamma \in \Gamma_j := \Gamma_{\varepsilon_j}$. By Mazur's lemma (cf. Theorem 2 in Chapter V.1 of [33]), we find functions $\hat{\rho}_i$, each a convex combination of $\rho_i, \rho_{i+1}, \ldots$, so that the sequence $\{\hat{\rho}_i\}$ converges to ρ in L^p. Now (13) also holds with ρ_i replaced with $\hat{\rho}_i$ for each $i \geq j$. By Fuglede's lemma (cf. [15]), (13) holds for ρ for p-almost every $\gamma \in \cup_j \Gamma_j$. Thus, using Proposition 3.2, we notice that (13) holds for 1-almost every rectifiable curve in $B(x_0, r_0)$; recall that we excluded the curves for which $\mathcal{H}^1(f(\gamma \cap E)) > 0$. Thus f has a 1-weak upper gradient in $L^1(B(x_0, r_0))$, and consequently an upper gradient in $L^1(B(x_0, r_0))$. In conclusion, $f \in W^{1,1}(B(x_0, r_0); Y)$, and the theorem is proven.

Remark 4.2 The above proof immediately gives the following variations on Theorem 4.1. First of all, if we consider a smaller exceptional set E which is of σ-finite $\mathcal{H}^{Q-p}$-measure, $1 < p < Q$, we can conclude (using Proposition 3.1 instead of Proposition 3.2) that $f \in W^{1,p}_{loc}(X; Y)$. In the borderline case $p = Q$, one can still conclude that $f \in W^{1,Q}_{loc}(X; Y)$ but one can only allow for a countable exceptional set E and the bound $h_f(x) \leq H$ needs to be assumed for each $x \in X \setminus E$. For a further improvement on this see Theorem 5.1 and Remarks 5.3 below.

The proof of Theorem 4.1 is not constructive in the sense that no explicit upper gradient is given. Classically, one can take the maximal streching

$$L_f(x) = \limsup_{r \to 0} \frac{L_f(x, r)}{r}$$

as an upper gradient. In our situation it turns out to be better to consider the volume derivative

$$\mu_f(x) = \lim_{r \to 0} \frac{\nu(f(B(x, r)))}{\mu(B(x, r))}.$$

Recall that, by the Lebesgue-Radon-Nikodym theorem [7], this limit exists almost everywhere and we have the estimate

$$\int_A \mu_f(x)\, d\mu \leq \nu(f(A))$$

for each measurable set $A \subset X$. Consequently,

$$\int_A u(f(x))\mu_f(x)\, d\mu \leq \int_{f(A)} u(y)\, d\nu \tag{14}$$

when $u \geq 0$ is continuous.

For technical reasons, we assume from now on a global condition on the measures μ and ν. We say that a measure μ on a metric space X is Q-regular if it is locally Q-regular with a universal constant C_μ and with $\delta = \operatorname{diam}(X)$, i.e.

$$\frac{1}{C_\mu} r^Q \leq \mu(B(x, r)) \leq C_\mu r^Q$$

holds for each x and all $0 < r < \operatorname{diam}(X)$.

Proposition 4.3 *Let (X, d_X, μ) and (Y, d_Y, ν) be Q-regular metric spaces, $Q > 1$. Assume also that X is proper. Let $f\colon X \to Y$ be a homeomorphism such that $h_f(x) \leq H < \infty$ for μ-almost every $x \in X$. Define*

$$g_f(x) = \begin{cases} H \limsup_{r\to 0} \left(\frac{\nu(f(B(x,r)))}{\mu(B(x,r))}\right)^{\frac{1}{Q}} & \text{when } h_f(x) \leq H \\ \infty & \text{when } h_f(x) > H \end{cases}.$$

If f is absolutely continuous on $\gamma\colon [0, L] \to X$, which is 1-Lipschitz, then

$$d_Y(f(\gamma(0)), f(\gamma(L))) \leq C \int_\gamma g_f\, ds, \tag{15}$$

where C depends only on the Q-regularity constants of μ, ν.

Proof. Notice that g_f is Borel measurable since h_f is Borel measurable.
Let $y \in \gamma(]0, L[)$. Then there are arbitrarily small $r_y > 0$ so that

$$f(B(y, r_y)) \subset B(f(y), L_f(y, r_y)) \subset B(f(y), 2h_f(y)l_f(y, r_y)).$$

We might have here that $h_f(y) = \infty$, but that will not harm us. Let $x \in]0, L[$ and write $y = \gamma(x)$. Because γ is 1-Lipschitz we conclude that

$$\begin{aligned} \operatorname{diam}(f(\gamma(]x - r_y, x + r_y[))) &\leq 4h_f(y)l_f(y, r_y) \\ &\leq C h_f(y)\nu(f(B(y, r_y))^{\frac{1}{Q}}, \end{aligned} \tag{16}$$

where we used the Q-regularity of ν.
Fix $\varepsilon > 0$. For $i \in \mathbb{Z}$, let

$$E_i = \{x \in]0, L[\colon 2^{i-1} < g_f(\gamma(x)) \leq 2^i\}.$$

Let $x \in E_i$. Using (16), the definition of g_f and the Q-regularity of μ, we may pick an arbitrarily small $r_y > 0$ such that

$$\operatorname{diam}(f(\gamma(]x - r_y, x + r_y[)) \leq C 2^i r_y.$$

Covering E_i appropriately, we obtain intervals $I_1^{(i)}, I_2^{(i)}, \ldots$ so that $E_i \subset \cup_j I_j^{(i)}$ and

$$\sum_j \operatorname{diam}(f(\gamma(I_j^i))) \leq C \int_{\gamma|E_i} g_f\, ds + \frac{\varepsilon}{2^i}.$$

Letting i run trought $\mathbb{Z}$, we end up with a collection $J_1, J_2, \ldots$ of intervals such that $\cup_i E_i \subset \cup_j J_j$ and

$$\sum_j \operatorname{diam}(f(\gamma(J_j))) \leq C \int_\gamma g_f\, ds + \varepsilon. \tag{17}$$

In order to obtain (15) we still need to consider the remaining parts of $]0, L[$ where g_f is either 0 or ∞.

Write $E^0 = \{x \in]0, L[\colon g_f(\gamma(x)) = 0\}$, and let $x \in E^0$. Then (16) implies that

$$\liminf_{r \to 0} \frac{\operatorname{diam}(f(\gamma(]x-r, x+r[)))}{r} = 0.$$

It follows that

$$\mathcal{H}^1(f(\gamma(E^0))) = 0. \tag{18}$$

Write $E^\infty = \{x \in]0, L[\colon g_f(\gamma(x)) = \infty\}$. If $\mathcal{H}^1(E^\infty) > 0$, then (15) is trivially true. Otherwise, $\mathcal{H}^1(E^\infty) = 0$, and the absolute continuity of f on γ shows that

$$\mathcal{H}^1(f(\gamma(E^\infty))) = 0. \tag{19}$$

The claim follows by combining (17), (18) and (19) and letting $\varepsilon \to 0$.

Remark 4.4 Combining Lemma 4.3 and Remark 4.2 it is easy to check that the uniform boundedness of h_f implies the so-called geometric definition (see [9]) for proper Q-regular spaces. Notice also that, in the Euclidean setting, it follows that f is quasiconformal if and only if there is a constant K so that $K\mu_f^{1/Q}$ is a Q-weak upper gradient of f.

5 $W^{1,Q}_{loc}$ and quasisymmetry

Let us recall that (X, d, μ) supports a p-Poincaré inequality, $1 \le p < \infty$ if there are two constants C and τ such that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le Cr \left(\frac{1}{\mu(B(x,\tau r))} \int_{B(x,\tau r)} g^p \, d\mu \right)^{1/p}$$

whenever $B(x, r)$ is a ball, u is continuous and g is an upper gradient of u. Here $u_{B(x,r)}$ is the average of u over $B(x, r)$. It then follows that this also holds when g is a p-weak upper gradient, see [22]. For more on Poincaré inequalities see [14], [12], [19].

We begin by improving on the regularity of f in Theorem 4.1.

Theorem 5.1 *Let (X, d_X, μ) and (Y, d_Y, ν) be Q-regular metric spaces, $Q > 1$, and suppose that $E \subset X$ has σ-finite $\mathcal{H}^{Q-1}$-measure. Assume also that X is proper and supports a 1-Poincaré inequality. If $f \colon X \to Y$ is a homeomorphism such that $h_f(x) < \infty$ for every $x \in X \setminus E$ and $h_f(x) \le H < \infty$ for μ-almost every $x \in X$, then $f \in W^{1,Q}_{loc}(X; Y)$.*

Proof. From Theorem 4.1 we know that $f \in W^{1,1}_{loc}(X; Y)$, and from Proposition 4.3 that $g_f \in L^Q_{loc}(X)$ is a 1-weak upper gradient. Because X supports a 1-Poincaré inequality, it follows that $f \in W^{1,Q}_{loc}(X; Y)$; see Proposition 4.4 in [22] and Theorem 6.11 in [15].

We continue with a quasisymmetry result. This result could be proven by combining the ideas from the proof of Theorem 4.1 with the techniques in [13], [14]. Instead of that we sketch a direct proof along classical lines using Theorem 4.1.

Theorem 5.2 *Let (X, d_X, μ) and (Y, d_Y, ν) be both proper, Q-regular metric spaces, $Q > 1$, and $E \subset X$ such that it has σ-finite $\mathcal{H}^{Q-1}$-measure. Assume also that X supports a 1-Poincaré inequality and that Y is linearly locally connected. If $f : X \to Y$ is a homeomorphism such that $h_f(x) < \infty$ for every $x \in X \setminus E$ and $h_f(x) \le H < \infty$ for almost every $x \in X$, then f is quasisymmetric, $f \in W^{1,Q}_{loc}(X; Y)$, and $L_f(x)^Q \le K\mu_f(x)$ almost everywhere for some constant K.*

Proof. Fix $B(x_0, r_0)$, and suppose that $L := L_f(x_0, r_0)$ is much larger than $l := l_f(x_0, r_0)$.

From Theorem 4.1 and Proposition 4.3 we know that f is absolutely continuous on 1-almost every path and that we may take g_f as defined in Proposition 4.3 as an upper gradient.

Suppose that E and F are continua so that $y_0 \in f(E) \subset B(y_0, Cl)$ and $f(F) \subset Y \setminus \overline{B}(y_0, L/C)$ for some $y_0 \in Y$. We define a function $u : X \to [0, \infty[$ by setting

$$u(x) = \frac{\log(L/Cd_Y(f(x), y_0))}{\log(L/C^2 l)}$$

in $A := f^{-1}(\overline{B}(y_0, L) \setminus B(y_0, l))$ and by extending u to all of X by setting $u = 0$ in $X \setminus f^{-1}(\overline{B}(y_0, L/C))$ and $u = 1$ in $f^{-1}(\overline{B}(y_0, Cl))$. It easily follows that $g(x) = \log(L/C^2 l)^{-1} d_Y(f(x), y_0)^{-1} g_f(x)\chi_A(x)$ is a 1-weak upper gradient of u.

Now

$$\int_X g^Q \, d\mu \le CH^Q \log(L/l)^{-Q} \int_A d(f(x), y_0)^{-Q} \mu_f(x) \, d\mu(x)$$

$$\le CH^Q \log(L/C^2 l)^{-Q} \int_{\overline{B}(y_0, L/C) \setminus B(y_0, Cl)} d(y, y_0)^{-Q} \, d\nu(y)$$

$$\le C \log(L/C^2 l)^{1-Q}$$

by (14) and the Q-regularity of Y.

On the other hand, the Poincaré inequality and Q-regularity assumptions on Y allow us to choose the continua E, F as above so that $\operatorname{diam}(E), \operatorname{diam}(F) \ge r_0$, $d(E, F) \le 2r_0$, where C is a constant only depending on the Q-regularity constant, Q, and the constants in the linear local connectivity condition. For all this see section 4 in [14] and 4.4 and 4.5 in [12]. Because g is a 1-weak upper gradient of u, we may apply the 1-Poincaré inequality to the pair u, g and Theorem 5.9 in [14] shows that

$$\int_X g^Q \, d\mu \ge C > 0.$$

A bound on L/l follows, as desired.

Next, from Theorem 5.1 we know that $f \in W^{1,Q}_{loc}(X; Y)$.

The final conclusion immediately follows from the uniform boundedness of $H_f(x, r)$.

Remarks 5.3 1) If we instead of a 1-Poincaré inequality on X assume a p-Poincaré inequality, $1 < p < Q$, then the conclusions of Theorem 5.1, 5.2 still hold provided we require that the exceptional set E have σ-finite $\mathcal{H}^{Q-p}$-measure. Notice that a Q-Poincaré inequality for a space as in the above theorem always implies a p-Poincaré inequality for some $p < Q$ by a very recent result in [19].

2) There is a local version of Theorem 5.2, whose formulation and proof we leave to the interested reader.

Proof of Theorem 1.2 A Carnot group of homogeneous dimension $Q > 1$ is proper, Q-regular, and supports a 1-Poincaré inequality. Thus the claim follows from Theorem 5.2, except for the inequality for the horizontal gradient. This inequality follows from the corresponding inequality in Theorem 5.2 and quasisymmetry [21], [29], [32].

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