Linear dilatation and absolute continuity

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Abstract

We show that already the local integrability of the linear dilatation of a homeomorphism guarantees that the homeomorphism is absolutely continuous on almost all lines parallel to the coordinate axes, under the assumption that the linear dilatation be finite outside a set of σ-finite $(n-1)$-measure.

1 Background and statement of results

Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, and $f \colon \Omega \to \Omega' \subset \mathbf{R}^n$ a homeomorphism. For $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$ we set

$$L_f(x,r) = \max\{|f(x) - f(y)| : |x-y| \leq r\},$$

$$l_f(x,r) = \min\{|f(x) - f(y)| : |x-y| \geq r\}.$$

The linear limsup-dilatation of f at x is defined as

$$H_f(x) = \limsup_{r \to 0} H_f(x,r)$$

where $H_f(x,r) = L_f(x,r)/l_f(x,r)$. Similarly we can define the linear liminf-dilatation $h_f(x)$ by replacing $\limsup$ with $\liminf$.

A well-known result of Gehring's [2], [3] says that if a homeomorphism f has its linear limsup-dilatation $H_f(x)$ uniformly bounded a.e. in $\Omega \setminus E$, where E has σ-finite $(n-1)$-measure, then f is a quasiconformal mapping. Similarly it has been showed in [6] that if the linear liminf-dilatation $h_f(x)$ is uniformly bounded a.e. in $\Omega \setminus E$, then f is still quasiconformal. For earlier results, see [4], [5]. In particular, in both cases f is ACL in Ω, which means that f is absolutely continuous on almost every line segment parallel to the coordinate axes in Ω. In [9], Tukia conjectured that the condition

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$m_2(\{H_f(x) > K\}) < CK^{-\alpha}$, for some $\alpha > 3$ and $K \geq K_0$, is sufficient for the ACL-property of a plane homeomorphism f. This was proven in [1] together with a space analogue. Furthermore, it was later showed in [7] that $H_f \in L^{\alpha}_{loc}(\Omega)$, $\alpha > n/(n-1)$, guarantees the ACL-property. In the above results, it was also assumed that H_f be finite outside a set of σ-finite $(n-1)$-measure, which seems crucial, see Remark 1.2 (b).

In this paper we will show that already $H_f \in L^1_{loc}(\Omega)$ is sufficient for the ACL-property. Before stating our results, let us introduce the following dilatations:

$$K_f(x) = \limsup_{r \to 0} \left(\frac{\operatorname{diam}(f(B(x,r)))^n}{|fB(x,r)|} \right)^{\frac{1}{n-1}},$$

and

$$k_f(x) = \liminf_{r \to 0} \left(\frac{\operatorname{diam}(f(B(x,r)))^n}{|fB(x,r)|} \right)^{\frac{1}{n-1}},$$

where $|A|$ denotes the Lebesgue measure of a set A. We noticed during our studies that these are more natural (and so more useful) for proving absolute continuity. At the points of differentiability with $l(Df(x)) = \min\{|Df(x)e| : |e| = 1\} > 0$ these dilatations are comparable to H_f and h_f, respectively. It is also easy to see that K_f and k_f are Borel functions.

Our main theorem is the following result.

Theorem 1.1 *Let $f \colon \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathbf{R}^n$ are domains, be a homeomorphism for which $k_f(x) < \infty$ outside a set S of σ-finite $(n-1)$-measure, and suppose that $k_f \in L^1_{loc}(\Omega)$. Then f belongs to $W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$.*

The above theorem gives us the ACL-property, since a continuous $W^{1,1}_{loc}$-mapping is ACL (see Proposition I.1.2 in [8]).

Remarks 1.2 (a) To see that Theorem 1.1 is sharp, consider the mapping $f \colon]0, \frac{1}{e}[\times \mathbf{R}^{n-1} \to f(]0, \frac{1}{e}[\times \mathbf{R}^{n-1})$, $f(x) = (1/\log(1/x_1), x_1 \sin(1/x_1) + x_2, x_3, \ldots, x_n)$. This mapping is a non-ACL homeomorphism of $\mathbf{R}^n$ which satisfies $k_f(x) \in L^s_{loc}$ for any $s < 1$.
(b) The condition of σ-finiteness of S is crucial. For example, if $g \colon [0,1] \to [0,1]$ is the Cantor staircase function, then $f \colon]0,1[\times]0,1[\to]0,2[\times]0,1[$ defined by $f(x,y) = (g(x) + x, y)$ is a homeomorphism with $k_f(z) = 1$ almost everywhere, but f is not ACL.

The following corollary summaries the conclusions obtained from Theorem 1.1 for various distortion functions.

Corollary 1.3 *Let $f\colon \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathbf{R}^n$ are domains, be a homeomorphism and suppose that S has σ-finite $(n-1)$-measure. Now each of the conditions below guarantees that $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$.*

1. *$K_f(x) < \infty$ outside S and $K_f \in L^1_{loc}(\Omega)$,*
2. *$k_f(x) < \infty$ outside S and $k_f \in L^1_{loc}(\Omega)$,*
3. *$H_f(x) < \infty$ outside S and $H_f \in L^1_{loc}(\Omega)$,*
4. *$h_f(x) < \infty$ outside S and $h_f \in L^{\frac{n}{n-1}}_{loc}(\Omega)$.*

Remarks 1.4 The mapping discussed in the first part of Remarks 1.2 shows the sharpness of the integrability assumptions in 1., 2., 3. Regarding 4., we do not know if already $h_f \in L^1_{loc}$ could be sufficient. This would follow if it were true that the requirement $h_f(x) < \infty$ outside S and $h_f \in L^1_{loc}(\Omega)$ would guarantee a.e. differentiability. We do not know if this could be the case. In any case, 4. already substantially improves on the known results from [6].

2 Proofs

Proof of Theorem 1.1. To prove that $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$ we first show that f is ACL. After that we show the local integrability of the partial derivatives, whose existence is guaranteed by the ACL-property.

Pick a closed cube $Q \subset\subset \Omega$ whose sides are parallel to the coordinate axes. Assume that $Q = Q_0 \times J_0$, where Q_0 is $(n-1)$-interval in $\mathbf{R}^{n-1}$, and $J_0 = [a, b] \subset \mathbf{R}$. In order to show that f is ACL it suffices to show that f is absolutely continuous on almost every line segment in Q, parallel to the coordinate axes, and by symmetry it is sufficient to consider segments parallel to the x_n-axis.

Next, for a Borel set $A \subset Q_0$, write

$$\Phi(A) := |f(A \times [a-d, b+d])| \le |f(Q+d)| < \infty,$$

where $d = \frac{1}{10}\mathrm{dist}(Q, \partial\Omega)$ and $Q + d = \{x \in \Omega : \mathrm{dist}(x, Q) \le d\}$. Then Φ is a finite Borel measure on Q_0, and hence it has a finite derivative $\Phi'(y)$ for almost all $y \in Q_0$. Denote by E_0 the set where Φ' does not exists or is not finite.

Next we consider the set $\mathcal{A} = \{I \subset J_0 : I$ is a finite union of closed intervals, whose interiors are mutually disjoint and whose end points are rational$\}$. This set is countable: (i) If we take all the intervals whose end points are rational, there is just a countable number of intervals. (ii) If we take all the pairs of intervals, whose endpoints are rational, there is just a countable number of pairs. (iii) If we take all triples of the same type we again have a countable numbers of triples, etc. Thus $\mathcal{A}$ is countable union of countable sets and so countable.

Now, for almost every $y \in Q_0$, we know by the Fubini theorem that

$$\int_{\{y\}\times[a-d,b+d]} k_f(z)\,dz_n < \infty.$$

Denote the set where the above fails by E_1. Let us define for every $I \subset \mathcal{A}$ a function $g_I\colon Q_0 \to \mathbf{R}$,

$$g_I(y) = \int_{\{y\}\times I} k_f(z)\,dz_n.$$

By the Fubini theorem, $g_I \in L^1(Q_0)$, and thus for almost every $y \in Q_0$

$$\lim_{r\to 0} \fint_{B^{n-1}(y,r)} g_I(z)\,dz = g_I(y).$$

Denote by E_I the set where this is not true. Now $E = E_0 \cup E_1 \cup (\cup_{I\in\mathcal{A}} E_I)$ has measure zero, because it is a countable union of sets of measure zero.

Fix $y \in Q_0 \setminus E$. We will prove that f is absolutely continuous on the segment $\{y\} \times J_0$ which will prove the claim.

Let $\{I_j\}_{j=1}^{l}$, $I_j = [a_j, b_j]$, be a union of closed intervals on J_0, whose interiors are mutually disjoint, and whose endpoints are rational numbers. Since f is continuous, for every $j = 1, \ldots, l$ there is δ_j such that

$$|f(y, a_j) - f(x)| < \frac{|f(y, a_j) - f(y, b_j)|}{4} \quad \text{when } |(y, a_j) - x| < \delta_j$$

and

$$|f(y, b_j) - f(x)| < \frac{|f(y, a_j) - f(y, b_j)|}{4} \quad \text{when } |(y, b_j) - x| < \delta_j.$$

Denote $\delta = \min_j\{\delta_j\}$.

Let $0 < r < \delta$ and let $\varepsilon > 0$. For each $k = 0, 1, 2, \ldots$, write

$$A_k = \{x \in B^{n-1}(y, r) \times \cup_j I_j : 2^k \le k_f(x) < 2^{k+1}\}.$$

Then A_k is a Borel set,

$$B^{n-1}(y, r) \times \cup_j I_j \setminus S = \cup_k A_k$$

and for every k there exists open U_k such that $A_k \subset U_k$ and

$$|U_k| \le |A_k| + \frac{\varepsilon}{2^{2k}}.$$

Fix k. Now for every $x \in A_k$ there is $r_x > 0$ such that

(i) $0 < r_x < \frac{1}{10}\min\{r, d, |a_j - b_j|\}$,

(ii) $\operatorname{diam}(fB_x)^{\frac{n}{n-1}} < 2^{k+1}|fB_x|^{\frac{1}{n-1}}$, and

(iv) $B_x \subset U_k$.

Here $B_x = B(x, r_x)$.

By the Besicovitch covering theorem we find balls $\overline{B}_1, \overline{B}_2, \ldots$ from balls $\overline{B}(x, r_x)$ so that

$$B^{n-1}(y,r) \times \cup_j I_j \subset \cup_j \overline{B}_j \subset B^{n-1}(y, 2r) \times [a-d, b+d]$$

and $\sum_j \chi_{\overline{B}_j}(x) \le C(n)$ for every $x \in \mathbf{R}^n$.

Let us define

$$\rho(x) = \left(\sum_{j=1}^{l} |f(y,a_j) - f(y,b_j)|\right)^{-1} \sum_i \frac{\operatorname{diam}(fB_i)}{\operatorname{diam}(B_i)} \chi_{2B_i}(x).$$

The function ρ is a Borel function, because it is a countable sum of (simple) Borel functions. In the following we denote $G = \sum_{j=1}^{l} |f(y,a_j) - f(y,b_j)|$.

We wish to estimate the volume integral of ρ. First of all

$$\int_{B^{n-1}(y,r)\times\cup_j I_j} \rho(x)\,dx \ge G^{-1} \int_{B^{n-1}(y,r)} \int_{\cup_j I_j} \sum_{B_i \cap (\{z\}\times\cup_j I_j)\neq\emptyset} \frac{\operatorname{diam}(fB_i)}{\operatorname{diam}(B_i)} \chi_{2B_i}(z, x_n)\,dx_n dz.$$

Notice that

$$\int_{\cup_j I_j} \chi_{2B_i}(z, x_n)\,dx_n \ge \frac{\operatorname{diam}(B_i)}{2}$$

for the balls B_i for which $B_i \cap (\{z\} \times \cup_j I_j) \neq \emptyset$. Moreover, for almost every $z \in B^{n-1}(y,r)$, the sets fB_i cover the $f(\{z\} \times \cup_j I_j)$ up to a countable set, because S has σ-finite $(n-1)$-measure (see Theorem 30.16 in [10]). Thus, since $r < \delta$, we have that

$$\sum_{B_i \cap (\{z\}\times\cup_j I_j)\neq\emptyset} \operatorname{diam}(fB_i) \ge \frac{1}{4}G \tag{1}$$

for almost every $z \in B^{n-1}(y,r)$.

So

$$\int_{B^{n-1}(y,r)\times\cup_j I_j} \rho(x)\,dx \ge C(n) r^{n-1}.$$

Next we establish an upper bound for this integral. Using the monotone convergence theorem, we obtain the estimate

$$\begin{aligned}\int_{B^{n-1}(y,r)\times\cup_j I_j} \rho(x)\,dx &\le C(n)G^{-1} \sum_i \operatorname{diam}(fB_i)|B_i|^{1-\frac{1}{n}} \\ &\le C(n)G^{-1} \left(\sum_i |fB_i|\right)^{\frac{1}{n}} \left(\sum_i \left(\frac{\operatorname{diam}(fB_i)^n}{|fB_i|}\right)^{\frac{1}{n-1}} |B_i|\right)^{\frac{n-1}{n}}.\end{aligned}$$

The last inequality is obtained by Hölders inequality.

For the first term we have that

$$\sum_j |fB_j| \le C(n)|f(B^{n-1}(y,2r)\times[a-d,b+d])| = C(n)\Phi(B^{n-1}(y,2r))$$

because the overlapping of the balls was bounded. The approximation of the second term is a little bit trickier. First notice that $|B_i| = (|B_i \cap A_k| + |B_i \setminus A_k|)$. Thus

$$\begin{aligned}\sum_i \Big(\frac{\operatorname{diam}(fB_i)^n}{|fB_i|}\Big)^{\frac{1}{n-1}}|B_i| &\le 2\sum_k \sum_{x_i\in A_k} 2^k|B_i\cap A_k| + 2\sum_k\sum_{x_i\in A_k} 2^k|B_i\setminus A_k| \\ &\le \int_{B^{n-1}(y,r)\times\cup_j I_j} k_f(x)dx + \varepsilon.\end{aligned}$$

Here x_i is the center of the ball B_i.

Combining the lower bound and the upper bound and remembering that ε was arbitrary, we obtain the inequality

$$\sum_{j=1}^{l}|f(y,a_j)-f(y,b_j)| \le C(n)\left(\frac{\Phi(B^{n-1}(y,2r))}{(2r)^{n-1}}\right)^{\frac{1}{n}}\left(⨍_{B^{n-1}(y,r)}\int_{\cup_j I_j} k_f(z,x_n)\,dx_n\,dz\right)^{\frac{n-1}{n}}.$$

Letting finally r tend to zero, we arrive at

$$\sum_{j=1}^{l}|f(y,a_j)-f(y,b_j)| \le C(n)(\Phi'(y))^{\frac{1}{n}}\left(\int_{\{y\}\times\cup_j I_j} k_f(x)\,dx_n\right)^{\frac{n-1}{n}}.$$

This estimate holds for rational a_i, b_i. By continuity, it then holds for all $a_i, b_i \in \mathbf{R}$. Thus f is absolutely continuous on $\{y\}\times J_0$.

To see that partial derivatives, which now exists a.e. , are locally integrable, notice that for a.e. x we have

$$|\partial_i f_j(x)| \le \liminf_{r\to 0}\frac{\operatorname{diam} fB(x,r)}{r},$$

when $i,j\in\{1,\dots,n\}$. Since

$$\frac{\operatorname{diam} fB(x,r)}{r} \le \left(\frac{|fB(x,r)|}{r^n}\right)^{\frac{1}{n}}\frac{\operatorname{diam} fB(x,r)}{|fB(x,r)|^{\frac{1}{n}}},$$

we conclude that for a.e. x we have

$$|\partial_i f_j(x)| \le \mu_f(x)^{\frac{1}{n}}k_f(x)^{\frac{n-1}{n}}, \qquad i,j\in\{1,\dots,n\},$$

where μ_f is the measure derivative of the measure $m(A) := |f(A)|$. Since the measure derivative and k_f are locally integrable, it follows by Hölder's inequality that the partial derivatives are locally integrable. $\square$

Remark 2.1 The assumption that S have σ-finite $(n-1)$-measure was only used to guarantee that (1) holds. In fact, this estimate follows even if we only know that the one-dimensional measure of $f(L \cap S)$ is zero for (almost every) line L parallel to the coordinate axes. Thus, instead of an exceptional set S of σ-finite $(n-1)$-measure, we could allow for any exceptional set S with this property.

Proof of Corollary 1.3. First, Part 2. is the previous theorem.

Next, 1. is a trivial corollary of 2.

For 3., we first notice that always, for every x, we have $K_f(x) \leq C(n)H_f(x)^{\frac{n}{n-1}}$, and thus K_f is finite outside a set of σ-finite $(n-1)$-measure. Secondly, since H_f is finite almost everywhere, our mapping is differentiable almost everywhere; for this see for example [7]. Now, in the set where f is differentiable and $\min_{|e|=1} |Df(x)e| > 0$, it is easy to see that $K_f(x) \leq C(n)H_f(x)$. Set $\tilde{S} = S \cup \{x \in \Omega : f$ differentiable at x with $|Df(x)| = 0\}$. Then the one-dimensional measure of $f(\tilde{S} \cap L)$ is zero for each line L. The claim follows from Theorem 1.1 and Remark 2.1.

Part 4. follows from Theorem 1.1 and the fact that always $k_f \leq C(n)h_f^{\frac{n}{n-1}}$. $\square$

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