SHARP GENERALIZED TRUDINGER INEQUALITIES VIA TRUNCATION

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Abstract. We prove that the generalized Trudinger inequalities into exponential and double exponential Orlicz spaces improve to inequalities on Orlicz-Lorentz spaces provided they are stable under truncation.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded domain. The classical Sobolev embedding theorem states that $W_0^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$ if $1 \leq p < n$ and $p^* = \frac{pn}{n-p}$. If $p > n$ then every function from $W_0^{1,p}(\Omega)$ is bounded (i.e. belongs to $L^\infty(\Omega)$) and in the limiting case $p = n$ it is known that every function from $W_0^{1,n}(\Omega)$ belongs to $L^q(\Omega)$ for every $1 \leq q < \infty$ but not neccesarily to $L^\infty(\Omega)$.

A famous result of Trudinger (see [23], [24]) states that the space $W_0^{1,n}(\Omega)$ is continuously embedded in the Orlicz space $\exp L^{\frac{n}{n-1}}(\Omega)$ (see Preliminaries for the definition of various Orlicz spaces): i.e. there exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ such that

$$\int_\Omega \exp\Big(\Big(\frac{|u(x)|}{C_1||\nabla u||_{L^n(\Omega)}}\Big)^{\frac{n}{n-1}}\Big)dx \leq C_2\mathcal{L}_n(\Omega) \tag{1.1}$$

for every $u \in W_0^{1,n}(\Omega)$. When Ω is sufficiently nice this means that there are $C_1 = C_1(n)$ and $C_2 = C_2(n)$ so that for every $u \in W^{1,n}(\Omega)$ we have

$$\inf_{c\in\mathbf{R}} \int_\Omega \exp\Big(\Big(\frac{|u(x) - c|}{C_1||\nabla u||_{L^n(\Omega)}}\Big)^{\frac{n}{n-1}}\Big)dx \leq C_2\mathcal{L}_n(\Omega). \tag{1.2}$$

It is known (see [12], [6] and [2]) that $\exp L^{\frac{n}{n-1}}(\Omega)$ is the smallest Orlicz space with this property. However, even sharper inequalities exist in other scales. By a result of Brézis and Wainger [1] and independently Hansson [11] (see also [17]) we know that

$$\inf_{c\in\mathbf{R}} \int_0^\infty \frac{t^{n-1}}{\log^{n-1}\Big(\frac{e\mathcal{L}_n(\Omega)}{\mathcal{L}_n(\{x\in\Omega:|u(x)-c|\geq t\})}\Big)}dt \leq C||\nabla u||^n_{L^n(\Omega)} \tag{1.3}$$

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for every $u \in W^{1,n}(\Omega)$ if Ω is sufficiently nice. This inequality can be also derived from capacitary estimates of Maz'ya [15]. The results in [8] and [3] tell us that this inequality gives us the smallest rearrangement invariant Banach function space $Y(\Omega)$ such that $W_0^{1,n}(\Omega)$ is continuously embedded into $Y(\Omega)$.

It is a surprising result of Koskela and Onninen [14] that if (1.2) is valid for every $u \in W^{1,n}(\Omega)$ then (1.3) is also valid for every $u \in W^{1,n}(\Omega)$. That is, with no additional requirement on Ω we have that the validity of the embedding (1.2) implies the validity of the sharper embedding (1.3). It is also proven in [14] that the Sobolev inequality $W^{1,p}(\Omega) \to L^{p^*}(\Omega)$, $1 \le p < n$, improves in the same way to an inequality of O'Neil [19] and Peetre [20].

The aim of this paper is to show that the same phenomenon occurs in the more general embeddings of Edmunds, Gurka and Opic [4] and Fusco, Lions and Sbordone [9]. Let $\Omega \subset \mathbf{R}^n$, $n \ge 2$, be a bounded domain and let $\alpha \in \mathbf{R}$, $\alpha < n-1$. It is shown in [4] and [9] (see also [2] and [13]) that there are constants C_1 and C_2 such that

$$\int_\Omega \exp\Big(\Big(\frac{|u(x)|}{C_1||\nabla u||_{L^n \log^\alpha L(\Omega)}}\Big)^{\frac{n}{n-1-\alpha}}\Big)dx \le C_2$$

for every $u \in W_0 L^n \log^\alpha L(\Omega)$ (see Preliminaries for the definitions of these spaces). If $\alpha > n-1$ then every function from $W_0 L^n \log^\alpha L(\Omega)$ is bounded and in the limiting case $\alpha = n-1$ we have the following embedding into double exponential Orlicz spaces: for every $\beta < n-1$ there are constants C_1 and C_2 such that

$$\int_\Omega \exp\exp\Big(\Big(\frac{|u(x)|}{C_1||\nabla u||_{L^n \log^{n-1} L \log^\beta \log L(\Omega)}}\Big)^{\frac{n}{n-1-\beta}}\Big)dx \le C_2$$

for every $u \in W_0 L^n \log^{n-1} L \log^\beta \log L(\Omega)$. For a further discussion about the limiting case $\beta = n-1$ see [5].

Following [14] we state our results in the generality which can be applied in the context of analysis on metric measure spaces. In what follows X will always denote a metric space equipped with a Borel measure μ and Ω will denote a measurable subset of X. The statement of [14, Theorem 1.1 *ii*)] is essentially the statement of the following theorem in the special case $\alpha = 0$.

Theorem 1.1. *Let $\Omega \subset X$ be a domain with $\mu(\Omega) < \infty$ and let $u, g : \Omega \to \mathbf{R}$. Fix $s \in (1,\infty)$ and $\alpha \in \mathbf{R}$, $\alpha < s-1$. Suppose that the inequality*

$$\inf_{c\in\mathbf{R}} \int_\Omega \exp\Big(\Big(\frac{|u(y)-c|}{C_1||g||_{L^s \log^\alpha L(\Omega)}}\Big)^{\frac{s}{s-1-\alpha}}\Big)d\mu(y) \le C_2 \tag{1.4}$$

is stable under truncation. Then

$$\inf_{c\in\mathbf{R}} \int_0^\infty \frac{t^{s-1}}{\log^{s-1-\alpha}\Big(\frac{e\mu(\Omega)}{\mu(\{x\in\Omega:|u(x)-c|\ge t\})}\Big)}dt < \infty. \tag{1.5}$$

Theorem 1.2. *Let $\Omega \subset X$ be a domain with $\mu(\Omega) < \infty$ and let $u, g : \Omega \to \mathbf{R}$. Fix $s \in (1, \infty)$ and $\beta \in \mathbf{R}$, $\beta < s-1$. Suppose that the inequality*

$$\inf_{c\in\mathbf{R}} \int_\Omega \exp\exp\Big(\Big(\frac{|u(y)-c|}{C_1||g||_{L^s \log^{s-1} L \log^\beta \log L(\Omega)}}\Big)^{\frac{s}{s-1-\beta}}\Big) d\mu(y) \le C_2 \tag{1.6}$$

is stable under truncation. Then

$$\inf_{c\in\mathbf{R}} \int_0^\infty \frac{t^{s-1}}{\log^{s-1-\beta}\Big(e + \log\Big(\frac{e\mu(\Omega)}{\mu(\{x\in\Omega: |u(x)-c|\ge t\})}\Big)\Big)} dt < \infty. \tag{1.7}$$

The requirement that the inequality (1.4) (resp. (1.6)) is stable under truncation means that for every $d \in \mathbf{R}$, $0 < t_1 < t_2 < \infty$ and $z \in \{-1, 1\}$ the pairs $v_{t_1}^{t_2}$, $g_{t_1}^{t_2} = g\chi_{\{t_1 < v \le t_2\}}$, where $v = z(u-d)$ and $v_{t_1}^{t_2} = \min\{\max\{0, v-t_1\}, t_2 - t_1\}$, also satisfies (1.4) (resp. (1.6)) :

$$\inf_{c\in\mathbf{R}} \int_\Omega \exp\Big(\Big(\frac{|v_{t_1}^{t_2}(y)-c|}{C_1||g_{t_1}^{t_2}||_{L^s \log^\alpha L(\Omega)}}\Big)^{\frac{s}{s-1-\alpha}}\Big) d\mu(y) \le C_2.$$

Notice that the function u clearly satisfies truncation property if $\Omega \subset \mathbf{R}^n$, $\mu = \mathcal{L}_n$ and $g = |\nabla u|$. For further applications of the powerful truncation technique which was first used in [16] we refer the reader to [15], [10] and references given there.

The validity of (1.5) and (1.7) is known in the Euclidean settings if we deal only with functions with zero traces (see [4], [8] and [3]). Again these spaces serve as the best rearrangement invariant space target of the embedding of $W_0 L^n \log^\alpha L(\Omega)$ and $W_0 L^n \log^{n-1} L \log^\beta \log L(\Omega)$. Our approach gives a new proof of these embeddings and we have additional information if we deal with functions that do not have a zero trace at the boundary.

To relate our statements (1.5) and (1.7) with the results in [1], [8] and [3] simply notice that for every decreasing differentiable function $\phi : [0, \infty) \to (0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = 0$ we have

$$\int_0^\infty (f_\mu^*(t))^s \phi'(t) dt = \int_0^\infty \phi\big(\mu(\{x \in \Omega : |f(x)| > r\})\big) s r^{s-1} dr.$$

Here f_μ^* denotes the non-increasing rearrangement of f with respect to the measure μ (see e.g. [22] for the definition and basic properties). The simple proof that (1.5) implies $u \in \exp L^{\frac{n}{n-1-\alpha}}(\Omega)$ (or (1.7) implies $u \in \exp\exp L^{\frac{n}{n-1-\beta}}(\Omega)$) can be caried out analogously to [1, proof of Theorem 3].

As we pointed out before, Theorems 1.1 and 1.2 are general enough so that they can be applied in the context of analysis on metric measure spaces (see [10], [14] and references given there for a possible range of applications), but our results give a new information even in the Euclidean setting. This will be briefly discussed in the last section.

2. Preliminaries

A function $\Phi : \mathbf{R}^+ \to \mathbf{R}^+$ is a Young function if $\Phi(0) = 0$, Φ is increasing and convex. We denote by $L^\Phi(\Omega)$ the corresponding Orlicz space with Young function Φ on a set Ω with measure μ. This space is equipped with the norm

$$||f||_{L^\Phi(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\ d\mu(x) \leq 1\Big\}\ .$$

For an introduction to Orlicz spaces see [21]. By $WL^\Phi(\Omega)$ we denote the set of functions f such that $|\nabla f| \in L^\Phi(\Omega)$ and by $W_0L^\Phi(\Omega)$ we denote the closure of $C_0^1(\Omega)$ in $WL^\Phi(\Omega)$.

For $s > 1$, $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}$ we fix an Young function $\Phi_{s,\alpha,\beta}$ such that $\Phi_{s,\alpha,\beta}(t) \sim t^s \log^\alpha(e+t) \log^\beta(e + \log(e+t))$. We denote by $L^s \log^\alpha L$ an Orlicz space corresponding to the Young function $\Phi_{s,\alpha,0}$ and by $L^s \log^\alpha L \log^\beta \log L$ an Orlicz space corresponding to the Young function $\Phi_{s,\alpha,\beta}$. Analogously given $\gamma > 0$ and $\delta > 0$ we fix Young functions $\Phi_\gamma(t)$ and $\Phi_\delta(t)$ such that $\Phi_\gamma(t) \sim \exp(t^\gamma)$ for $t \geq 1$ and $\Phi_\delta(t) \sim \exp(\exp(t^\delta))$ for $t \geq 1$ and we denote by $\exp L^\gamma$ (resp. $\exp\exp L^\delta$) an Orlicz space corresponding to the Young function Φ_γ (resp. Φ_δ).

We say that an Young function Φ satisfies Δ_2-condition if there is $C > 0$ such that $\Phi(2t) < C\Phi(t)$ for every $t > 0$. If Φ satisfies Δ_2-condition then (see [21, Proposition 6, p. 77])

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{||f||_{L_\Phi}}\Big) d\mu(x) = 1. \tag{2.1}$$

Note that the function $\Phi_{s,\alpha,\beta}$ satisfies Δ_2-condition.

Let Φ be a convex function and let $h : S \to \mathbf{R}$ be a non-negative function. Then we can use the following version of the Jensen's inequality:

$$\frac{1}{\mu(S)} \int_S h(x)dx \leq \Phi^{-1}\Big(\frac{1}{\mu(S)} \int_S \Phi(h(x))dx\Big). \tag{2.2}$$

We will also employ the following simple lemma.

Lemma 2.1. *Let ν be a finite measure on a set Y. If $w \geq 0$ is a ν-measurable function such that $\nu(\{y \in Y : w(y) = 0\}) \geq \frac{\nu(Y)}{2}$, then, for every $t > 0$*

$$\nu(\{y \in Y : w(y) > t\}) \leq 2 \inf_{c \in \mathbf{R}} \nu(\{y \in Y : |w(y) - c| > t/2\}).$$

We will often abbreviate the set $\{x \in \Omega : f(x) \geq a\}$ ($\{x \in \Omega : f(x) \geq a\}$ $\ldots$) to $\{f \geq a\}$ ($\{f \leq a\}$ $\ldots$).

By C we will denote various positive constants that may depend on s, α, β, C_1, C_2, K, $||g||_{Y(\Omega)}$ and $||f||_{Y(\Omega)}$, where $Y = L^s \log^\alpha L(\Omega)$ in the proof of Theorem 1.1 and $Y = L^s \log^{s-1} L \log^\beta \log L(\Omega)$ in the proof of Theorem 1.2. These constants may vary from expression to expression as usual.

We denote by $\mathcal{L}_n$ the n-dimensional Lebesgue measure. We write $h(t) \sim g(t)$ if there is a constant $C > 1$ such that $\frac{1}{C} h(t) \le g(t) \le C h(t)$ for every t. Sometimes we abbreviate $\int_\Omega f(x) d\mu(x)$ to $\int_\Omega f d\mu$ and we write $\fint_\Omega u d\mu := \frac{1}{\mu(\Omega)} \int_\Omega u d\mu$.

3. Proof of Theorem 1.1

Lemma 3.1. *Let $s \in (1, \infty)$ and $\alpha \in \mathbf{R}$. Suppose that the functions $f_k : \Omega \to \mathbf{R}$ have pairwise disjoint supports and that $f = \sum_{k=1}^{\infty} f_k \in L^s \log^\alpha L(\Omega)$. If $\alpha > 0$ we further assume that*

$$\log\Big(\frac{1}{||f_k||_{L^s \log^\alpha L(\Omega)}}\Big) < C \log\Big(\frac{e\mu(\Omega)}{\mu(\{f_k \neq 0\})}\Big). \tag{3.1}$$

Then

$$\sum_{k=1}^{\infty} ||f_k||^s_{L^s \log^\alpha L(\Omega)} < \infty.$$

Proof. Without loss of generality we may assume that $||f||_{L^s \log^\alpha L(\Omega)} = 1$. Otherwise, we will replace f_k by $\frac{f_k}{||f||_{L^s \log^\alpha L(\Omega)}}$ and the constant C in (3.1) we will replace by $C + \max\{0, \log ||f||_{L^s \log^\alpha L(\Omega)}\}$. Other conditions are not affected by this change.

Denote $\lambda_k = ||f_k||_{L^s \log^\alpha L(\Omega)}$. First let us assume that $\alpha \le 0$. From (2.1), $\alpha \le 0$ and $\lambda_k \le 1$ we have

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k^s &\le C \sum_{k=1}^{\infty} \int_\Omega f_k^s \log^\alpha\big(e + \frac{f_k}{\lambda_k}\big) d\mu \\ &\le C \sum_{k=1}^{\infty} \int_\Omega f_k^s \log^\alpha(e + f_k) d\mu = C \int_\Omega f^s \log^\alpha f d\mu \le C. \end{aligned} \tag{3.2}$$

From now on let $\alpha > 0$. We claim that

$$\log\big(e + \frac{1}{\lambda_k}\big) \le C \log\Big(e + \frac{1}{\mu(\{f_k \neq 0\})} \int_\Omega f_k^s \log^\alpha(e + f) d\mu\Big). \tag{3.3}$$

From (2.1), $\lambda_k \le 1$ and the simple inequality we obtain

$$\lambda_k^s \le C \int_\Omega f_k^s \log^\alpha(e + \frac{f_k}{\lambda_k}) d\mu \le C \frac{1}{\lambda_k^\alpha} \int_\Omega f_k^s \log^\alpha(e + f_k) d\mu$$

and hence

$$-\log\Big(\int_\Omega f_k^s \log^\alpha(e + f_k) d\mu\Big) \le C \log(e + \frac{1}{\lambda_k}).$$

Together with (3.1) this implies (3.3).

Fix a convex function Φ such that $\Phi(t) \sim t\log^{\alpha}(e+t)$. Then $\Phi^{-1}(t) \sim t\log^{-\alpha}(e+t)$ and therefore we may use Jensen's inequality (2.2) for the function $h = f_k^s$ and $S = \{f_k \neq 0\}$ to conclude that

$$\int_S f_k^s \leq C\Big(\int_S f_k^s \log^{\alpha}(e+f_k)d\mu\Big)\log^{-\alpha}\Big(e + \frac{1}{\mu(\{f_k\neq 0\})}\int_S f_k^s\log^{\alpha}(e+f_k)d\mu\Big). \tag{3.4}$$

From (3.3) and (3.4) we have

$$\begin{aligned}\int_\Omega f_k^s \log^{\alpha}\big(e+\frac{f_k}{\lambda_k}\big)d\mu &\leq C\int_\Omega f_k^s\log^{\alpha}\big(e+f_k\big)d\mu + C\log^{\alpha}\big(e+\frac{1}{\lambda_k}\big)\int_\Omega f_k^s d\mu\\ &\leq C\int_\Omega f_k^s \log^{\alpha}(e+f_k)d\mu.\end{aligned}$$

and hence we can finish the proof similarly to (3.2). $\square$

Proof of Theorem 1.1. Part of our arguments will follow the ideas of [14, Theorem 1.1 *ii*)]. For the completeness we give some details.

Choose $d \in \mathbf{R}$ such that

$$\mu(\{u \geq d\}) \geq \frac{\mu(\Omega)}{2} \text{ and } \mu(\{u\leq d\}) \geq \frac{\mu(\Omega)}{2}.$$

Let $v_+ = \max\{u-d, 0\}$ and $v_- = -\min\{u-d,0\}$. In what follows v will denote either v_+ or v_-. Fix $0 < t_1 < t_2 < \infty$. From (1.4), truncation property and $\exp t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$ we have

$$\inf_{c\in\mathbf{R}}\Big(\int_\Omega |v_{t_1}^{t_2} - c|^{\frac{sk}{s-1-\alpha}} d\mu\Big)^{\frac{s-1-\alpha}{sk}} \leq C(k!)^{\frac{s-1-\alpha}{sk}}||g_{t_1}^{t_2}||_{L^s\log^{\alpha}L(\Omega)} \tag{3.5}$$

for every $k \in \mathbf{N}$. From Lemma 2.1, $C^{1/k} \sim C$ for $k \geq 1$ and the weak form of (3.5) we obtain

$$\begin{aligned} t[\mu(\{v_{t_1}^{t_2} \geq t\})]^{\frac{s-1-\alpha}{sk}} &\leq C\inf_{c\in\mathbf{R}}\frac{t}{2}[\mu(\{|v_{t_1}^{t_2} - c| \geq t/2\})]^{\frac{s-1-\alpha}{sk}} \leq\\ &\leq C\big(\mu(\Omega)\big)^{\frac{s-1-\alpha}{sk}}(k!)^{\frac{s-1-\alpha}{sk}}||g_{t_1}^{t_2}||_{L^s\log^{\alpha}L(\Omega)}\end{aligned}$$

for all $k \in \mathbf{N}$ and every $t > 0$. Since $(k!)^{\frac{1}{k}} \sim k$ we have

$$t\Big(\frac{\mu(\{v_{t_1}^{t_2}\geq t\})}{e\mu(\Omega)}\Big)^{\frac{s-1-\alpha}{sm}} \leq Cm^{\frac{s-1-\alpha}{s}}||g_{t_1}^{t_2}||_{L^s\log^{\alpha}L(\Omega)} \tag{3.6}$$

for all $m \geq 1$ and every $t > 0$.

Fix $i \in \mathbf{N}$ and let $t = t_i = 2^i$, $t_{i+1} = 2^{i+1}$ and $m = \log(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})})$. From $\{v_{2^i}^{2^{i+1}} \geq 2^i\} = \{v \geq 2^{i+1}\}$, $A^{\frac{1}{\log\frac{1}{A}}} = e^{-1}$ and (3.6) we have

$$\frac{2^i}{\log^{\frac{s-1-\alpha}{s}}\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big)} \leq C||g_{2^i}^{2^{i+1}}||_{L^s\log^{\alpha}L(\Omega)}. \tag{3.7}$$

We raise this estimate to power s and sum over i and we obtain

$$
\begin{aligned}
\int_2^\infty \frac{t^{s-1}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v\geq t\})}\big)}dt &\leq C\sum_{i=0}^\infty \frac{2^{si}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big)} \\
&\leq C\sum_{i=0}^\infty ||g_{2^i}^{2^{i+1}}||^s_{L^s\log^\alpha L(\Omega)}.
\end{aligned} \tag{3.8}
$$

For $\alpha\leq 0$ we may use Lemma 3.1 to conclude that the last sum is finite. From (3.8) for v_+ and v_- we have

$$
\begin{aligned}
\inf_{c\in\mathbf{R}}\int_0^\infty \frac{t^{s-1}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{x\in\Omega:|u(x)-c|\geq t\})}\big)}dt &\leq \int_0^\infty \frac{t^{s-1}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{x\in\Omega:|u(x)-d|\geq t\})}\big)}dt \leq \\
\leq C\Big(\int_0^\infty \frac{t^{s-1}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v_+\geq t\})}\big)}dt &+ \int_0^\infty \frac{t^{s-1}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v_-\geq t\})}\big)}dt\Big) < \infty.
\end{aligned} \tag{3.9}
$$

Now let us return to the case $0<\alpha<s-1$. The only thing we need for finishing the proof similarly as above is

$$
\sum_{i=0}^\infty \frac{2^{si}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big)} < \infty. \tag{3.10}
$$

Set $S_i=\{v\geq 2^i\}$,

$$
G=\Big\{i\in\mathbf{N}_0:\log\big(\frac{e\mu(\Omega)}{\mu(S_{i+1})}\big)<K4^{\frac{s}{s-1-\alpha}}\log\big(\frac{e\mu(\Omega)}{\mu(S_i)}\big)\Big\} \text{ and } B=\mathbf{N}_0\setminus G,
$$

where $K\geq 1$ is a constant big enough such that $0\in G$. For $i\in G$ we can use (3.6) for $m=1$, $t=t_i=2^i$, $t_{i+1}=2^{i+1}$ and the definition of G to conclude that

$$
\begin{aligned}
\log\big(\frac{1}{||g_{2^i}^{2^{i+1}}||_{L^s\log^\alpha L(\Omega)}}\big) &\leq C\log\big(\frac{e\mu(\Omega)}{\mu(S_{i+1})}\big) \\
&\leq C\log\big(\frac{e\mu(\Omega)}{\mu(S_i)}\big)\leq C\log\big(\frac{e\mu(\Omega)}{\mu(\{g_{2^i}^{2^{i+1}}\neq 0\})}\big).
\end{aligned}
$$

This verifies assumption (3.1) and therefore Lemma 3.1 and (3.7) give us

$$
\sum_{i\in G}\frac{2^{si}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big)} < C\sum_{i\in G}||g_{2^i}^{2^{i+1}}||^s_{L^s\log^\alpha L(\Omega)}<\infty. \tag{3.11}
$$

For $i\in G$ let us define $B_i=\big\{j\in B: j>i \text{ and } \{i+1,i+2,\dots,j\}\subset B\big\}$. From the definition of B, simple induction and (3.11) we have

(3.12)
$$\begin{aligned}\sum_{j\in B}\frac{2^{sj}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{j+1}\})}\big)}&\leq\sum_{i\in G}\sum_{j\in B_i}\frac{2^{sj}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(S_{j+1})}\big)}\\ &\leq C\sum_{i\in G}\sum_{j=i+1}^{\infty}\frac{2^{sj}}{4^{s(j-i)}\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(S_{i+1})}\big)}\\ &\leq C\sum_{i\in G}\frac{2^{si}}{\log^{s-1-\alpha}\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big)}\sum_{j=i+1}^{\infty}\frac{1}{2^{s(j-i)}}<\infty.\end{aligned}$$

From (3.11) and (3.12) we obtain (3.10) and the proof is finished. $\square$

4. Proof of Theorem 1.2

The strategy of this section is similar to the previous one, but we give some details for the convenience of the reader.

Lemma 4.1. *Let $s\in(1,\infty)$ and $\beta\in\mathbf{R}$. Suppose that the functions $f_k:\Omega\to\mathbf{R}$ have pairwise disjoint supports and that $f=\sum_{k=1}^{\infty}f_k\in L^s\log^{s-1}L\log^{\beta}\log L(\Omega)$. We further assume that*

$$\log\Big(\frac{1}{||f_k||_{L^s\log^{s-1}L\log^{\beta}\log L(\Omega)}}\Big)<C\log\Big(\frac{e\mu(\Omega)}{\mu(\{f_k\neq 0\})}\Big). \tag{4.1}$$

Then

$$\sum_{k=1}^{\infty}||f_k||^s_{L^s\log^{s-1}L\log^{\beta}\log L(\Omega)}<\infty.$$

Proof. Denote $Y=L^s\log^{s-1}L\log^{\beta}\log L$ and $\lambda_k=||f_k||_{Y(\Omega)}$. Without loss of generality we will suppose that $||f||_{Y(\Omega)}=1$. We claim that

(4.2)
$$\log\Big(e+\frac{1}{\lambda_k}\Big)\leq C\log\Big(e+\frac{1}{\mu(\{f_k\neq 0\})}\int_{\Omega}f_k^s\log^{s-1}(e+f)\log^{\beta}(e+\log(e+f))d\mu\Big).$$

From (2.1), $\lambda_k\leq 1$ and an elementary inequality we obtain

$$\begin{aligned}\lambda_k^s&\leq C\int_{\Omega}f_k^s\log^{s-1}(e+\frac{f_k}{\lambda_k})\log^{\beta}\big(e+\log\big(e+\frac{f_k}{\lambda_k}\big)\big)d\mu\\ &\leq C\frac{1}{\lambda_k}\int_{\Omega}f_k^s\log^{s-1}(e+f_k)\log^{\beta}(e+\log(e+f_k))d\mu\end{aligned}$$

and hence

$$-\log(\int_{\Omega}f_k^s\log^{s-1}(e+f_k)\log^{\beta}(e+\log(e+f_k))d\mu)\leq C\log\big(e+\frac{1}{\lambda_k}\big).$$

Together with (4.1) this implies (4.2).

Fix a convex function Φ such that $\Phi(t)\sim t\log^{s-1}(e+t)\log^{\beta}(e+\log(e+t))$. Then $\Phi^{-1}(t)\sim t\log^{-(s-1)}(e+t)\log^{-\beta}(e+\log(e+t))$ and therefore we may use Jensen's inequality (2.2) for the function $h=f_k^s$ and $S=\{f_k\neq 0\}$ to conclude that

(4.3)
$$\begin{aligned}&\int_S f_k^s \leq C\int_S f_k^s\log^{s-1}(e+f_k)\log^{\beta}(e+\log(e+f_k))d\mu\\
&\cdot\log^{-(s-1)}\Big(e+\frac{1}{\mu(\{f_k\neq 0\})}\int_S f_k^s\log^{s-1}(e+f_k)\log^{\beta}(e+\log(e+f_k))d\mu\Big)\\
&\cdot\log^{-\beta}\Big(e+\log\big(e+\frac{1}{\mu(\{f_k\neq 0\})}\int_S f_k^s\log^{s-1}(e+f_k)\log^{\beta}(e+\log(e+f_k))d\mu\big)\Big).\end{aligned}$$

The function $t\to t^{s-1}\log^{\beta}(e+t)$ is increasing for big enough t and therefore (4.2) gives us

$$\begin{aligned}&\log^{s-1}\big(e+\frac{1}{\lambda_k}\big)\log^{\beta}\big(e+\log\big(e+\frac{1}{\lambda_k}\big)\big)\\
&\leq C\log^{s-1}\Big(e+\frac{1}{\mu(\{f_k\neq 0\})}\int_\Omega f_k^s\log^{s-1}(e+f_k)\log^{\beta}(e+\log(e+f_k))d\mu\Big)\\
&\cdot\log^{\beta}\Big(e+\log\Big(e+\frac{1}{\mu(\{f_k\neq 0\})}\int_\Omega f_k^s\log^{s-1}(e+f_k)\log^{\beta}(e+\log(e+f_k))d\mu\Big)\Big).\end{aligned}$$

Thus the elementary inequality and (4.3) imply

$$\begin{aligned}\int_\Omega f_k^s\log^{s-1}\big(e+\frac{f_k}{\lambda_k}\big)&\log^{\beta}\big(e+\log\big(e+\frac{f_k}{\lambda_k}\big)\big)d\mu\\
&\leq C\int_\Omega f_k^s\log^{s-1}\big(e+f_k\big)\log^{\beta}\big(e+\log\big(e+f_k\big)\big)d\mu\\
&+C\log^{s-1}\big(e+\frac{1}{\lambda_k}\big)\log^{\beta}\big(e+\log\big(e+\frac{1}{\lambda_k}\big)\big)\int_\Omega f_k^s d\mu\\
&\leq C\int_\Omega f_k^s\log^{s-1}\big(e+f_k\big)\log^{\beta}(e+\log(e+f_k))d\mu.\end{aligned}$$

Now (2.1) implies

$$\begin{aligned}\sum_{k=1}^{\infty}\lambda_k^s&\leq C\sum_{k=1}^{\infty}\int_\Omega f_k^s\log^{s-1}\big(e+\frac{f_k}{\lambda_k}\big)\log^{\beta}\big(e+\log\big(e+\frac{f_k}{\lambda_k}\big)\big)d\mu\\
&\leq C\sum_{k=1}^{\infty}\int_\Omega f_k^s\log^{s-1}\big(e+f_k\big)\log^{\beta}(e+\log(e+f_k))d\mu\\
&=C\int_\Omega f^s\log^{s-1}\big(e+f\big)\log^{\beta}(e+\log(e+f))d\mu\leq C.\end{aligned}$$

$\square$

Proof of Theorem 1.2. We choose $d \in \mathbf{R}$ such that

$$\mu(\{u \geq d\}) \geq \frac{\mu(\Omega)}{2} \text{ and } \mu(\{u \leq d\}) \geq \frac{\mu(\Omega)}{2}$$

and let $v_+ = \max\{u-d, 0\}$ and $v_- = -\min\{u-d, 0\}$. We fix $0 < t_1 < t_2 < \infty$ and we denote $Y = L^s \log^{s-1} L \log^\beta \log L$. From (1.6), truncation property and

$$\exp\exp t = \sum_{k=0}^{\infty} \frac{\exp(kt)}{k!} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{k^l}{k!\, l!} t^l$$

we have

$$\inf_{c\in\mathbf{R}} \Big(\int_\Omega |v_{t_1}^{t_2} - c|^{\frac{sl}{s-1-\beta}} d\mu \Big)^{\frac{s-1-\beta}{sl}} \leq C \Big(\frac{k!\, l!}{k^l}\Big)^{\frac{s-1-\beta}{sl}} ||g_{t_1}^{t_2}||_{Y(\Omega)} \tag{4.4}$$

for every $k, l \in \mathbf{N}$. From Lemma 2.1 and the weak form of (4.4) we obtain

$$t[\mu(\{v_{t_1}^{t_2} \geq t\})]^{\frac{s-1-\beta}{sl}} \leq C\big(\mu(\Omega)\big)^{\frac{s-1-\beta}{sl}} \Big(\frac{k!\, l!}{k^l}\Big)^{\frac{s-1-\beta}{sl}} ||g_{t_1}^{t_2}||_{Y(\Omega)}$$

for all $k, l \in \mathbf{N}$ and every $t > 0$. Since $(k!)^{\frac{1}{l}} \sim ((k+1)!)^{\frac{1}{l}}$ if $k \leq l$ we obtain

$$t\Big(\frac{\mu(\{v_{t_1}^{t_2} \geq t\})}{e\mu(\Omega)}\Big)^{\frac{s-1-\beta}{sb}} \leq C\Big(\frac{a^{\frac{a}{b}} b}{a}\Big)^{\frac{s-1-\beta}{s}} ||g_{t_1}^{t_2}||_{Y(\Omega)} \tag{4.5}$$

for every $b \geq 1$, $1 \leq a \leq b$ and $t > 0$.

Fix $i \in \mathbf{N}$ and let $t_i = t = 2^i$, $t_{i+1} = 2^{i+1}$, $b = \log(\frac{e\mu(\Omega)}{\mu(\{v \geq 2^{i+1}\})})$ and $a = b\log^{-1}(e+b)$. From $A^{\frac{1}{\log\frac{1}{A}}} = e^{-1}$, $(b\log^{-1}(e+b))^{\frac{b\log^{-1}(e+b)}{b}} \sim C$ for $b \geq 1$, $\{v_{2^i}^{2^{i+1}} \geq 2^i\} = \{v \geq 2^{i+1}\}$ and (4.5) we have

$$\frac{2^i}{\log^{\frac{s-1-\beta}{s}}\Big(e + \log\big(\frac{e\mu(\Omega)}{\mu(\{v \geq 2^{i+1}\})}\big)\Big)} \leq C||g_{2^i}^{2^{i+1}}||_{Y(\Omega)}. \tag{4.6}$$

Set $S_i = \{v \geq 2^i\}$,

$$G = \Big\{ i \in \mathbf{N}_0 : \log\big(e + \log\big(\frac{e\mu(\Omega)}{\mu(S_{i+1})}\big)\big) < K 4^{\frac{s}{s-1-\alpha}} \log\big(e + \log\big(\frac{e\mu(\Omega)}{\mu(S_i)}\big)\big)\Big\}$$

and $B = \mathbf{N}_0 \setminus G$, where $K \geq 1$ is big enough such that $0 \in G$.

From Lemma 2.1 we obtain

$$\mu(\{v \geq 2^{i+1}\}) = \mu(\{v_{2^i}^{2^{i+1}} \geq 2^i\}) \leq 2 \inf_{c\in\mathbf{R}} \mu(\{|v_{2^i}^{2^{i+1}} - c| \geq 2^{i-1}\}).$$

Hence we can use (1.6) and the truncation property for $t_1 = 2^i$ and $t_2 = 2^{i+1}$ to obtain

$$\mu(\{v \geq 2^{i+1}\}) \exp\exp\Big(\Big(\frac{2^{i-1}}{C||g_{2^i}^{2^{i+1}}||_{Y(\Omega)}}\Big)^{\frac{s}{s-1-\beta}}\Big) \leq C_2$$

Thus for $i \in G$ we have

$$\frac{1}{||g_{2^i}^{2^{i+1}}||_{Y(\Omega)}} \leq C \log^{\frac{s-1-\beta}{s}}(e+\log(\frac{C}{\mu(S_{i+1})})) \leq C \log^{\frac{s-1-\beta}{s}}(e+\log(\frac{C}{\mu(\{g_{2^i}^{2^{i+1}} \neq 0\})}))$$

This verifies assumption (4.1) and therefore Lemma 4.1 and (4.6) give us

$$\sum_{i\in G} \frac{2^{si}}{\log^{s-1-\beta}\big(e+\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big)\big)} < C \sum_{i\in G} ||g_{2^i}^{2^{i+1}}||_{Y(\Omega)}^s < \infty. \tag{4.7}$$

For $i \in G$ let us define $B_i = \big\{j \in B : j > i \text{ and } \{i+1, i+2, \dots, j\} \subset B\big\}$. Analogously to (3.12) we obtain from the definition of B, simple induction and (4.7) that

$$\begin{aligned}\sum_{j\in B} &\frac{2^{sj}}{\log^{s-1-\beta}\big(e+\log\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{j+1}\})}\big)\big)} \leq \\ &\leq C \sum_{i\in G} \sum_{j=i+1}^{\infty} \frac{2^{sj}}{4^{s(j-i)}\log^{s-1-\beta}\big(e+\log\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big)\big)} < \infty.\end{aligned} \tag{4.8}$$

From (4.7) and (4.8) we obtain

$$\int_2^{\infty} \frac{t^{s-1}}{\log^{s-1-\beta}(e+\log(\frac{e\mu(\Omega)}{\mu(\{v\geq t\})}))} dt \leq C \sum_{i=0}^{\infty} \frac{2^{si}}{\log^{s-1-\beta}(e+\log\big(\frac{e\mu(\Omega)}{\mu(\{v\geq 2^{i+1}\})}\big))} < \infty$$

and therefore we can finish the proof similarly to (3.9). □

5. The Euclidean setting

A domain $\Omega \subset \mathbf{R}^n$ is called an c_0-John domain, $0 < c_0 \leq 1$, if Ω is bounded and there exists $x_0 \in \Omega$ such that each $x \in \Omega$ can be joined to x_0 inside Ω by a rectifiable curve $\gamma : [0, l] \to \Omega$, parametrized by arc length such that the distance to the boundary satisfies

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq c_0 t,\ \ t \in [0, l].$$

Each Lipschitz domain is a John domain and so is a Koch snowflake domain, [18].

Given a domain $\Omega \subset \mathbf{R}^n$ and $u : \Omega \to \mathbf{R}$ we denote $u_\Omega = \frac{1}{\mathcal{L}_n(\Omega)} \int_\Omega u$. The proof of [7, Theorem 3.2] tells us that if Ω is a c_0-John domain, $\alpha < 0$ and

$$\int_\Omega |\nabla u(x)|^n \log^\alpha(e + |\nabla u(x)|) dx \leq 1$$

then we can find constants $C_1 = C(\alpha, c_0, \mathcal{L}_n(\Omega), n)$ and $C_2 = C(\alpha, c_0, n, \mathcal{L}_n(\Omega))$, such that

$$\int_\Omega \exp\Big(\Big(\frac{|u(x) - u_\Omega|}{C_1}\Big)^{\frac{n}{n-1-\alpha}}\Big) dx \leq C_2.$$

This clearly gives us (1.4) and therefore we have the following corollary of Theorem 1.1:

Corollary 5.1. *Let $\Omega \subset \mathbf{R}^n$ be an c_0-John domain for some $0 < c_0 \leq 1$. Let $\alpha < 0$ and $\nabla u \in L^n \log^\alpha L(\Omega)$. Then*

$$\inf_{c\in\mathbf{R}} \int_0^\infty \frac{t^{s-1}}{\log^{s-1-\alpha}\Big(\frac{e\mathcal{L}_n(\Omega)}{\mathcal{L}_n(\{x\in\Omega:|u(x)-c|\geq t\})}\Big)}dt < \infty.$$

The main application of our results in the Euclidean settings is the fact, that we can always have a better embedding in the domain Ω, if the embeddings into exponential or double exponential Orlicz space is valid in Ω.

Let us note that the original proof of the exponential embedding of Trudinger (1.1) was based on the following idea: First estimate a value of the constant of the embedding

$$||u||_{L^p(\Omega)} \leq \tilde{C}(p)||\nabla u||_{L^n(\Omega)}$$

for large p. Then use the exponential series

$$\int_\Omega \exp\Big(\Big(\frac{u(x)}{C_1||\nabla u||_{L^n(\Omega)}}\Big)^{\frac{n}{n-1}}\Big)dx = \sum_{k=0}^\infty \frac{1}{k!}\Big(\frac{||u||_{L^{\frac{nk}{n-1}}(\Omega)}}{C_1||\nabla u||_{L^n(\Omega)}}\Big)^{\frac{nk}{n-1}}$$

and the knowledge on $\tilde{C}(\frac{kn}{n-1})$ to show that the series converges if we choose C_1 sufficiently big. Our proof tells us that the similar strategy would be possible for the proof in the general case, i.e. $\alpha < n-1$. Indeed, we have used (1.4) only to deduce (3.5), which is equivalent (note that functions of the form $v_{t_1}^{t_2}$ are dense in $WL^n \log^\alpha L(\Omega)$) to an estimate of the constant of the embedding

$$||u||_{L^p(\Omega)} \leq \tilde{C}(p,\alpha)||\nabla u||_{L^n \log^\alpha L(\Omega)}.$$

From this fact only we were able to deduce (1.5) which is even a stronger property than $u \in \exp L^{\frac{n}{n-1-\alpha}}(\Omega)$.

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