Weakly Differentiable Mappings Between Manifolds

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Abstract

We study Sobolev classes of weakly differentiable mappings $f : \mathbb{X} \to \mathbb{Y}$ between compact Riemannian manifolds without boundary. These mappings need not be continuous. They actually possess less regularity than the mappings in $\mathscr{W}^{1,n}(\mathbb{X},\ \mathbb{Y})$, $n = \dim \mathbb{X}$. The central themes being discussed are:

- smooth approximation of those mappings
- integrability of the Jacobian determinant

The approximation problem in the category of Sobolev spaces between manifolds $\mathscr{W}^{1,p}(\mathbb{X},\ \mathbb{Y})$, $1 \leqslant p \leqslant n$, has been recently settled in [2], [3], [20], [21]. However, the point of our results is that we make no topological restrictions on manifolds $\mathbb{X}$ and $\mathbb{Y}$. We characterize, essentially all, classes of weakly differentiable mappings which satisfy the approximation property. The novelty of our approach is that we were able to detect tiny sets on which the mappings are continuous. These sets give rise to the so-called web like structure of $\mathbb{X}$ associated with the given mapping $f : \mathbb{X} \to \mathbb{Y}$.

The integrability theory of Jacobians in a manifold setting is really different than one might a priori expect based on the results in the Euclidean space. To our surprise, the case when the target manifold $\mathbb{Y}$ admits only trivial cohomology groups $H^\ell(\mathbb{Y})$, $1 \leqslant \ell < n = \dim \mathbb{Y}$,

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like n-sphere, is more difficult than the nontrivial case in which $\mathbb{Y}$ has at least one non-zero ℓ-cohomology. The necessity of topological constraints on the target manifold is a new phenomenon in the theory of Jacobians.

Contents

1 Introduction

Sobolev theory on Riemannian manifolds has come into widespread usage in modern geometry and topology. It also continues to be of great importance in nonlinear partial differential equations (PDE's for short), variational problems, like those in the theory of harmonic maps [23], [34] or quasiconformal deformations [29], [32], nonlinear elasticity, continuum mechanics, and much more. Looking ahead, we have attempted in this paper to present such mappings with all their nuances and possible applications.

The primary objects of our study are weakly differentiable mappings:

$$f : \mathbb{X} \to \mathbb{Y} \tag{1}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are smooth compact oriented Riemannian manifolds without boundary, $\dim \mathbb{X} = n \geqslant 2$ and $\dim \mathbb{Y} = m \geqslant 2$. One might say that C. B. Morrey [40] was the first to consider such mappings. The Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ can be defined in a myriad of ways that are not always equivalent. In our approach we appeal to the celebrated theorem of J. Nash [44], which ensures that $\mathbb{Y}$ can be $\mathscr{C}^\infty$-isometrically imbedded in some Euclidean space $\mathbb{R}^N$. Let us assume that $\mathbb{Y} \subset \mathbb{R}^N$, for simplicity. This being so, we say that $f = (f^1, ..., f^N) : \mathbb{X} \to \mathbb{R}^N$ belongs to the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ if each coordinate function $f^i : \mathbb{X} \to \mathbb{R}$ lies in the usual Sobolev space $\mathscr{W}^{1,p}(\mathbb{X})$, and $f(x) \in \mathbb{Y}$ for almost every $x \in \mathbb{X}$. We do not reserve any particular notation of the Riemannian tensors on $\mathbb{X}$ and $\mathbb{Y}$, as these tensors will be fixed for the duration of this paper. The volume elements on $\mathbb{X}$ and $\mathbb{Y}$, denoted by $dx \in \mathscr{C}^\infty(\wedge^n \mathbb{X})$ and $dy \in \mathscr{C}^\infty(\wedge^m \mathbb{Y})$, will be the ones induced by the orientation and the metric tensors. In this way $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \leqslant p < \infty$, becomes a complete metric subspace of the linear space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$.

In the Riemannian manifolds setting it is not clear at all whether smooth mappings $f \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ are dense in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, a question raised by J. Eells and L. Lemaire [11]. This is trivially the case for $p > n$. R. Schoen and K. Uhlenbeck [46], [47] showed that the answer is also positive when $p = n$. That is all we can have in the category of the Sobolev spaces $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, unless additional topological conditions are imposed on the manifolds $\mathbb{X}$ and $\mathbb{Y}$ [20], [21]. For example, in the same paper R. Schoen and K. Uhlenbeck [47] demonstrate that $\mathscr{C}^\infty(\mathbb{S}^n, \mathbb{S}^{n-1})$ is not dense in $\mathscr{W}^{1,p}(\mathbb{S}^n, \mathbb{S}^{n-1})$ for every $n - 1 \leqslant p < n$. While it is not clear at this point, the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$, with $n = \dim \mathbb{X} \geqslant 2$, will be the borderline case for many more classes of weakly differentiable mappings. Other related papers are [17], [18]. Sobolev spaces with exponents $1 < p < n$ are natural in the theory of harmonic mappings [11], [34], [46] and other related areas. However, properties of these mappings are very different from those in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. This difference lies fairly deep in the concept of the topological degree. If $\dim \mathbb{X} = \dim \mathbb{Y} = n$, then a smooth mapping $f : \mathbb{X} \to \mathbb{Y}$ has well defined degree

$$\deg(f;\mathbb{X},\mathbb{Y}) = \frac{1}{\operatorname{vol}\mathbb{Y}}\int_{\mathbb{X}} \mathcal{J}(x,f)\,dx \tag{2}$$

where $\mathcal{J}(x,f)$ stands for the Jacobian determinant of f. It is evident that this formula makes sense also for mappings in $\mathscr{W}^{1,n}(\mathbb{X},\,\mathbb{Y})$. But it is less obvious whether it relates to topological properties of such mappings. Indeed it does, thanks to the density of $\mathscr{C}^\infty(\mathbb{X},\,\mathbb{Y})$ in $\mathscr{W}^{1,n}(\mathbb{X},\,\mathbb{Y})$. One might try to extend formula (2) to mappings $f\in\mathscr{W}^{1,p}(\mathbb{X},\,\mathbb{Y})$, $1<p<n$. For example, by assuming that the Jacobian is integrable. This attempt will fail simply because there is no way to control the integral of the Jacobian by means of the p-norms of the differential of f. Actually, as f runs over $\mathscr{W}^{1,p}(\mathbb{X},\,\mathbb{Y})$, $1<p<n$, the integrals at (2) assume every real number.

In spite of these examples, we are still able to build a viable theory of weakly differentiable mappings slightly less regular than those in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X},\,\mathbb{Y})$. One representative example is the Orlicz-Sobolev space $\mathscr{W}^{1,P}(\mathbb{X},\,\mathbb{Y})$ of mapping $f:\mathbb{X}\to\mathbb{Y}$ whose differential $Df:\mathbf{T}\mathbb{X}\to\mathbf{T}\mathbb{Y}$ satisfies

$$\int_{\mathbb{X}} P(|Df(x)|)\,dx<\infty, \qquad P(t)=\frac{t^n}{\log(e+t)} \tag{3}$$

Let us emphasize, without getting into some technical details, that our theory will actually work for other Orlicz-Sobolev spaces $\mathscr{W}^{1,P}(\mathbb{X},\,\mathbb{Y})$. But we must assume that P grows fast enough to satisfy the so-called *divergence condition*

$$\int_1^\infty \frac{P(t)}{t^{n+1}}\,dt=\infty \tag{4}$$

These classes, although appearing rather restrictive, contain $\mathscr{W}^{1,n}(\mathbb{X},\,\mathbb{Y})$. However, they are typically smaller than the intersection of all the spaces $\mathscr{W}^{1,p}(\mathbb{X},\,\mathbb{Y})$, $1\leqslant p<n$.

$$\mathscr{W}^{1,n}(\mathbb{X},\,\mathbb{Y})\subset\mathscr{W}^{1,P}(\mathbb{X},\,\mathbb{Y})\subsetneq\bigcap_{1\leqslant p<n}\mathscr{W}^{1,p}(\mathbb{X},\,\mathbb{Y}) \tag{5}$$

We learn the necessity of the divergence condition (4) from the radial projection

$$f_\circ:\mathbb{B}^n\to\mathbb{S}^{n-1}, \quad f_\circ(x)=\frac{x}{|x|} \tag{6}$$

As observed by R. Schoen and K. Uhlenbeck [47], $f_\circ$ cannot be approximated by smooth mapping $f_j : \mathbb{B}^n \to \mathbb{S}^{n-1}$ in the metric of $\mathscr{W}^{1,p}(\mathbb{B}^n, \mathbb{S}^{n-1})$ with any $p \geqslant n-1$. This example, together with an analogue on manifolds without boundary, receives a thorough discussion in Section 3.1. Let us find out what we should assume on P to prevent $f_\circ$ from being a member of $\mathscr{W}^{1,P}(\mathbb{B}^n, \mathbb{S}^{n-1})$. The differential of $f_\circ$ belongs to the Marcinkiewicz space $\mathscr{L}^n_{\text{weak}}(\mathbb{B}^n)$. Precisely, $|Df_\circ(x)| = |x|^{-1}$ and hence

$$\int_{|Df_\circ|>t} dx = \frac{|\mathbb{B}^n|}{t^n}, \quad 1 \leqslant t < \infty \tag{7}$$

Integration in polar coordinates shows that

$$\int_{\mathbb{B}} P(|Df_\circ(x)|)\, dx = |\mathbb{S}^{n-1}| \int_1^\infty \frac{P(t)}{t^{n+1}}\, dt \tag{8}$$

Thus $f_\circ \notin \mathscr{W}^{1,P}(\mathbb{B}^n)$ if and only if P satisfies the divergence condition. In this way $f_\circ$ will lay beyond the range of our theory.

In Section 5.6 we find the closure of $\mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ in the Marcinkiewicz class $\mathscr{W}^{1,n}_{\text{weak}}(\mathbb{X}, \mathbb{Y})$.

THEOREM 1.1. *The closure of $\mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ in the metric topology of the Marcinkiewicz class $\mathscr{W}^{1,n}_{\text{weak}}(\mathbb{X}, \mathbb{Y})$ consists of mappings $f \in \mathscr{W}^{1,1}(\mathbb{X}, \mathbb{Y})$ such that*

$$\lim_{t\to\infty} t^n \int_{|Df|>t} dx = 0 \tag{9}$$

This property is slightly stronger than $|Df| \in \mathscr{L}^n_{\text{weak}}(\mathbb{X})$. Moreover, for every $0 \leqslant \alpha < n$, (9) is equivalent to

$$\lim_{t\to\infty} t^{n-\alpha} \int_{|Df|>t} |Df(x)|^\alpha\, dx = 0 \tag{10}$$

In the spirit of Theorem 1.1, we give here the most general approximation result.

THEOREM 1.2. *Every weakly differentiable mapping* $f : \mathbb{X} \to \mathbb{Y}$, *dim* $\mathbb{X} = n$, *satisfying*

$$\liminf_{t\to\infty} t^{n-p} \int_{|Df|>t} |Df(x)|^p \, dx = 0 \,, \qquad n-1 < p < n \tag{11}$$

can be approximated by smooth mappings $f_j : \mathbb{X} \to \mathbb{Y}$ *in the metric topology of* $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$.

This seemingly insignificant replacement of "lim" in (10) by "lim inf", has far reaching consequences. Among them is the density of $\mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ in the Orlicz-Sobolev spaces $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$. In addition to (4), we impose some minor technical assumptions on P, see Theorem 5.2.

One further category of mappings appears in a natural way; the *grand Sobolev space*, denoted by $\mathrm{G}\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. It consists of mappings

$$f \in \bigcap_{1\leqslant s<n} \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$$

such that

$$\| Df \|_{n)} \stackrel{\text{def}}{=\!=} \sup_{0<\epsilon\leqslant n-1} \left(\epsilon \int_{\mathbb{X}} |Df(x)|^{n-\epsilon} dx \right)^{\frac{1}{n-\epsilon}} < \infty \tag{12}$$

For instance, this space contains $\mathscr{W}^{1,n}_{\text{weak}}(\mathbb{X}, \mathbb{Y})$. Rather than discuss this space in full details, let us introduce a subspace $\mathrm{V}\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y}) \subset \mathrm{G}\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ characterized by

$$\lim_{\epsilon\to 0} \epsilon \int_{\mathbb{X}} |Df(x)|^{n-\epsilon} \, dx = 0 \tag{13}$$

Our consideration of this subspace is motivated by:

THEOREM 1.3. *Smooth mappings are dense in* $V\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$.

All known proofs of the density of smooth mappings in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ are based on the embedding of $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ into VMO$(\mathbb{X}, \mathbb{Y})$ -the space of mappings of vanishing mean oscillation [7], [8]. It turns out, however, that our spaces $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$ and $\mathrm{V}\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ for which we prove density results do not admit embeddings in $\mathrm{VMO}(\mathbb{X},\ \mathbb{Y})$, so we had to use a completely different idea. The proofs of smooth approximation involve an interesting new device, the so-called web like structure on $\mathbb{X}$. A web on $\mathbb{X}$ is a compact set $\mathbb{F} \subset \mathbb{X}$ of measure zero whose complement consists of a finite number of components, disjoint open connected sets called meshes of the web.

Given $f : \mathbb{X} \to \mathbb{Y}$, as in Theorem 1.2, there exist webs $\mathbb{F} \subset \mathbb{X}$, with meshes as small as we wish, so that f restricted to $\mathbb{F}$ is continuous. Moreover, oscillations of f over the boundary of every mesh of the web can be arbitrarily small too. For this reason we say that f has vanishing web oscillations. The presence of small oscillations of mappings in $\mathscr{W}^{1,n}(\mathbb{X},\ \mathbb{Y})$, $\mathrm{V}\mathscr{W}^{1,n}(\mathbb{X},\ \mathbb{Y})$ and other Sobolev subclasses of $\mathscr{W}^{1,p}(\mathbb{X},\ \mathbb{Y})$ with exponent p below the dimension of $\mathbb{X}$ seems to be important in future applications.

Now that we have the approximation theorems we will be able to give meaning to usually divergent integrals of the Jacobian of $f : \mathbb{X} \to \mathbb{Y}$, $\dim \mathbb{X} = \dim \mathbb{Y} = n$. Like in the Euclidean case [1], [42], [14], [29] this leads to a definition of the distributional Jacobian. In various situations, however, the manifold setting is really different than one might a priori expect. Manifolds of the same deRham cohomology groups as $\mathbb{S}^n$ will be named *rational homology spheres*. This class of manifolds contains all homology spheres, manifolds whose integral homology groups are the same as those of the sphere, though these two classes are not the same. Indeed, for $p > 1$ the lens spaces $L(p,q)$ are 3-dimensional rational homology spheres, but their integral homology[1] groups are different than those of $\mathbb{S}^3$, see e.g. Proposition 21.28 in [5]. To our surprise the case when the target manifold $\mathbb{Y}$ is a rational homology sphere is more difficult than all other cases. What makes a difference is that in these other cases every n-form $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$ decomposes into wedge products of closed forms of degree smaller than n, namely:

$$\omega = \sum_{i=1}^{K} \alpha_i \wedge \beta_i \tag{14}$$

[1]In what follows by cohomology we will always mean deRham cohomology

where

$$\begin{cases} \alpha_i \in \mathscr{C}^\infty(\wedge^{\ell_i}\mathbb{Y}) \cap \ker d, & \ell_i, k_i \in \{1, 2, ..., n-1\} \\ \beta_i \in \mathscr{C}^\infty(\wedge^{k_i}\mathbb{Y}) \cap \ker d, & \ell_i + k_i = n \end{cases} \tag{15}$$

Such is not the case of the n-sphere $\mathbb{Y} = \mathbb{S}^n$. These forms, once pulled back via a mapping $f : \mathbb{X} \to \mathbb{Y}$, bring us to analogous decomposition of $f^\sharp\omega$ in $\mathbb{X}$:

$$f^\sharp\omega = \sum_{i=1}^{K} f^\sharp\alpha_i \wedge f^\sharp\beta_i \tag{16}$$

Under suitable regularity of the mapping f, the pullbacks $f^\sharp\alpha_i$ and $f^\sharp\beta_i$ are closed forms. At this point, a careful reader may observe that the dimension of $\mathbb{Y}$ is immaterial if we confine ourselves to pullbacks of the wedge products at (14), with $k_i + \ell_i = n \leqslant \dim \mathbb{Y}$. We refer to (14) as Cartan forms, named after H. Cartan who studies similar decompositions of differential forms. These ideas fit into even larger framework. The pullback at (16) is just a special case of a Cartan form on the domain manifold $\mathbb{X}$. Precisely, the n-form $\Lambda = f^\sharp\omega$ on $\mathbb{X}$ admits Cartan's decomposition as well:

$$\Lambda = \sum_{i=1}^{K} \Phi_i \wedge \Psi_i \tag{17}$$

where

$$\begin{cases} \Phi_i \in \mathscr{L}^{p_i}(\wedge^{\ell_i}\mathbb{X}) \cap \ker d, & \ell_i, k_i \in \{1, 2, ..., n-1\} \\ \Psi_i \in \mathscr{L}^{r_i}(\wedge^{k_i}\mathbb{X}) \cap \ker d, & \ell_i + k_i = n \end{cases} \tag{18}$$

If we assume that $f \in \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$ for some $s \geqslant n-1$, then $f^\sharp\alpha_i = \Phi_i \in \mathscr{L}^{\frac{s}{\ell_i}}(\wedge^{\ell_i}\mathbb{X})$ and $f^\sharp\beta_i = \Psi \in \mathscr{L}^{\frac{s}{k_i}}(\wedge^{k_i}\mathbb{X})$.

The integral $\int_{\mathbb{X}} \mathcal{J}(x, f)\, dx = \int_{\mathbb{X}} \Lambda$ will exist in somewhat weak sense. It will actually converge if $\mathcal{J}(x, f) \geqslant 0$, basically by Monotone Convergence Theorem. We refer to mappings with nonnegative Jacobian as *orientation preserving.*

The situation is dramatically different if one cannot decompose $f^\sharp\omega$ into wedge products. To illustrate, we give the following rather striking result.

THEOREM 1.4. *Every Orlicz-Sobolev class $\mathscr{W}^{1,P}(\mathbb{S}^n, \mathbb{S}^n)$, with $P(t) = o(t^n)$, contains an orientation preserving mapping whose Jacobian is not integrable.*

Such mappings are constructed in Section 3. In other words, if the target manifold $\mathbb{Y}$ is a rational homology sphere, then the classical integral formula for the degree of f fails in every Orlicz-Sobolev class below $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$.

It is evident from the Sobolev imbedding theorem that two mappings $f, g \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, with $p > n = \dim \mathbb{X}$, which are sufficiently close in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, are homotopic. Here $\mathbb{X}$ and $\mathbb{Y}$ may have different dimension and need not be even orientable. B. White [53], [54] proved a stronger result according to which every two continuous mappings in the space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ are homotopic, provided they are sufficiently close in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. This and the fact that smooth mappings are dense in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$, allow us to define homotopy classes in the space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$.

These homotopy classes suggest several natural questions like the following one:

Question **1.1.** Are there any topological conditions on $\mathbb{X}$ and $\mathbb{Y}$ under which every two mappings $f, g \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y}) \cap \mathscr{C}(\mathbb{X}, \mathbb{Y})$, $n-1 < p < n = \dim \mathbb{X}$, sufficiently close in the metric of $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, are homotopic?

This is not away the case for continuous mappings in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{S}^n)$. However, the answer is in the affirmative if $\pi_n(\mathbb{Y}) = 0$, where we even do not require that $\dim \mathbb{Y} = n$. If in addition $\pi_{n-1}(\mathbb{Y}) = 0$, then smooth mappings are dense in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$. Note also that the homotopy condition is of a different nature than the cohomological one. Indeed, it is not difficult to construct a manifold $\mathbb{Y}$ of dimension n with nontrivial cohomology group $H^\ell(\mathbb{Y})$ for some $1 \leqslant \ell < n$ and such that $\pi_n(\mathbb{Y}) \neq 0$. This shows that, continuity of the degree can not be deduced from the homotopy equivalence of mappings under the assumption that $\pi_n(\mathbb{Y}) = 0$.

The difference between trivial and nontrivial target space in terms of its ℓ-cohomologies $1 \leqslant \ell < n$, can also be observed in the borderline case

$\mathscr{W}^{1,n}(\mathbb{X},\ \mathbb{Y})$. Two well known results in $\mathbb{R}^n$ are worthwhile to consider on manifolds. The first result is: if the Jacobian of a mapping in $\mathscr{W}^{1,n}(\mathbb{X},\ \mathbb{Y})$ is nonnegative then it belongs to the Zygmund space $\mathscr{L}\log\mathscr{L}(\mathbb{X})$. The second is: if $\mathbb{Y}$ is not a rational homology sphere, then there also is a uniform bound, namely

$$\int_{\mathbb{X}} \mathcal{J}(x,f)\log\left(e+\frac{\mathcal{J}(x,f)}{\int_{\mathbb{X}}\mathcal{J}(z,f)\,dz}\right)\,dx \ \preccurlyeq\ \int_{\mathbb{X}}|Df|^n \tag{19}$$

Remark. Throughout this paper we use the symbol $\preccurlyeq$ to signify that the inequality holds with certain positive constant in the right hand side. This constant, referred to as *implied constant*, will vary from line to line. In many instances the reader may recognize the parameters on which the implied constant depends on. If not, we will explicitly specify those parameters. For example in (19) the implied constant depends only on the manifolds $\mathbb{X}$ and $\mathbb{Y}$.

The $\mathscr{L}\log\mathscr{L}$-integrability of $\mathcal{J}(x,f)$ will remain true in case of the rational homology sphere space but the arguments will be completely different. Unexpectedly, the uniform bound (19) will be lost. If the Jacobian changes sign then it belong to the Hardy space $\mathscr{H}^1(\mathbb{X})$, a well known result by R. Coifman, P. Lions, Y. Meyer and S. Semmes [10] in $\mathbb{R}^n$, see also [30], [33]. Again, in manifold setting the arguments establishing $\mathscr{H}^1$-regularity of the Jacobian will be more subtle. We will obtain a uniform bounds only when the target manifold has a nontrivial cohomology, $H^\ell(\mathbb{Y})\neq 0$ for some $1\leqslant \ell < n$. Precisely, we have

$$\|\, f^\sharp\omega\,\|_{\mathscr{H}^1(\mathbb{X})} \ \preccurlyeq\ \int_{\mathbb{X}}|Df|^n \tag{20}$$

These and many more results will be discussed at full length in the text after background material is introduced.

2 Preliminaries Concerning Manifolds

This section is written to provide notation and to serve as brief introduction to the $\mathscr{L}^p$-theory of differential forms. The general references here are [9],

[40], [48], [24] and [32], [52].

2.1 Manifolds

While many geometric constructions in $\mathbb{R}^n$ can be transferred to the Riemannian manifolds, the sometimes cumbersome technical details are often new. Many unfamiliar differences will be explicitly emphasized here. Those differences sometimes only technical, sometimes delicate and important, are scattered throughout the research journals. Although, most of these facts will be left unproven in this text, we state them clearly so that they are available for a routine verification.

Our ambient space for the geometric analysis will be an oriented compact (without boundary) smooth Riemannian manifold $\mathbb{X}$ of dimension $n \geqslant 2$.

2.1.1 Legitimate balls

Making precise estimates demands that we must work with one atlas $\mathcal{A}$ consisting of a finite number of coordinate charts $(\Omega, \kappa) \in \mathcal{A}$, where $\kappa : \Omega \to \mathbb{R}^n$ is a $\mathscr{C}^\infty$-diffeomorphism of an open region $\Omega \subset \mathbb{X}$ *onto* $\mathbb{R}^n$. Let us fix, once for ever, such an atlas $\mathcal{A}$ and call it the reference atlas. Having $\mathcal{A}$, we introduce the so-called *reliable* radius of the manifold $\mathbb{X}$ and denote it by $R = R_{\mathbb{X}}$. It is a positive number such that all concentric geodesic balls $\mathbb{B}(x,r) \subset \mathbb{B}(x,4r) \subset \mathbb{X}$ lay in one coordinate region Ω of the atlas $\mathcal{A}$, provided $r \leqslant R_{\mathbb{X}}$. We then refer to such $\mathbb{B}(x,r)$ as a *legitimate* ball in $\mathbb{X}$. The point to this concept is that estimates on a legitimate ball will reduce equivalently to analogous estimates in the Euclidean space. We mention now that the legitimate balls $\mathbb{B} = \mathbb{B}(x,r) \subset \mathbb{X}$ share basic properties of the Euclidean balls. In particular,

$$|\mathbb{B}| \preccurlyeq (\operatorname{diam} \mathbb{B})^n \preccurlyeq |\mathbb{B}| \tag{21}$$

Here the implied constant depends only on the manifold $\mathbb{X}$.

2.1.2 Whitney covering

The familiar decomposition of an open set $\Omega \subset \mathbb{R}^n$ into Whitney cubes will be adapted to manifolds. While cubes are perfect regions for constructing various partitions of $\mathbb{R}^n$, there are serious geometric and combinatorial obstacles to do the same on manifolds. We have chosen to work with the legitimate balls instead of cubes. In general, it is impossible to partition a manifold with balls. Nevertheless, we will work with finite coverings by balls in which the overlaping number depends only on the manifold $\mathbb{X}$.

PROPOSITION **2.1.** *Given a non-empty open set $\Omega \subsetneq \mathbb{X}$ and its complement $\mathbb{F} = \mathbb{X} \setminus \Omega$. There exists a collection $\mathcal{F} = \{\mathbb{B}_1, \mathbb{B}_2, ...\}$ of legitimate balls $\mathbb{B}_i \subset \mathbb{X}$ such that*

1) $\mathbb{B}_i \subset 2\mathbb{B}_i \subset \Omega, \quad i = 1, 2, ...$

2) $\Omega = \bigcup_{i=1}^{\infty} \mathbb{B}_i$

3) $\sum_{i=1}^{\infty} \chi_{2\mathbb{B}_i}(x) \preccurlyeq 1$ *for all* $x \in \Omega$

4) $\operatorname{diam} \mathbb{B}_i \preccurlyeq \operatorname{dist}(\mathbb{B}_i, \mathbb{F}) \preccurlyeq \operatorname{diam} \mathbb{B}_i$ *for all* $i = 1, 2...$

Hereafter, χ denotes the characteristic function and $2\mathbb{B}$ stands for the ball of the same center as $\mathbb{B}$ but with radius 2 times larger.

2.2 The Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$

The various classes of mappings $f : \mathbb{X} \to \mathbb{Y}$ in this paper are defined based on the classical Sobolev theory of real valued functions. Note that $\mathscr{W}^{1,p}(\mathbb{X})$ is a Banach space equipped with the norm

$$\| f \|_{\mathscr{W}^{1,p}} = \| f \|_{\mathscr{L}^1} + \| Df \|_{\mathscr{L}^p} \tag{22}$$

We adopt the well known results in $\mathbb{R}^n$ to our manifold setting, see for instance [45].

LEMMA **2.1.** *Smooth functions in $\mathscr{C}^\infty(\mathbb{X})$ are dense in $\mathscr{W}^{1,p}(\mathbb{X})$, $1 \leqslant p < \infty$.*

Lemma **2.2.** [Poincaré Inequality] *For every set* $\mathbb{E}$ *of a positive measure in* $\mathbb{X}$ *and* $f \in \mathscr{W}^{1,p}(\mathbb{X})$ *we have*

$$\int_{\mathbb{X}} |f - f_{\mathbb{E}}|^p \leqslant C_{\mathbb{E}} \int_{\mathbb{X}} |Df(x)|^p \, dx$$

The constant $C_{\mathbb{E}}$ actually depends only on the measure of $\mathbb{E}$. As usual, the integral average of f over the set $\mathbb{E}$ is denoted by

$$f_{\mathbb{E}} = \fint_{\mathbb{E}} f(x)\, dx = \frac{1}{|\mathbb{E}|} \int_{\mathbb{E}} f(x)\, dx$$

Next, the local variant of Poincaré inequality reads as:

Lemma **2.3.** *For every legitimate ball* $\mathbb{B} = \mathbb{B}(a, r)$ *we have*

$$\int_{\mathbb{B}} |f - f_{\mathbb{B}}|^p \preccurlyeq r^p \int_{\mathbb{B}} |Df(x)|^p \, dx \quad \textit{whenever } f \in \mathscr{W}^{1,p}(\mathbb{X})$$

As a matter of fact this inequality is true for all geodesic balls in $\mathbb{X}$, but we will need the inequality only for legitimate balls. Regarding the implied constant, we emphasize that it depends neither on f nor on the radius r.

Now given two Riemannian manifolds $\mathbb{X}$ and $\mathbb{Y}$, we shall consider the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ of mappings whose tangent map (differential)

$$Df(x) : T_x\mathbb{X} \to T_y\mathbb{Y}, \quad y = f(x) \tag{23}$$

is $\mathscr{L}^p$-integrable. Our description and certainly a rigorous definition of $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ relies on an imbedding $\mathbb{Y} \subset \mathbb{R}^N$ [44].

THEOREM 2.1. ***(J. Nash)*** *Every* $\mathscr{C}^\infty$*-smooth Riemannian manifold* $\mathbb{Y}$ *can be* $\mathscr{C}^\infty$*-isometrically imbedded in some Euclidean space* $\mathbb{R}^N$*.*

The reader is also referred to M. L. Gromov and V. A. Rohlin [16] for an account of the imbedding problem. The Nash theorem allows us to consider $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ as a subclass of a linear space of mappings $f : \mathbb{X} \to \mathbb{R}^N$ such that $f(x) \in \mathbb{Y}$ at almost every $x \in \mathbb{X}$. The metric topology in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ will be inherited from the associated norm topology in the linear space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$. In this way the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ becomes a complete metric space. In what follows we shall tacitly exploit the fact that $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ is also closed under weak topology of $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$.

2.3 Differential forms

Throughout this paper we let $\mathscr{C}^\infty(\wedge^\ell \mathbb{X})$, $0 \leqslant \ell \leqslant n = \dim \mathbb{X}$, denote the space of smooth ℓ-forms on $\mathbb{X}$. Two differential operators on forms will be of particular interest to us. First is the exterior derivative

$$d : \mathscr{C}^\infty(\wedge^\ell \mathbb{X}) \to \mathscr{C}^\infty(\wedge^{\ell+1} \mathbb{X}) \tag{24}$$

Second is the formal adjoint of d, also called Hodge codifferential

$$d^* = (-1)^{n\ell+1} * d* : \mathscr{C}^\infty(\wedge^{\ell+1} \mathbb{X}) \to \mathscr{C}^\infty(\wedge^\ell \mathbb{X}) \tag{25}$$

where $* : \mathscr{C}^\infty(\wedge^\ell \mathbb{X}) \to \mathscr{C}^\infty(\wedge^{n-\ell} \mathbb{X})$ denotes the Hodge star duality operator. Here, we conveniently introduce $\mathscr{C}^\infty(\wedge^\ell \mathbb{X}) = 0$, whenever $\ell < 0$ or $\ell > n$. Note that $** = (-1)^{\ell(n-\ell)}$ on $\mathscr{C}^\infty(\wedge^\ell \mathbb{X})$. The point-wise scalar product of $\alpha, \beta \in \mathscr{C}^\infty(\wedge^\ell \mathbb{X})$ is given by $\langle \alpha, \beta \rangle \, dx = \alpha \wedge * \in \mathscr{C}^\infty(\wedge^n \mathbb{X})$. Hence

$$\int_{\mathbb{X}} \langle \alpha, \beta \rangle = \int_{\mathbb{X}} \alpha \wedge * \beta \tag{26}$$

The duality between d and d^* is emphasized in the integration by parts

$$\int_{\mathbb{X}} \langle d\varphi, \psi \rangle = \int_{\mathbb{X}} \langle \varphi, d^* \psi \rangle \tag{27}$$

for $\varphi \in \mathscr{C}^\infty(\wedge^\ell \mathbb{X})$ and $\psi \in \mathscr{C}^\infty(\wedge^{\ell+1} \mathbb{X})$. Now a differential form $\varphi \in \mathscr{L}^p(\wedge^\ell \mathbb{X})$ is said to be closed in the sense of distributions if $\int_{\mathbb{X}} \langle \varphi, d^*\psi \rangle = 0$ for every test form $\psi \in \mathscr{C}^\infty(\wedge^{\ell+1} \mathbb{X})$, write it as $d\varphi = 0$. Similarly, we define what it means for ψ to be coclosed and write it as $d^*\psi = 0$. Forms of the type $d\alpha$, with $\alpha \in \mathscr{W}^{1,p}(\wedge^{\ell-1} \mathbb{X})$, are called exact while those of type $d^*\beta$, with $\beta \in \mathscr{W}^{1,p}(\wedge^{\ell+1} \mathbb{X})$, are called coexact. It results from the identities $d \circ d = 0$ and $d^* \circ d^* = 0$ that the ℓ-forms $d\alpha \in \mathscr{L}^p(\wedge^\ell \mathbb{X})$ and $d^*\beta \in \mathscr{L}^p(\wedge^\ell \mathbb{X})$ are closed and coclosed, respectively. Finally, those forms $h \in \mathscr{L}^p(\wedge^\ell \mathbb{X})$ which are closed and coclosed will be called harmonic fields of degree ℓ. We denote by $\mathcal{H}(\wedge^\ell \mathbb{X})$ the space of all harmonic fields of degree ℓ and regard it as well known that this space is finite dimensional and consists of $\mathscr{C}^\infty$-smooth forms. Being so, all possible norms on $\mathcal{H}(\wedge^\ell \mathbb{X})$ are equivalent. For instance,

$$\| h \|_{\mathscr{L}^\infty(\wedge^\ell \mathbb{X})} \preccurlyeq \| h \|_{\mathscr{L}^1(\wedge^\ell \mathbb{X})} \tag{28}$$

and further,

$$\| h \|_{\mathscr{L}^\infty(\wedge^\ell \mathbb{X})} \preccurlyeq \left(\int_{\mathbb{X}} |h|^p \right)^{\frac{1}{p}} \qquad \text{for all } p > 0 \tag{29}$$

In relation to the imbedding $\mathscr{L}^1_{\text{weak}}(\wedge^\ell \mathbb{X}) \subset \mathscr{L}^p(\wedge^\ell \mathbb{X})$, with $0 < p < 1$, we record the following estimate

$$\| h \|_{\mathscr{L}^\infty(\wedge^\ell \mathbb{X})} \preccurlyeq \left(\int_{\mathbb{X}} |h|^p \right)^{\frac{1}{p}} \preccurlyeq |\mathbb{X}|^{\frac{1-p}{p}} \sup \left\{ t \int_{|h|>t} dx; \;\; t > 0 \right\} \tag{30}$$

as is easily verified, see (113).

2.3.1 Sobolev classes of differential forms

Four spaces of differential forms have a special place in our studies. These spaces are:

- The Sobolev space of closed forms:

$$\mathscr{L}^p(\wedge^\ell \mathbb{X}) \cap \ker d$$

- The Sobolev space of coclosed forms:

$$\mathscr{L}^p(\wedge^\ell \mathbb{X}) \cap \ker d^*$$

- The Sobolev space of exact forms:

$$d\mathscr{W}^{1,p}(\wedge^{\ell-1} \mathbb{X}) = \left\{ d\alpha; \; \alpha \in \mathscr{W}^{1,p}(\wedge^{\ell-1} \mathbb{X}) \right\} \subset \mathscr{L}^p(\wedge^\ell \mathbb{X}) \cap \ker d$$

- The Sobolev space of coexact forms:

$$d^*\mathscr{W}^{1,p}(\wedge^{\ell+1} \mathbb{X}) = \left\{ d^*\beta; \; \beta \in \mathscr{W}^{1,p}(\wedge^{\ell+1} \mathbb{X}) \right\} \subset \mathscr{L}^p(\wedge^\ell \mathbb{X}) \cap \ker d^*$$

It is far from being evident, although it is true, that for $1 < p < \infty$ all four of these classes are complete linear subspaces of $\mathscr{L}^p(\wedge^\ell \mathbb{X})$ [32]. In each of those classes the corresponding subspace of smooth forms is dense.

2.3.2 Hodge decomposition

Decomposition of a differential form $\omega \in \mathscr{L}^p(\wedge^\ell \mathbb{X})$ into exact, coexact and harmonic component will play essential role in our proofs.

THEOREM 2.2. [Hodge decomposition] *For* $1 < p < \infty$ *and* $\ell = 0, 1, ..., n$ *we have the following direct sum decomposition*

$$\mathscr{L}^p(\wedge^\ell \mathbb{X}) = d\mathscr{W}^{1,p}(\wedge^{\ell-1}\mathbb{X}) \oplus d^*\mathscr{W}^{1,p}(\wedge^{\ell+1}\mathbb{X}) \oplus \mathcal{H}(\wedge^\ell \mathbb{X}) \tag{31}$$

Accordingly, every $\omega \in \mathscr{L}^p(\wedge^\ell \mathbb{X})$ *can be decomposed as*

$$\omega = d\alpha + d^*\beta + h \tag{32}$$

where

$$\|\,\alpha\,\|_{\mathscr{W}^{1,p}(\wedge^{\ell-1}\mathbb{X})} + \|\,\beta\,\|_{\mathscr{W}^{1,p}(\wedge^{\ell+1}\mathbb{X})} + \|\,h\,\|_{\mathscr{L}^\infty(\wedge^\ell \mathbb{X})} \preccurlyeq \|\,\omega\,\|_{\mathscr{L}^p(\wedge^\ell \mathbb{X})} \tag{33}$$

We restrain ourselves to only a few comments about the case when ω is a closed form. It follows from the uniqueness of this decomposition that the coexact component of ω vanishes. That is,

$$\omega = d\alpha + h, \quad \text{for } \omega \in \mathscr{L}^p(\wedge^\ell \mathbb{X}) \cap \ker d \tag{34}$$

In fact, this is none other than the $\mathscr{L}^p$-setting of the deRham cohomology:

Every closed form in $\mathscr{L}^p(\wedge^\ell \mathbb{X}) \cap \ker d$ *is exact modulo harmonic fields.*

In other words, each deRham ℓ-cohomology class of $\mathbb{X}$ is uniquely represented by a harmonic field. In symbols $H^\ell(\mathbb{X}) \cong \mathcal{H}(\wedge^\ell \mathbb{X})$. The harmonic component at (34) can be explicitly expressed as $h = \mathbf{T}\omega$, where $\mathbf{T}$ is a Calderón-Zygmund type operator acting on $\mathscr{L}^p(\wedge^\ell \mathbb{X})$. As this operator is weak $(1,1)$-type, a uniform $\mathscr{L}^1_{\text{weak}}$-estimate combined with (30) yields

$$\|\,h\,\|_{\mathscr{L}^\infty(\wedge^\ell \mathbb{X})} \preccurlyeq \sup\left\{ t \int_{|h|>t} dx; \;\; t > 0 \right\} \preccurlyeq \|\,\omega\,\|_{\mathscr{L}^1(\wedge^\ell \mathbb{X})} \tag{35}$$

We reiterate that the implied constant depends on the volume of $\mathbb{X}$.

2.3.3 Pullback

Our applications of the $\mathscr{L}^p$-theory of differential forms pertain largely to the pullbacks via a mapping $f : \mathbb{X} \to \mathbb{Y}$ of differential forms in the target space. Let $\alpha \in \mathscr{C}^\infty(\wedge^\ell \mathbb{Y})$ and $f \in \mathscr{W}^{1,p}(\mathbb{X},\, \mathbb{Y})$, $p \geqslant \ell \geqslant 1$. The pullback $f^\sharp \alpha$ lies in $\mathscr{L}^{\frac{p}{\ell}}(\wedge^\ell \mathbb{X})$, because of the point-wise inequality

$$\left| f^\sharp \alpha \right| \preccurlyeq |Df|^\ell \tag{36}$$

We point out that the commutation rule

$$f^\sharp \circ d = d \circ f^\sharp \tag{37}$$

requires some regularity of f. For instance it holds for $f \in \mathscr{W}^{1,p}(\mathbb{X},\, \mathbb{Y})$ provided $p \geqslant \ell+1$. The exterior derivative in the right hand side is understood in the sense of distributions, while the left hand side is a form in $\mathscr{L}^{\frac{p}{\ell+1}}(\wedge^{\ell+1}\mathbb{X})$.

2.3.4 Partition of unity

The arguments for our proofs as well as the definition of the Hardy space $\mathscr{H}^1(\mathbb{X})$ will involve a device of regularization. For such purposes partition of unity is needed. To assemble local estimates into global ones we shall make use of partitions of unity with small supports on $\mathbb{X}$.

Given any locally finite covering $\mathfrak{F}$ of $\mathbb{X}$, a smooth partition of unity subordinate to $\mathfrak{F}$ is a collection $\left\{\varphi_{\mathbb{U}};\;\; \mathbb{U} \in \mathfrak{F}\right\}$ of functions $\varphi_{\mathbb{U}} \in \mathscr{C}_0^\infty(\mathbb{U})$ such that

- $0 \leqslant \varphi_{\mathbb{U}}(x) \leqslant 1$
- $\sum\limits_{\mathbb{U}\in\mathfrak{F}} \varphi_{\mathbb{U}}(x) = 1 \qquad$ for all $x \in \mathbb{X}$

The existence of partitions of unity is well known. We illustrate it with an example that will be used later on. Given differential forms $\alpha \in \mathscr{L}^p(\wedge^\ell \mathbb{X})$ and $d\beta \in \mathscr{W}^{1,p}(\wedge^{\ell-1}\mathbb{X})$, with the aid of a partition of unity we write

$$\alpha = \sum_{\mathbb{U}\in\mathfrak{F}} \alpha_{\mathbb{U}} \overset{\text{def}}{=\!=} \sum_{\mathbb{U}\in\mathfrak{F}} \varphi_{\mathbb{U}}\, \alpha \tag{38}$$

$$d\beta = \sum_{\mathbb{U}\in\mathfrak{F}} d\beta_{\mathbb{U}} \stackrel{\text{def}}{=\!=} \sum_{\mathbb{U}\in\mathfrak{F}} d(\varphi_{\mathbb{U}}\ \beta) \tag{39}$$

In this sum each term $d\beta_{\mathbb{U}}$ is an exact form. On the contrary, if α is closed the terms $\alpha_{\mathbb{U}}$ are no longer closed forms. In spite of this inconvenience the above decomposition is still the most practical one. The point to it is that the exterior derivative

$$d\alpha_{\mathbb{U}} = \varphi_{\mathbb{U}}\ d\alpha + d\varphi_{\mathbb{U}} \wedge \alpha \tag{40}$$

enjoys the same regularity as $d\alpha$.

2.3.5 Cartan forms

Let $\mathbb{Y}$ be a $\mathscr{C}^\infty$-smooth oriented closed Riemannian manifold of dimension $m \geqslant 2$. Recall that $\mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$, $1 \leqslant \ell \leqslant m$, is a module over the ring $\mathscr{C}^\infty(\mathbb{Y})$. The first thing we wish to discuss here is that $\mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$ is finitely generated by exact differential forms, say

$$d\Xi_1,\ d\Xi_2,\ ...,\ d\Xi_M \tag{41}$$

where $\Xi_i \in \mathscr{C}^\infty(\wedge^{\ell-1}\mathbb{Y})$. This simply means that every $\gamma \in \mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$ can be written as

$$\sum_{i=1}^M \lambda_i\, d\Xi_i \qquad \text{where } \lambda \in \mathscr{C}^\infty(\mathbb{Y}) \tag{42}$$

In general, one cannot guarantee that the generators $d\Xi_1,\ d\Xi_2,\ ...,\ d\Xi_M$ will be linearly independent at each point $y \in \mathbb{Y}$. Therefore, the decomposition at (42) need not be unique. Our goal is to select the generators carefully and give an explicit formula for the coefficients $\lambda_i \in \mathscr{C}^\infty(\mathbb{Y})$ in terms of $\gamma \in \mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$.

PROPOSITION **2.2.** *There exist differential forms $\Xi_i \in \mathscr{C}^\infty(\wedge^{\ell-1}\mathbb{Y})$ and $\Gamma_i \in \mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$, $i = 1, 2, ..., M$, such that every $\gamma \in \mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$ admits the decomposition*

$$\gamma = \sum_{i=1}^M \lambda_i\, d\Xi_i \qquad \textit{where } \lambda_i = \langle\gamma, \Gamma_i\rangle \tag{43}$$

The symbol $\langle\ ,\ \rangle$ stands for the point-wise scalar product of differential forms. Precisely, using the Hodge star operator, $\lambda_i\, dy = \lambda_i \wedge (*\Gamma_i)$. We take particular note of the fact that the functions λ_i depend on γ in a linear fashion.

It is useful to begin with a finite atlas of local charts (Ω_k, κ_k) on $\mathbb{Y}$, $k = 1, 2, ..., K$, such that each mapping

$$\kappa_k = (\kappa_k^1, \kappa_k^2, ..., \kappa_k^m) : \ \Omega_k \to \mathbb{R}^m$$

is a diffeomorphism of an open region $\Omega_k \subset \mathbb{Y}$ onto $\mathbb{R}^m$ and $\bigcup_{k=1}^{K} \Omega_k = \mathbb{Y}$. Let us state it in this way:

$$d\kappa_k^1 \wedge d\kappa_k^2 \wedge ... \wedge d\kappa_k^m \neq 0 \qquad \text{on } \Omega_k$$

Upon obvious modifications (multiply by a suitable bump function) we produce a system of mappings, again denote by κ_k, such that

- Each κ_k is defined on the entire manifold $\mathbb{Y}$ and maps it smoothly into $\mathbb{R}^m$.

- The Jacobian determinant of κ_k, which we define by the rule

  $$\mathscr{J}_k(y)\, dy = d\kappa_k^1 \wedge d\kappa_k^2 \wedge ... \wedge d\kappa_k^m$$

  satisfies:

  $$\mathscr{J}_k(y) \geqslant 1 \qquad \text{for } y \in \Omega_k$$

In this extension of κ_k the new open sets Ω_k are actually slightly smaller than the original ones, though they still cover the manifold $\mathbb{Y}$. To each $k = 1, 2, ..., K$ and ℓ-tuple I; $1 \leqslant i_1 < i_2 < ... < i_\ell \leqslant m$ there corresponds a differential form

$$\Xi_k^I = \kappa_k^{i_1}\ d\kappa_k^{i_2} \wedge ... \wedge d\kappa_k^{i_\ell} \in \mathscr{C}^\infty(\wedge^{\ell-1}\mathbb{Y}) \tag{44}$$

Now, the exact forms we had in mind can be defined as

$$d\Xi_k^I = d\kappa_k^{i_1} \wedge d\kappa_k^{i_2} \wedge ... \wedge d\kappa_k^{i_\ell} \in \mathscr{C}^\infty(\wedge^{\ell}\mathbb{Y}) \tag{45}$$

They will serve as generators of the module $\mathscr{C}^\infty(\wedge\mathbb{Y})$ over the ring $\mathscr{C}^\infty(\mathbb{Y})$. Fix a partition of unity $\{\varphi_k\}_{k=1,2,...,K}$ subordinate to the cover $\{\Omega_k\}_{k=1,2,...,K}$. We introduce differential forms $\Gamma^I_k \in \mathscr{C}^\infty_\circ(\wedge^\ell\Omega_k)$, by the rule

$$*\Gamma^I_k = \frac{\varphi_k\, d\Xi^{I'}_k}{\mathcal{J}_k} \quad \text{where} \qquad \begin{matrix} I\colon 1\leqslant i_1<i_2<...<i_\ell\leqslant m \\ k=1,2,...,K \end{matrix} \tag{46}$$

The superscript I' stands for an ordered complementary $(m-\ell)$-tuple. Precisely, $I' = (i'_1, i'_2, ..., i'_{m-\ell})$ is ordered in such a way that $(i_1,...,i_\ell, i'_1,...,i'_{m-\ell})$ constitutes an even permutation of $(1,2,...,m)$. In this way $d\Xi^{I'}_k = d\kappa^{i'_1}_k \wedge ... \wedge d\kappa^{i'_{m-\ell}}_k$ is a smooth $(m-\ell)$-form on $\mathbb{Y}$. The Hodge star operator $* : \mathscr{C}^\infty(\wedge^{m-\ell}\mathbb{Y}) \to \mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$ converts it onto an ℓ-form, making $\Gamma^I_k \in \mathscr{C}^\infty_\circ(\wedge^\ell\Omega_k)$. To prove the decomposition formula at (43) we recall that $\varphi_k\gamma$, being a form in $\mathscr{C}^\infty(\wedge^\ell\Omega_k)$, can be uniquely written as

$$\varphi_k\,\gamma = \sum_{1\leqslant i_1<...<i_\ell\leqslant m} \alpha^{i_1...i_\ell}_k\, d\kappa^{i_1}_k \wedge d\kappa^{i_2}_k \wedge ... \wedge d\kappa^{i_\ell}_k \quad \text{where } \alpha^{i_1...i_\ell}_k \in \mathscr{C}^\infty(\Omega_k) \tag{47}$$

To compute the coefficients $\alpha^{i_1,...,i_\ell}_k$ we wedge both side with the complementary product

$$d\Xi^{I'}_k = d\kappa^{i'_1}_k \wedge d\kappa^{i'_2}_k \wedge ... \wedge d\kappa^{i'_{m-\ell}}_k \in \mathscr{C}^\infty(\wedge^\ell\mathbb{Y})$$

The exterior multiplication annihilates all terms except the one corresponding to $I; 1\leqslant i_1 < ... < i_\ell \leqslant m$

$$\varphi_k\,\gamma \wedge d\Xi^{I'}_k = \alpha^{i_1...i_\ell}_k(y)\; \mathcal{J}_k(y)\, dy$$

Applying Hodge star operator this equation reads as

$$\alpha^{i_1...i_\ell}_k = *\left(\gamma \wedge \frac{\varphi_k\; d\Xi^{I'}_k}{\mathcal{J}_k}\right) = \left\langle \gamma, \Gamma^I_k \right\rangle \tag{48}$$

Hence

$$\varphi_k\gamma = \sum_I \left\langle \gamma, \Gamma^I_k \right\rangle\, d\Xi^I_k, \qquad k=1,2,...,K \tag{49}$$

Summing up we arrive at the desired decomposition

$$\gamma = \sum_{k=1}^{K} \sum_{I} \left\langle \gamma, \Gamma_k^I \right\rangle \, d\Xi_k^I \tag{50}$$

To complete the proof we need only rename the indices. Precisely, we abbreviate the multi-index ${}^{I}_{k}$ to a single letter $i = 1, 2, ..., M = K \binom{m}{\ell}$.

Uniform bounds of the functions λ_i in terms of γ follow directly from the formula (43). Let us record the following one

$$\| \lambda_i \|_{\mathscr{C}^1(\mathbb{Y})} \preccurlyeq \| \gamma \|_{\mathscr{C}^1(\mathbb{Y})} \tag{51}$$

The decomposition, as constructed above, is at intrinsic interest in regard to the following representation of the exact forms on $\mathbb{Y}$.

Proposition **2.3.** *Every exact n-form $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$, $2 \leqslant n \leqslant \dim \mathbb{Y}$ can be written as*

$$\omega = \sum_{i=1}^{M} d\lambda_i \wedge d\Xi_i \tag{52}$$

where λ_i are smooth functions on $\mathbb{Y}$.

Proof. We write $\omega = d\gamma$, for some $\gamma \in \mathscr{C}^\infty(\wedge^{n-1}\mathbb{Y})$. With the aid of Proposition 2.2 we decompose γ as

$$\gamma = \sum_{i=1}^{K} \lambda_i \, d\Xi_i$$

and differentiate to obtain

$$d\gamma = \sum_{i=1}^{K} d\lambda_i \wedge d\Xi_i$$

as desired.

Concerning estimates of λ_i in terms of ω, we observe that our decomposition depends on the choice of γ. For this reason it is desirable to introduce the following norm

$$[\![\omega]\!]_\infty = \inf \left\{ \| \gamma \|_{\mathscr{C}^1(\wedge^{n-1}\mathbb{Y})}; \;\; d\gamma = \omega \right\} \tag{53}$$

With this definition at hand we can achieve the following estimate

$$\| \lambda_i \|_{\mathscr{C}^1(\mathbb{Y})} \preccurlyeq [\![\omega]\!]_\infty \tag{54}$$

Proposition 2.3 gives rise to a class of the so called Cartan forms, named after H. Cartan who studied similar differential forms.

Definition **2.1.** [Cartan forms] An n-form $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$, $n \leqslant m = \dim \mathbb{Y}$, is said to be a Cartan form if it can be decomposed as

$$\omega = \sum_{i=1}^{M} \alpha_i \wedge \beta_i \tag{55}$$

where $\alpha_i \in \mathscr{C}^\infty(\wedge^{\ell_i} \mathbb{Y}) \cap \ker d$ and $\beta_i \in \mathscr{C}^\infty(\wedge^{k_i} \mathbb{Y}) \cap \ker d$. Here we assume that $k_i, \ell_i \geqslant 1$ and $k_i + \ell_i = n$.

Thus the exact forms in $\mathscr{C}^\infty(\wedge^n \mathbb{Y})$, with $2 \leqslant n \leqslant \dim \mathbb{Y}$, are Cartan forms. We take up this topic here by assuming from now on that $\dim \mathbb{Y} = \dim \mathbb{X} = n$.

Corollary **2.1.** *Every n-form $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$, $\dim \mathbb{Y} = n$, whose integral over $\mathbb{Y}$ vanishes is exact. Consequently, ω is a Cartan form of the type*

$$\omega = \sum_{i=1}^{M} d\lambda_i \wedge d\Xi_i \tag{56}$$

Corollary **2.2.** *If the manifold $\mathbb{Y}$ of dimension n admits at least one Cartan n-form with non-vanishing integral, then all n-forms on $\mathbb{Y}$ are Cartan forms.*

Corollary **2.3.** [decomposition of n-forms] *Every $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$, dim $\mathbb{Y} = n$, can be written as*

$$\omega = \left(\fint_{\mathbb{Y}} \omega \right) dy + \sum_{i=1}^{M} d\lambda_i \wedge d\Xi_i \tag{57}$$

Next we bring on stage the manifolds which are cohomologically simple. Recall that $\mathbb{Y}$ is a rational homology sphere if all its cohomology groups $H^\ell(\mathbb{Y})$, with $1 \leqslant \ell < n$ vanish. In this case Cartan's n-forms

$$\omega = \sum_{i=1}^{M} \alpha_i \wedge \beta_i \tag{58}$$

are necessarily exact and as such have integral zero over $\mathbb{Y}$. Indeed, every closed form α_i is exact so is each wedge product $\alpha_i \wedge \beta_i$, $i = 1, ..., M$. In other words, the condition $H^\ell(\mathbb{Y}) \neq 0$, for some $1 \leqslant \ell < n$, is necessary in order to find a Cartan form on $\mathbb{Y}$ with non-vanishing integral. Our next result shows that this condition is also sufficient.

Suppose $H^\ell(\mathbb{Y}) \neq 0$ for some $1 \leqslant \ell < n$. Hodge-deRham theory tells us that there exists a nonzero harmonic field $h \in \mathcal{H}(\wedge^\ell \mathbb{Y})$. Consider the n-form

$$\omega = h \wedge *h = |h|^2 \, dy \tag{59}$$

where $*h \in \mathscr{C}^\infty(\wedge^{n-\ell}\mathbb{Y})$ is Hodge dual to h. By the definition of $\mathcal{H}(\wedge^\ell \mathbb{Y})$ the form h is both closed and coclosed. We then see that $*h$ is also closed. Thus ω is a Cartan form. That ω has non-vanishing integral follows from the formula

$$\int_{\mathbb{Y}} \omega = \int_{\mathbb{Y}} h \wedge *h = \int_{\mathbb{Y}} |h|^2 \, dy \neq 0 \tag{60}$$

We end this section by combining these later observations with Corollary 2.2.

PROPOSITION **2.4.** *Let* $\dim \mathbb{Y} = n$*. Then every* $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$ *is a Cartan form if and only if* $H^\ell(\mathbb{Y}) \neq 0$*, for some* $1 \leqslant \ell < n$ *i.e. if* $\mathbb{Y}$ *is not rational homology sphere.*

2.4 Mollifiers and smoothing operator

For the duration of this paper we fix a nonnegative function $\mathbf{\Phi} \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ supported in the closed unit ball and having integral 1. For example

$$\mathbf{\Phi} = C_n \begin{cases} \exp \frac{1}{|x|^2 - 1} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqslant 1 \end{cases} \tag{61}$$

where C_n is a constant. The one parameter family

$$\mathbf{\Phi}_t(x) = \frac{1}{t^n}\mathbf{\Phi}\left(\frac{x}{t}\right), \quad t > 0 \tag{62}$$

defines an approximation of the Dirac mass at the origin; $\mathbf{\Phi}_t$ are called mollifiers.

Given $u \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, the mollification of u is the family of functions $u_t \in \mathscr{C}^\infty(\mathbb{R}^n)$, $t > 0$, defined by the convolution formula

$$u_t(x) = \int_{\mathbb{R}^n} \mathbf{\Phi}_t(x - z)u(z)\, dz \tag{63}$$

Various bounds for a function $u \in \mathscr{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, imply the same bounds for u_t. Basic properties of the mollification are listed below:

(i) $\lim_{t\to 0} u_t(x) = u(x)$ for almost every $x \in \mathbb{R}^n$

(ii) If u is continuous then u_t converges to u uniformly on compact subsets

(iii) Mollification preserves the $\mathscr{L}^p$-bounds; more specifically,

$$\| u_t \|_{\mathscr{L}^p(\mathbb{R}^n)} \preccurlyeq \| u \|_{\mathscr{L}^p(\mathbb{R}^n)}, \quad 1 \leqslant p \leqslant \infty$$

$$\| du_t \|_{\mathscr{L}^p(\mathbb{R}^n)} \preccurlyeq \| du \|_{\mathscr{L}^p(\mathbb{R}^n)}, \quad 1 \leqslant p \leqslant \infty$$

(iv) For $1 \leqslant p < \infty$ and $u \in \mathscr{W}^{1,p}(\mathbb{R}^n)$, we have

$$\lim_{t\to 0} \| u_t - u \|_{\mathscr{L}^p(\mathbb{R}^n)} = 0$$

$$\lim_{t\to 0} \| u_t - u \|_{\mathscr{W}^{1,p}(\mathbb{R}^n)} = 0$$

The implied constants in (iii) are actually equal to 1, but not on manifolds latter on. It is well known that $\lim_{t\to 0} u_t(x) = u(x)$ at every Lebesgue point of u, regardless of the generating mollifier Φ. We can even take Φ to be the average of the characteristic function of the unit ball:

$$\mathbf{\Phi}(x) = \frac{\chi_{\mathbb{B}}(x)}{|\mathbb{B}|} \tag{64}$$

For this choice of $\mathbf{\Phi}$ the mollifications, denoted by $u_r(x)$, are the familiar $\mathscr{L}^1$-averages over the balls $\mathbb{B}(x,r)$:

$$u_r(x) = \fint_{\mathbb{B}(x,r)} u(z)\, dz \tag{65}$$

Next we invoke the reference atlas $\mathcal{A}$ on $\mathbb{X}$, which we already fixed in Section 2.1.1. Let us also fix a partition of unity $\{\varphi_\Omega\}_{\Omega\in\mathcal{A}}$ subordinate to $\mathcal{A}$. Thus, to every Ω there corresponds a coordinate mapping $\kappa : \Omega \xrightarrow{onto} \mathbb{R}^n$. Now the mollification operator in $\mathbb{R}^n$ can also be defined on $\mathbb{X}$, by the rule

$$f_t(x) = \sum_{\Omega\in\mathcal{A}} \varphi_\Omega(x) \int_\Omega \mathbf{\Phi}_t(\kappa(x) - \kappa(z))\mathcal{J}(z,\kappa) f(z)\, dz \tag{66}$$

Each integral term is a smooth function on Ω, equal to $(f\circ\kappa^{-1})_t \circ \kappa \in \mathscr{C}^\infty(\Omega)$. We shall write this formula in a compact form as

$$f_t(x) = \int_{\mathbb{X}} K_t(x,z) f(z)\, dz \tag{67}$$

where $K_t : \mathbb{X}\times\mathbb{X} \to \mathbb{R}_+$ is $\mathscr{C}^\infty$-smooth for all $t > 0$.

Clearly, all basic properties of the mollification listed in (i-iv) remain valid on a manifold $\mathbb{X}$ if we restrict ourselves to sufficiently small parameters t, say

$$0 < t \leqslant t_{\mathbb{X}} \tag{68}$$

where $t_{\mathbb{X}}$ depends on $\mathbb{X}$, the atlas $\mathcal{A}$ and the partition of unity. As those were fixed, our upper bound $t_{\mathbb{X}}$ is also fixed for the duration of this text. Note that the proof of (iii) relies on Lemma 2.2 with $\mathbb{E} = \mathbb{X}$.

Mollification procedure usually enlarges the support of f. For example, if $\operatorname{supp} f \subset \mathbb{U}$, then $\operatorname{supp} f_t \subset \mathbb{U}_{t'}$, where

$$\mathbb{U}_{t'} = \big\{x \in \mathbb{X};\ \operatorname{dist}(x,\mathbb{U}) < t'\big\} \quad \text{and} \quad t \preccurlyeq t' \preccurlyeq t$$

The implied constants in the inequalities $t \preccurlyeq t' \preccurlyeq t$ depend only on $\mathbb{X}$. This can be seen from the equation

(v) $K_t(x,z) = 0, \quad$ whenever $\operatorname{dist}(x,z) \succcurlyeq t$

Moreover, if $f : \mathbb{X} \to \mathbb{R}^N$ is constant then $f_t : \mathbb{X} \to \mathbb{R}^N$ is also constant for all $t > 0$, which is immediate from the identity

(vi) $\int_{\mathbb{X}} K_t(x,z)\, dz \equiv 1, \quad$ for all $t > 0$

Finally, combining (v) and (vi) we obtain

(vii) $\operatorname*{osc}_{\mathbb{U}} f_t \preccurlyeq \operatorname*{ess\,osc}_{\mathbb{U}'} f \qquad t \preccurlyeq t' \preccurlyeq t$

Perhaps, the definition of the essential oscillation of a measurable function $f : \mathbb{V} \to \mathbb{R}^N$ is in order. The symbol $\operatorname*{ess\,osc}_{\mathbb{V}} f$ stands for the infimum of all $\delta > 0$ such that the set $\{(x_1, x_2) \in \mathbb{V} \times \mathbb{V};\ |f(x_1) - f(x_2)| > \delta\}$ has measure zero in $\mathbb{V} \times \mathbb{V}$.

2.5 Maximal operators

The well-developed study of maximal functions has an analogue for Riemannian manifolds. Here we shall frame the definitions and basic properties in this setting, some pending a discussion in subsequent sections.

2.5.1 The Hardy-Littlewood maximal operator

Given $1 \leqslant p < \infty$ and $h \in \mathscr{L}^p(\mathbb{X}, \mathbf{V})$, where $\mathbf{V}$ is a finite dimensional normed space, we define

$$\mathbf{M}_p h(x) = \sup \left\{ \left(\fint_{\mathbb{B}} |h(x)|^p \, dx \right)^{\frac{1}{p}} ; \quad x \in \mathbb{B} \subset \mathbb{X} \right\} \tag{69}$$

The supremum runs over all metric balls $\mathbb{B} \subset \mathbb{X}$ containing a given point x. Since the entire manifold $\mathbb{X}$ is also a ball we see that for every $x \in \mathbb{X}$

$$\mathbf{M}_p h(x) \geqslant \left(\fint_{\mathbb{X}} |h|^p \right)^{\frac{1}{p}} \tag{70}$$

Lebesgue Differentiation Theorem tells us that

$$|h(x)| \leqslant \mathbf{M}_p h(x) \quad \text{for a.e. } x \in \mathbb{X} \tag{71}$$

Also note, by using Hölder's inequality, that the function $p \to \mathbf{M}_p h(x)$ is increasing. For notational convenience we omit the subscript p if it equals 1. Thus

$$\mathbf{M}_p h = (\mathbf{M}|h|^p)^{\frac{1}{p}} \tag{72}$$

PROPOSITION **2.5.** [WEAK TYPE ESTIMATE] *For every $h \in \mathscr{L}^p(\mathbb{X}, \mathbf{V})$ the maximal function $\mathbf{M}_p h$ belongs to the Marcinkiewicz class $\mathscr{L}^p_{weak}(\mathbb{X})$ and we have*

$$\int_{\mathbf{M}_p h > 2t} dx \preccurlyeq \frac{1}{t^p} \int_{|h|>t} |h(x)|^p \, dx \tag{73}$$

for all $t > 0$.

We will not prove this proposition here. The arguments establishing (73) are very similar to those used in the Euclidean setting. Let us point out that the main tool is Vitali type covering lemma, which is true in any separable metric space. But we are not involved in such generality. The interested reader may try to consult [13, 2.8.4-2.8.6].

As a consequence of the weak-type estimate at (73) and of sublinearity of $\mathbf{M}_s$ we obtain

COROLLARY **2.4.** *Let $\{h_j\}$ converge to h in $\mathscr{L}^s(\mathbb{X}, \mathbf{V})$, $s \geq 1$, then $\{\mathbf{M}_s h_j\}$ contains a subsequence converging to $\mathbf{M}_s h$ almost everywhere.*

Of course, $\mathbf{M}_p$ is a sublinear bounded operator in $\mathscr{L}^\infty(\mathbb{X}, \mathbf{V})$. By interpolation, we infer strong type estimates.

PROPOSITION **2.6.** *The maximal operator $\mathbf{M}_p : \mathscr{L}^s(\mathbb{X}, \mathbf{V}) \to \mathscr{L}^s(\mathbb{X})$ is bounded for all $p < s \leqslant \infty$. Precisely, we have*

$$\| \mathbf{M}_p h \|_s \preccurlyeq \| h \|_s \tag{74}$$

Reviewing the maximal function $\mathbf{M}f$ in relation to the mollifiers f_t we first notice that

$$|f_t(x)| \preccurlyeq \fint_{\mathbb{B}(x,r)} |f(z)| \, dz \tag{75}$$

where $t \leqslant r \preccurlyeq t$. In particular,

$$|f_t(x)| \preccurlyeq \mathbf{M}f(x) \quad \text{for every } x \in \mathbb{X} \tag{76}$$

As for the differential Df_t, we have the following estimate

$$|Df_t(x)| \preccurlyeq \fint_{\mathbb{B}(x,r)} |f(z)|\, dz + \fint_{\mathbb{B}(x,r)} |Df(z)|\, dz \tag{77}$$

This also applies to $f(x) - C$, with C being any constant. It has to be noted that $(f - C)_t = f_t - C$, hence

$$|Df_t(x)| \preccurlyeq \fint_{\mathbb{B}(x,r)} |f(z) - C|\, dz + \fint_{\mathbb{B}(x,r)} |Df(z)|\, dz \tag{78}$$

Poincaré inequality gives an estimate of the first integral in terms of Df.

$$\fint_{\mathbb{B}(x,r)} |f(z) - C|\, dz \preccurlyeq r \fint_{\mathbb{B}(x,r)} |Df(z)|\, dz \preccurlyeq \fint_{\mathbb{B}(x,r)} |Df(z)|\, dz$$

Finally, as the integral averages do not exceed the maximal function, we obtain

$$|Df_t(x)| \preccurlyeq \mathbf{M}(Df)(x) \quad \text{for all } x \in \mathbb{X} \tag{79}$$

2.5.2 The Fefferman-Stein operator and the Hardy space

The one parameter family of $\mathscr{C}^\infty$-smooth functions $K_t : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$, $0 < t \leqslant t_{\mathbb{X}}$, introduced in Section 2.4, will be employed to define another maximal operator on $\mathbb{X}$. For an n-form $\omega \in \mathscr{L}^1(\wedge^n \mathbb{X})$ we define a function $\omega_t \in \mathscr{C}^\infty(\mathbb{X})$ by the rule

$$\omega_t(x) = \int_{\mathbb{X}} K_t(x, \cdot)\, \omega, \quad 0 < t \leqslant t_{\mathbb{X}} \tag{80}$$

If $\omega = h(z)\, dz$, where h is an integrable function and dz is the Riemannian volume element on $\mathbb{X}$, we also write

$$h_t(x) = \int_{\mathbb{X}} K_t(x, z)\, h(z)\, dz \tag{81}$$

Then $\lim_{t\to 0} h_t(x) = h(x)$ at the Lebesgue points of h. This gives a way to the concept of the maximal operator; replace $\lim_{t\to 0}$ by $\sup_{t>0}$. Recall that the Hardy-Littlewood maximal function of $h \in \mathscr{L}^1(\mathbb{X})$ is defined by

$$(\mathbf{M}h)(x) = \sup \left\{ \fint_{\mathbb{B}} |h(z)|\, dz; \quad x \in B \subset \mathbb{X} \right\} \succcurlyeq \sup_{0<t\leqslant t_{\mathbb{X}}} |h|_t(x) \tag{82}$$

where the supremum runs over all metric balls containing the given point $x \in \mathbb{X}$. More sensitive on various cancellations is the Fefferman-Stein maximal function

$$(\mathcal{M}h)(x) = \sup_{0<t\leqslant t_{\mathbb{X}}} |h_t(x)|, \quad h \in \mathscr{L}^1(\wedge^n \mathbb{X}) \tag{83}$$

Let us emphasize explicitly that here we first mollify h and then take the absolute value of it. Clearly, we have

$$\mathcal{M}h(x) \preccurlyeq \mathbf{M}h(x) \tag{84}$$

but not conversely. As a note of additional interest, the maximal operator $\mathcal{M}$ can be defined on Schwartz distributions due to the smoothness of the generating function $\mathbf{\Phi}$, see [50].

Definition **2.2.** [Hardy space] An n-form $h \in \mathscr{L}^1(\wedge^n \mathbb{X})$ is said to be in the Hardy space $\mathscr{H}^1(\wedge^n \mathbb{X})$ if $\mathcal{M}h \in \mathscr{L}^1(\mathbb{X})$.

We see that $\mathscr{H}^1(\wedge^n \mathbb{X})$ is a Banach space with respect to the norm

$$\| h \|_{\mathscr{H}^1(\mathbb{X})} = \int_{\mathbb{X}} \mathcal{M}h \tag{85}$$

We refer to [51] for yet another approach to $\mathscr{H}^1$-spaces on manifolds. Now, we recall very briefly the Zygmund space $\mathscr{L}\log\mathscr{L}(\mathbb{X})$. It consists of functions $h : \mathbb{X} \to \mathbb{R}$ such that

$$\| h \|_{\mathscr{L}\log\mathscr{L}} = \int_{\mathbb{X}} |h(x)| \log\left(e + \frac{|h(x)|}{\int_{\mathbb{X}} |h|} \right) dx < \infty \tag{86}$$

see also Section 4.3. It is known that $\mathscr{L}\log\mathscr{L}(\mathbb{X}) \subset \mathscr{H}^1(\mathbb{X})$. Indeed, for $h \in \mathscr{L}\log\mathscr{L}(\mathbb{X})$, $|\mathcal{M}h| \leqslant |\mathbf{M}h| \in \mathscr{L}^1(\mathbb{X})$, by Stein's Theorem [49]. Conversely, any non-negative function in $\mathscr{H}^1(\mathbb{X})$ lies in $\mathscr{L}\log\mathscr{L}(\mathbb{X})$, see [50].

3 Examples

In this section we go through some well known and some new examples which provide a view on weakly differentiable mapping. In this vein the following modification of the example by R. Schoen and K. Uhlenbeck [46] proves to be most desirable.

3.1 The longitude projection

Consider the n-sphere $\mathbb{S}^n$ in the Euclidean space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. We write the point $x \in \mathbb{S}^n$ as $x = (\mathbf{z}, x_{n+1})$, where $\mathbf{z} \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$ are coupled by the equation $|\mathbf{z}|^2 + |x_{n+1}|^2 = 1$. The projection along the longitude lines of $\mathbb{S}^n$ onto its equatorial sphere $\mathbb{S}^{n-1} = \{y \in \mathbb{R}^n; \quad |y| = 1\}$ is defined by the rule

$$f : \mathbb{S}^n \setminus \{\mathfrak{n}, \mathfrak{s}\} \to \mathbb{S}^{n-1}, \quad f(\mathbf{z}, x_{n+1}) = \frac{\mathbf{z}}{|\mathbf{z}|} \tag{87}$$

Thus f is not defined at the north pole $\mathfrak{n} = (\mathbf{0}, 1)$ and the south pole $\mathfrak{s} = (\mathbf{0}, -1)$. Elementary computation shows that

$$|Df(x)| = \frac{1}{|\mathbf{z}|} \tag{88}$$

Therefore, $f \in \mathscr{W}^{1,p}(\mathbb{S}^n, \mathbb{S}^{n-1})$ for all $1 \leqslant p < n$ but not for $p = n$. Actually, its differential lies in the Marcinkiewicz space $\mathscr{L}^n_{\text{weak}}(\mathbb{X})$. Precisely, we have

$$\frac{1}{t^n} \preccurlyeq \int_{|Df|>t} dx \preccurlyeq \frac{1}{t^n}, \quad \text{for} \quad t \geqslant 1 \tag{89}$$

Arguing by analytic methods of topological degree we find that f cannot be approximated by smooth mappings

$$f_j = (f_j^1, f_j^2, ..., f_j^n) : \mathbb{S}^n \to \mathbb{S}^{n-1}$$

in the metric of $\mathscr{W}^{1,n-1}(\mathbb{S}^n, \mathbb{S}^{n-1})$. Indeed, looking for a contradiction we examine the wedge products

$$df_j^1 \wedge ... \wedge df_j^n \in \mathscr{C}^\infty(\wedge^n \mathbb{S}^n)$$

The 1-forms $df_j^1, df_j^2, ..., df_j^n$ are linearly dependent (at each point) because of the relation $|f_j^1|^2 + ... + |f_j^n|^2 = 1$, which yields that $f_j^1\, df_j^1 + ... + f_j^n\, df_j^n \equiv 0$. Hence, the wedge products $df_j^1 \wedge ... \wedge df_j^n$ are identically equal to zero. Now, for every $\varphi \in \mathscr{C}^\infty(\mathbb{S}^n)$ the integration by parts yields

$$0 = \int_{\mathbb{S}^n} \varphi\, df_j^1 \wedge ... \wedge df_j^n = - \int_{\mathbb{S}^n} f_j^1\, d\varphi \wedge df_j^2 \wedge ... \wedge df_j^n$$

The interested reader may recognize that the right hand side defines the so-called distributional wedge product [24], [29]. These latter integrands possess sufficient degree of integrability to pass the limit under the integral sign as $j \to \infty$. Elementary computation then shows that

$$|\mathbb{B}^n|\, [\varphi(\mathfrak{n}) - \varphi(\mathfrak{s})] = \int_{\mathbb{S}^n} f^1 d\varphi \wedge df^2 ... \wedge df^n = \lim_{j\to\infty} \int_{\mathbb{S}^n} f_j^1 d\varphi \wedge df_j^2 ... \wedge df_j^n = 0,$$

which gives a clear contradiction if we choose $\varphi(\mathfrak{n}) \neq \varphi(\mathfrak{s})$.

In this example f fails to be uniformly continuous near the poles. Furthermore, the oscillation of f on any $(n-1)$-surface surrounding one of those poles stays away from zero, no matter how close to the pole the surface is. And this is exactly what we shall try to avoid. But the precise results must wait until the relevant concepts will be introduced.

3.2 Spherical coordinates

Let $x = (x_1, x_2, ..., x_{n+1})$ be a point in the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $x_1^2 + ... + x_{n+1}^2 = 1$. Spherical coordinates $(\mathbf{z}, \theta)$ on $\mathbb{S}^n$ can be introduced by setting

$$(x_1, x_2, ..., x_{n+1}) = (\mathbf{z} \sin\theta, \cos\theta) \tag{90}$$

where $\mathbf{z}$ lies in the equatorial sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $0 \leqslant \theta \leqslant \pi$ is the latitude angle, $x_{n+1} = \cos\theta$. Singularities occur at the north pole $\mathfrak{n} = (0, ..., 0, 1)$, $\theta = 0$, and at the south pole $\mathfrak{s} = (0, ..., 0, -1)$, $\theta = \pi$. At those poles we cannot determine the equatorial point $\mathbf{z} \in \mathbb{S}^{n-1}$. The usual volume element on $\mathbb{S}^n$ takes the following form in spherical coordinates

$$dx = |\sin\theta|^{n-1}\, d\theta\, d\mathbf{z} \tag{91}$$

where $d\mathbf{z}$ stands for the standard volume element on $\mathbb{S}^{n-1}$. Thus, in particular

$$\omega_n = |\mathbb{S}^n| = \int_{\mathbb{S}^n} dx = \omega_{n-1} \int_0^\pi \sin^{n-1}\theta \, d\theta \tag{92}$$

For $0 \leqslant \alpha < \beta \leqslant \pi$ we shall consider the spherical slice

$$\mathbb{S}_\alpha^\beta = \left\{ (\mathbf{z}\cos\theta, \sin\theta); \quad \mathbf{z} \in \mathbb{S}^{n-1} \text{ and } \alpha \leqslant \theta \leqslant \beta \right\} \tag{93}$$

Its n- dimensional volume can be estimated by

$$|\mathbb{S}_\alpha^\beta| = \omega_{n-1} \int_\alpha^\beta \sin^{n-1}\theta \, d\theta \leqslant \frac{\omega_{n-1}}{n} (\beta^n - \alpha^n) = |\mathbb{B}^n| (\beta^n - \alpha^n) \tag{94}$$

3.3 Winding around the longitude circles

Given a spherical slice $\mathbb{S}_\alpha^\beta$, $0 \leqslant \alpha < \beta \leqslant \pi$, let $\gamma : [\alpha, \beta] \to [0, \infty)$ be an increasing function. The mapping $f : \mathbb{S}_\alpha^\beta \to \mathbb{S}^n$ is defined using spherical coordinates by the rule

$$f(\mathbf{z}\cos\theta, \sin\theta) = (\mathbf{z}\cos\gamma(\theta), \sin\gamma(\theta)) \quad \text{for} \quad \alpha \leqslant \theta \leqslant \beta \tag{95}$$

is called winding map. Note that $(\mathbf{z}\cos\theta, \sin\theta)$ and $(\mathbf{z}\cos\gamma(\theta), \sin\gamma(\theta))$ lay on the same longitude circles.

We calculate the Jacobian determinant $\mathcal{J}(x, f)$ at the points where the derivative $\gamma'(\theta)$ exists. Observe that the linear tangent map $Df(x) : T_x\mathbb{S}^n \to T_{f(x)}\mathbb{S}^n$ is stretching in the longitude direction by $\gamma'(\theta)$ and in all the equatorial directions by the factor $\left|\frac{\sin\gamma(\theta)}{\sin\theta}\right|$. Hence the Jacobian determinant of f at $x = (\mathbf{z}\cos\theta, \sin\theta)$ is the product of those stretching factors.

$$\mathcal{J}(x, f) = \gamma'(\theta) \left| \frac{\sin\gamma(\theta)}{\sin\theta} \right|^{n-1} \geqslant 0 \tag{96}$$

The operator norm of the linear map $Df(x)$ is precisely equal to:

$$|Df(x)| = \max \left\{ \gamma'(\theta), \ \left| \frac{\sin\gamma(\theta)}{\sin\theta} \right| \right\}, \quad x = (\mathbf{z}\sin\theta, \cos\theta) \tag{97}$$

3.4 A mapping of infinite degree

Consider a sequence of latitude angles

$$\theta_0 = \pi > \frac{\pi}{2} \geqslant 2\theta_1 > \theta_1 \geqslant 2\theta_2 > \theta_2 \geqslant ...2\theta_k > \theta_k \geqslant 2\theta_{k+1} > \theta_{k+1} ... > 0 \quad (98)$$

Additional conditions on these angles will be imposed later on. The sphere $\mathbb{S}^n$ is divided into countable number of spherical slices

$$\mathbb{S}^n = \bigcup_{k=0}^{\infty} \mathbb{S}^{\theta_k}_{2\theta_{k+1}} \cup \bigcup_{k=1}^{\infty} \mathbb{S}^{2\theta_k}_{\theta_k} \quad (99)$$

We now construct an infinite covering $f : \mathbb{S}^n \to \mathbb{S}^n$ by the rule

- $f : \mathbb{S}^{\theta_k}_{2\theta_{k+1}} \to \mathbb{S}^n$ is the identity for $k = 0, 1, ...$
- $f : \mathbb{S}^{2\theta_k}_{\theta_k} \to \mathbb{S}^n$ is the latitude winding for $k = 1, 2, ...$

$$\begin{aligned} f(\mathbf{z}\cos\theta, \sin\theta) &= (\mathbf{z}\cos\gamma_k(\theta), \sin\gamma_k(\theta)) \\ \gamma_k(\theta) &= \theta + \frac{2\pi(\theta - \theta_k)}{\theta_k} \end{aligned} \quad (100)$$

Let us observe that $f : \mathbb{S}^{2\theta_k}_{\theta_k} \to \mathbb{S}^n$ is the identity on the boundary of $\mathbb{S}^{2\theta_k}_{\theta_k}$. Precisely, we have $\gamma_k(\theta_k) = \theta_k$ and $\gamma_k(2\theta_k) = 2\theta_k + 2\pi$. Furthermore, f maps all points of latitude $\theta = 3\pi\theta_k/(2\pi + \theta_k)$ into the south pole. It maps points of latitude $\theta = 4\pi\theta_k/(2\pi + \theta_k)$ into the north pole. Outside those latitude spheres f is a local diffeomorphism. Since the image of $\mathbb{S}^{2\theta_k}_{\theta_k}$ covers the whole sphere $\mathbb{S}^n$ we estimate the integral of the Jacobian determinant of f as

$$\int_{\mathbb{S}^{2\theta_k}_{\theta_k}} \mathcal{J}(x, f)\, dx \geqslant 2\omega_n \quad (101)$$

for every $k = 1, 2, ...$ We then conclude that the Jacobian is not integrable

$$\int_{\mathbb{S}^n} \mathcal{J}(x, f)\, dx \geqslant \sum_{k=1}^{\infty} 2\omega_n = \infty \quad (102)$$

Because of this f does not belong to the Sobolev class $\mathscr{W}^{1,n}(\mathbb{S}^n, \mathbb{S}^n)$. It is desirable to see which Orlicz-Sobolev classes contain this mapping f. We therefore need to estimate the differential of f. On each spherical slice $\mathbb{S}^{\theta_k}_{2\theta_{k+1}}$ the norm of Df equals 1, whereas for $(\mathbf{z}\cos\theta, \sin\theta)$ in $\mathbb{S}^{2\theta_k}_{\theta_k}$ formula (97) yields

$$\begin{aligned} |Df(\mathbf{z},\theta)| &= \max\left\{1+\frac{2\pi}{\theta_k},\ \left|\frac{\sin\gamma_k(\theta)}{\sin\theta}\right|\right\} \\ &\leqslant \max\left\{1+\frac{2\pi}{\theta_k},\ \frac{1}{\sin\theta}\right\} \leqslant \frac{8}{\theta_k}\,, \end{aligned} \tag{103}$$

because $\theta_k \leqslant \theta < 2\theta_k$. Also observe that the volume of $\mathbb{S}^{2\theta_k}_{\theta_k}$ does not exceed $2^n n^{-1}\omega_{n-1}\theta_k^n$, by the formula at (94).

Now let $P : [0,\infty) \to [0,\infty)$ be any Orlicz function that exhibits slower growth than t^n. Precisely, we assume that

$$\liminf_{t\to\infty}\ t^{-n}P(t) = 0 \tag{104}$$

The reader may consult Section 4.3 for the definition of the Orlicz function. We find that

$$\int_{\mathbb{S}^{2\theta_k}_{\theta_k}} P(|Df(x)|)\,dx \leqslant |\ \mathbb{S}^{2\theta_k}_{\theta_k}|\ P(\tfrac{8}{\theta_k}) \leqslant \tfrac{2^n\omega_{n-1}}{n}P(\tfrac{8}{\theta_k}) \tag{105}$$

As $|Df(x)| = 1$ on the remaining spherical slices, it follows that

$$\int_{\mathbb{S}^n} P(|Df(x)|)\,dx \leqslant \omega_n\,P(1) + \sum_{k=1}^{\infty}\lambda(\theta_k) \tag{106}$$

where

$$\lambda(\epsilon) = \tfrac{2^n\omega_{n-1}}{n}\ \epsilon^n P(\tfrac{8}{\epsilon}) \tag{107}$$

Since

$$\liminf_{\epsilon\to 0}\ \lambda(\epsilon) = 0,$$

one can find a sequence of latitude angles satisfying (98) such that

$$\sum_{k=1}^{\infty}\lambda(\theta_k) < \infty$$

We then conclude with the following result.

THEOREM 3.1. *For every Orlicz function satisfying (104) there exists an orientation preserving mapping $f : \mathbb{S}^n \to \mathbb{S}^n$ in the Orlicz-Sobolev class $\mathscr{W}^{1,P}(\mathbb{S}^n, \mathbb{S}^n)$ whose Jacobian determinant is not integrable.*

4 Some Classes of Functions

We begin with the Riemannian volume element dx on $\mathbb{X}$. However, our considerations in this section pertain to more abstract setting. Perhaps the reader has already observed that we have reserved capital script letters for all types of function spaces, with few exceptions. Thus, let $(\mathbb{X}, dx)$ be a finite measure space and $0 < p < \infty$. The Lebesgue $\mathscr{L}^p$-space, denoted by $\mathscr{L}^p(\mathbb{X})$, is a complete metric space. The metric is induced by the non-linear functional

$$\| F \|_p = \| F \|_{\mathscr{L}^p(\mathbb{X})} = \left(\int_{\mathbb{X}} |F(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \tag{108}$$

see Section 4.3 and formula (136).

Before making generalizations we single out the weak-L^p space, which is also known as Marcinkiewicz space.

4.1 Marcinkiewicz space $\mathscr{L}^p_{\mathbf{weak}}(\mathbb{X})$

This space consists of functions satisfying

$$[F]_p \stackrel{\text{def}}{=\!=} \sup_{t \geqslant 0} \left(t^p \int_{|F|>t} dx \right)^{\frac{1}{p}} < \infty \tag{109}$$

Clearly, we have

$$[F]_p \leqslant \| F \|_p, \qquad \text{hence} \quad \mathscr{L}^p(\mathbb{X}) \subset \mathscr{L}^p_{\text{weak}}(\mathbb{X}) \tag{110}$$

It is evident that $[\ ]_p$ is not a norm, and $\|\ \|_p$ is also not a norm in $\mathscr{L}^p(\mathbb{X})$ when $0 < p < 1$. For every $0 \leqslant \alpha < p$, Fubini's theorem yields

$$\begin{aligned} t^p \int_{|F|>t} dx &\leqslant t^{p-\alpha} \int_{|F|>t} |F(x)|^\alpha \, dx \\ &= t^p \int_{|F|>t} dx + t^{p-\alpha} \int_{|F|>t} (|F(x)|^\alpha - t^\alpha) \, dx \\ &\leqslant [F]_p^p + t^{p-\alpha} \int_{|F|>t} \left(\int_t^{|F(x)|} \alpha \tau^{\alpha-1} \right) d\tau \, dx \\ &= [F]_p^p + t^{p-\alpha} \int_t^\infty \alpha \tau^{\alpha-p-1} \left(\tau^p \int_{|F|>\tau} dx \right) d\tau \\ &\leqslant \frac{p}{p-\alpha} [F]_p^p \end{aligned} \tag{111}$$

Taking the supremum over $t \geqslant 0$, we obtain

$$[F]_p \leqslant [F]_{\alpha,p} \stackrel{\text{def}}{=\!=} \sup_{t \geqslant 0} \left(t^{p-\alpha} \int_{|F|>t} |F|^\alpha \right)^{\frac{1}{p}} \leqslant \sqrt[p]{\tfrac{p}{p-\alpha}}\, [F]_p \tag{112}$$

In other words, $\mathscr{L}^p_{\text{weak}}(\mathbb{X})$ is characterized by the inequality $[F]_{\alpha,p} < \infty$ for some or, equivalently, for all $0 \leqslant \alpha < p$. The Lebesgue space $\mathscr{L}^p(\mathbb{X})$ corresponds to $\alpha = p$, in which $\| F \|_p = [F]_{p,p}$. It follows from the above estimates that $\mathscr{L}^p_{\text{weak}}(\mathbb{X}) \subset \mathscr{L}^\alpha(\mathbb{X})$, for every $0 < \alpha < p$. As a mater of fact, we have the estimate

$$\| F \|_\alpha \preccurlyeq |\mathbb{X}|^{\frac{1}{\alpha} - \frac{1}{p}} [F]_p \tag{113}$$

To this end consider the inequalities

$$\begin{aligned} \int_{\mathbb{X}} |F|^\alpha &\leqslant \int_{|F| \leqslant t} |F(x)|^\alpha \, dx + \int_{|F|>t} |F(x)|^\alpha \, dx \\ &\leqslant t^\alpha |\mathbb{X}| + \frac{p}{p-\alpha} t^{\alpha-p} [F]_p^p \end{aligned} \tag{114}$$

by (112). Now the estimate (113) follows if we take $t = |\mathbb{X}|^{-1/p}[F]_p$.

We shall now place the Marcinkiewicz class $\mathscr{L}^p_{\text{weak}}(\mathbb{X})$ in a family $\mathscr{L}^{\alpha,p}(\mathbb{X})$ of the so-called very weak Lebesgue spaces, see [27], [14].

4.2 The space $\mathscr{L}^{\alpha,p}(\mathbb{X})$

Given $0 \leqslant \alpha < p$, the space $\mathscr{L}^{\alpha,p}(\mathbb{X})$ consists of measurable functions $F = F(x)$ such that

$$\{F\}_{\alpha,p} \stackrel{\text{def}}{=\!=} \liminf_{t\to\infty} \left(t^{p-\alpha} \int_{|F|>t} |F(x)|^\alpha \, dx \right)^{\frac{1}{p}} < \infty \tag{115}$$

One should be a little cautious because $\mathscr{L}^{\alpha,p}(\mathbb{X})$ is not a linear space, though $F \in \mathscr{L}^{\alpha,p}(\mathbb{X})$ implies $\lambda F \in \mathscr{L}^{\alpha,p}(\mathbb{X})$ for every $\lambda \in \mathbb{R}$. Precisely, $\{\lambda F\}_{\alpha,p} = |\lambda| \, \{F\}_{\alpha,p}$. It is clear that $\mathscr{L}^p(\mathbb{X}) \subset \mathscr{L}^{\alpha,p}(\mathbb{X})$ for every $0 < \alpha < p$. On the other hand,

$$\mathscr{L}^{\alpha,p}(\mathbb{X}) \not\subseteq \bigcup_{s>\alpha} \mathscr{L}^s(\mathbb{X}) \tag{116}$$

see Section 4.5. For $0 \leqslant \alpha \leqslant \beta < p$ we have the following chain of inclusions

$$\mathscr{L}^p_{\text{weak}}(\mathbb{X}) \subset \mathscr{L}^{\beta,p}(\mathbb{X}) \subset \mathscr{L}^{\alpha,p}(\mathbb{X}) \subset \mathscr{L}^p_{\text{Weak}}(\mathbb{X}) \tag{117}$$

This latter new space $\mathscr{L}^p_{\text{Weak}}(\mathbb{X}) = \mathscr{L}^{0,p}(\mathbb{X})$ consists of functions satisfying

$$\liminf_{t\to\infty} \left(t^p \int_{|F|>t} dx \right)^{\frac{1}{p}} = \{F\}_{0,p} < \infty \tag{118}$$

The nuance is that we have replaced sup in the definition of $\mathscr{L}^p_{\text{weak}}(\mathbb{X})$ by lim inf. Finally, we introduce the subclass $\mathscr{L}^{\alpha,p}_\circ(\mathbb{X}) \subset \mathscr{L}^{\alpha,p}(\mathbb{X})$, $0 \leqslant \alpha < p$, by requiring that $\{F\}_{\alpha,p} = 0$.

4.2.1 Spherical averages

Let $n - 1 < \alpha < n$ and $F \in \mathscr{L}^{\alpha,n}_\circ(\mathbb{X})$. For a given point $x \in \mathbb{X}$, we shall look closely at the expressions

$$F_x(r) = r \left(\fint_{\mathbb{S}(x,r)} |F(y)|^\alpha \, dy \right)^{\frac{1}{\alpha}} \tag{119}$$

as functions defined for almost every $r \in (0, R]$, where R will be a small number. As usually, $\mathbb{S}(x,r) = \partial\mathbb{B}(x,r)$ denotes the geodesic sphere in $\mathbb{X}$

centered at x and with radius r. When applied to $F = |Df|$ these integral expressions represent average stretchings of the deformation $f : \mathbb{X} \to \mathbb{Y}$. One important feature of the space $\mathscr{L}_\circ^{\alpha,n}(\mathbb{X})$ is that $F_x(r)$ takes arbitrarily small values as $r \to 0$. Precise statement reads as:

PROPOSITION **4.1.** *For every $\epsilon > 0$ the set of radii $r \in (0, R]$ such that $F_x(r) \leqslant \epsilon$ has positive linear measure.*

Proof. Recall that $F \in \mathscr{L}_\circ^{\alpha,n}(\mathbb{X})$ has the property

$$\liminf_{t\to\infty} t^{n-\alpha} \int_{|F|>t} |F(x)|^\alpha \, dx = 0 \tag{120}$$

Assume, to the contrary, that there exists $\epsilon > 0$ such that

$$F_x(r) = r \left(\fint_{\mathbb{S}(x,r)} |F(y)|^\alpha \, dy \right)^{\frac{1}{\alpha}} > \epsilon \tag{121}$$

for almost every $r \in (0, R]$. For simplicity of the exposition we assume that $\mathbb{S}(x,r)$ are spheres in $\mathbb{R}^n$ and $x = 0$. The general case reduces to this Euclidean one by using the normal coordinates. In this coordinate system small geodesic spheres centered at x become the Euclidean spheres centered at 0, see [34, definition 1.4.4.]. Inequality (121) translates into the following estimate

$$\begin{aligned} \frac{\omega_{n-1}\, \rho^{n-\alpha}}{n-\alpha} &= \omega_{n-1} \int_0^\rho \frac{dr}{r^{\alpha-n+1}} < \frac{1}{\epsilon^\alpha} \int_0^\rho \left(\int_{|x|=r} |F(x)|^\alpha \, dx \right) dr \\ &= \frac{1}{\epsilon^\alpha} \int_{|x|\leqslant\rho} |F(x)|^\alpha \, dx \end{aligned} \tag{122}$$

for every $0 < \rho < R$. We split $|F|$ into two parts, say $|F| = F_1 + F_2$

$$F_1(x) = \begin{cases} |F(x)| & \text{if } \; |F(x)| \leqslant \frac{\epsilon}{2\,|x|} \\ 0 & \text{if } \; |F(x)| > \frac{\epsilon}{2\,|x|} \end{cases} \tag{123}$$

$$F_2(x) = \begin{cases} 0 & \text{if } \; |F(x)| \leqslant \frac{\epsilon}{2\,|x|} \\ |F(x)| & \text{if } \; |F(x)| > \frac{\epsilon}{2\,|x|} \end{cases} \tag{124}$$

To simplify typing we introduce the parameter $t = \frac{\epsilon}{2\rho}$ and proceed as follows:

$$\begin{aligned}
\frac{\omega_{n-1}\,\rho^{n-\alpha}}{n-\alpha} &< \frac{1}{\epsilon^\alpha}\int_{|x|\leqslant\rho} |F(x)|^\alpha\,dx \\
&= \frac{1}{\epsilon^\alpha}\int_{|x|\leqslant\rho} |F_1(x)|^\alpha\,dx + \frac{1}{\epsilon^\alpha}\int_{|x|\leqslant\rho} |F_2(x)|^\alpha\,dx \\
&\leqslant \frac{1}{2^\alpha}\int_{|x|\leqslant\rho} \frac{dx}{|x|^\alpha} + \frac{1}{\epsilon^\alpha}\int_{|F(x)|>t} |F(x)|^\alpha\,dx \\
&= \frac{\omega_{n-1}\,\rho^{n-\alpha}}{2^\alpha(n-\alpha)} + \frac{(2\,\rho)^{n-\alpha}}{\epsilon^n} t^{n-\alpha}\int_{|F(x)|>t} |F(x)|^\alpha\,dx
\end{aligned}$$

The first term is absorbed by the left hand side, so we arrive at the estimate

$$\frac{(2^\alpha-1)\omega_{n-1}\,\epsilon^n}{2^n(n-\alpha)} \leqslant t^{n-\alpha}\int_{|F|>t} |F(x)|^\alpha\,dx \tag{125}$$

This estimate is in contradiction with (120) once we let $t = \frac{\epsilon}{2\rho}$ go to infinity.

4.2.2 Special sequences

By the definition, every $F \in \mathscr{L}^{\alpha,p}(\mathbb{X})$ admits a sequence $\{t_i\}_{i=1,2,\dots}$ of positive numbers increasing to infinity such that

$$\sup_{i\geqslant 1}\left(t_i^{p-\alpha}\int_{|F|>t_i} |F(x)|^\alpha\,dx\right)^{\frac{1}{p}} < \infty \tag{126}$$

If $F \in \mathscr{L}^p_{\mathrm{weak}}(\mathbb{X})$ then every sequence $\{t_i\}_{i=1,2,\dots}$ has this property. Now, a sequence of positive numbers $\{t_i\}_{i=1,2,\dots}$ increasing to infinity will be referred to as *special sequence* for F if

$$\lim_{i\to\infty}\left(t_i^{p-\alpha}\int_{|F|>t_i} |F(x)|^\alpha\,dx\right)^{\frac{1}{p}} = 0 \tag{127}$$

Thus the notation $F \in \mathscr{L}^{\alpha,p}_\circ(\mathbb{X})$ simply means that F admits a special sequence. It is not difficult to see that if $F, G \in \mathscr{L}^{\alpha,p}_\circ(\mathbb{X})$ have common special sequence $\{t_i\}$ then the nontrivial linear combination $H = \lambda F + \mu G$

also lies in $\mathscr{L}_\circ^{\alpha,p}(\mathbb{X})$. Indeed, special sequence for H consists of the number $\tau_i = (|\lambda| + |\mu|)t_i$, by the following computation:

$$\left(\tau_i^{p-\alpha}\int_{|H|>\tau_i}|H(x)|^\alpha\,dx\right)^{\frac{1}{p}} \leqslant$$

$$(|\mu|+|\lambda|)\left[\left(t_i^{p-\alpha}\int_{|F|>t_i}|F(x)|^\alpha\,dx\right)^{\frac{1}{p}} + \left(t_i^{p-\alpha}\int_{|G|>t_i}|G(x)|^\alpha\,dx\right)^{\frac{1}{p}}\right]$$

However, the linear structure is lost in $\mathscr{L}_\circ^{\alpha,p}(\mathbb{X})$ because two different functions may not have a common special sequence.

The reader may wish to recall the function $F(x) = |Df(x)|$ defined at (88). It belongs to the Marcinkiewicz class $\mathscr{L}^n_{\text{weak}}(\mathbb{X})$, but fails to satisfy the condition $\{F\}_{0,n} = 0$. In Section 8.2 we shall make use of special sequences to introduce the concept of the so-called weak integrals. It will not matter which special sequence we choose, they all yield the same weak integral. It is therefore natural and important to know in which of the function spaces we always have special sequences. Let us begin with the Orlicz classes.

4.3 The Orlicz space $\mathscr{L}^P(\mathbb{X})$

In this section we briefly review basic concepts of the theory of Orlicz spaces. Naturally, it also gives us an opportunity to discuss the notation used in this text.

Definition **4.1.** The term *Orlicz function* pertains to any infinitely differentiable function $P : \mathbb{R}_+ \to \mathbb{R}_+$ which is strictly increasing and satisfies

$$P(0) \stackrel{\text{def}}{=\!=} \lim_{t\to 0} P(t) = 0 \tag{128}$$

$$P(\infty) \stackrel{\text{def}}{=\!=} \lim_{t\to \infty} P(t) = \infty \tag{129}$$

Definition **4.2.** *The Orlicz space* is a collection of measurable functions $u : \mathbb{X} \to \mathbf{V}$, such that

$$\int_{\mathbb{X}} P\left(\tfrac{|u(x)|}{k}\right)\,dx < \infty \tag{130}$$

for some positive $k = k(u)$. This space will be denoted by $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$.

Here and in what follows $\mathbf{V}$ is a finite dimensional normed space. If $\mathbf{V} = \mathbb{R}$ we simply write $\mathscr{L}^P(\mathbb{X})$. Note explicitly that the usual convexity of P will not always be required in this paper. Also, if we fix a basic for $\mathbf{V}$ then $u \in \mathscr{L}^P(\mathbb{X}, \mathbf{V})$ if and only if its coordinate functions belong to $\mathscr{L}^P(\mathbb{X})$. It is easy to see that $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$ is a linear space. Clearly $\mathscr{L}^p(\mathbb{X}, \mathbf{V})$ is the Orlicz space for $P(t) = t^p$. Of special importance to us will be the Orlicz spaces $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$ which are slightly larger than $\mathscr{L}^p(\mathbb{X})$, $1 \leqslant p < \infty$. More precisely, our standing assumption upon P will be the so-called *divergence condition*

$$\int_1^\infty \frac{P(t)}{t^{p+1}}\, dt = \infty, \qquad \text{for instance } P(t) = \frac{t^p}{\log(e+t)} \tag{131}$$

Actually, p will be the dimension of $\mathbb{X}$. We shall see in Section 4.5 that

Under the divergence condition at (131) we have the inclusion

$$\mathscr{L}^P(\mathbb{X}) \subset \mathscr{L}_\circ^{\alpha,p}(\mathbb{X}) \quad \textit{for all} \ \ 0 \leqslant \alpha < p \tag{132}$$

The Orlicz space is equipped with the *Luxemburg functional* (no triangle inequality) defined by

$$\| u \|_P = \inf \left\{ k > 0; \ \int_{\mathbb{X}} P\left(\tfrac{|u(x)|}{k}\right)\, dx \leqslant 1 \right\} \tag{133}$$

Of course, if a function u vanishes almost everywhere in $\mathbb{X}$ then $\| u \|_P = 0$. Otherwise, the infimum in (133) is attained at exactly one value $k = \| u \|_P > 0$. As a note of warning, it can happen that

$$\int_{\mathbb{X}} P\left(\tfrac{|u(x)|}{\| u \|_P}\right)\, dx < 1 \tag{134}$$

To see this take $P(t) = e^t - 1$, $\mathbb{X} = (0,1]$ and $u(x) = -\log(x + x\,\log^2 x) \geqslant 0$. Indeed, elementary computation reveals that

$$\int_0^1 \left(e^{|u(x)|} - 1\right) dx = \frac{\pi}{2} - 1 < 1\,, \quad \text{whereas } \| u \|_P = 1$$

because

$$\int_0^1 \left(e^{\frac{|u(x)|}{k}} - 1\right) dx = \infty \quad \text{for all } k < 1$$

On the other hand for $u \not\equiv 0$ we have the equation

$$\int_{\mathbb{X}} P\left(\frac{|u(x)|}{\|u\|_P}\right) dx = 1\ , \qquad \text{provided } \|u\|_P > 0$$

whenever the defining Orlicz function satisfies a *doubling condition.* This simply means that

$$P(2t) \leqslant \Bbbk\, P(t) \tag{135}$$

for some $\Bbbk > 1$ and all $t \geqslant 0$. We call $\Bbbk$ the *doubling constant.*

Recall that $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$ is a complete linear metric space in which the distance between u and v is measured by

$$\operatorname{dist}_P[u,v] = \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})}[u,v] \stackrel{\text{def}}{=\!=} \inf\left\{\rho > 0;\ \int_{\mathbb{X}} P\left(\frac{|u(x)-v(x)|}{\rho}\right) dx \leqslant \rho\right\} \tag{136}$$

Clearly if $\operatorname{dist}_P[u,v] > 0$ then

$$\int_{\mathbb{X}} P\left(\frac{|u(x)-v(x)|}{\rho_0}\right) dx \leqslant \rho_0 \quad \text{for } \rho_0 = \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})}[u,v] \tag{137}$$

with the possibility of equality to occur sometimes. However, if P satisfies the doubling condition, then the equality always holds.

The triangle inequality $\operatorname{dist}_P[u,v] \leqslant \operatorname{dist}_P[u,w] + \operatorname{dist}_P[w,v]$ follows directly from the elementary inequality

$$\frac{|a+b|}{\rho_1+\rho_2} \leqslant \max\left\{\frac{|a|}{\rho_1}, \frac{|b|}{\rho_2}\right\}, \quad \rho_1, \rho_2 > 0$$

when we apply it to $a = u - w$ and $b = w - v$. The arguments establishing completeness of the space $\mathscr{L}^P(\mathbb{X})$ are much the same as in the case of the space $\mathscr{L}^p(\mathbb{X})$, with $p \geqslant 1$.

Taking $P(t) = t^p$, where $p > 0$, we recover the well know fact that $\mathscr{L}^p(\mathbb{X}, \mathbf{V})$ is a complete linear metric space with respect to the distance

$$\operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})}[u,v] = \|u-v\|_p^{p/(p+1)}$$

However, for $p \geqslant 1$ the exponent $p/(p+1)$ is unnecessary in order to make a metric in $\mathscr{L}^p(\mathbb{X})$. For $p \geqslant 1$ the usual distance function $\| u - v \|_p$ has many advantages such as homogeneity. Unfortunately, if $0 < p < 1$, this nice expression $\| u - v \|_p$ fails to satisfy the triangle inequality; $\mathscr{L}^p(\mathbb{X})$ is not a Banach space.

If a sequence u_j converges to u in $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$ then also $\lim\limits_{j\to\infty} \| u_j - u \|_{_P} = 0$. This follows from the inequality

$$\| u - v \|_{_P} \leqslant \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})} [u, v] \,, \qquad \text{provided } \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})} [u, v] \leqslant 1 \tag{138}$$

Another useful observation is that if functions $u_j \in \mathscr{L}^P(\mathbb{X}, \mathbf{V})$, $j = 1, 2, ...$ are supported on a common set of finite measure and converge uniformly to u then $u \in \mathscr{L}^P(\mathbb{X}, \mathbf{V})$. Moreover,

$$\lim_{j\to\infty} \| u_j - u \|_{_P} \leqslant \lim_{j\to\infty} \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})} [u_j, u] = 0 \tag{139}$$

Proposition **4.2.** *Let $\mathbb{X}$ be a finite measure space. The closure of bounded functions in $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$ is a linear subspace given by*

$$\mathscr{L}^P_\infty(\mathbb{X}, \mathbf{V}) = \left\{ u; \quad \int_{\mathbb{X}} P\left(\tfrac{|u(x)|}{k}\right) \, dx < \infty, \qquad \textit{for every } k > 0 \right\} \tag{140}$$

In fact we have even more precise result. Given $u \in \mathscr{L}^P(\mathbb{X}, \mathbf{V})$, where $\mathbb{X}$ is a finite measure space, its distance to $\mathscr{L}^\infty(\mathbb{X}, \mathbf{V}) \subset \mathscr{L}^P(\mathbb{X}, \mathbf{V})$ can be computed by the rule

$$\operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})} [u, \mathscr{L}^\infty] = \inf \left\{ k > 0; \quad \int_{\mathbb{X}} P\left(\tfrac{|u(x)|}{k}\right) \, dx < \infty \right\} \tag{141}$$

If the space $\mathbb{X}$ enjoys some differentiable structure we find the following corollary.

Corollary **4.1.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ (or any Riemannian n-manifold). Then the closure of $\mathscr{C}^\infty(\Omega, \mathbf{V}) \cap \mathscr{L}^\infty(\Omega, \mathbf{V})$ equals $\mathscr{L}^P_\infty(\Omega, \mathbf{V})$.*

It is to be noted that, in general, bounded functions are not dense in $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$. An example that illustrates this possibility is furnished by $P(t) = e^t - 1$, which defines the so-called *exponential class*

$$\operatorname{Exp}\mathscr{L}(\mathbb{X}, \mathbf{V}) = \left\{ u;\ \int_{\mathbb{X}} e^{\frac{|u(x)|}{k}}\, dx < \infty, \quad \text{for some } k = k(u) > 0 \right\} \tag{142}$$

Fortunately for our analysis, $\mathscr{L}^P_\infty(\mathbb{X}, \mathbf{V}) = \mathscr{L}^P(\mathbb{X}, \mathbf{V})$ whenever the defining Orlicz function satisfies a doubling condition.

We refer to an Orlicz function P as *Young function* if its second derivative is non-negative, i.e. P is convex. In this case $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$ is a Banach space and $\| \ \|_P$ satisfies also the triangle inequality. That is why we shall use the term Luxemburg norm for $\| \ \|_P$, in the convex case. It compares with the distance function rather nicely:

$$\operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})}[u, v] \leqslant \sqrt{\| u - v \|_P}, \text{ provided } \| u - v \|_P \leqslant 1 \tag{143}$$

This follows from the inequality $P(\epsilon t) \leqslant \epsilon P(t)$ for all $0 \leqslant \epsilon \leqslant 1$ and $t \geqslant 0$; a simple consequence of convexity. Inequalities (138) and (143) imply that

$$\lim_{j \to 0} \| u_j - u \|_P = 0 \quad \text{iff} \quad \lim_{j \to 0} \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})}[u_j, u] = 0 \tag{144}$$

Note that even in this convex case the density of bounded functions (Proposition 4.2) in $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$ still requires the doubling condition (135) as the example of exponential class demonstrates.

In many situations, when we speak of the space $\mathscr{L}^P(\mathbb{X}, \mathbf{V})$, we do not need to explicitly specify the defining function $P = P(t)$; only its behavior for large values of t will be significant to us. To effectively handle this case we make the following definition. Given $\Phi \in C[0, \infty)$ (not necessarily increasing), an Orlicz function $P = P(t)$ is said to be equivalent to Φ if there exists $\lambda > 1$ such that

$$\tfrac{1}{\lambda} P\left(\tfrac{t}{\lambda}\right) \leqslant \Phi(t) \leqslant \lambda P(\lambda t), \qquad \text{for all } t \geqslant 0 \tag{145}$$

We write it as $P \approx \Phi$. Note we do not claim here that a given $\Phi \in C[0, \infty)$ admits an equivalent Orlicz function. If two Orlicz functions P and Q are equivalent to each other, with the parameter λ, then

$$\frac{1}{\lambda} \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})}[u,v] \leqslant \operatorname*{dist}_{\mathscr{L}^Q(\mathbb{X})}[u,v] \leqslant \lambda \operatorname*{dist}_{\mathscr{L}^P(\mathbb{X})}[u,v] \tag{146}$$

Hence $\mathscr{L}^P(\mathbb{X}, \mathbf{V}) = \mathscr{L}^Q(\mathbb{X}, \mathbf{V})$, as metric spaces. In particular, two Orlicz functions equivalent to a given Φ yield the same metric space. When $\mathbb{X}$ has finite measure it will suffice to assume that (145) holds for only large values of t, in symbols $P \sim \Phi$. Here Φ need not be even defined for all t. In this finite measure case we still have $\mathscr{L}^P(\mathbb{X}, \mathbf{V}) = \mathscr{L}^Q(\mathbb{X}, \mathbf{V})$, whenever $P \sim \Phi$ and $Q \sim \Phi$.

Some Orlicz spaces play special role in the theory of Jacobians. The Zygmund spaces already have standard notation which we want to recall here:

- $\mathscr{L} \log \mathscr{L}(\mathbb{X}) = \mathscr{L}^P(\mathbb{X}), \quad P = t \log(e+t)$
- $\mathscr{L} \log^{-1} \mathscr{L}(\mathbb{X}) = \mathscr{L}^P(\mathbb{X}), \quad P = \frac{t}{\log(e+t)}$
- $\mathscr{L}^p \log^\alpha \mathscr{L}(\mathbb{X}) = \mathscr{L}^P(\mathbb{X}), \quad P \sim t^p \log^\alpha(e+t),\ 0 < p < \infty$ and $\alpha \in \mathbb{R}$
- $\mathscr{L}^p \log\log \mathscr{L}(\mathbb{X}) = \mathscr{L}^P(\mathbb{X}), \quad P \sim t^p \log\log(e^e + t),\ 0 < p < \infty$

4.4 Grand G$\mathscr{L}^p$-space

Let $1 < p < \infty$, we consider functions $F \in \bigcap_{1 \leqslant s < p} \mathscr{L}^s(\mathbb{X})$ furnished with the norm

$$\| F \|_{p)} = \sup_{0 < \epsilon \leqslant p-1} \left(\epsilon \int_{\mathbb{X}} |F(x)|^{p-\epsilon}\, dx \right)^{\frac{1}{p-\epsilon}} < \infty \tag{147}$$

This gives us a Banach space, denoted by G$\mathscr{L}^p(\mathbb{X})$, which is even larger than the Marcinkiewicz class

$$\mathscr{L}^p(\mathbb{X}) \subset \mathscr{L}^p_{\text{weak}}(\mathbb{X}) \subset \mathrm{G}\mathscr{L}^p(\mathbb{X}) \tag{148}$$

It is important to realize that $\mathscr{L}^p(\mathbb{X})$ is not dense in G$\mathscr{L}^p(\mathbb{X})$. Its closure consists of functions having "vanishing p-modulus", see [31], [27]

$$\lim_{\epsilon \to 0} \epsilon \int_{\mathbb{X}} |F(x)|^{p-\epsilon}\, dx = 0 \tag{149}$$

We denote this space by $\mathrm{V}\mathscr{L}^p(\mathbb{X})$. As before, the function $F(x) = |DF(x)|$ defined at (88) lies in $\mathrm{G}\mathscr{L}^n(\mathbb{X})$ but not in $\mathrm{V}\mathscr{L}^n(\mathbb{X})$; a cause for the lack of smooth approximation later on. Let us record two more inclusions (see the next section)

$$\mathrm{G}\mathscr{L}^p(\mathbb{X}) \quad \subset \quad \mathscr{L}^{\alpha,p}(\mathbb{X}) \tag{150}$$

$$\mathrm{V}\mathscr{L}^p(\mathbb{X}) \quad \subset \quad \mathscr{L}_\circ^{\alpha,p}(\mathbb{X}) \tag{151}$$

for every $0 \leqslant \alpha < p$.

4.5 Relations between spaces

In this section we will show that

$$\mathrm{V}\mathscr{L}^p(\mathbb{X}) \subset \bigcup_P \mathscr{L}^P(\mathbb{X}) = \mathscr{L}_\circ^{\alpha,p}(\mathbb{X}) \quad 0 < \alpha < p \tag{152}$$

where the union runs over all Orlicz functions P satisfying

- divergence condition
$$\int_1^\infty \frac{P(t)}{t^{p+1}}\, dt = \infty \tag{153}$$

- growth condition

$$[t^{-\alpha} P(t)]' \geqslant 0, \quad \text{for large values of } t \tag{154}$$

We would like to remind the reader that the divergence condition (153) alone is too weak to guarantee that the functions in $\mathscr{L}^P(\mathbb{X})$ belong to $\mathscr{L}^\alpha(\mathbb{X})$. To fill up this gap we have to impose the additional condition (154).

It is well known that $\mathscr{L}^p_{\text{weak}}(\mathbb{X}) \subset \bigcap_{s<p} \mathscr{L}^s(\mathbb{X})$. Here we shall demonstrate that the space $\mathscr{L}_\circ^{\alpha,p}(\mathbb{X})$ is not contained in $\bigcap_{s<p} \mathscr{L}^s(\mathbb{X})$. Even more, we will construct a function which lies in $\mathscr{L}^{\alpha,p}(\mathbb{X})$ but not in $\mathscr{L}^s(\mathbb{X})$, for any $s > \alpha$. Of course this example also will show that the inclusion $\mathscr{L}^{\beta,p}(\mathbb{X}) \subset \mathscr{L}^{\alpha,p}(\mathbb{X})$, where $\alpha < \beta$, is proper, because $\mathscr{L}^{\alpha,p}(\mathbb{X}) \subset \mathscr{L}^\alpha(\mathbb{X})$.

Let us notice that

$$\mathscr{L}^{\alpha,p}_{\circ}(\mathbb{X}) = \Big\{F \in \mathscr{L}^{\alpha}(\mathbb{X})\colon \inf_{t\to\infty} t^{p-\alpha} \int_{|F|>t} |F(x)|^{\alpha}\, dx = 0\Big\} \tag{155}$$

as this infimum, if vanishes, coincides with lim inf.

Proposition **4.3.** *Suppose that $u \in \mathrm{V}\mathscr{L}^p(\mathbb{X})$. Then $u \in \mathscr{L}^{\alpha,p}_{\circ}(\mathbb{X})$ for all $0 < \alpha < p$.*

Proof. Let $\epsilon > 0$ and consider the function $\Psi_\epsilon(t) = t^{-\epsilon-1}$, for all $t > 0$. First we observe that

$$\epsilon \int_1^{\infty} \Psi_\epsilon(t)\, dt = 1 \tag{156}$$

Next we perform the following computation, by using Fubini's Theorem:

$$\begin{aligned} \inf_{t\geqslant 1}\ t^{p-\alpha}\int_{|u|>t} |u(x)|^{\alpha}\, dx &\leqslant \epsilon \int_1^{\infty} \Psi_\epsilon(t)\, t^{p-\alpha} \int_{|u|>t} |u(x)|^{\alpha}\, dx\, dt \\ &= \epsilon \int_{|u|>1} |u(x)|^{\alpha} \int_1^{|u(x)|} t^{p-\alpha-\epsilon-1}\, dt\, dx \\ &\leqslant \frac{\epsilon}{p-\alpha-\epsilon} \int_{\mathbb{X}} |u(x)|^{p-\epsilon}\, dx \end{aligned} \tag{157}$$

Letting ϵ go to zero, the claim follows by (155). □

Proposition **4.4.** *Fix $0 < \alpha < p$. Suppose that P is an Orlicz-function satisfying* (153) *and* (154). *Let $u : \mathbb{X} \to \mathbb{R}$ be a measurable function such that*

$$\int_{\mathbb{X}} P(|u(x)|)\, dx < \infty \tag{158}$$

Then

$$u \in \mathscr{L}^{\alpha,p}_{\circ}(\mathbb{X}) \tag{159}$$

Proof. First we observe that the condition (154) implies $u \in \mathscr{L}^{\alpha}(\mathbb{X})$. Consider the non-negative function $\Psi = t^{\alpha-p}[t^{-\alpha}P(t)]'$, with large values of t;

say $t \geqslant A$. Then

$$\begin{aligned}\int_A^\infty \Psi(t)\,dt &= \left.\frac{P(t)}{t^p}\right|_A^\infty + (p-\alpha)\int_A^\infty \frac{P(t)}{t^{p+1}}\,dt \\ &\geqslant -\frac{P(A)}{A^p} + (p-\alpha)\int_A^\infty \frac{P(t)}{t^{p+1}}\,dt = \infty \qquad (160)\end{aligned}$$

Next we pick up $T > A$, and compute by using Fubini's Theorem:

$$\begin{aligned}\left(\int_A^T \Psi(t)\,dt\right)\inf_{t\geqslant A} t^{p-\alpha}\int_{|u|>t}|u(x)|^\alpha\,dx &\leqslant \int_A^T \Psi(t)\,t^{p-\alpha}\left(\int_{|u|>t}|u|^\alpha\right)\,dt \\ &\leqslant \int_{|u|>A}|u|^\alpha\int_A^{|u|}\Psi(t)t^{p-\alpha}\,dt \\ &\leqslant \int_{|u|>A}P(|u|) \leqslant \int_{\mathbb{X}}P(|u|) \quad (161)\end{aligned}$$

Letting T tend to infinity, the claim follows from (158), (160) and (155). $\square$

Proposition 4.5. *Suppose that* $u \in \mathscr{L}_\circ^{\alpha,p}(\mathbb{X})$. *Then there exists an Orlicz function* P *satisfying conditions* (153) *and* (154) *such that*

$$u \in \mathscr{L}^P(\mathbb{X})$$

Proof. We find a special sequence $\{t_k\}$ such that

$$1 < t_1 < t_1 + 1 < t_2 < t_2 + 1 < t_3 < \dots$$

and

$$t_k^{p-\alpha}\int_{|u|>t_k}|u(x)|^\alpha\,dx \leq 2^{-k}$$

For each $k = 1, 2, \dots$ we find a smooth nonnegative bump function η_k on $(0,\infty)$ with support in (t_k, t_k+1), such that

$$\int_{t_k}^{t_k+1}\eta_k(s)\,ds = (p-\alpha)(t_k+1)^{p-\alpha}$$

We set

$$\eta = \sum_{k=1}^\infty \eta_k$$

and

$$P(t) = t^\alpha \int_0^t \eta(s)\,ds, \qquad t \in [0,\infty)$$

Then

$$\begin{aligned}
\int_0^\infty \frac{P(t)}{t^{p+1}}\,dt &= \int_0^\infty t^{\alpha-p-1}\frac{P(t)}{t^\alpha}\,dt = \int_0^\infty t^{\alpha-p-1}\Big(\int_0^t \eta(s)\,ds\Big)\,dt \\
&= \int_0^\infty \Big(\int_s^\infty t^{\alpha-p-1}\eta(s)\,dt\Big)\,ds = (p-\alpha)^{-1}\int_0^\infty s^{\alpha-p}\eta(s)\,ds \\
&= (p-\alpha)^{-1}\sum_{k=1}^\infty \int_{t_k}^{t_k+1} s^{\alpha-p}\eta_k(s)\,ds \geq \sum_{k=1}^\infty 1 = \infty
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{X}} P(|u(x)|)\,dx &= \int_{\mathbb{X}} |u(x)|^\alpha \Big(\int_0^{|u(x)|} \eta(s)\,ds\Big)\,dx \\
&= \int_0^\infty \eta(s)\Big(\int_{|u|>s} |u(x)|^\alpha\,dx\Big)\,ds \\
&= \sum_{k=1}^\infty \int_{t_k}^{t_k+1} \eta_k(s)\Big(\int_{|u|>s} |u(x)|^\alpha\,dx\Big)\,ds \\
&\leq \sum_{k=1}^\infty \Big(\int_{|u|>t_k} |u(x)|^\alpha\,dx\Big)\int_{t_k}^{t_k+1} \eta_k(s)\,ds \\
&\leq C\sum_{k=1}^\infty t_k^{p-\alpha}\int_{|u|>t_k} |u(x)|^\alpha\,dx \\
&\leq C2^{-k}
\end{aligned}$$

which shows that $u \in \mathscr{L}^P(\mathbb{X})$. □

EXAMPLE **4.1.** We prove strictness of the inclusion in (152) and simultaneously validate the claim (116). There exists a function $F : \mathbb{I} = [0,1] \to [0,\infty)$ in the space $\mathscr{L}^{\alpha,p}_\circ(\mathbb{I})$, so that $F \notin \bigcup_{s>\alpha}\mathscr{L}^s(\mathbb{I})$. It then follows that $F \notin \mathrm{V}\mathscr{L}^p(\mathbb{I})$.

Construction. We define $I_1 = 0,\ a_1 = 0,\ t_1 = 1$

$$
\begin{aligned}
t_k &= 2^{k^k} \\
I_k &= t_k^{-\alpha} t_{k-1}^{\alpha-p} 2^{-k} \qquad\qquad k = 2, 3, \dots, \\
a_k &= a_{k-1} + I_k
\end{aligned}
$$

and denote

$$\mathbb{I}_k = [a_{k-1}, a_k), \qquad k = 2, 3, \dots$$

so that

$$I_k = a_k - a_{k-1} = |\mathbb{I}_k|$$

Then we set

$$F(x) = \sum_{k=2}^{\infty} t_k \chi_{\mathbb{I}_k}(x) \qquad x \in \mathbb{I} = [0, 1].$$

Notice that

$$I_k \le 2^{-k}$$

and thus $\bigcup_k \mathbb{I}_k \subset \mathbb{I}$. First we show that F belongs to $\mathscr{L}_\circ^{\alpha,p}(\mathbb{I})$. The sequence $\{t_k\}_{k=1}^{\infty}$ is special. Indeed, for $i > k \ge 2$

$$t_k^{p-\alpha} I_i t_i^{\alpha} = t_k^{p-\alpha} t_{i-1}^{\alpha-p} 2^{-i} \le 2^{-i}$$

Hence

$$t_k^{p-\alpha} \int_{F > t_k} |F(x)|^{\alpha}\, dx = t_k^{p-\alpha} \sum_{i=k+1}^{\infty} I_i\, t_i^{\alpha} \le \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k} \to 0 \qquad (162)$$

as k goes to infinity. Now, we show that F does not lie in $\bigcup_{s>\alpha} \mathscr{L}^s(\mathbb{I})$. Indeed,

$$\int_0^1 |F(x)|^s\, dx = \sum_{k=1}^{\infty} t_k^s I_k = \sum_{k=1}^{\infty} t_k^{s-\alpha} t_{k-1}^{\alpha-p} 2^{-k} \qquad (163)$$

Using elementary inequality

$$
\begin{aligned}
\log_2 t_k^s I_k &= (s-\alpha)k^k - (p-\alpha)(k-1)^{k-1} - k \\
&\ge (s-\alpha)k^k - (p-\alpha)k^{k-1} - k^{k-1} \\
&= \Big((s-\alpha)k - (p-\alpha+1)\Big)k^{k-1} \to \infty
\end{aligned}
$$

we obtain

$$\int_0^1 |F(x)|^s \, dx = \infty$$

as desired.

4.6 Sobolev classes

A Sobolev mapping $f : \mathbb{X} \to \mathbb{R}^N$ is a vector field $f = (f^1, f^2, ..., f^N)$ whose coordinate functions lay in the usual Sobolev space $\mathscr{W}^{1,p}(\mathbb{X})$, $1 \leqslant p \leqslant \infty$. As for the mappings $f : \mathbb{X} \to \mathbb{Y}$ between manifolds it has been increasingly acknowledged that the introduction of the Riemannian structure on both $\mathbb{X}$ and $\mathbb{Y}$ is necessary to build a viable theory. Broadly speaking the presence of this additional structure involves no loss of generality and at the same time it pays off handsomely in geometric insights. Adopting the imbedding theorem of J. Nash simplifies matters substantially. Thus we assume that the target manifold $\mathbb{Y}$ is $\mathscr{C}^\infty$-isometrically imbedded in some Euclidean space $\mathbb{R}^N$. Now the term Sobolev mapping $f : \mathbb{X} \to \mathbb{Y}$ pertains to a measurable function $f : \mathbb{X} \to \mathbb{R}^N$ such that $f(x) \in \mathbb{Y}$ for a.e. $x \in \mathbb{X}$.

4.6.1 The Orlicz-Sobolev space $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$

This space consists of weakly differentiable functions $f : \mathbb{X} \to \mathbb{Y} \subset \mathbb{R}^N$ such that

$$\| f \|_{1,P} = \int_{\mathbb{X}} |f(x)| \, dx + \| Df \|_P < \infty \tag{164}$$

We should stress that in order to speak of weakly differentiable mapping $f : \mathbb{X} \to \mathbb{Y}$ we must ensure that $|Df| \in \mathscr{L}^1(\mathbb{X})$. This requires $P(t)$ to grow at least linearly, namely $P(t) \succcurlyeq t$. With this assumption in place $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$ becomes a complete metric space with respect to the distance

$$\text{dist}_{1,P}[f, g] = \| f - g \|_1 + \text{dist}_P[Df, Dg] \tag{165}$$

Of course, the distance function depends on the imbedding $\mathbb{Y} \subset \mathbb{R}^N$ but different imbeddings yield the same topology in $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$. The weak topology in the linear spaces $\mathscr{L}^P(\mathbb{X}, \mathbb{R}^N)$ makes no sense in the nonlinear class

$\mathscr{L}^P(\mathbb{X},\mathbb{Y})$. But we can speak of weak convergence in $\mathscr{W}^{1,P}(\mathbb{X},\mathbb{Y})$. In what follows we will be interested in the Orlicz-Sobolev classes $\mathscr{W}^{1,P}(\mathbb{X},\mathbb{Y})$ such that

$$\int_1^\infty \frac{P(t)\,dt}{t^{n+1}} = \infty \tag{166}$$

Additional conditions, such as (135) or (154), will also be imposed when necessary.

4.6.2 The Sobolev classes $\mathbf{G}\mathscr{W}^{1,n}(\mathbb{X},\mathbb{R}^N)$ and $\mathbf{V}\mathscr{W}^{1,n}(\mathbb{X},\mathbb{R}^N)$

As always the dimension of the domain manifold $\mathbb{X}$ equals $n \geqslant 2$. The grand Sobolev space $\mathrm{G}\mathscr{W}^{1,n}(\mathbb{X},\mathbb{R}^N)$ consists of the vector functions $f : \mathbb{X} \to \mathbb{R}^N$ such that

$$\| f \|_{1,n)} = \| f \|_1 + \sup_{0<\epsilon\leqslant n-1} \left(\epsilon \int_{\mathbb{X}} |Df(x)|^{n-\epsilon}\,dx \right)^{\frac{1}{n-\epsilon}} < \infty \tag{167}$$

This is a norm which makes $\mathrm{G}\mathscr{W}^{1,n}(\mathbb{X},\mathbb{R}^N)$ a Banach space. The closure of $\mathscr{C}^\infty(\mathbb{X},\mathbb{R}^N)$ in this space denoted by $\mathrm{V}\mathscr{W}^{1,n}(\mathbb{X},\mathbb{R}^N)$ is characterized precisely by the condition

$$\lim_{\epsilon\to 0} \epsilon \int_{\mathbb{X}} |Df(x)|^{n-\epsilon}\,dx = 0 \tag{168}$$

However, the density of $\mathscr{C}^\infty(\mathbb{X},\mathbb{Y})$ in $\mathrm{V}\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$ will require some work, if the target manifold $\mathbb{Y}$ is not a vector space. We shall establish this important fact in Section 5.8.

5 Smooth Approximation

The first of the main questions that faces us is to whether smooth mappings $f : \mathbb{X} \to \mathbb{Y}$ are dense in the given Sobolev class. As we have said, this approximation problem has already a remarkable history, J. Eells and L. Lemaire [11] first consider $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$, with $p > n = \dim\mathbb{X}$. By virtue of the embedding theorem such mappings are continuous. A general fact is that whenever $f : \mathbb{X} \to \mathbb{Y}$ happens to be continuous the usual mollification followed by the projection of a tubular neighborhood of $\mathbb{Y}$ gives the desired approximation of f, see also [7] for the related ideas concerning the space $\mathrm{VMO}(\mathbb{X}, \mathbb{Y})$-mappings with vanishing mean oscillations. But the true difficulty shows up below the dimension of $\mathbb{X}$, $1 \leqslant p < n$. The prevailing idea of our approach is that in the Sobolev classes slightly below $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ we were able to detect certain sets (refered to as webs) on which a given map is continuous.

5.1 Web like structures

We repeat from Introduction that a web on $\mathbb{X}$ is a compact set $\mathbb{F} \subset \mathbb{X}$ of Lebesgue measure zero whose complement consists of finite number of components $\mathbb{U}_i$, $i = 1, ..., I$ (disjoint open connected sets). We call them meshes of the web; thus,

$$\mathbb{X} \setminus \mathbb{F} = \bigcup_{i=1}^{I} \mathbb{U}_i = \bigcup_{\mathbb{U} \in \mathfrak{W}} \mathbb{U} \tag{169}$$

Let the collection of meshes be denoted by $\mathfrak{W} = \{\mathbb{U}_i; \ \ i = 1, ..., I\}$. In what follows we will spin webs on $\mathbb{X}$ with arbitrarily small meshes. the precise term for this is

$$\text{fine-diameter } (\mathbb{F}) = \max\left\{\operatorname{diam} \mathbb{U}; \quad \mathbb{U} \in \mathfrak{W}\right\} \tag{170}$$

It seems that in the perspective we will need only consider "regular web structures", meshes being Lipschitz or even more regular domains. In this paper the web will be no other than a finite union of geodesic spheres in $\mathbb{X}$.

5.2 Vanishing web oscillations

We shall investigate Sobolev mappings $f \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$ which have continuous trace along the web. Precisely this means that there exists a continuous function $\varphi : \mathbb{X} \to \mathbb{R}^N$ such that $f - \varphi \in \mathscr{W}_0^{1,p}(\mathbb{U}, \mathbb{R}^N)$, for every mesh $\mathbb{U} \in \mathfrak{W}$. Equivalently,

$$f \in \varphi + \mathscr{W}_0^{1,p}(\mathbb{X} \setminus \mathbb{F}, \ \mathbb{R}^N) \tag{171}$$

where $\mathscr{W}_0^{1,p}(\mathbb{U},\mathbb{R}^N)$ is the closure of $\mathscr{C}_0^\infty(\mathbb{U},\mathbb{R}^N)$ in $\mathscr{W}^{1,p}(\mathbb{U},\mathbb{R}^N)$. Note that the above assumptions imply $\varphi\in\mathscr{W}^{1,p}(\mathbb{X},\mathbb{R}^N)$. Now, we say that a mapping $f\in\mathscr{W}^{1,p}(\mathbb{X},\mathbb{R}^N)$ has vanishing web oscillations if to every $\epsilon>0$ there corresponds a web $\mathbb{F}\subset\mathbb{X}$ such that:

1) fine-diameter($\mathbb{F}$) $\leqslant \epsilon$

2) f has continuous trace along $\mathbb{F}$, say $\varphi\in\mathscr{W}^{1,p}(\mathbb{X},\mathbb{R}^N)\cap\mathscr{C}(\mathbb{X},\mathbb{R}^N)$

For every mesh $\mathbb{U}\in\mathfrak{W}$, we have

3) $\operatorname{osc}(f,\partial\mathbb{U})\overset{\text{def}}{=\!=}\max\big\{|\varphi(x_1)-\varphi(x_2)|;\quad x_1,x_2\in\partial\mathbb{U}\big\}<\epsilon$

It is interesting to observe that in this definition the Sobolev exponent p plays no role if $\mathbb{F}$ is regular web. But we shall not use this observation.

5.3 Statements of the results

Our first theorem shows that the smooth approximation in $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$, $1\leqslant p\leqslant n$, is still possible for discontinuous mappings provided they have vanishing web oscillations. We shall see later that the vanishing web oscillations always occur in Sobolev spaces slightly weaker than $\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$.

THEOREM 5.1. *Suppose that $f\in\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$, $p>1$, has vanishing web oscillations. Then there exist mappings $f_j\in\mathscr{C}^\infty(\mathbb{X},\mathbb{Y})$, converging to f in $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$, such that*

$$|Df_j| \preccurlyeq \mathbf{M}(Df),\ \textit{almost everywhere in } \mathbb{X} \tag{172}$$

where the implied constant depends only on $\mathbb{X}$ and $\mathbb{Y}$.

Recall that $\mathbf{M}$ is the Hardy-Littlewood maximal operator on $\mathbb{X}$. This theorem, although only auxiliary, will be the key to many more convergence results. Let us emphasize that $\mathbf{M}(Df)$ enjoys (in general) the same degree of integrability as $|Df|$, hence passing to the limit will be achieved by the Lebesgue Dominated Convergence Theorem.

The longitude projection in Section 3.1 demonstrates that the vanishing web oscillations fail in the Marcinkiewicz class $\mathscr{W}^{1,n}_{\text{weak}}(\mathbb{X}, \mathbb{Y})$. However, the situation is completely different if we assume instead of (89) that

$$\lim_{t\to\infty} t^n \int_{|Df|>t} dx = 0 \tag{173}$$

Such mappings indeed have vanishing web oscillations. As a consequence of Theorem 5.1 we will obtain Theorem 1.1.

It is certainly curious that the vanishing web oscillations occur under slightly weaker assumption that (173). These weaker assumptions are stated in (11). Theorem 1.2 will be a consequence of this observation as well.

This idea applies with great effectiveness to many Orlicz-Sobolev classes $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$ in which the defining function $P : [0, \infty) \to [0, \infty)$ satisfies the divergence condition

$$\int_1^\infty \frac{P(t)}{t^{n+1}}\, dt = \infty \tag{174}$$

Here is a typical example of such functions

$$P(t) = \frac{t^n}{\log(e+t)\log\log(e^e+t)\ldots\log..\log(e^{e^{\cdot^{\cdot}}}+t)}$$

It is probably worth mentioning that the divergence condition at (174) is critical for many more phenomena in geometric PDEs. Among them are: the $\mathscr{L}^1$-integrability of Jacobians [31], [14], [39] monotonicity of Sobolev functions [27] and compactness of mappings with finite distortion [28]. For precise statement concerning smooth approximation in $\mathscr{W}^{1,P}(\mathbb{X},\ \mathbb{Y})$ we need to impose two additional technical assumptions:

$$\text{the function} \quad t^{-\alpha}P(t) \quad \text{with some } \alpha > n-1 \quad \text{is nondecreasing} \tag{175}$$

and the doubling condition

$$P(2t) \leqslant \Bbbk P(t) \quad \text{for some } \Bbbk > 1 \text{ and all } t \geqslant 0 \tag{176}$$

THEOREM 5.2. *Let hypothesis (174), (175) and (176) hold. Then the space* $\mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ *is dense in the metric topology* $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$.

We will emphasize that in practice the technical assumptions (175) and (175) will always be satisfied.

5.4 Proof of Theorem 5.1

We divide the proof into 5 steps.

5.4.1 Step 1- *Truncations*

Let ϵ be any positive number. We consider a web $\mathbb{F}$ of fine-diameter ϵ such that f has continuous trace φ on $\mathbb{F}$ and

$$\operatorname{osc}(f, \partial\mathbb{U}) \leqslant \epsilon, \quad \text{for every mesh } \mathbb{U} \in \mathfrak{W} \tag{177}$$

Given any mesh $\mathbb{U} \in \mathfrak{W}$, we pick up a point

$$a \in f(\partial\mathbb{U}) \subset \mathbb{Y} \subset \mathbb{R}^N \tag{178}$$

and consider a map

$$T_\epsilon \circ f : \mathbb{U} \to \mathbb{R}^N, \tag{179}$$

where $T_\epsilon : \mathbb{R}^N \to \mathbb{R}^N$, called truncation operator, is given by

$$T_\epsilon y = a + (y - a)\lambda(|y - a|), \tag{180}$$

$$\lambda(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant 2\epsilon \\ \frac{4(t-\epsilon)\epsilon}{t^2} & \text{for } t \geqslant 2\epsilon \end{cases} \tag{181}$$

It is immediate that $0 \leqslant \lambda(t) \leqslant 1$ and

$$|T_\epsilon y - a| \leqslant 4\epsilon \quad \text{for every } y \in \mathbb{R}^N \tag{182}$$

As $T_\epsilon \in \mathscr{C}^1(\mathbb{R}^N, \mathbb{R}^N)$, we see that $T_\epsilon f \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$. Using chain rule we compute the Hilbert-Schmidt norm of the $N \times n$-matrix $D(T_\epsilon f) \in \mathbb{R}^{N\times n}$

$$|D(T_\epsilon f)|^2 = \lambda^2 |Df|^2 + \left(\frac{2\,\lambda\,\lambda'}{|f-a|} + \lambda'\,\lambda'\right) |(D^* f)\,(f-a)|^2 \tag{183}$$

where λ and its derivative λ' are computed at $t = |f(x) - a|$. Recall from algebra that the Hilbert-Schmidt norm of $Df \in \mathbb{R}^{N\times n}$ is given by

$$|Df|^2 = \text{Trace } (D^*f \; Df) \tag{184}$$

where $D^*f \in \mathbb{R}^{n\times N}$ is transpose of Df. Hence the vector $(D^*f)f \in \mathbb{R}^n$. It is important to realize that the last term in (183) is non-positive and so we can ignore it to obtain

$$|D(T_\epsilon f)| \leqslant |Df| \quad \text{almost everywhere in } \; \mathbb{U} \tag{185}$$

We also observe that

$$T_\epsilon f - f \in \mathscr{W}_0^{1,p}(\mathbb{U}, \mathbb{R}^N) \tag{186}$$

Indeed, since f has continuous trace along $\mathbb{F}$ there exist $u \in \mathscr{W}_0^{1,p}(\mathbb{U}, \mathbb{R}^N)$ and a continuous mapping $\varphi \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$, such that $f = \varphi + u$ on $\mathbb{U}$. We approximate u by mappings $u_j \in \mathscr{C}_0^\infty(\mathbb{U}, \mathbb{R}^N)$. By continuity of the truncation operator $T_\epsilon : \mathscr{W}^{1,p}(\mathbb{U}, \mathbb{R}^N) \to \mathscr{W}^{1,p}(\mathbb{U}, \mathbb{R}^N)$ we conclude that

$$\begin{aligned} f - T_\epsilon f &= f - T_\epsilon \big[\lim_j (\varphi + u_j) \big] = \varphi + u - \lim_j T_\epsilon(\varphi + u_j) \\ &= u + \lim_j [\varphi - T_\epsilon(\varphi + u_j)] \in \mathscr{W}_0^{1,p}(\mathbb{U}, \mathbb{R}^N) \end{aligned} \tag{187}$$

This follows from the observation that $\varphi - T_\epsilon(\varphi + u_j)$ vanishes near $\partial\mathbb{U}$. To see this we notice that $\varphi(\partial\mathbb{U})$ lies in the ball $\mathbb{B}(a, \epsilon) \subset \mathbb{R}^N$, by (177). Since φ is continuous, the image of a neighborhood of $\partial\mathbb{U}$ lies in $\mathbb{B}(a, 2\epsilon)$. It only remains to notice that T_ϵ in $\mathbb{B}(a, 2\epsilon)$.

Finally, we perform truncation of f over every mesh $\mathbb{U} \in \mathfrak{W}$ and denote the resulting mapping by $f^\epsilon : \mathbb{X} \to \mathbb{R}^N$. It follows from (186) that

$$f^\epsilon \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N) \tag{188}$$

and

$$|Df^\epsilon(x)| \leqslant |Df(x)| \quad \text{a.e.} \quad x \in \mathbb{X} \tag{189}$$

It is important that there is no constant in the right hand side. Unfortunately, the image of $\mathbb{X}$ under f^ϵ is not longer in the target manifold $\mathbb{Y}$. However, in making truncation we gain small oscillations. Precisely, we have

$$|f^\epsilon(x_1) - f^\epsilon(x_2)| \leqslant 8\,\epsilon \quad \text{for all } x_1, x_2 \in \mathbb{U} \in \mathfrak{W} \tag{190}$$

by (182). This holds for the original f only when $x_1, x_2 \in \partial\mathbb{U}$

Remark 5.1. Before leaving this step of the proof, let us remark that we could introduce somewhat simpler, though only Lipschitz continuous, truncation operator. However, we prefer the $\mathscr{C}^1$-truncation to the Lipschitz one in order to apply chain rule.

5.4.2 Step 2.- *Truncations converge in* $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$

Now we investigate the limit of f^ϵ as $\epsilon \to 0$. First, by using Poincaré inequality (the version with zero traces) we see that for every $\mathbb{U} \in \mathfrak{W}$

$$\begin{aligned} \int_{\mathbb{U}} |f^\epsilon - f|^p \quad &\preccurlyeq \quad (\operatorname{diam} \mathbb{U})^p \int_{\mathbb{U}} |Df^\epsilon - Df|^p \\ &\preccurlyeq \quad (\operatorname{diam} \mathbb{U})^p \int_{\mathbb{U}} |Df|^p \end{aligned} \tag{191}$$

by (185). Since the fine-diameter of $\mathbb{F}$ is no larger than ϵ, we may add those estimates for all meshes, to obtain

$$\| f^\epsilon - f \|_{\mathscr{L}^p(\mathbb{X})} \preccurlyeq \epsilon \, \| Df \|_{\mathscr{L}^p(\mathbb{X})} \tag{192}$$

Hence

$$\lim_{\epsilon \to 0} f^\epsilon = f \quad \text{in} \quad \mathscr{L}^p(\mathbb{X}, \mathbb{R}^N) \tag{193}$$

Next, we infer from (189) that

$$\lim_{\epsilon \to 0} Df^\epsilon = Df, \qquad \text{in weak topology of} \quad \mathscr{L}^p(\mathbb{X}, \mathbb{R}^{N \times n}) \tag{194}$$

It is at this point important that the estimate at (189) holds without a constant. Lower semicontinuity of the p-norm yields

$$\| Df \|_p \leqslant \liminf_{\epsilon \to 0} \; \| Df^\epsilon \|_p \leqslant \limsup_{\epsilon \to 0} \; \| Df^\epsilon \|_p \leqslant \| Df \|_p \tag{195}$$

Hence,

$$\lim_{\epsilon \to 0} \, \| Df^\epsilon \|_p = \| Df \|_p \tag{196}$$

By virtue of uniform convexity of $\mathscr{L}^p(\mathbb{X}, \mathbb{R}^{N\times n})$ we conclude that

$$\lim_{\epsilon \to 0} Df^\epsilon = Df, \quad \text{strongly in} \quad \mathscr{L}^p(\mathbb{X}, \mathbb{R}^{N\times n}) \tag{197}$$

as desired.

Remark 5.2. A fact worth noticing is that f^ϵ is not converging uniformly, unless f is continuous. The reason is that in reality the meshes in the web $\mathfrak{W}$ are significantly smaller than the oscillations of f^ϵ. If they were comparable then the limit mapping would be even Lipschitz continuous.

5.4.3 Step 3.-*Mollification*

The truncated mappings $f^\epsilon : \mathbb{X} \to \mathbb{R}^N$ are not smooth, but they have small local oscillations. We now mollify each f^ϵ, as discussed in Section 2.4. The mollified mappings will be denoted by $f^\epsilon_t \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{R}^N)$, for $0 < t \leqslant t_\mathbb{X}$. The reader may wish consult (68) for the definition of the upper bound $t_\mathbb{X}$. Hence

(i)

$$\lim_{t \to 0} \| f^\epsilon_t - f^\epsilon \|_{\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)} = 0 \tag{198}$$

(ii) It follows by (79) and by (189) again, that

$$|Df^\epsilon_t| \preccurlyeq \mathbf{M}(Df^\epsilon) \leqslant \mathbf{M}(Df), \quad \text{for all } 0 < t \leqslant t_\mathbb{X} \tag{199}$$

Given small $\epsilon > 0$ we shall restrict the mollifying parameter $0 < t \leqslant t_\mathbb{X}$ to an interval $0 < t \leqslant t_\epsilon$. The upper bound t_ϵ is determined by requiring the following:

(iii) For every mesh $\mathbb{U} \in \mathfrak{M}$ and $0 < t \leqslant t_\epsilon$ it holds

$$\operatorname*{osc}_{\mathbb{U}} f^\epsilon_t \preccurlyeq \operatorname*{ess\,osc}_{\mathbb{U}_{t'}} f^\epsilon \leqslant 24\epsilon \quad \text{where } t \preccurlyeq t' \preccurlyeq t \tag{200}$$

consult formula (vii) of Section 2.4.

The reasoning for the last inequality is as follows. Once t is sufficiently small so is t'. We can choose it small enough to ensure that every $\mathbb{U}_{t'}$ intersects only those meshes of the web which touch $\mathbb{U}$. Then, by triangle inequality, we see that $\operatorname{ess\,osc}_{\mathbb{U}}[f^\epsilon] \leqslant 3 \cdot 8\epsilon$ because of (190).

5.4.4 Step 4.-*Convergence of the mollified truncations*

It is immediate from (193), (194) and (198) that

$$\lim_{\epsilon \to 0} f^\epsilon_{t_\epsilon} = f \quad \text{in} \quad \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N) \tag{201}$$

We also infer from (199) that

$$\left| D f^\epsilon_{t_\epsilon} \right| \preccurlyeq \mathbf{M}(Df) \tag{202}$$

5.4.5 Step 5.-*Projection onto* $\mathbb{Y}$

In the final step we project the values of f^ϵ_t smoothly onto $\mathbb{Y}$. The actual calculation is reduced to a tubular neighborhood of $\mathbb{Y}$ of sufficiently small width; say

$$\mathbb{Y}_h = \left\{ y \in \mathbb{R}^N; \quad \operatorname{dist}(y, \mathbb{Y}) < h \right\} \tag{203}$$

Note that the closest point projection

$$\Pi : \ \mathbb{Y}_h \to \mathbb{Y} \tag{204}$$

is a map of class $\mathscr{C}^\infty(\mathbb{Y}_h, \mathbb{Y})$. Now, the approximating sequence $\{f_j\}$ of smooth mappings converging to f is obtained as $f_j = \Pi(f^{\epsilon_j}_{t_j})$, where $\epsilon_j \to 0$ and the mollifying parameters $t_j \to 0$, are chosen accordingly. For the proof of Theorem 5.1 we need only show that the mappings $\Pi\left(f^{\epsilon_j}_{t_j}\right) \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ converge to f in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^N)$. First notice that each $f^{\epsilon_j}_{t_j}$ maps $\mathbb{X}$ into $\mathbb{Y}_h$ maps if we choose sufficiently small ϵ_j. This follows from the inequality (200) and the fact that $f^\epsilon(\mathbb{X})$ lies in a small tubular neighborhood of $\mathbb{Y}$. As $j \to \infty$ the mappings $f^{\epsilon_j}_{t_j}$ are arbitrarily close to f^{ϵ_j} at some points in each mesh $\mathbb{U}$

due to (198). For abbreviation, we let $\Pi'(y)$ stand for the differential of Π at $y \in \mathbb{Y}_h$. The remaining reasoning goes without further special comment.

$$
\begin{aligned}
\| f - \Pi f_\epsilon \|_{\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})} &= \| \Pi f - \Pi f_\epsilon \|_{\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})} \\
&= \| \Pi f - \Pi f_\epsilon \|_p + \| D(\Pi f) - D(\Pi f_\epsilon) \|_p \\
&\preccurlyeq \| f - f_\epsilon \|_p + \| \Pi'(f) \circ Df - \Pi'(f_\epsilon) \circ Df_\epsilon \|_p \\
&\preccurlyeq \| f - f_\epsilon \|_p + \| \Pi'(f_\epsilon) \circ (Df - Df_\epsilon) \|_p \\
&\quad + \| \Pi'(f) - \Pi'(f_\epsilon) \circ Df \|_p \\
&\to 0 + 0 + 0 = 0
\end{aligned} \tag{205}
$$

Here in the last step we have made appeal to Dominated Convergence Theorem.

5.5 Spinning a web on $\mathbb{X}$

In this subsection we consider a Sobolev mapping $f : \mathbb{X} \to \mathbb{Y}$ whose differential lies in the very weak Lebesgue space $\mathscr{L}^{\alpha,n}(\mathbb{X}, \mathbb{R}^{N\times N})$, where $n-1 < \alpha < n$, see formula (115) for the definition of $\mathscr{L}^{\alpha,n}$. Our goal is to build webs on $\mathbb{X}$ which capture arbitrarily small oscillations of f. Precise statement is contained in the following

Proposition **5.1.** *Given $\epsilon > 0$, there exists a finite family $\mathfrak{W} = \{\mathbb{U}_\nu;\quad \nu = 1, ..., K\}$ of mutually disjoint open sets $\mathbb{U}_\nu \subset \mathbb{X}$, with diam $\mathbb{U}_\nu \leqslant \epsilon$, whose union $\mathbb{M} = \bigcup_{\nu=1}^K \mathbb{U}_\nu$ has full measure, and there exists a continuous mapping $\varphi : \mathbb{X} \to \mathbb{R}^N$ such that*

(i)

$$f - \varphi \in \mathscr{W}_0^{1,\alpha}(\mathbb{U}_\nu, \mathbb{R}^N)$$

(ii)

$$\operatorname*{osc}_{\partial \mathbb{U}_\nu} \varphi \stackrel{\text{def}}{=\!=} \max \left\{ |\varphi(x_1) - \varphi(x_2)|;\quad x_1, x_2 \in \mathbb{U}_\nu \right\} \leqslant \epsilon$$

for all $\nu = 1, 2, ..., K$.

It is automatic from (i) that $\varphi \in \mathscr{W}^{1,\alpha}(\mathbb{X}, \mathbb{R}^N)$. We shall then consider the web $\mathbb{F} = \mathbb{F}_\epsilon = \mathbb{X} - \cup_{\nu=1}^{K} \mathbb{U}_\nu$. The key ingredient needed for the construction of such webs will be the following

LEMMA **5.1.** [OSCILLATIONS ON SPHERES] *Let* $h \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{R}^N)$ *and* $R \leqslant R_{\mathbb{X}}$ *(reliable radius for* $\mathbb{X}$ *see Section 2.1.1.). Then for every* $a \in \mathbb{X}$ *and* $r \in (0, R]$ *we have*

$$\operatorname*{osc}_{\mathbb{S}(a,r)} h \preccurlyeq r \left(\fint_{\mathbb{S}(a,r)} |Dh(x)|^\alpha \, dx \right)^{\frac{1}{\alpha}} \tag{206}$$

provided $\alpha > n - 1$.

This is none other than a spherical variant of the imbedding inequality. That is why the Sobolev exponent α is required to be greater than the dimension of the sphere.

Proof of Proposition 5.1. We have now all requisites needed for the proof of Proposition 5.1. Given a sequence $\{f_j\}$ of mappings $f_j \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{R}^N)$, $j = 1, 2, ...,$ converging to f in $\mathscr{W}^{1,\alpha}(\mathbb{X}, \mathbb{R}^N)$. Fix a positive number $R \leqslant \min\{\epsilon, R_{\mathbb{X}}\}$, so that we can use the oscillation Lemma 5.1. Accordingly,

$$\begin{aligned} \operatorname*{osc}_{\mathbb{S}(x,r)} g &\leq C_\alpha(\mathbb{X}) r \left(\fint_{\mathbb{S}(x,r)} |Dg|^\alpha \right)^{\frac{1}{\alpha}}, \\ \sup_{\mathbb{S}(x,r)} |g| &\leq \inf_{\mathbb{S}(x,r)} |g| + \operatorname*{osc}_{\mathbb{S}(x,r)} g \\ &\leq \left(\fint_{\mathbb{S}(x,r)} |g|^\alpha \right)^{\frac{1}{\alpha}} + C_\alpha(\mathbb{X}) r \left(\fint_{\mathbb{S}(x,r)} |Dg|^\alpha \right)^{\frac{1}{\alpha}} \end{aligned} \tag{207}$$

whenever $0 < r \leqslant R$ and $g \colon \mathbb{X} \to \mathbb{R}^N$ is a smooth function. On the other hand, since $|Df| \in \mathscr{L}^{\alpha,n}(\mathbb{X})$, we may appeal to Proposition 4.1 to ensure the inequalities

$$C_\alpha(\mathbb{X}) \, r \left(\fint_{\mathbb{S}(x,r)} |Df|^\alpha \right)^{\frac{1}{\alpha}} \leqslant \frac{\epsilon}{4} \tag{208}$$

for some radii r in a set of positive linear measure in $(0, R]$. Next, Fubini's theorem tells us that

$$\lim_{j \to \infty} \int_{\mathbb{S}(x,r)} \left(|Df_j - Df|^\alpha + |f_j - f|^\alpha \right) = 0 \tag{209}$$

for almost every r in $(0, R]$. When confronted with (208), this gives at least one radius $r = r_x \in (0, R]$ for which (208) and (209) hold. We consider the covering $\mathbb{X} = \bigcup_{x\in\mathbb{X}} \mathbb{B}(x, r_x)$ by geodesic open balls. Since $\mathbb{X}$ is compact, a finite collection of these balls will also cover $\mathbb{X}$. We assort this finite collection further to obtain a sequence, denoted by $\mathbb{B}_1, ..., \mathbb{B}_k$, $\mathbb{B}_i = \mathbb{B}(x_i, r_i)$, such that

(i)
$$\mathbb{B}_1 \cup ... \cup \mathbb{B}_k = \mathbb{X}$$

(ii) *No ball in the sequence is contained in the other one.*

Having these selected balls at hand we now define a web $\mathbb{F} = \mathbb{F}_\epsilon$ to be the union of the spheres $\mathbb{S}_i = \partial \mathbb{B}_i$, $i = 1, 2, ..., k$. Then the meshes $\mathbb{U}_1, ..., \mathbb{U}_K$ are the connected components of $\mathbb{X} \setminus \mathbb{F}$. Note, for curiosity, that $K \leqslant 2^k$. Now, we pass to a subsequence (labelled again as f_j) such that

$$\|f_j - f_{j-1}\|_{\mathscr{W}^{1,\alpha}(\mathbb{X})} \leq 2^{-j}$$
$$\left(\fint_{\mathbb{S}_i} |f_j - f|^\alpha\right)^{\frac{1}{\alpha}} + C_\alpha(\mathbb{X}) r \left(\fint_{\mathbb{S}_i} |Df_j - Df|^\alpha\right)^{\frac{1}{\alpha}} \leqslant 2^{-j-3}\epsilon, \qquad i = 1, \dots, k \tag{210}$$

We define a truncated sequence $\{\varphi_j\}$ by

$$\varphi_1 = f_1,$$
$$\varphi_j - \varphi_{j-1} = T_{2^{-j}\epsilon}(f_j - f_{j-1}), \qquad j = 2, 3, \dots$$

where $T_{2^{-j}\epsilon}$ is the truncation operator defined in (180) with $2^{-j}\epsilon$ in place of ϵ and $a = 0$. The properties of the truncation operator ensure that

$$\sup_{\mathbb{X}} |\varphi_j - \varphi_{j-1}| \leq 2^{-j+2}\epsilon \tag{211}$$

see inequality (182). Next we apply (207) to the mapping $f_j - f_{j-1}$ in place of g and in view of (210) we obtain

$$\sup_{\mathbb{S}_i} |f_j - f_{j-1}| \leq 2^{-j-2}\epsilon \tag{212}$$

This latter estimate combined with formulas (180) and (181) show that $T_{2^{-j}\epsilon}(f_j - f_{j-1}) = f_j - f_{j-1}$ on every $\mathbb{S}_i$ $i = 1, ..., k$. In particular, we see that

$$\varphi_j = f_j \qquad \text{on } \mathbb{F} \tag{213}$$

Appealing to (189) we have

$$\|D(\varphi_j - \varphi_{j-1})\|_{\mathscr{L}^\alpha(\mathbb{X})} \leq \|D(f_j - f_{j-1})\|_{\mathscr{L}^\alpha(\mathbb{X})} \leq 2^{-j} \tag{214}$$

By (211) and (214), the sequence $\{g_j\}$ is a Cauchy sequence both in $\mathscr{W}^{1,\alpha}(\mathbb{X})$ and $\mathscr{C}(\mathbb{X})$. We define φ to be the uniform limit of the sequence $\{\varphi_j\}$. Notice that $\varphi_j - f_j$ are Lipschitz continuous functions on $\mathbb{X}$ and vanish on $\mathbb{F}$, by (213). As these functions converge to $\varphi - f$ in $\mathscr{W}^{1,\alpha}(\mathbb{X})$ we deduce that $\varphi - f \in \mathscr{W}_0^{1,\alpha}(\mathbb{U})$ for each connected component $\mathbb{U}$ of $\mathbb{X} \setminus \mathbb{F}$. It remains to estimate the oscillation of φ on the web. We infer from (207), (208), (209) and (213) that

$$\begin{aligned} \operatorname*{osc}_{\mathbb{S}_i} \varphi &\leq \liminf_j \operatorname*{osc}_{\mathbb{S}_i} \varphi_j \leq C_\alpha(\mathbb{X}) r_i \liminf_j \left(\fint_{\mathbb{S}_i} |D\varphi_j|^\alpha \right)^{\frac{1}{\alpha}} \\ &\leq C_\alpha(\mathbb{X})\, r_i \left(\fint_{\mathbb{S}_i} |Df|^\alpha \right)^{\frac{1}{\alpha}} \leq \frac{\epsilon}{4} \end{aligned} \tag{215}$$

Now, each $\mathbb{U}_\nu$ lies in a ball $\mathbb{B}$ from the family $\{\mathbb{B}_1, ..., \mathbb{B}_k\}$. In particular,

$$\operatorname{diam} \mathbb{U}_\nu \leqslant \epsilon \tag{216}$$

Further, $\partial \mathbb{U}_\nu$ consists of certain fragments of the spheres $\mathbb{S}_1, ..., \mathbb{S}_k$, those spheres that intersect $\mathbb{B}$. By condition (ii), every such sphere intersects $\partial\mathbb{B}$. Consequently, given two points $x_1, x_2 \in \partial\mathbb{U}_\nu$, say $x_1 \in \mathbb{S}_{i_1}$ and $x_2 \in \mathbb{S}_{i_2}$, we can find $a_1 \in \mathbb{S}_{i_1} \cap \partial\mathbb{B}$ and $a_2 \in \mathbb{S}_{i_2} \cap \partial\mathbb{B}$ and conclude by triangle inequality, that

$$\begin{aligned} |\varphi(x_1) - \varphi(x_2)| &\leqslant |\varphi(x_1) - \varphi(a_1)| + |\varphi(a_1) - \varphi(a_2)| \\ &\quad + |\varphi(a_2) - \varphi(x_2)| \\ &\leqslant \operatorname*{osc}_{\mathbb{S}_{i_1}} \varphi + \operatorname*{osc}_{\mathbb{S}} \varphi + \operatorname*{osc}_{\mathbb{S}_{i_2}} \varphi \\ &\leqslant \epsilon \end{aligned} \tag{217}$$

by (215). □

5.6 Proof of Theorems 1.1 and 1.2

The distance in the space $\mathscr{W}^{1,n}_{\text{weak}}(\mathbb{X},\mathbb{Y})$ will be denoted by

$$\text{Dist}[f,g] = \| f - g \|_{\mathscr{L}(\mathbb{X})} + \sup_{t\geqslant 0}\left(t^n \int_{|Df-Dg|>t} dx\right)^{\frac{1}{n+1}} \tag{218}$$

It is obvious that (173) holds for smooth mappings and remains valid in the closure of $\mathscr{C}^\infty(\mathbb{X},\mathbb{Y})$. The only non-trivial fact is that every $f \in \mathscr{W}^{1,n}_{\text{weak}}(\mathbb{X},\mathbb{Y})$ can be approximated by smooth mappings. In view of the inclusion at (117) we see that $|Df|$ satisfies (10), for every $0 \leqslant \alpha < n$. Proposition 5.1 tells us that f has vanishing web oscillations. Then by virtue of Theorem 5.1, there exist smooth mappings $f_j : \mathbb{X} \to \mathbb{Y}$ converging to f in every $\mathscr{W}^{1,\alpha}(\mathbb{X},\mathbb{Y})$, $1 \leqslant \alpha < n$. It is important that $|Df_j|$ are dominated by the maximal function $\mathbf{M}(Df)$, that is, independently of $j = 1, 2, ...$, see (172). Since $\mathbf{M}$: $\mathscr{L}^n_{\text{weak}} \to \mathscr{L}^n_{\text{weak}}$ is bounded, it follows that $\{f_j\}$ is bounded in $\mathscr{W}^{1,n}_{\text{weak}}(\mathbb{X},\mathbb{Y})$. With the aid of Lebesgue Dominated Convergence Theorem we conclude that

$$\lim_{j\to\infty} \text{Dist}[f_j, f] = 0$$

completing the proof of Theorem 1.1.

The reader may see from Proposition 5.1 that (11) implies vanishing web oscillation. Theorem 1.2 then follows from Theorem 5.1.

5.7 Proof of Theorem 5.2

Given $f \in \mathscr{W}^{1,P}(\mathbb{X},\mathbb{Y})$ we find, as before, a sequence $f_j \in \mathscr{C}^\infty(\mathbb{X},\mathbb{Y})$ converging to f in $\mathscr{W}^{1,1}(\mathbb{X},\mathbb{Y})$, with differentials Df_j controlled point-wise by the maximal function of Df as in the inequality (172). In particular, $\{Df_j\}$ contains a subsequence converging point-wise almost everywhere to Df. Since $\mathbf{M}$: $\mathscr{L}^P \to \mathscr{L}^P$ is bounded, again by Lebesgue Dominated Convergence Theorem we conclude that this subsequence converges to f in the metric topology of $\mathscr{W}^{1,P}(\mathbb{X},\mathbb{Y})$.

5.8 Proof of Theorem 1.3

The distance function in $\mathrm{V}\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$ is the one induced by the norm at (167). This space is contained in every $\mathscr{L}^p(\mathbb{X},\mathbb{R}^N)$, $1 \leqslant p < n$. Also $\mathbf{M}(Df) \in \mathrm{V}\mathscr{L}^n(\mathbb{X})$. The rest of the proof runs in much the same way as above.

6 $\mathscr{L}^1$-Estimates of the Jacobian

Let $f : \mathbb{X} \to \mathbb{Y}$ be a Sobolev mapping, where we assume that $n = \dim \mathbb{X} \leqslant \dim \mathbb{Y} = m$. To every $\mathscr{C}^\infty$-smooth n-form $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$ there corresponds its pullback $f^\sharp \omega \in \mathscr{L}(\wedge^n \mathbb{X})$ via f. The point-wise estimate

$$|f^\sharp \omega| \preccurlyeq |Df|^n \tag{219}$$

gives us at least some idea how to control the degree of integrability of the pullback $f^\sharp\omega$ in terms of $|Df|$. Surprisingly, if $d\omega = 0$, then $f^\sharp\omega$ may enjoy higher degree of integrability than $|Df|^n$. This phenomenon, first observed in [41] and [10] for mappings in $\mathscr{W}^{1,n}(\mathbb{R}^n,\mathbb{R}^n)$, has come to play a central role in modern calculus of variations, nonlinear elasticity and the geometric function theory. Our integral estimates in this paper are sharp generalizations of these results in the manifold setting. If we wish not to make any topological assumption on the target manifold then we need to restrict ourselves to the pullbacks of Cartan n-forms

$$\omega = \sum_{i=1}^{K} \alpha_i \wedge \beta_i, \qquad \begin{matrix} d\alpha_i = d\beta_i = 0 \quad \deg\,\alpha_i + \deg\,\beta_i = n \\ \deg\,\alpha_i \geqslant 1 \quad \text{and} \quad \deg\,\beta_i \geqslant 1 \end{matrix} \tag{220}$$

The wedge product structure of the terms will be critical for our arguments. For $f \in \mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$ we have a linear functional that operates on the test function $\varphi \in \mathscr{C}^\infty(\mathbb{X})$ by the rule

$$\begin{aligned} \left(f^\sharp\omega\right)[\varphi] \quad &= \quad \int_{\mathbb{X}} \varphi\left(f^\sharp\omega\right) = \int_{\mathbb{X}} \varphi \sum_{i=1}^{K} \left(f^\sharp\alpha_i\right) \wedge \left(f^\sharp\beta_i\right) \\ &\preccurlyeq \quad \|\,\varphi\,\|_\infty \int_{\mathbb{X}} |Df(x)|^n \, dx \end{aligned} \tag{221}$$

In other words, $f^\sharp\omega$ can be viewed as a Schwartz distribution of order zero. The differential forms $f^\sharp\alpha_i$ and $f^\sharp\beta_i$ are closed and, therefore, exact modulo harmonic fields. One of the useful analytic advantages of this idea is that $f^\sharp\omega$ can be defined as a Schwartz distribution for all mappings

$$f \in \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y}), \quad \text{with } s = \frac{n^2}{n+1} \tag{222}$$

see Section 6.1, for details. In particular, $f_j : \mathbb{X} \to \mathbb{Y}$ are smooth mappings converging to f in $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$ then the pullback $f^\sharp\omega$ be computed by the formula

$$(f^\sharp\omega)[\varphi] = \lim_{j\to\infty} \int_{\mathbb{X}} \varphi\,(f_j^\sharp\omega) \tag{223}$$

We call $s = \frac{n^2}{n+1}$ the *critical exponent*. This turns out to be exponent the smallest one that we can prove existence of the limit at (223).

In this larger context $f^\sharp\omega$ will be a distribution of order 1, more precisely

$$\left(f^\sharp\omega\right)[\varphi] \;\preccurlyeq\; \left(\,\|\,\varphi\,\|_\infty + \|\,d\varphi\,\|_\infty\right)\,\|\,Df\,\|^n_{\mathscr{L}^s(\mathbb{X})} \tag{224}$$

for every $\varphi \in \mathscr{C}^\infty(\mathbb{X})$. The computation in the forthcoming section 6.1 should be compared with the analogous situation in the Euclidean case. Later, it will prove handy to express the distribution $f^\sharp\omega$ by integrals of the point-wise Jacobian $\mathcal{J}_\omega(x, f)$, defined by

$$\mathcal{J}_\omega(x, f)\,dx = \sum_{i=1}^{K} \left(f^\sharp\alpha_i\right) \wedge \left(f^\sharp\beta_i\right) \tag{225}$$

To this effect we notice that $d(f^\sharp\alpha_i) = 0$ and $d(f^\sharp\beta_i) = 0$. It suggests that we must consider even more general situation.

6.1 Weak wedge products

Consider closed differential forms $\Phi \in \mathscr{L}^p(\wedge^l\mathbb{X}) \cap \ker d$ and $\Psi \in \mathscr{L}^r(\wedge^k\mathbb{X}) \cap \ker d$, where $1 \leqslant k, l < n$, $k + l = n$ and $1 < p, r < \infty$. First assume that p

and r are Hölder conjugate. Thus $\Phi \wedge \Psi$ is integrable. It defines a Schwartz distribution of order zero

$$(\Phi \wedge \Psi)[\eta] = \int_{\mathbb{X}} \eta\,(\Phi \wedge \Psi) \quad \text{for } \eta \in \mathscr{C}^\infty(\mathbb{X}) \tag{226}$$

But we can do much better if we apply the $\mathscr{L}^p$-cohomology theory, see Section 2.3.2. Accordingly, every closed form of class $\mathscr{L}^p(\wedge^l \mathbb{X})$ is exact modulo harmonic fields. Precisely,

$$\Phi = d\varphi + \vartheta \tag{227}$$

where the exact component $d\varphi \in \mathscr{L}^p(\wedge^l \mathbb{X})$ and the harmonic field $\vartheta \in \mathscr{C}^\infty(\wedge^l \mathbb{X})$ are determined by the equation

$$\varphi = \mathbf{E}\Phi \quad \text{and} \quad \vartheta = \mathbf{H}\Phi \tag{228}$$

Here both $\mathbf{E}$ and $\mathbf{H}$ are linear integral operators. Precisely,

$$\mathbf{E} : \mathscr{L}^p(\wedge^l \mathbb{X}) \to \mathscr{W}^{1,p}(\wedge^{l-1} \mathbb{X}) \ , \qquad 1 < p < \infty \tag{229}$$

and

$$\mathbf{H} : \mathscr{L}^p(\wedge^l \mathbb{X}) \to \mathscr{C}^\infty(\mathbb{X}) \ , \qquad 1 \leqslant p < \infty \tag{230}$$

Using the decomposition $\Phi = d\varphi + \vartheta$, we obtain another formula for the action at $\Phi \wedge \Psi$ on the test function $\eta \in \mathscr{C}^\infty(\mathbb{X})$;

$$\begin{aligned} (\Phi \wedge \Psi)[\eta] &= \int_{\mathbb{X}} \eta\,(d\varphi \wedge \Psi) + \int_{\mathbb{X}} \eta\,(\vartheta \wedge \Psi) \\ &= \int_{\mathbb{X}} \eta\,(\vartheta \wedge \Psi) - \int_{\mathbb{X}} d\eta \wedge (\varphi \wedge \Psi) \end{aligned} \tag{231}$$

It is important to realize that this latter integral converges whenever $\varphi \wedge \Psi$ is integrable. Assume now that $1 \leqslant p, r < \infty$ is a Sobolev conjugate pair, that is

$$\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{n} \tag{232}$$

This condition implies that one of the exponents is less than n, for instance $1 \leqslant p < n$. Then, by Sobolev imbedding, we find that $\varphi \in \mathscr{L}^{\frac{np}{n-p}}(\wedge^{l-1}\mathbb{X})$. The exponent $\frac{np}{n-p}$ is exactly Hölder conjugate to r, ensuring that $\varphi \wedge \Psi \in \mathscr{L}^1(\wedge^{n-1}\mathbb{X})$.

Remark 6.1. The reader may wish to argue in much the same way for analogous formula in case when $1 \leqslant r < n$.

We are now ready to make the definition.

DEFINITION **6.1.** (DISTRIBUTIONAL WEDGE PRODUCT) Let

$$\Phi \in \mathscr{L}^p(\wedge^\ell \mathbb{X}) \cap \ker d \quad \text{ and } \quad \Psi \in \mathscr{L}^r(\wedge^k \mathbb{X}) \cap \ker d$$

where $1 \leqslant p, r < \infty$ are Sobolev conjugate exponents. The distribution $\Phi \wedge \Psi$ operates on the test function $\eta \in \mathscr{C}^\infty(\mathbb{X})$ by the rule

$$(\Phi \wedge \Psi)[\eta] \stackrel{\text{def}}{=\!=} \lim_{j \to \infty} \int_{\mathbb{X}} \eta \, (\Phi_j \wedge \Psi_j) \tag{233}$$

where $\Phi_j \in \mathscr{C}^\infty(\wedge^\ell \mathbb{X})$ and $\Psi_j \in \mathscr{C}^\infty(\wedge^k \mathbb{X})$ are closed forms converging to Φ and Ψ in $\mathscr{L}^p(\wedge^\ell \mathbb{X})$ and $\mathscr{L}^r(\wedge^k \mathbb{X})$, respectively.

Remark 6.2. For this definition let us recall that closed forms in $\mathscr{C}^\infty(\wedge^l \mathbb{X})$ are dense in $\mathscr{L}^p(\wedge^l \mathbb{X}) \cap \ker d$, see Section 2.3.1. Also notice that the limit at (233) does not depend on the choice of the sequences $\{\Phi_j\}$ and $\{\Psi_j\}$.

Remark 6.3. Our arguments above also show that if $\frac{1}{p} + \frac{1}{r} < 1 + \frac{1}{n}$ then the limit at (233) still exists when Φ_j and Ψ_j converge to Φ and Ψ weakly in $\mathscr{L}^p(\wedge^\ell \mathbb{X})$ and $\mathscr{L}^r(\wedge^k \mathbb{X})$, respectively.

It is now obvious how to define the distributional pullback. Suppose we are given a Cartan form as in (220). For $f \in \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$, with the critical exponent $s = \frac{n^2}{n+1}$, we consider the closed forms

$$\Phi_i = f^\sharp \alpha_i \in \mathscr{L}^{s/\ell_i}(\wedge^{\ell_i} \mathbb{X}) \cap \ker d \tag{234}$$

$$\Psi_i = f^\sharp \beta_i \in \mathscr{L}^{s/k_i}(\wedge^{k_i} \mathbb{X}) \cap \ker d \tag{235}$$

where we observe that

$$\frac{\ell_i}{s} + \frac{k_i}{s} = 1 + \frac{1}{n} \tag{236}$$

Thus $\Phi_i \wedge \Psi_i$ can be regarded as a Schwartz distribution. The distributional pullback of ω is then defined by

$$(f^\sharp \omega)[\eta] = \sum_{i=1}^{K} (\Phi_i \wedge \Psi_i)[\eta] \tag{237}$$

6.2 Distributional Jacobian

It is reasonable to ask how the distributional pullback relates to the point-wise Jacobian. The answer is obvious if $f \in \mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$, we simply have $\left(f^\sharp\omega\right)[\varphi] = \int_{\mathbb{X}} \varphi(x)\, \mathcal{J}_\omega(x,f)\, dx$. However one can go slightly below this regularity assumption.

THEOREM 6.1. *Let $f : \mathbb{X} \to \mathbb{Y}$, $n = dim\mathbb{X} \leqslant dim\mathbb{Y}$, be a Sobolev map satisfying*

$$\liminf_{t\to\infty}\; t^{\frac{n}{n+1}} \int_{|Df|>t} |Df(x)|^{\frac{n^2}{n+1}}\, dx = 0 \tag{238}$$

Then there are measurable sets $\mathbb{X}_1 \subset \mathbb{X}_2 \subset ... \subset \mathbb{X}$ whose union is $\mathbb{X}$ such that the distributional pullback of every Cartan form $\omega \in \mathscr{C}^\infty(\wedge^n\mathbb{Y})$ takes the form

$$\left(f^\sharp\omega\right)[\varphi] = \lim_{j\to\infty} \int_{\mathbb{X}_j} \varphi(x)\mathcal{J}_\omega(x,f)\, dx, \qquad \varphi \in \mathscr{C}^\infty(\mathbb{X}) \tag{239}$$

The reader is warned that the sets $\mathbb{X}_1, \mathbb{X}_2, ...$ are chosen for a specific map f, the limit at (239) may not exist for other sets. As a matter of fact $\mathbb{X}_j$ will be carefully selected from the level sets of the maximal function of $|Df|^{\frac{n^2}{n+1}}$. In particular, the point-wise Jacobian will be bounded on each of those sets, making the integrals at (239) will defined.

At this stage we are able to give meaning to the so-called weak integral of the Jacobian that is $(f^\sharp\omega)[1]$. Formula (239) gives

$$\left(f^\sharp\omega\right)[1] \overset{\text{def}}{=\!=} \lim_{j\to\infty} \int_{\mathbb{X}_j} \mathcal{J}_\omega(x,f)\, dx \tag{240}$$

The following corollary is straightforward by a monotone convergence argument.

THEOREM 6.2. *If, in addition to the conditions stated in Theorem 6.1, the Jacobian is nonnegative then it is integrable and coincides with the distribution $f^\sharp\omega$. Precisely, we have*

$$\int_{\mathbb{X}} \varphi(x)\mathcal{J}_\omega(x,f)\, dx = \left(f^\sharp\omega\right)[\varphi] \tag{241}$$

for all $\varphi \in \mathscr{C}^\infty(\mathbb{X})$.

Passing to the limit under the integral sign at (239) is perfectly justified whenever the point-wise Jacobian is integrable over $\mathbb{X}$. Thus, we also have the following variant of Theorem 6.2.

THEOREM 6.3. *Under the conditions stated in Theorem 6.1, if $\mathcal{J}_\omega(\cdot, f) \in \mathscr{L}^1(\mathbb{X})$, then the point-wise Jacobian coincides with the distribution $f^\sharp\omega$. Precisely, this means that*

$$\int_{\mathbb{X}} \varphi(x)\mathcal{J}_\omega(x,f)\,dx = \left(f^\sharp\omega\right)[\varphi] \tag{242}$$

for all $\varphi \in \mathscr{C}^\infty(\mathbb{X})$.

In what follows we refer to Sobolev mappings having non-negative Jacobians $\mathcal{J}_\omega(x,f)$, with respect to the Riemannian volume form $\omega = dy$, as orientation preserving. We reserve the notation $\mathcal{J}(x,f)$ for the Jacobian if $\omega = dy$. Theorem 6.2 may fail if ω is not a Cartan form, which is the case of the volume form on the n-sphere $\mathbb{S}^n$, see Theorem 3.1. In the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ the orientation preserving mappings satisfy

$$\int_{\mathbb{X}} \mathcal{J}(x,f)\log\left(e + \frac{\mathcal{J}(x,f)}{\int_{\mathbb{X}}\mathcal{J}(z,f)\,dz}\right)dx \;\preccurlyeq\; \|\,\mathbf{M}\mathcal{J}(\cdot,f)\,\|_{\mathscr{L}^1(\mathbb{X})} \;\preccurlyeq\; \int_{\mathbb{X}} |Df|^n \tag{243}$$

This simply means that $\mathcal{J}(\cdot,f)$ belongs to the Zygmund space $\mathscr{L}\log\mathscr{L}(\mathbb{X})$, see S. Müller [41] for the Euclidean case. In our manifold setting, in which the target space is not a rational homology sphere this result will follow from the forthcoming $\mathscr{H}^1$-estimates.

We come now to perhaps the most surprising phenomenon. It is true that if $f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{J}_\omega(x,f) \geqslant 0$, then $\mathcal{J}_\omega$ belongs to $\mathscr{L}\log\mathscr{L}(\mathbb{X})$ regardless whether ω is a Cartan form or not. However, the uniform bound at (243) will be lost for non Cartan forms.

6.3 Proof of Theorem 6.1

Although our main objective is to prove Theorem 6.1, the arguments we shall use here can be set in more general context, which might be of independent interest. This more general consists in replacing the forms $f^\sharp\alpha_i$ and $f^\sharp\beta_i$ at (225) by arbitrary closed forms.

6.3.1 An integral estimate of wedge products

LEMMA **6.1.** *Let* $\Phi \in \mathscr{C}^\infty(\wedge^\ell\mathbb{X})$ *and* $\Psi \in \mathscr{C}^\infty(\wedge^k\mathbb{X})$, $k = 1, 2, ..., k+\ell = n$ *be closed differential forms and let* $p, q \geqslant 1$, *satisfy the Sobolev relation* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{n}$. *Then for every nonempty open set* $\Omega \subsetneq \mathbb{X}$ *and every test function* $\eta \in \mathscr{C}^\infty(\Omega)$, *we have*

$$\left| \int_\Omega \eta\,(\Phi \wedge \Psi) \right| \preccurlyeq \|\,\eta\,\|_{\mathscr{C}^1(\Omega)}\, \|\,\mathbf{M}_p\Phi\,\|_{\mathscr{L}^\infty(\mathbb{X}\setminus\Omega)}\, \|\,\mathbf{M}_r\Psi\,\|_{\mathscr{L}^\infty(\mathbb{X}\setminus\Omega)} \tag{244}$$

Here, as always, the implied constant depends only on the manifold $\mathbb{X}$.

Proof. Because of the relation $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{n}$ one of these exponents does not exceed $\frac{2n}{n+1}$. Suppose that

$$1 \leqslant p \leqslant \frac{2n}{n+1} < n \tag{245}$$

We consider Whitney's covering of Ω by legitimate balls $\mathbb{B}_i$, $i = 1, 2, ...$, as in Proposition 2.1. Next we construct a partition of unity, non-negative functions $\eta_i \in \mathscr{C}_0^\infty(2\mathbb{B}_i)$ whose sum equals 1 on Ω and such that $|d\eta_i| \preccurlyeq (\operatorname{diam}\mathbb{B}_i)^{-1}$ for $i = 1, 2, ...$ We now calculate as follows:

$$\int_\Omega \eta\,(\Phi \wedge \Psi) = \sum_{i=1}^{\infty} \int_{2\mathbb{B}_i} \eta_i\, \eta\,(\Phi \wedge \Psi) \tag{246}$$

Since $2\mathbb{B}_i$ is a legitimate ball, Poincaré Lemma tells us that the closed form Φ is also exact on $2\mathbb{B}_i$. As a matter of fact, using Sobolev theory of differential forms [32], we find a differential form $\varphi_i \in \mathscr{C}^\infty(\wedge^{\ell-1}2\mathbb{B}_i)$ whose $\mathscr{W}^{1,p}$-norm is controlled by $\mathscr{L}^p$-norm of Φ, and such that $d\varphi_i = \Phi$. Then, by Sobolev-Poincaré inequality, we obtain

$$\left(\fint_{2\mathbb{B}_i} |\varphi_i|^{\frac{np}{n-p}} \right)^{\frac{n-p}{np}} \preccurlyeq (\operatorname{diam}\mathbb{B}_i) \left(\fint_{2\mathbb{B}_i} |\Phi|^p \right)^{\frac{1}{p}} \tag{247}$$

Next, we integrate (246) by parts and use Hölder's inequality with exponents $\frac{np}{n-p}$ and r,

$$\begin{aligned} \left| \int_{2\mathbb{B}_i} \eta_i \, \eta \, (\Phi \wedge \Psi) \right| &= \left| \int_{2\mathbb{B}_i} d(\eta_i \, \eta) \, (\varphi_i \wedge \Psi) \right| \\ &\preccurlyeq \| \eta \|_{\mathscr{C}^1(\Omega)} (\operatorname{diam} \mathbb{B}_i)^{-1} \int_{2\mathbb{B}_i} |\varphi_i| |\Psi| \\ &\preccurlyeq \| \eta \|_{\mathscr{C}^1(\Omega)} \frac{|\mathbb{B}_i|}{\operatorname{diam} \mathbb{B}_i} \left(\fint_{2\mathbb{B}_i} |\varphi_i|^{\frac{np}{n-p}} \right)^{\frac{n-p}{np}} \left(\fint_{2\mathbb{B}_i} |\Psi|^r \right)^{\frac{1}{r}} \\ &\preccurlyeq \| \eta \|_{\mathscr{C}^1(\Omega)} \, |\mathbb{B}_i| \left(\fint_{2\mathbb{B}_i} |\Phi|^p \right)^{\frac{1}{p}} \left(\fint_{2\mathbb{B}_i} |\Psi|^r \right)^{\frac{1}{r}} \end{aligned} \tag{248}$$

At this point it is important to observe that if we enlarge the ball $2\mathbb{B}_i$ by a suitable factor (depending only on $\mathbb{X}$) then it will touch the set $\mathbb{X} \setminus \Omega$. This is immediate from the property 4) listed in Proposition 2.1. In other words, there is $\lambda = \lambda(\mathbb{X})$ such that $2\mathbb{B}_i \subset \lambda\mathbb{B}_i$ and $\lambda\mathbb{B}_i \setminus \Omega \neq \emptyset$. We infer from this observation that

$$\left(\fint_{2\mathbb{B}_i} |\Phi|^p \right)^{\frac{1}{p}} \preccurlyeq \left(\fint_{\lambda\mathbb{B}_i} |\Phi|^p \right)^{\frac{1}{p}} \leqslant \| \mathbf{M}_p \Phi \|_{\mathscr{L}^\infty(\mathbb{X} \setminus \Omega)} \tag{249}$$

Similarly,

$$\left(\fint_{2\mathbb{B}_i} |\Psi|^p \right)^{\frac{1}{p}} \preccurlyeq \| \mathbf{M}_r \Psi \|_{\mathscr{L}^\infty(\mathbb{X} \setminus \Omega)} \tag{250}$$

Therefore, for each ball $2\mathbb{B}_i$ we can write

$$\left| \int_{2\mathbb{B}_i} \eta_i \, \eta \, (\Phi \wedge \Psi) \right| \preccurlyeq |\mathbb{B}_i| \, \| \eta \|_{\mathscr{C}^1(\Omega)} \| \mathbf{M}_p \Phi \|_{\mathscr{L}^\infty(\mathbb{X} \setminus \Omega)} \| \mathbf{M}_r \Psi \|_{\mathscr{L}^\infty(\mathbb{X} \setminus \Omega)} \tag{251}$$

As the overlaping number for the covering $\{2\mathbb{B}_i\}_{i=1,2,\dots}$ depends only on $\mathbb{X}$, we see that $\sum_{i=1}^\infty |\mathbb{B}_i| \preccurlyeq |\Omega|$. Finally, combining (246) and (251) we conclude with the desired estimate at (244).

6.3.2 Point-wise Jacobian versus distributional Jacobian

Here is the first of our estimates which relates the point-wise Jacobian with distributional Jacobian.

LEMMA **6.2.** *Suppose that* $f \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ *and* $\lambda > 0$*. Then*

$$\left| (f^\sharp \omega)[\eta] - \int_{\mathbb{X}\setminus\Omega} \eta(x)\, \mathcal{J}_\omega(x,f)\, dx \right| \preccurlyeq \lambda^n |\Omega| \, \| \eta \|_{\mathscr{C}^1(\mathbb{X})} \tag{252}$$

where $\Omega = \{x; \ (\mathbf{M}_s Df)(x) > \lambda\}$ *and* $s = \frac{n^2}{n+1}$.

Proof. The left hand side of the inequality (252) reduces to

$$\left| \sum_{i=1}^{K} \int_\Omega \eta\,(\Phi_i \wedge \Psi_i) \right| \tag{253}$$

where we consider the closed forms $\Phi_i = f^\sharp \alpha_i \in \mathscr{C}^\infty(\wedge^{\ell_i}\mathbb{X})$ and $\Psi_i = f^\sharp \beta_i \in \mathscr{C}^\infty(\wedge^{k_i}\mathbb{X})$. Using Lemma 6.1 we find that

$$\left| \sum_{i=1}^{K} \int_\Omega \eta\,(\Phi_i \wedge \Psi_i) \right| \leqslant \sum_{i=1}^{K} \left| \int_\Omega \eta\,(\Phi_i \wedge \Psi_i) \right|$$

$$\preccurlyeq |\Omega| \, \| \eta \|_{\mathscr{C}^1(\Omega)} \sum_{i=1}^{K} \| \mathbf{M}_{p_i} \Phi_i \|_{\mathscr{L}^\infty(\mathbb{X}\setminus\Omega)} \| \mathbf{M}_{r_i} \Psi_i \|_{\mathscr{L}^\infty(\mathbb{X}\setminus\Omega)} \tag{254}$$

where $p_i = \frac{s}{\ell_i}$ and $r_i = \frac{s}{k_i}$. Next we observe that

$$|\Phi_i|^{p_i} = |f^\sharp \alpha_i|^{p_i} \preccurlyeq |Df|^{\ell_i p_i} = |Df|^s \tag{255}$$

and

$$|\Psi_i|^{r_i} = |f^\sharp \beta_i|^{r_i} \preccurlyeq |Df|^{k_i r_i} = |Df|^s \tag{256}$$

Finally, inequality (252) follows readily from the point-wise estimates

$$\mathbf{M}_{p_i} \Phi_i \preccurlyeq (\mathbf{M}_s Df)^{\ell_i} \leqslant \lambda^{\ell_i} \tag{257}$$

and

$$\mathbf{M}_{r_i} \Psi_i \preccurlyeq (\mathbf{M}_s Df)^{k_i} \leqslant \lambda^{k_i} \tag{258}$$

where $\ell_i + k_i = n$.

6.3.3 Proof of Theorem 6.1

First observe that our mapping f also satisfies inequality (252) for all but a countable number of parameters $\lambda > 0$. Indeed, by Theorem 1.2 there exist mappings $f_j \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ converging to f in $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$, $s = \frac{n^2}{n+1}$. We need only justify a passage to the limit in the following inequalities:

$$\left| (f_j^\sharp \omega)[\eta] - \int_{\mathbf{M}_s Df_j \leqslant \lambda} \eta(x)\, \mathcal{J}_\omega(x, f_j)\, dx \right| \preccurlyeq \lambda^n \,\|\, \eta \,\|_{\mathscr{C}^1(\mathbb{X})} \int_{\mathbf{M}_s Df_j > \lambda} dx \qquad (259)$$

We recall that λ is a regular value of h if the set $\{x;\ h(x) = \lambda\}$ has measure zero. We also point out that the non regular values are always countable. Since, for any $\varepsilon > 0$,

$$\int_{\mathbf{M}_s Df_j > \lambda} dx \le \int_{\mathbf{M}_s Df > \lambda - \varepsilon} dx + \int_{\mathbf{M}_s |Df - Df_j| > \varepsilon} dx \qquad (260)$$

by the weak-type estimate at (73) we obtain

$$\limsup_{j \to \infty} \int_{\mathbf{M}_s Df_j > \lambda} dx \le \int_{\mathbf{M}_s Df > \lambda} dx \qquad (261)$$

for each regular value λ. Of course, $(f_j^\sharp \omega)[\eta] \to (f^\sharp \omega)[\eta]$, by the definition of the distributional pullback. To deal with the integral in the left hand side we write it as

$$\int_{\mathbb{X}} \eta(x)\, \mathcal{J}_\omega(x, f_j) \chi_j(x)\, dx \qquad (262)$$

where χ_j are characteristic functions of the level sets $\{x; \mathbf{M}_s Df_j(x) \leqslant \lambda\}$. Since the integrands are uniformly bounded by $\lambda^n \,\|\, \eta \,\|_{\mathscr{L}^\infty(\mathbb{X})}$, we can apply the Lebesgue Dominated Convergence Theorem. To this effect we observe that $\mathcal{J}_\omega(x, f_j) \to \mathcal{J}_\omega(x, f)$ a.e. We need only verify that χ_j converge a.e. to χ-the characteristic function of $\{x;\ \mathbf{M}_s Df \leqslant \lambda\}$. This is true for all regular values of $\mathbf{M}_s Df$.

LEMMA **6.3.** *Given measurable functions $h_j : \mathbb{X} \to \mathbb{R}$, converging to h almost everywhere. Then for every regular value λ, we have*

$$\lim_{j \to \infty} \chi_j = \chi \qquad a.e. \qquad (263)$$

where χ_j and χ are the characteristic functions of the level sets $\{x;\ \ h_j(x) \leqslant \lambda\}$ and $\{x;\ \ h(x) \leqslant \lambda\}$, respectively.

We recall that λ is a regular value of h if the set $\{x;\ \ h(x) = \lambda\}$ has measure zero. We also point out that the non regular values are always countable. We can apply this lemma to $h_j = \mathbf{M}_s Df_j$ and $h = \mathbf{M}_s Df$ in view of Corollary 2.4.

Having disposed with the inequality

$$\left|(f^\sharp\omega)[\eta] - \int_{\mathbf{M}_s Df \leqslant \lambda} \eta(x)\, \mathcal{J}_\omega(x,f)\, dx\right| \preccurlyeq \lambda^n \,\|\, \eta \,\|_{\mathscr{C}^1(\mathbb{X})} \int_{\mathbf{M}_s Df > \lambda} dx \tag{264}$$

we now fix a special sequence $\{t_j\}$ for $|Df|$. Recall that t_j are numbers increasing to infinity such that

$$\lim_{j\to\infty} t_j^{\frac{n}{n+1}} \int_{|Df|>t_j} |Df(x)|^{\frac{n^2}{n+1}}\, dx = 0 \tag{265}$$

There are many such sequences. We may choose a one that consists of regular values of $\frac{1}{2}\mathbf{M}_s Df$. Now we are in a position to define the sets $\mathbb{X}_j$

$$\mathbb{X}_j = \Big\{x;\ \ (\mathbf{M}_s Df)(x) \leqslant 2t_j\Big\} \qquad s = \frac{n^2}{n+1} \tag{266}$$

We make use of the estimate in Lemma 6.2 with $\lambda = 2t_j$; these are regular values of $\mathbf{M}_s Df$.

$$\begin{aligned} \left|(f^\sharp\omega)[\eta] - \int_{\mathbb{X}_j} \eta(x)\, \mathcal{J}_\omega(x,f)\, dx\right| \quad &\preccurlyeq \quad \|\, \eta \,\|_{\mathscr{C}^1(\mathbb{X})}\, t_j^n \int_{\mathbf{M}_s Df > 2t_j} dx \\ &\preccurlyeq \quad \|\, \eta \,\|_{\mathscr{C}^1(\mathbb{X})}\, t_j^{n-s} \int_{|Df|>t_j} |Df|^s \end{aligned} \tag{267}$$

The latter follows by weak type inequality stated in Proposition (2.5). Letting t_j go to infinity we conclude with (239). The proof of Theorem 6.1is complete.

The interested reader may see that the above arguments also work for differential forms. Let us state this more general variant of Theorem 6.1 without proof.

THEOREM 6.4. *Given a Cartan form* $\Lambda = \sum_{i=1}^{K} \Phi_i \wedge \Psi_i$, *where* $\Phi_i \in \mathscr{L}^{p_i}(\wedge^{\ell_i}\mathbb{X}) \cap \ker d$ *and* $\Psi_i \in \mathscr{L}^{r_i}(\wedge^{k_i}\mathbb{X}) \cap \ker d$, $1 \leqslant k_i, \ell_i < n$, $k_i + \ell_i = n$. *Here each pair* (p_i, r_i) *consists of Sobolev conjugate exponents. Suppose that*

$$\liminf_{t\to\infty} t^{\frac{1}{n}} \int_{H>t} H(x)\, dx = 0 \tag{268}$$

where $H = \sum_{i=1}^{K} |\Phi_i|^{p_i \ell_i} + |\Psi_i|^{r_i k_i}$. *Then there are measurable sets* $\mathbb{X}_1 \subset \mathbb{X}_2 \subset \ldots \subset \mathbb{X}$ *whose union is* $\mathbb{X}$ *such that*

- Λ *is integrable over each* $\mathbb{X}_i$
- $$\Lambda[\eta] \overset{\text{def}}{=\!=} \sum_{i=1}^{K} (\Phi_i \wedge \Psi_i)[\eta] = \lim_{j\to\infty} \int_{\mathbb{X}_j} \eta\, \Lambda \tag{269}$$

  for every $\eta \in \mathscr{C}^\infty(\mathbb{X})$.

7 $\mathscr{H}^1$-Estimates

In this section we formulate and prove the sharpest possible result concerning $\mathscr{H}^1$-regularity of the Jacobian, see [43] for somewhat different ideas. Before, we need some auxiliary material.

7.1 The Hausdorff content

Let $s > 0$ and $\mathbb{E} \subset \mathbb{R}^n$. The s-content of $\mathbb{E}$ is defined by the rule

$$\mathfrak{h}^s(\mathbb{E}) = \inf \sum_{j=1}^{\infty} (\operatorname{diam} \mathbb{B}_j)^s, \tag{270}$$

where the infimum is taken over all sequences of balls $\mathbb{B}_j \subset \mathbb{R}^n$ covering the set $\mathbb{E}$.

LEMMA **7.1.** *Let* $u \in \mathscr{C}_0^\infty(\Omega)$, $\mathbb{B} = \mathbb{B}(a, R) \subset \mathbb{R}^n$ *and* $0 \leqslant n - s < p \leqslant n$. *Then*

$$\mathfrak{h}^s(\{x \in \mathbb{B};\ |u(x)| \geqslant 1\}) \preccurlyeq R^{p+s} \fint_{\mathbb{B}} |\nabla u(x)|^p\, dx \tag{271}$$

For the proof see [22, p. 45]. We infer from this lemma the following useful corollary.

Corollary **7.1.** *Let $\frac{n^2}{n+1} < p < n$, and $\mathbb{B} = \mathbb{B}(a,\varrho)$ be a legitimate ball in $\mathbb{X}$, dim $\mathbb{X} = n$. Then every compact set $\mathbb{E} \subset \mathbb{B}$ can be split into a finite number of mutually disjoint compact sets $\mathbb{E}_1, ..., \mathbb{E}_k$ such that*

$$\sum_{i=1}^{k} \textit{diam}\, \mathbb{E}_i \preccurlyeq \varrho^{n+1} \left(\fint_{\mathbb{B}} |\nabla u(x)|^p \, dx \right)^{\frac{n}{p}} \tag{272}$$

for every test function $u \in \mathscr{C}(\overline{B}) \cap \mathscr{W}^{1,p}(\mathbb{B})$ satisfying $u \leqslant 0$ on $\partial\mathbb{B}$ and $u \geqslant 1$ on $\mathbb{E}$.

Proof. As the concentric balls $\mathbb{B} \subset 3\mathbb{B}$ lay in a coordinate region Ω, we may change the variables via the diffeomorphism $\kappa : \Omega \xrightarrow{onto} \mathbb{R}^n$, reducing the problem to the Euclidean space. We now apply Lemma 7.1 with $s = \frac{p}{n} < 1$. Clearly, $n - s < p < n$ and

$$\mathfrak{h}^s(\mathbb{E}) \preccurlyeq \varrho^{p+s} \fint_{\mathbb{B}} |\nabla u|^p \tag{273}$$

There exists a finite cover of $\mathbb{E}$ by balls $\mathbb{B}_1, ..., \mathbb{B}_m$ such that

$$\sum_{j=1}^{m} (\operatorname{diam} \mathbb{B}_j)^s \preccurlyeq \varrho^{p+s} \fint_{\mathbb{B}} |\nabla u|^p \tag{274}$$

Let $\mathbb{C}_1, \mathbb{C}_2, ..., \mathbb{C}_k$ be connected components of $\bigcup_{j=1}^{m} \mathbb{B}_j$, and denote $\mathbb{E}_i = \mathbb{E} \cap \mathbb{C}_i$, $i = 1, ..., k$. Obviously, $\mathbb{E}_i$ are mutually disjoint compact sets whose union is $\mathbb{E}$. The rest is the following elementary computation

$$\begin{aligned} \sum_{i=1}^{k} \operatorname{diam} \mathbb{E}_i &\leqslant \sum_{i=1}^{k} \operatorname{diam} \mathbb{C}_i \leqslant \sum_{j=1}^{m} \operatorname{diam} \mathbb{B}_j \\ &\leqslant \left[\sum_{j=1}^{m} (\operatorname{diam} \mathbb{B}_j)^s \right]^{\frac{1}{s}} \preccurlyeq \varrho^{n+1} \left(\fint_{\mathbb{B}} |\nabla u|^p \right)^{\frac{n}{p}} \end{aligned} \tag{275}$$

as claimed. Later we shall choose the following exponents

$$p = \frac{2n^2}{2n+1}, \qquad \text{so that} \quad s = \frac{2n}{2n+1} \tag{276}$$

7.2 The $\mathscr{H}^1$-Theorem

We can now state the main result. We assume that $2 \leqslant n = \dim \mathbb{X} \leqslant \dim \mathbb{Y} = m$. We denote by $\mathscr{B}(g, \rho)$ the ball centered at g and with radius ρ in the space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$, namely

$$\mathscr{B}(g, \rho) = \Big\{ f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y}) : \ \| f - g \|_{\mathscr{W}^{1,n}} < \rho \Big\}.$$

THEOREM 7.1. *Let $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$ be either a Cartan form or a closed form and $g \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. Then there exist a constant $C(\omega, g)$ and a radius $\epsilon > 0$ such that*

$$\| f^\sharp \omega \|_{\mathscr{H}^1(\mathbb{X})} \leqslant \ C(\omega, g) \int_{\mathbb{X}} |Df(x)|^n \, dx \tag{277}$$

for all $f \in \mathscr{B}(g, \epsilon)$. Moreover, the pullback operator

$$^\sharp\omega : \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y}) \to \mathscr{H}^1(\wedge^n \mathbb{X}) \tag{278}$$

is continuous.

Remark 7.1. If $\omega = \sum \alpha_i \wedge \beta_i$ is a Cartan form, as in (220), then in fact

$$\| f^\sharp \omega \|_{\mathscr{H}^1(\mathbb{X})} \preccurlyeq \ \| \omega \| \int_{\mathbb{X}} |Df(x)|^n \, dx \tag{279}$$

for all $f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ where

$$\| \omega \| \ = \sum \| \alpha_i \|_{\mathscr{L}^\infty(\mathbb{Y})} \| \beta_i \|_{\mathscr{L}^\infty(\mathbb{Y})} \tag{280}$$

Remark 7.2. Theorem 7.1 will be established by proving the following inequality for the Fefferman-Stein maximal function of $f^\sharp\omega$:

$$\mathcal{M}(f^\sharp \omega) \ \preccurlyeq \ C(\omega, g) \left[\mathbf{M}_p(Df) + \| Df \|_n \right]^n \tag{281}$$

where

$$\mathbf{M}_p(Df) = (\mathbf{M}|Df|^p)^{\frac{1}{p}} \, , \quad p = \frac{2n^2}{2n+1} \tag{282}$$

Indeed, estimate at (277) is then straightforward by the maximal theorem. As for the continuity of the pullback $^\sharp\omega$ we argue as follows. Let $f_j \to g$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. We need to show that

$$\mathcal{M}(f_j^\sharp\omega - g^\sharp\omega) \to 0 \quad \text{in } \mathscr{L}^1(\mathbb{X}) \tag{283}$$

We may assume that $f_j \in \mathscr{B}(g, \epsilon)$. Therefore, we have the uniform bound

$$\sup_{t>0} \left| \int_{\mathbb{X}} K_t(x, \cdot)(f_j^\sharp\omega - f^\sharp\omega) \right|$$
$$\preccurlyeq \; (\mathbf{M}_p(Df_j) + \mathbf{M}_p(Df) + \| Df_j \|_n + \| Df \|_n)^n \tag{284}$$

for every $x \in \mathbb{X}$. In the right hand side we have a sequence of functions converging in $\mathscr{L}^1(\mathbb{X})$ to $2^n \, (\mathbf{M}_p(Df) + \| Df \|_n)^n$. By virtue of the Lebesgue Dominated Convergence Theorem we shall have established (283) if we prove that

$$\sup_{t>0} \left| \int_{\mathbb{X}} K_t(x, \cdot)(f_j^\sharp\omega - f^\sharp\omega) \right| \to 0 \tag{285}$$

for almost every $x \in \mathbb{X}$. To this end, we estimate the supremum in terms of the Hardy-Littlewood maximal function. We are reduced to proving that

$$\mathbf{M}(f_j^\sharp\omega - f^\sharp\omega) \to 0 \quad \text{ a.e. in } \mathbb{X} \tag{286}$$

But this is well known, since $f_j^\sharp\omega \to f^\sharp\omega$ in $\mathscr{L}^1(\mathbb{X})$.

7.2.1 Step 1.-The case of Cartan forms

We give here the reasoning for the inequality (279).

It suffices to consider one wedge product $\omega = \alpha \wedge \beta$, with $\alpha \in \mathscr{C}^\infty(\wedge^l \mathbb{Y}) \cap \ker d$ and $\beta \in \mathscr{C}^\infty(\wedge^{n-l}\mathbb{Y}) \cap \ker d$, and we may assume that $\frac{n}{2} \leqslant l < n$. Clearly

$$f^\sharp\omega = f^\sharp\alpha \wedge f^\sharp\beta \tag{287}$$

Here we have sufficient degree of regularity to ensure that the factors are closed forms. These factors satisfy:

$$\left| f^\sharp\alpha \right| \; \preccurlyeq \; \| \alpha \|_\infty |Df|^l \in \mathscr{L}^{\frac{n}{l}}(\mathbb{X}) \tag{288}$$

$$\left|f^\sharp\beta\right| \preccurlyeq \|\beta\|_\infty |Df|^{n-l} \in \mathscr{L}^{\frac{n}{n-l}}(\mathbb{X}) \tag{289}$$

The Hodge theory of the deRham cohomology tells us that

$$f^\sharp\alpha = d\gamma + \chi, \quad \chi \in \mathscr{W}^{1,\frac{n}{l}}(\wedge^{l-1}\mathbb{X}) \tag{290}$$

where χ is a harmonic field of degree l representing the cohomology class of $f^\sharp\alpha$. Harmonic fields, being $\mathscr{C}^\infty$-smooth, are harmless. They form a finite dimensional space and we have nice bounds, such as

$$\|\chi\|_\infty \preccurlyeq \|\chi\|_s \preccurlyeq \|f^\sharp\alpha\|_s \preccurlyeq \|\alpha\|_\infty \|Df\|_{ls}^l \tag{291}$$

provided $1 < s < \infty$. Taking $s = \frac{n}{l}$ we obtain

$$\|\chi\|_\infty \preccurlyeq \|\alpha\|_\infty \|Df\|_n^l \tag{292}$$

Accordingly, we split $f^\sharp\omega$ as

$$f^\sharp\omega = d\gamma \wedge (f^\sharp\beta) + \chi \wedge (f^\sharp\beta) \tag{293}$$

The latter term poses no difficulty as it belongs to $\mathscr{L}^2(\wedge^n\mathbb{X})$. Indeed, we have

$$\begin{aligned} \|\chi \wedge (f^\sharp\beta)\|_2 &\preccurlyeq \|\chi\|_\infty \|f^\sharp\beta\|_2 \\ &\preccurlyeq \|\alpha\|_\infty \|\beta\|_\infty \|Df\|_n^l \, \| |Df|^{n-l} \|_2 \\ &\preccurlyeq \|\alpha\|_\infty \|\beta\|_\infty \|Df\|_n^n \end{aligned} \tag{294}$$

Note that we have actually a point-wise estimate of the maximal function

$$\begin{aligned} \mathcal{M}(\chi \wedge f^\sharp\beta) &\preccurlyeq \|\chi\|_\infty \, \mathbf{M}(f^\sharp\beta) \\ &\preccurlyeq \|\chi\|_\infty \|\beta\|_\infty \, \mathbf{M}(|Df|^{n-l}) \\ &\preccurlyeq \|\alpha\|_\infty \|\beta\|_\infty \|Df\|_n^l \, [\mathbf{M}_p(|Df|)]^{n-l} \\ &\preccurlyeq \|\alpha\|_\infty \|\beta\|_\infty \big[\|Df\|_n + \mathbf{M}_p(|Df|)\big]^n \end{aligned} \tag{295}$$

We will now concern ourselves with the estimates of the maximal function of $d\gamma \wedge (f^\sharp\beta)$. Before jumping to the computation let us observe that the exact l-form $d\gamma$ is not affected if we add any closed form to γ. We begin with the following variant of the Poincaré-Sobolev inequality for differential forms [32].

$$\left(\int_{\mathbb{B}} |\gamma-\gamma_0|^{\frac{ns}{n-s}}\right)^{\frac{n-s}{ns}} \preccurlyeq \left(\int_{\mathbb{B}} |d\gamma|^s\right)^{\frac{1}{s}}, \quad 1 \leqslant s < n, \tag{296}$$

where $\mathbb{B} = \mathbb{B}(x,t)$ is a legitimate ball in $\mathbb{X}$, $0 < t < R_{\mathbb{X}}$, and γ_0 is a suitable closed form on $\mathbb{B}$. As far as integration is concerned we also notice that the mollifying kernels $\zeta \to K_t(x,\zeta)$ are supported in the ball $\mathbb{B} = \mathbb{B}(x,t')$, where t' is comparable with t by a factor depending only on $\mathbb{X}$.

Remark 7.3. For notational convenience one could introduce new kernel

$$K'_t(x,\zeta) = K_{t'}(x,\zeta)$$

so that the function $\zeta \to K'_t(x,\zeta)$ would be supported in $\mathbb{B} = \mathbb{B}(x,t)$. Instead of doing this we simply assume that $\zeta \to K_t(x,\zeta)$ is supported in $\mathbb{B}(x,t)$.

Integration by parts yields

$$\begin{aligned} \left|(d\gamma\wedge f^\sharp\beta)_t(x)\right| &= \left|\int_{\mathbb{B}} K_t(x,\cdot)d(\gamma-\gamma_0)\wedge f^\sharp\beta\right| \\ &\preccurlyeq \frac{\|\beta\|_\infty}{t^{n+1}}\int_{\mathbb{B}}|\gamma-\gamma_0||Df|^{n-l} \\ &\preccurlyeq \frac{\|\beta\|_\infty}{t^{n+1}}\left(\int_{\mathbb{B}}|\gamma-\gamma_0|^{\frac{ns}{n-s}}\right)^{\frac{n-s}{ns}}\left(\int_{\mathbb{B}}|Df|^{\frac{ns(n-l)}{ns-n+s}}\right)^{\frac{ns-n+s}{ns}} \end{aligned} \tag{297}$$

We take $s = \frac{n^2}{l(n+1)}$ to obtain

$$\left|(d\gamma\wedge f^\sharp\beta)_t(x)\right| \preccurlyeq \|\beta\|_\infty \left(⨍_{\mathbb{B}}|d\gamma|^s\right)^{\frac{1}{s}}\left(⨍_{\mathbb{B}}|Df|^{\frac{n^2}{n+1}}\right)^{\frac{(n-l)(n+1)}{n^2}} \tag{298}$$

Directly from the decomposition $f^\sharp\alpha = d\gamma + \chi$ we see that

$$\begin{aligned} |d\gamma| &\leqslant |f^\sharp\alpha| + \|\chi\|_\infty \\ &\preccurlyeq \|\alpha\|_\infty |Df|^l + \|\alpha\|_\infty \|Df\|_n^l \end{aligned} \tag{299}$$

Hence

$$\left(\fint_{\mathbb{B}} |d\gamma|^s\right)^{\frac{1}{s}} \preccurlyeq \|\alpha\| \left[\left(\fint_{\mathbb{B}} |Df|^{\frac{n^2}{n+1}}\right)^{\frac{n+1}{n^2}} + \|Df\|_n\right]^l \tag{300}$$

Note that $\left(\fint_{\mathbb{B}} |Df|^{\frac{n^2}{n+1}}\right)^{\frac{n+1}{n^2}} \leqslant \mathbf{M}_p(Df)$, since $p = \frac{2n^2}{2n+1} > \frac{n^2}{n+1}$. This combined with (300), (296) and (298) results in the following estimate

$$\left|(d\gamma \wedge f^\sharp\beta)_t\right| \preccurlyeq \|\alpha\|_\infty \|\beta\|_\infty \Big[\|Df\|_n + \mathbf{M}_p(Df)\Big]^n \tag{301}$$

By virtue of the previously established inequality for $\mathcal{M}(\chi \wedge f^\sharp\beta)$, we obtain

$$\left|(f^\sharp\omega)_t\right| \preccurlyeq \|\alpha\|_\infty \|\beta\|_\infty \Big[\|Df\|_n + \mathbf{M}_p(Df)\Big]^n \tag{302}$$

for $0 < t \leqslant R_{\mathbb{X}}$. In summary, if $\omega = \alpha \wedge \beta$ is a Cartan form, then

$$\mathcal{M}(f^\sharp\omega) \preccurlyeq \|\alpha\|_\infty \|\beta\|_\infty \Big[\|Df\|_n + \mathbf{M}_p(Df)\Big]^n \tag{303}$$

Hence, Remark 7.1 follows by the maximal theorem.

7.2.2 Step 2.-The case of closed forms

Now, we pass to proving Theorem 7.1 for closed forms.

We aim to prove the following inequality

$$\left|\int_{\mathbb{X}} K_t(a,x)\mathcal{J}_\omega(x,f)\,dx\right| \preccurlyeq C(\omega,g)\Big[\mathbf{M}_p(|Df|)(a) + \|Df\|_n\Big]^n \tag{304}$$

for all $t > 0$, $a \in \mathbb{X}$ and $f \in \mathscr{B}(g,\epsilon)$. Here $\mathcal{J}_\omega(\cdot,f)$ stands for the pointwise pullback $f^\sharp\omega$.

We shall work with small balls $\mathbb{B}(a,t) \subset \mathbb{X}$; say with radii $t \leqslant R = R(g)$. We now list all the conditions on R and ϵ needed in the sequel. First condition is:

$$R \leqslant R_{\mathbb{X}}. \tag{305}$$

Another restriction on ϵ and R results from the following lemma

Lemma 7.2. [Oscillation lemma] *Let $h \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ and let $\mathbb{B}(a, R)$ be a legitimate ball in $\mathbb{X}$. Then for every $0 < t < 2t \leqslant R$ there exists $r \in (t, 2t)$ such that*

$$\underset{\partial\mathbb{B}(a,r)}{osc}\ h \ \preccurlyeq\ r\left(⨍_{\mathbb{B}(a,2r)} |Dh|^p\right)^{\frac{1}{p}} \tag{306}$$

where $n - 1 < p = \frac{2n^2}{n+1} < n$. By Hölder's inequality

$$\left(\underset{\partial\mathbb{B}(a,r)}{osc}\ h\right)^n \leqslant C(n, \mathbb{X})^n \int_{\mathbb{B}(a,R)} |Dh|^n \tag{307}$$

We want these oscillations to be smaller than $R_{\mathbb{Y}}$. For this reason we must confine ourselves to $R < R_{\mathbb{X}}$ and ϵ small enough so that

$$C(n, \mathbb{X})^n \int_{\mathbb{B}(a,R)} |Dg(x)|^n\, dx \leqslant \left(\frac{1}{2} R_{\mathbb{Y}}\right)^n \tag{308}$$

for all $a \in \mathbb{X}$ and

$$C(n, \mathbb{X})\epsilon < \frac{1}{2} R_{\mathbb{Y}} \tag{309}$$

Next we wish that the integrals $\int_{\mathbb{B}(a,R)} |Df|^n$, with $a \in \mathbb{X}$ and $f \in \mathscr{B}(g, \epsilon)$ will be sufficiently small. We have

$$\begin{aligned} \left(\int_{\mathbb{B}(a,R)} |Df|^n\, dx\right)^{1/n} &\leq \left(\int_{\mathbb{B}(a,R)} |Dg|^n\, dx\right)^{1/n} \\ &\quad + \left(\int_{\mathbb{B}(a,R)} |Df - Dg|^n\, dx\right)^{1/n} \\ &\leq \left(\int_{\mathbb{B}(a,R)} |Dg|^n\, dx\right)^{1/n} + \epsilon \end{aligned} \tag{310}$$

for each $a \in \mathbb{X}$.

A theorem of B. White (Theorem 2. p. 135 in [53]) states that for each $g \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ there exists $\rho > 0$ such that if $f_1, f_2 \in \mathscr{B}(g, 2\rho)$ are smooth for $i = 1, 2$, then f_1 and f_2 are homotopic. The requirement

$$C(\mathbb{Y})\left(\left(\int_{\mathbb{B}(a,R)} |Dg|^n\, dx\right)^{1/n} + \epsilon\right) \leq \rho \tag{311}$$

where the constant $C(\mathbb{Y})$ will ve determined later, see (321), will be the last condition on R and ϵ.

The above conditions at (305)–(311) determine the numbers $R = R(g) > 0$ and $\epsilon = \epsilon(g) > 0$. The estimates occurring in the rest of this section will be explicit given $R = R(g)$ and $\epsilon = \epsilon(g)$. Returning to the inequality (HK40) let us temporarily fix both $f \in \mathscr{B}(g, \epsilon)$ and the parameter $0 < t < R(g)$. It involves no loss of generality in assuming that $f \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$, simply because $\mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ is dense in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. It remains to prove the inequality

$$\left| \int_{\mathbb{X}} K_t(a,x) \mathcal{J}_\omega(x,f) \right| \preccurlyeq \left[\mathbf{M}_p(|Df|)(a) \right]^n \tag{312}$$

where the implied constant depends on R which we have already determined for the given function g.

We consider a fixed finite covering of $\mathbb{Y}$ by legitimate balls of radius $T := R_{\mathbb{Y}}$. By the oscillation inequality at (306) we find a radius $r \in (t, 2t)$ such that

$$\operatorname*{osc}_{\partial \mathbb{B}(a,r)} f \leqslant R_{\mathbb{Y}} = T \tag{313}$$

which is immediate from (307). We look at the image of $f : \partial\mathbb{B}(a,r) \to \mathbb{Y}$. It intersects some legitimate ball $\mathbb{B}(b,T) \subset \mathbb{Y}$ from the above mentioned fixed finite family and by (313),

$$f(\partial \mathbb{B}(a,r)) \subset \mathbb{B}(b, 2T). \tag{314}$$

Recall that there is a coordinate chart $(\Omega, \kappa) \in \mathcal{A}$ onto $\mathbb{R}^m$ such that $\mathbb{B}(b, 4T) \subset \Omega$. We may assume that $\kappa(b) = 0$. Consider a cut-off function $\eta \in \mathscr{C}^\infty(\mathbb{Y})$ with support in $\mathbb{B}(b, 4T)$ and equal to 1 on a neighborhood of $\mathbb{B}(b, 3T)$. The form $\kappa^{-1}\sharp\omega$ is closed in $\mathbb{R}^m$ and thus there exists a form $\gamma \in \mathscr{C}^\infty(\wedge^{n-1}\mathbb{R}^m)$ such that $d\gamma = \kappa^{-1}\sharp\omega$. Consider the form

$$\tilde{\omega} = d(\eta\kappa\sharp\gamma)$$

Then $\tilde{\omega}$ is exact and coincides with ω on $\mathbb{B}(b, 3T)$. Since the legitimate ball is selected from the given finite family, all quantities related to η and κ depend only on $\mathbb{Y}$, and all quantities related to $\tilde{\omega}$ depend only on $\mathbb{Y}$ and ω. Define

$$\tilde{f}(x) = \begin{cases} \kappa^{-1}\Big(\kappa(z) + \eta(f(x))\big(\kappa(f(x)) - \kappa(z)\big)\Big) & f(x) \in \Omega \\ b & f(x) \notin \Omega \end{cases}$$

where z is a point of $\mathbb{Y}$ which is nearest to the mean value of f. Then

$$\mathcal{J}_\omega(x, \tilde{f}) = \mathcal{J}_{\tilde{\omega}}(x, f)$$

and by (303) we have

$$\begin{aligned} \Big|\int_{\mathbb{X}} K_t(a,x)\mathcal{J}_\omega(x,\tilde{f})\,dx\Big| &\le C(\omega)\Big[\,\|\,D\tilde{f}\,\|_n + \mathbf{M}_p(D\tilde{f})(a)\Big]^n \\ &\le C(\omega)\Big[\,\|\,Df\,\|_n + \|\,f - z\,\|_n + \mathbf{M}_p(Df)(a)\Big]^n \\ &\le C(\omega)\Big[\,\|\,Df\,\|_n + \mathbf{M}_p(Df)(a)\Big]^n \end{aligned} \tag{315}$$

where in the last step we have used a version of the Poincaré inequality.

It remains to achieve the estimate

$$\Big|\int_{\mathbb{X}} K_t(a,x)\Big(\mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f)\Big)\,dx\Big| \le C(\omega, g)\Big(\mathbf{M}_p(Df)(a)\Big)^n \tag{316}$$

Let us look closely at the set

$$\mathbb{E} := \mathbb{B}(a,r) \cap f^{-1}\Big(\mathbb{Y} \setminus \mathbb{B}(b, 3T)\Big) \supset \mathbb{B}(a,r) \cap \Big\{\mathcal{J}_\omega(x,\tilde{f}) \neq \mathcal{J}_\omega(x,f)\Big\}$$

We first notice that $f(\partial\mathbb{B}(a,r))$ lies in $\mathbb{B}(b, 2T)$ by (314). Thus $\mathbb{E}$ is compact subset of the ball $\mathbb{B}(a,r)$. The function

$$u = \frac{|f - b|}{T} - 2$$

is negative on $\partial\mathbb{B}(a,r)$ and assumes values $\geqslant 1$ on the set $\mathbb{E}$. By Corollary 7.1 we can split $\mathbb{E}$ into mutually disjoint compact sets $\mathbb{E}_1, \ldots, \mathbb{E}_k$ such that

$$\begin{aligned} \sum_{i=1}^k \operatorname{diam}\mathbb{E}_i \;&\leqslant\; C_{\mathbb{X}}\, r^{n+1}\left(\fint_{\mathbb{B}(a,r)} |\nabla u|^p\right)^{\frac{n}{p}} \\ &\preccurlyeq\; t^{n+1}\Big[\mathbf{M}_p(|Df|)(a)\Big]^n \end{aligned} \tag{317}$$

We accordingly split the integral at (316) as:

$$\begin{aligned}\int_{\mathbb{B}(a,t)} K_t(a,x)\Big(\mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f))\Big)\,dx \\ = \sum_{i=1}^{k}\int_{\mathbb{E}_i} K_t(a,x)\Big(\mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f))\Big)\,dx\end{aligned} \tag{318}$$

An important point to make here is that

$$\int_{\mathbb{E}_i} \mathcal{J}_\omega(x,\tilde{f})\,dx = \int_{\mathbb{E}_i} \mathcal{J}_\omega(x,f)\,dx, \quad \text{for} \ \ i = 1,2,...,k \tag{319}$$

To see this we consider the following functions

$$f_i = \begin{cases} \tilde{f} & \text{on } \mathbb{E}_i \\ f & \text{on } \mathbb{X}\setminus\mathbb{E}_i \end{cases} \tag{320}$$

Then f_i are smooth. Using (311) we obtain

$$\|\, f - f_i \,\|_{1,n} \leq C(\mathbb{Y})\Big(\int_{B(a,r)} |Df|^n\Big)^{1/n} \tag{321}$$

Thus both functions f and f_i belong to $\mathcal{B}(g,2\rho)$. This, by the definition of ρ, implies that f and f_i are homotopic. Hence

$$\int_{\mathbb{X}} \mathcal{J}_\omega(x,f_i)\,dx = \int_{\mathbb{X}} \mathcal{J}_\omega(x,f)\,dx$$

which proves the claim (319). Similarly we obtain the estimate

$$\int_{\mathbb{E}_i} \big|\mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f)\big|\,dx \ \preccurlyeq \ \int_{B(a,r)} |Df|^n \ \preccurlyeq \ \rho \tag{322}$$

To make use of the formulas (317) we pick up some points $x_i \in \mathbb{E}_i$ and express the left term of (316) as

$$\begin{aligned}\int_{\mathbb{B}(a,t)} K_t(a,x)\Big(\mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f)\Big)\,dx \\ = \sum_{i=1}^{k}\int_{\mathbb{E}_i} \Big[K_t(a,x) - K_t(a,x_i)\Big]\Big(\mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f)\Big)\,dx\end{aligned} \tag{323}$$

Next, we use the following inequalities for the mollifiers

$$|K_t(a,x) - K_t(a,x_i)| \preccurlyeq \frac{|x - x_i|}{t^{n+1}} \leqslant \frac{\operatorname{diam} \mathbb{E}_i}{t^{n+1}} \tag{324}$$

for all $x \in \mathbb{E}_i$. They follows routinely from (66). Finally, by (317) and (322) we conclude with the desired estimate

$$\begin{aligned} &\Big| \int_{\mathbb{B}(a,t)} K_t(a,x) \Big(\mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f) \Big)\, dx \Big| \\ &\quad \preccurlyeq \sum_{i=1}^{k} \frac{\operatorname{diam} \mathbb{E}_i}{t^{n+1}} \int_{\mathbb{E}_i} \big| \mathcal{J}_\omega(x,\tilde{f}) - \mathcal{J}_\omega(x,f) \big|\, dx \qquad (325) \\ &\quad \preccurlyeq \Big[\mathbf{M}_p(Df)(a) \Big]^n \end{aligned}$$

completing the proof of (316) and thus of Theorem 7.1.

8 Degree Theory

$\mathscr{L}^1$-estimates of the Jacobian and related wedge products lead to an analytic degree theory of weakly differentiable mappings. Readers interested in this topic will find it profitable to consult Brezis and Nirenberg [7], [8] and also [4], [6], [12], [19], [15]. Analytic approach to the degree of smooth mappings begins with a choice of a closed form $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$. This form must have non-vanishing integral, which may not be possible within the class of Cartan forms;

$$\omega = \sum_{i=1}^{K} \alpha_i \wedge \beta_i, \quad d\alpha_i = d\beta_i = 0, \quad \int_{\mathbb{Y}} \omega \neq 0, \tag{326}$$

Unluckily, such is the case $\mathbb{Y} = \mathbb{S}^n$. On the other hand we need Cartan forms in order to employ Theorems 6.1, 6.2 and 6.3. More generally, if $\mathscr{H}^l(\mathbb{Y}) = 0$ for all $1 \leqslant l < n$, Cartan's forms are exact and, therefore, have integral zero. This being so, we must assume that $\mathscr{H}^l(\mathbb{Y}) \neq 0$, for some $1 \leqslant l < n$.

8.1 Definition of the degree via weak integrals

There are several approaches to the degree of Sobolev mappings that are each of considerable interest. We shall give first the most general one.

DEFINITION **8.1.** Let $\dim \mathbb{X} = \dim \mathbb{Y} = n$. The notation and hypothesis being as in Theorem 6.1, we define the degree of $f : \mathbb{X} \to \mathbb{Y}$ by the rule

$$\deg(f; \mathbb{X}, \mathbb{Y}) = \lim_{j \to \infty} \int_{\mathbb{X}_j} \mathcal{J}_\omega(x, f)\, dx = (f^\sharp \omega)[1] \tag{327}$$

where $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$ has integral 1 over $\mathbb{Y}$.

Absence of ω in the notation for the degree is justified by Theorem 8.1 below.

From differential topology we know that the notion of the topological degree of a mapping $f : \mathbb{X} \to \mathbb{Y}$ of class $\mathscr{C}^1(\mathbb{X}, \mathbb{Y})$ coincides with the integral of $\mathcal{J}_\omega(x, f)$ and, therefore, is an integer. Basic characteristics of $\deg(f; \mathbb{X}, \mathbb{Y})$; that justify the name degree, are listed in the following theorem.

THEOREM 8.1. *With the reference to Definition 8.1, we have*

(i) Different choices of the Cartan forms with integral 1 yield the same limit at (327).

(ii) If smooth mappings $f_k : \mathbb{X} \to \mathbb{Y}$, converge to f in $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$, with the critical exponent $s = \frac{n^2}{n+1}$, then

$$\deg(f; \mathbb{X}, \mathbb{Y}) = \lim_{k \to \infty} \deg(f_k; \mathbb{X}, \mathbb{Y})$$

Moreover, such a smooth approximation of f always exists.

(iii) The degree is an integer

(iv) If the Jacobian $\mathcal{J}_\omega(x, f)$ is non-negative, then it is integrable and we have

$$\deg(f; \mathbb{X}, \mathbb{Y}) = \int_{\mathbb{X}} \mathcal{J}_\omega(x, f)\, dx$$

(v) *If* $\deg(f;\mathbb{X},\mathbb{Y}) \neq 0$, *then the image of any set of full measure is dense in* $\mathbb{Y}$

Proof. The proof is simply an adaptation of ideas of analytic degree theory of smooth mappings. To prove statement (i) we fix two Cartan forms $\omega, \theta \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$ whose integral is equal to 1. Thus

$$\lim_{j\to\infty} \int_{\mathbb{X}_j} \mathcal{J}_\omega(x,f)\, dx = (f^\sharp \omega)[1] \tag{328}$$

and

$$\lim_{j\to\infty} \int_{\mathbb{X}_j} \mathcal{J}_\theta(x,f)\, dx = (f^\sharp \theta)[1] \tag{329}$$

by Theorem 6.1. Now the problem reduces to showing that $(f^\sharp\omega)[1] = (f^\sharp\theta)[1]$. Thanks to Theorem 1.2, we can approximate f by smooth mappings in the metric topology of $\mathscr{W}^{1,s}(\mathbb{X},\mathbb{Y})$. Since the nonlinear functionals $f \to (f^\sharp\omega)[1]$ and $f \to (f^\sharp\theta)[1]$ are continuous in $\mathscr{W}^{1,s}(\mathbb{X},\mathbb{Y})$, we are further reduced to showing that $(f^\sharp\omega - f^\sharp\theta)[1] = 0$, whenever $f \in \mathscr{C}^\infty(\mathbb{X},\mathbb{Y})$. To this end we observe that the differential form $\omega - \theta \in \mathscr{C}^\infty(\wedge^n\mathbb{Y})$ is exact; that is $\omega - \theta = d\alpha$ for some $\alpha \in \mathscr{C}^\infty(\wedge^{n-1}\mathbb{Y})$. This is because the integral of $\omega - \theta$ over $\mathbb{Y}$ vanishes and $\mathcal{H}^n(\mathbb{Y}) \simeq \mathbb{R}$. The rest is folklore, $f^\sharp(d\alpha) = d(f^\sharp\alpha)$ and by Stokes' theorem

$$(f^\sharp\omega - f^\sharp\theta)[1] = \int_{\mathbb{X}} d(f^\sharp\alpha) = 0 \tag{330}$$

The property (ii) is immediate from Theorem 1.2. Then to see (iii), we need only recall that the degree of a smooth mapping is an integer. Also (iv) follows readily from Theorem 6.2.

As for the statement (v), consider a set $\mathbb{X}'$ of full measure in $\mathbb{X}$. Let us assume, to the contrary, that $f : \mathbb{X}' \to \mathbb{Y}$ omits an open set $\mathbb{V} \subset \mathbb{Y}$. Fix a Cartan form $\omega \in \mathscr{C}_0^\infty(\wedge^n\mathbb{V})$ whose integral over $\mathbb{Y}$ equals 1; for instance, $\omega = \lambda(y)\, dy$ with $\lambda \in \mathscr{C}_0^\infty(\mathbb{V})$. Thus $\mathcal{J}_\omega(x,f) = 0$ almost everywhere in $\mathbb{X}'$, hence in $\mathbb{X}$ as well. Being so, the limit at (327) is equal to zero, contradicting the assumption that $\deg(f;\mathbb{X},\mathbb{Y}) \neq 0$.

8.2 Weak integrals

Our next objective is to investigate properties of the degree function $f \to \deg(f; \mathbb{X}, \mathbb{Y})$ defined on mappings $f \in \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$, $s = \frac{n^2}{n+1}$, such that

$$\liminf_{t \to \infty} t^{n-s} \int_{|Df|>t} |Df|^s = 0 \tag{331}$$

We assume here that $\mathbb{Y}$ has nontrivial l-cohomology for some $1 \leqslant l < n = \dim \mathbb{Y} = \dim \mathbb{X}$. As a preliminary step we consider a nonlinear functional $\mathcal{J}_\omega : \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y}) \to \mathbb{R}$, defined by

$$\mathcal{J}_\omega[f] = (f^\sharp \omega)[1] \tag{332}$$

where $\omega \in \mathscr{C}^\infty(\wedge^n \mathbb{Y})$ is fixed. We call it *weak integral* of the Jacobian.

8.2.1 Continuity of the weak integral

Surprisingly, $\mathcal{J}_\omega$ is continuous even in the metric topology of $\mathscr{W}^{1,n-1}(\mathbb{X}, \mathbb{Y})$.

Lemma **8.1.** *Suppose* $f_\nu \in \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$ *converge to*

$$f \in \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}^{1,n-1}(\mathbb{X}, \mathbb{Y})$$

in the metric of $\mathscr{W}^{1,n-1}(\mathbb{X}, \mathbb{Y})$. *Then*

$$\lim_{\nu \to \infty} \mathcal{J}_\omega[f_\nu] = \mathcal{J}_\omega[f] \tag{333}$$

8.2.2 $\mathscr{L}^1$-Estimate of the weak integral

Before jumping to the proof of Lemma 8.1 let us state another surprising result, which will be the key ingredient.

Lemma **8.2.** *Given* $\Phi \in \mathscr{L}^p(\wedge^l \mathbb{X}) \cap \ker d$ *and* $\Psi \in \mathscr{L}^r(\wedge^k \mathbb{X}) \cap \ker d$, $1 \leqslant k, l < n$, $k + l = n$, *where* $1 \leqslant p, r < \infty$ *are Sobolev conjugate exponents. We have*

$$|(\Phi \wedge \Psi)[1]| \preccurlyeq \| \Phi \|_{\mathscr{L}^1(\mathbb{X})} \| \Psi \|_{\mathscr{L}^1(\mathbb{X})} \tag{334}$$

This estimate is not always true if we replace 1 by arbitrary test function $\eta \in \mathscr{C}^\infty(\mathbb{X})$.

Proof. By the definition of the distributional wedge product, given at (233), it will be enough to prove (334) for smooth forms. In this case, we have

$$(\Phi \wedge \Psi)[1] = \int_{\mathbb{X}} \Phi \wedge \Psi \tag{335}$$

If one of the factors Φ or Ψ is exact then so is their wedge product. In this case the integral vanishes, so there is nothing to estimate. But this is not always the case. Fortunately, closed forms are exact modulo harmonic fields, which we consider as harmless terms. Precisely, we proceed as follows:

$$\Phi = d\varphi + h, \quad h \in \mathcal{H}^l(\mathbb{X}) \text{ and } \varphi \in \mathscr{C}^\infty(\wedge^{l-1}\mathbb{X}) \tag{336}$$

Although we may not have good estimates of $d\varphi$ in terms of Φ, we do have, however, good estimates of the harmonic component. Luckily, $d\varphi$ disappears after we integrate at (335):

$$\int_{\mathbb{X}} \Phi \wedge \Psi = \int_{\mathbb{X}} h \wedge \Psi \tag{337}$$

by Stokes' theorem. Hence

$$|(\Phi \wedge \Psi)[1]| \leqslant \| h \|_{\mathscr{L}^\infty(\mathbb{X})} \| \Psi \|_{\mathscr{L}^1(\mathbb{X})} \tag{338}$$

The rest of the proof relies on the regularity properties of the harmonic fields, see inequality (35). Accordingly,

$$\| h \|_\infty \preccurlyeq \sup_{t>0} t\,|\{x; \ \ |h(x)| > t\}| \preccurlyeq \| \Phi \|_1 \tag{339}$$

as desired.

8.2.3 Proof of Lemma 8.1

As we have already observed in Lemma 2.4 every $\omega \in \mathscr{C}^\infty(\wedge^n\mathbb{Y})$ is a Cartan form, say

$$\omega = \sum_{i=1}^{K} \alpha_i \wedge \beta_i \tag{340}$$

Hence $(f_\nu^\sharp \omega)[1] = \sum_{i=1}^K (\Phi_i^\nu \wedge \Psi_i^\nu)[1]$, where both $\Phi_i^\nu = f_\nu^\sharp \alpha_i$ and $\Psi_i^\nu = f_\nu^\sharp \beta_i$ are closed forms of degree $1 \leqslant l_i < n$ and $1 \leqslant k_i < n$, respectively. Similarly, $(f^\sharp \omega)[1] = \sum_{i=1}^K (\Phi_i \wedge \Psi_i)[1]$. First observe the point-wise inequalities $|\Phi_i^\nu| \preccurlyeq |Df_\nu|^{l_i}$ and $|\Psi_i^\nu| \preccurlyeq |Df_\nu|^{k_i}$, $l_i + k_i = n$. Thus $\Phi_i^\nu \in \mathscr{L}^{p_i}(\wedge^{l_i}\mathbb{X})$ and $\Psi_i^\nu \in \mathscr{L}^{r_i}(\wedge^{k_i}\mathbb{X})$, with a Sobolev conjugate pair of exponents $p_i = \frac{s}{l_i}$ and $r_i = \frac{s}{k_i}$, $\frac{1}{p_i} + \frac{1}{r_i} = \frac{n}{s} = 1 + \frac{1}{n}$. We need to show that

$$\lim_{\nu \to \infty} (\Phi_i^\nu \wedge \Psi_i^\nu)[1] = (\Phi_i \wedge \Psi_i)[1] \tag{341}$$

for every $i = 1, 2, ..., K$. Using telescoping decomposition, this reduces to two equations:

$$\lim_{\nu \to \infty} \Big[(\Phi_i^\nu - \Phi_i) \wedge \Psi_i^\nu \Big][1] = 0 \tag{342}$$

and

$$\lim_{\nu \to \infty} \Big[(\Phi_i \wedge (\Psi_i^\nu - \Psi_i) \Big][1] = 0 \tag{343}$$

We will only demonstrate the proof of (342); the other being similar is omitted. By Lemma 8.2, we have

$$\begin{aligned} \Big| \Big[(\Phi_i^\nu - \Phi_i) \wedge \Psi_i^\nu \Big][1] \Big| &\preccurlyeq \| \Phi_i^\nu - \Phi_i \|_{\mathscr{L}^1(\mathbb{X})} \| \Psi_i^\nu \|_{\mathscr{L}^1(\mathbb{X})} \\ &\preccurlyeq \| f_\nu^\sharp \alpha_i - f_\nu^\sharp \alpha \|_{\mathscr{L}^1(\mathbb{X})} \| Df_\nu \|^{k_i}_{\mathscr{L}^{k_i}(\mathbb{X})} \end{aligned} \tag{344}$$

Since $k_i \leqslant n-1$ the last factor is bounded by $\| Df_\nu \|^{k_i}_{n-1}$. Next observe the point-wise inequality

$$\big| f_\nu^\sharp \alpha_i - f_\nu^\sharp \alpha \big| \preccurlyeq |Df_\nu - Df| \big(|Df_\nu| + |Df| \big)^{l_i - 1} + |f_\nu - f| \, |Df|^{l_i} \tag{345}$$

This can be easily verified using local coordinates. The $\mathscr{L}^1$-norm of the first term in the right hand side of (345) is controlled by

$$\| Df_\nu - Df \|_{n-1} \Big(\| Df_\nu \|_{n-1} + \| Df \|_{n-1} \Big)^{l_i - 1}$$

Simple application of Hölder's inequality shows that integral of the second term is bounded by

$$\int_{\mathbb{X}} |f_\nu - f| \, |Df|^{l_i} \preccurlyeq \left(\int_{\mathbb{X}} |f_\nu - f|^{n-1} |Df|^{n-1} \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{X}} |Df|^{n-1} \right)^{\frac{l_i - 1}{n-1}} \tag{346}$$

We conclude with the following inequality

$$\left| \, (f_\nu^\sharp \omega - f^\sharp \omega)[1] \, \right| \tag{347}$$
$$\preccurlyeq \;\; \Big(\| \, |f_\nu - f| \, |Df| \, \|_{n-1} + \| \, Df_\nu - Df \, \|_{n-1} \Big) \left(\int_{\mathbb{X}} |Df_\nu|^{n-1} + |Df|^{n-1} \right)$$

Finally, let ν go to infinity. The integral stays bounded and the term $\| \, Df_\nu - Df \, \|_{n-1}$ goes to zero. Also $(f_\nu - f) \, |Df|$ goes to zero in $\mathscr{L}^{n-1}$ by the Lebesgue Convergence Theorem. Hence $\lim\limits_{\nu \to \infty} (f_\nu^\sharp \omega) \, [1] = (f^\sharp \omega) \, [1]$, as desired.

8.3 Stability of the degree

Next we are concerned with the fundamental question of the degree theory; how close should the mappings $f, g \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ be in order to ensure that they have the same degree. We shall measure the distance using the metric of the Sobolev space $\mathscr{W}^{1,q}(\mathbb{X}, \mathbb{Y})$ with $q > n-1$. We also assume, as always, that the l-cohomology of the target space is nontrivial for some $1 \leqslant l < n$.

THEOREM 8.2. *Given $M > 0$ and $q > n-1$, there exists $\epsilon = \epsilon(\mathbb{X}, \mathbb{Y})$ such that if two mappings $f, g \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ satisfy*

$$\| \, f \, \|_{\mathscr{W}^{1,q}} + \| \, g \, \|_{\mathscr{W}^{1,q}} \leqslant M \qquad \textit{and} \qquad \| \, f - g \, \|_{\mathscr{W}^{1,q}} \leqslant \epsilon \tag{348}$$

Then $\deg(f; \mathbb{X}, \mathbb{Y}) = \deg(g; \mathbb{X}, \mathbb{Y})$.

Proof. The reader may carefully reexamine the proof of (347) to observe that we have actually proven the following estimate

$$\left| \, (f^\sharp \omega - g^\sharp \omega)[1] \, \right|$$
$$\preccurlyeq \;\; \Big(\| \, |f - g| \, |Df| \, \|_{n-1} + \| \, Df - Dg \, \|_{n-1} \Big) \Big(\| \, Df \, \|_{n-1}^{n-1} + \| \, Dg \, \|_{n-1}^{n-1} \Big)$$

whenever $f, g \in \mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$ and $q > n-1$. Since the target space is bounded, it folows

$$\left| \, (f^\sharp \omega - g^\sharp \omega)[1] \, \right| \;\preccurlyeq\; \| \, f - g \, \|_{\mathscr{W}^{1,q}} \Big(\| \, f \, \|_{\mathscr{W}^{1,q}} + \| \, g \, \|_{\mathscr{W}^{1,q}} \Big)^{n-1} \tag{349}$$

This proves Theorem 8.2.

Remark 8.1. Theorem 8.2 also holds for mappings $f, g \in \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$, provided they both satisfy condition (238). This is because we could approximate them by smooth mappings.

8.4 The degree in Orlicz and grand Sobolev spaces

Finally, our discussion is narrowed to Orlicz-Sobolev and to grand-Sobolev classes of mappings $f : \mathbb{X} \to \mathbb{Y}$, $\dim \mathbb{X} = \dim \mathbb{Y} = n$, where $\mathcal{H}^l(\mathbb{Y}) \neq 0$ for some $1 \leqslant l < n$. Recall that these classes hold smooth approximation property, by Theorems 5.2 and 1.3.

Let P satisfy the hypothesis of Theorem 5.2. As a corollary of Theorem 8.2, we conclude:

THEOREM 8.3. *The degree function*

$$\deg : \mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y}) \to \{..., -2, -1, 0, 1, 2, ...\} \tag{350}$$

is uniformly continuous on every bounded subclass of $\mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$.

Speaking of the category of grand Sobolev spaces, let us recall that

$$\lim_{\epsilon \to 0} \epsilon \int_{\mathbb{X}} |Df(x)|^{n-\epsilon} dx = 0 \tag{351}$$

whenever $f \in V\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. For such mappings we have yet another interesting integral formula for the degree

$$\deg(f; \mathbb{X}, \mathbb{Y}) = \lim_{\epsilon \to 0} \int_{\mathbb{X}} \frac{\mathcal{J}_\omega(x, f)\, dx}{|\mathcal{J}_\omega(x, f)|^\epsilon} \tag{352}$$

simply because this limit coincides with $(f^\sharp \omega)[1]$, see [15], [31]. The advantage of this latter formula is that we neither approximate f by smooth mappings nor approximate $\mathbb{X}$ by the sets $\mathbb{X}_j$. This formula might be extremely useful in numerical treatment of the degree theory. Indeed, as $\deg(f; \mathbb{X}, \mathbb{Y})$ is an integer, we only need to compute the limit at (352), with sufficient accuracy to ensure that the error is less than $\frac{1}{2}$. Explicit estimates of the error in terms of ϵ are also available.

One particular Orlicz-Sobolev subspace of $V\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$ deserves mentioning here. This is the class of weakly differentiable mapping $f:\mathbb{X}\to\mathbb{Y}$ whose differential lies in the Zygmund class $\mathscr{L}^n\log^{-1}\mathscr{L}(\mathbb{X})$; that is

$$\int_{\mathbb{X}}\frac{|Df(x)|^n\,dx}{\log(e+|Df(x)|)}<\infty \tag{353}$$

9 Mappings of Finite Distortion

Recently there have been considerable advances made in the study of mappings of finite distortion between the domains in $\mathbb{R}^n$. The reader interested in these developments is referred to [27], [35], [36], [28] and the recent monograph [29]. What we want to point out here is the extent to which those results are true in the Riemannian manifold setting.

DEFINITION **9.1.** Let $\dim\mathbb{X}=\dim\mathbb{Y}=n$. A Sobolev mapping $f:\mathbb{X}\to\mathbb{Y}$ is said to have finite distortion if

(i) The Jacobian determinant $\mathcal{J}(x,f)\,dx=f^\sharp(dy)$ is integrable

(ii) There is a measurable function $K=K(x)\geqslant 1$, finite almost everywhere, such that f satisfies the distortion inequality

$$|Df(x)|^n\leqslant K(x)\,\mathcal{J}(x,f)\quad\text{for almost every } x\in\mathbb{X} \tag{354}$$

We emphasize that in many natural situations the condition (i) is automatic. Such is the case when f is a local homeomorphism. More generally, $\mathcal{J}(x,f)\in\mathscr{L}^1(\mathbb{X})$ if the cardinality of the set $\{x\in\mathbb{X};\ f(x)=y\}$ is an integrable function in $y\in\mathbb{Y}$. Foundational analysis of mappings of finite distortion relies on integration of the Jacobian. In order to fully benefit from the estimates and the degree formulas we must stay close to the natural Sobolev class $\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$. Thanks to $\mathscr{L}^1$-estimates in Section 6 we may consider unbounded distortion $K=K(x)$. It turns out that the following integral condition on K has interesting implications

$$\int_{\mathbb{X}}e^{\Phi(K(x))}\,dx<\infty\,,\qquad\text{where }\int_1^\infty\frac{\Phi(t)}{t^2}\,dt=\infty \tag{355}$$

This implies, via the distortion inequality, that $f \in \mathscr{W}^{1,P}(\mathbb{X}, \mathbb{Y})$, where P satisfies the hypotheses of Theorem 8.2. To be precise, we should mention here that one also needs $\Phi(t) \succcurlyeq \log t$. This additional condition plays rather technical role, since in practice $\Phi(t)$ behaves more or less like the linear function. For instance, $\Phi(t) = \lambda t$ or $\Phi(t) = \lambda t \log^{-1}(e+t)$, $\lambda > 0$. As a consequence of our investigation of the pullback of Sobolev mappings we are able to carry out this program on manifolds.

THEOREM 9.1. *Let $f : \mathbb{X} \to \mathbb{Y}$ be a non-constant mapping of finite distortion $K = K(x)$ satisfying (355). Then*

- *f is continuous, open, and discrete.*
- *The measure of $\mathbb{E} \subset \mathbb{X}$ is zero if and only if $f(\mathbb{E}) \subset \mathbb{Y}$ has measure zero.*
- *Given $\lambda > 0$, $C > 0$ and $d \in \{1, 2, ...\}$, the family of mappings $f : \mathbb{X} \to \mathbb{Y}$ such that*
$$\int_{\mathbb{X}} e^{\lambda K(x)}\, dx \leqslant C \tag{356}$$
$$\deg(f; \mathbb{X}, \mathbb{Y}) \leqslant d \tag{357}$$
  is compact with respect to uniform convergence.
- *If for sufficiently large $\lambda = \lambda(n)$*
$$\int_{\mathbb{X}} e^{\lambda K(x)}\, dx < \infty$$
  then $f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$.

We shall not prove this theorem, it can be found in [25], [26], [27], [28], [35], [36], [37], and [38].

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