An example concerning the zero set of the Jacobian

Janne Kauhanen

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain. We say that a mapping $f\colon \Omega \to \mathbb{R}^n$ has *finite distortion* if f belongs to the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^n)$, the Jacobian determinant $J_f = \det(Df)$ of f (Df being the differential matrix of f) is integrable in Ω, and if there is a measurable function $K \geq 1$, finite almost everywhere, such that

$$|Df(x)|^n \leq K(x) J_f(x) \qquad \text{a.e. } x \in \Omega. \tag{1.1}$$

The distortion inequality (1.1) guarantees that $J_f \geq 0$ a.e. and that Df vanishes a.e. in the zero set of J_f. The smallest function K satisfying (1.1) is then defined by

$$K_f(x) = \begin{cases} \dfrac{|Df(x)|^n}{J_f(x)} & \text{if } J_f(x) \neq 0, \\ 1 & \text{if } J_f(x) = 0. \end{cases} \tag{1.2}$$

The motivation for employing mappings of finite distortion partially arises from nonlinear elasticity. In modeling deformations of elastic bodies the property $J_f > 0$ a.e. (or $|f(E)| = 0 \Rightarrow |E| = 0$) is very desirable. For further discussion and references see [4].

Theorem 1.1. ([4]) *Let $f\colon \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying the distortion inequality* (1.1) *such that $K \in L^{1/(n-1)}(\Omega)$. If the multiplicity of f is essentially bounded and f is not constant, then for any $E \subset \Omega$ we have*

$$|f(E)| = 0 \implies |E| = 0$$

and $J_f > 0$ a.e. in Ω.

The integrability condition on K is known to be sharp in the L^p-scale:

Theorem 1.2. ([2, 6.5.6]) *There is a Lipschitz-homeomorphism* $f\colon \Omega \to \Omega$ *of finite distortion* K *such that* J_f *vanishes in a set* $E \subset \Omega$ *of positive measure,* $|f(E)| = 0$ *and* $K \in L^p(\Omega)$ *for all* $p < 1/(n-1)$*.*

In this paper we will prove that Theorem 1.1 is sharp in any Orlicz-scale:

Theorem 1.3. *Let* $\alpha\colon [1,\infty) \to [1,\infty)$ *be a continuous, strictly increasing function such that* $\lim_{t\to\infty} \alpha(t) = \infty$ *and such that*

$$t \mapsto \frac{t^{1/(n-1)}}{\alpha(t)}$$

is increasing. Then there exists a homeomorphism $f\colon \Omega \to \Omega$ *of finite distortion* K *such that* J_f *vanishes in a set* $E \subset \Omega$ *of positive measure,* $|f(E)| = 0$ *and*

$$\int_\Omega \frac{K(x)^{1/(n-1)}}{\alpha(K(x))}\,dx < \infty.$$

The mapping that we will construct for Theorem 1.3 is not, in general, in a much bigger space than $W^{1,1}(\Omega, \mathbb{R}^n)$. However, we will show that even for Lipschitz mappings the integrability condition on K in Theorem 1.1 is sharp up to a logarithm:

Theorem 1.4. *There exists a Lipschitz-homeomorphism* $f\colon \Omega \to \Omega$ *of finite distortion* K *such that* J_f *vanishes in a set* $E \subset \Omega$ *of positive measure,* $|f(E)| = 0$ *and*

$$\int_\Omega \frac{K(x)^{1/(n-1)}}{\log^{1+\epsilon}(e + K(x))}\,dx < \infty$$

for all $\epsilon > 0$*.*

2 The construction of mapping f

Similarly to [3] (see also [2] and [5]) we will construct a homeomorphism $f\colon Q_0 \to Q_0$ ($Q_0 = [0,1]^n$) such that a regular Cantor set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q_v \subset Q_0$$

of positive measure gets mapped onto a regular Cantor set

$$E' = \bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q'_v \subset Q_0$$

of zero measure.

2.1 The Cantor sets E and E'

The cubes Q_v and Q'_v are defined as follows. Let $\|x\| = \max_i |x_i|$ denote the cubic norm of $x \in \mathbb{R}^n$. We will denote by

$$Q(z, r) = \{x \in \mathbb{R}^n \colon \|z - x\| \le r\}$$

the closed cube with center z and radius r. Let $V = \{1, 2, \dots, 2^n\}$. Then the sets $V^k = V \times \cdots \times V$, $k \in \{1, 2, \dots\}$, will serve as the sets of indices for our construction. If $w \in V^{k-1}$, we denote

$$V^k[w] = \{v \in V^k : \ v_j = w_j, \ j = 1, \dots, k-1\}.$$

Divide Q_0 into 2^n closed subcubes $P_v = Q(z_v, 1/4)$, $v \in V^1$. Then for each $v \in V^1$ let $Q_v = Q(z_v, r_1)$ be a concentric closed subcube of P_v with radius $r_1 < 1/4$. The rest of the cubes are defined inductively: if the cubes P_w and $Q_w = Q(z_w, r_{k-1})$ have been defined for each $w \in V^{k-1}$, then for all $w \in V^{k-1}$ divide Q_w into 2^n closed subcubes $P_v = Q(z_v, r_{k-1}/2)$, $v \in V^k[w]$. Then for each $v \in V^k$ let $Q_v = Q(z_v, r_k)$ be a concentric closed subcube of P_v with radius $r_k < r_{k-1}/2$. The cubes $Q'_v = Q(z'_v, r'_k)$, $v \in V^k$, are defined in a similar manner.

At this point, we want to keep the construction as general as possible. Therefore we choose

$$r_k = \varphi(k) 2^{-k-1}$$

and

$$r'_k = \psi(k) 2^{-k-1},$$

where $\varphi : [1, \infty) \to (1/2, 1)$ and $\psi : [1, \infty) \to (0, 1)$ are strictly decreasing functions such that $\lim\limits_{t \to \infty} \psi(t) = 0$. Then

$$|E| = \lim_{k \to \infty} \Big| \bigcup_{v \in V^k} Q_v \Big| = \lim_{k \to \infty} 2^{nk} (r_k)^n > 0$$

and

$$|E'| = \lim_{k \to \infty} \Big| \bigcup_{v \in V^k} Q'_v \Big| = \lim_{k \to \infty} 2^{nk} (r'_k)^n = 0.$$

We also make the following technical assumptions on φ and ψ: assume that φ and ψ are continuously differentiable, φ' and ψ' are increasing and

$$\frac{-\psi'(t)}{\psi(t)} \gtrsim -\varphi'(t-1). \tag{2.1}$$

Here and subsequently, we denote $a \lesssim b$ ($a \gtrsim b$) if there exists a constant $c > 0$ depending only on constant parameters (such as n) such that $a \le cb$ ($b \le ca$), and $a \approx b$ if $a \lesssim b$ and $b \lesssim a$.

Note that the assumption (2.1) is harmless: we cannot have

$$\frac{-\psi'(t)}{\psi(t)} \lesssim -\varphi'(t-1)$$

since otherwise

$$1 > \int_1^\infty\!\!\!\!\!\!\!\!/\;\;(-\varphi(t)) = \int_1^\infty (-\varphi'(t))\,dt \gtrsim \int_2^\infty \frac{-\psi'(t)}{\psi(t)}\,dt = \int_2^\infty\!\!\!\!\!\!\!\!/\;\;-\log\psi(t) = \infty.$$

2.2 The mapping f

Let us first define the piecewise continuously differentiable homeomorphisms $f_k\colon Q_0 \to Q_0$ such that f_k maps each annulus $P_v \setminus Q_v$, $v \in V^k$, onto the annulus $P'_v \setminus Q'_v$ and fixes the boundary ∂Q_0. Let $f_0 = \mathrm{id}|Q_0$ and for $k \in \{1, 2, \ldots\}$ we set

$$f_k(x) = \begin{cases} f_{k-1}(x) & \text{if } x \notin \bigcup_{v\in V^k} P_v, \\ f_{k-1}(z_v) + a_k(x - z_v) + b_k \dfrac{x - z_v}{\|x - z_v\|} & \\ & \text{if } x \in P_v \setminus Q_v, \quad v \in V^k, \\ f_{k-1}(z_v) + c_k(x - z_v) & \text{if } x \in Q_v, \quad v \in V^k, \end{cases}$$

where a_k, b_k and c_k are chosen so that f_k is continuous:

$$\begin{aligned} a_k r_k + b_k &= r'_k, \\ a_k r_{k-1}/2 + b_k &= r'_{k-1}/2, \\ c_k r_k &= r'_k. \end{aligned} \tag{2.2}$$

It follows that

$$a_k = \frac{r'_{k-1}/2 - r'_k}{r_{k-1}/2 - r_k} = \frac{\psi(k-1) - \psi(k)}{\varphi(k-1) - \varphi(k)}. \tag{2.3}$$

Define $f = \lim_{k\to\infty} f_k$. It is easily seen that f is a one-to-one mapping such that $f(E) = E'$. Continuity of f follows from the uniform convergence of the sequence (f_k): for any $x \in Q_0$ and $l \geq j \geq 1$ we have

$$|f_l(x) - f_j(x)| \lesssim r'_j \to 0$$

as $j \to \infty$. Since $f\colon Q_0 \to Q_0$ is continuous one-to-one mapping and Q_0 is compact, it follows that f is a homeomorphism.

3 The analytic properties of the mapping f

Let $x \in \operatorname{int} P_v \setminus Q_v$, $v \in V^k$. Denote $r = \|x - z_v\|$. Then

$$r_k < r < \frac{r_{k-1}}{2} = \frac{\varphi(k-1)}{\varphi(k)} r_k < 2r_k,$$

whence $r \approx r_k$. We estimate $|Df(x)|$ and $J_f(x)$ using [3, Lemma 5.1]. Since

$$f(x) = f_{k-1}(z_v) + \big(a_k\|x - z_v\| + b_k\big)\frac{x - z_v}{\|x - z_v\|},$$

denoting $\rho(r) = a_k r + b_k$ we see that

$$|Df(x)| \approx \max\left\{\frac{\rho(r)}{r},\ |\rho'(r)|\right\} \approx \begin{cases} a_k + b_k/r_k = r'_k/r_k, & \text{if } b_k \geq 0,\\ a_k, & \text{if } b_k \leq 0,\end{cases}$$

$$\approx \begin{cases} \psi(k), & \text{if } b_k \geq 0,\\ \frac{\psi(k-1)-\psi(k)}{\varphi(k-1)-\varphi(k)}, & \text{if } b_k \leq 0.\end{cases}$$

If $b_k \geq 0$, then the tangential derivative dominates in the annulus $P_v \setminus Q_v$, $v \in V^k$, and if $b_k \leq 0$, then the radial derivative dominates. From (2.1) it follows that the radial derivative dominates:

$$\frac{\psi(k-1) - \psi(k)}{\varphi(k-1) - \varphi(k)} \geq \frac{-\psi'(k)}{-\varphi'(k-1)} \gtrsim \psi(k).$$

Therefore

$$|Df(x)| \approx \frac{\psi(k-1) - \psi(k)}{\varphi(k-1) - \varphi(k)}.$$

For the Jacobian we have the estimate

$$\begin{aligned} J_f(x) &\approx \frac{\rho'(r)\rho(r)^{n-1}}{r^{n-1}} \gtrsim a_k(a_k + b_k/r_k)^{n-1} = a_k\left(\frac{r'_k}{r_k}\right)^{n-1}\\ &\approx \frac{\psi(k-1) - \psi(k)}{\varphi(k-1) - \varphi(k)}\psi(k)^{n-1}, \end{aligned}$$

whence

$$K_f(x) = \frac{|Df(x)|^n}{J_f(x)} \lesssim \left(\frac{\psi(k-1) - \psi(k)}{\varphi(k-1) - \varphi(k)}\,\frac{1}{\psi(k)}\right)^{n-1}. \tag{3.1}$$

Note that $|Df(x)|$ is essentially bounded in $Q_0 \setminus E$ if and only if

$$\frac{\psi(k-1) - \psi(k)}{\varphi(k-1) - \varphi(k)} \leq C < \infty. \tag{3.2}$$

The following Proposition can be proven by following the argument in [2, p. 131].

Proposition 3.1. *f is Lipschitz if and only if* (3.2) *holds.*

Therefore $f \in W^{1,1}(Q_0, \mathbb{R}^n)$ if (3.2) holds. Next we will show that actually this is always the case at least when E' is small enough.

Lemma 3.2. *If $\dim_{\mathcal{H}}(E') < 1$, then $f \in W^{1,1}(Q_0, \mathbb{R}^n)$.*

Proof. The measure of the union of k-level annulae in the construction of E is

$$\Big|\bigcup_{v\in V^k}(P_v\setminus Q_v)\Big| = 2^{nk}\big((r_{k-1}/2)^n-(r_k)^n\big)\approx\varphi(k-1)^n-\varphi(k)^n$$
$$\approx\varphi(k-1)-\varphi(k). \tag{3.3}$$

Here, the last estimate follows from the assumption $1/2\le\varphi(t)\le 1$. Now,

$$\begin{aligned}\int_{Q_0\setminus E}|Df| &\approx \sum_k(\varphi(k-1)-\varphi(k))\frac{\psi(k-1)-\psi(k)}{\varphi(k-1)-\varphi(k)}\\ &= \sum_k(\psi(k-1)-\psi(k))=\psi(1)<\infty.\end{aligned}\tag{3.4}$$

Therefore, on $\mathcal{H}^{n-1}$-a.e. line segment $L\subset Q_0$, parallel to i^{th} coordinate axis, f is locally Lipschitz in $L\setminus E$, $D_if(x)=(\partial_if^1(x),\dots,\partial_if^n(x))$ exists for $\mathcal{H}^1$-a.e. $x\in L\setminus E$ and

$$\int_{L\setminus E}|D_if|\,d\mathcal{H}^1<\infty. \tag{3.5}$$

Fix such L. We will show that f is absolutely continuous on L. If f is Lipschitz-homeomorphism on an interval $I\subset L\setminus E$, then

$$\mathcal{H}^1(f(I))=\int_I|D_if|\,d\mathcal{H}^1. \tag{3.6}$$

We conclude that (3.6) holds for each open set $I\subset L\setminus E$. Fix $\epsilon>0$. Choose disjoint open intervals $I_j\subset L$. Since $\mathcal{H}^1(f(E))=0$ and $I_j\setminus E$ is open, we have by (3.6)

$$\begin{aligned}\mathcal{H}^1(f(I_j)) &= \mathcal{H}^1(f(I_j\cap E))+\mathcal{H}^1(f(I_j\setminus E))\\ &= \int_{I_j\setminus E}|D_if|\,d\mathcal{H}^1,\end{aligned}\tag{3.7}$$

whence

$$\sum_j\mathcal{H}^1(f(I_j))=\sum_j\int_{I_j\setminus E}|D_if|\,d\mathcal{H}^1=\int_{\cup_j(I_j\setminus E)}|D_if|\,d\mathcal{H}^1.$$

By absolute continuity of the integral with respect to the measure there exists $\delta>0$ such that

$$\mathcal{H}^1\Big(\bigcup_j(I_j\setminus E)\Big)\le\sum_j\mathcal{H}^1(I_j)<\delta\quad\Longrightarrow\quad\sum_j\mathcal{H}^1(f(I_j))<\epsilon.$$

Thus f is absolutely continuous on L. Therefore $D_i f$ exists $\mathcal{H}^1$-a.e. in L and (3.6) holds for all intervals $I \subset L$ (see e.g. [1, 2.9.20, 2.9.22 and 2.10.13]). As in (3.7), we see that

$$\int_L |D_i f| \, d\mathcal{H}^1 = \int_{L \setminus E} |D_i f| \, d\mathcal{H}^1,$$

whence

$$\int_{L \cap E} |D_i f| \, d\mathcal{H}^1 = 0$$

and thus $D_i f = 0$ $\mathcal{H}^1$-a.e. in $L \cap E$. We conclude that f is absolutely continuous on $\mathcal{H}^{n-1}$ almost all lines parallel to coordinate axes, f has all partial derivatives a.e. in Q_0 and $Df = 0$ a.e. in E. Together with (3.4) this implies that $f \in W^{1,1}(Q_0, \mathbb{R}^n)$. □

Lemma 3.3. *If f is Lipschitz or $\dim_{\mathcal{H}}(E') < 1$, then $J_f \in L^1(Q_0)$ and $J_f = 0$ a.e. in E.*

Proof. If $f \in W^{1,1}(Q_0, \mathbb{R}^n)$ is a Lipschitz-homeomorphism, then

$$\int_A J_f = |f(A)| \tag{3.8}$$

for all measurable sets $A \subset Q_0$ (see [1, 3.2.3 (1)]), and the claim follows. If $\dim_{\mathcal{H}}(E') < 1$, then $Df = 0$ (and thus $J_f = 0$) a.e. in E by the proof of Lemma 3.2. Since

$$Q_0 \setminus E = \bigcup_{k=1}^{\infty} \Big(Q_0 \setminus \bigcup_{v \in V^k} Q_v \Big) =: \bigcup_{k=1}^{\infty} A_k$$

and each $f|A_k$ is a Lipschitz-homeomorphism, applying (3.8) we see that $J_f \in L^1(Q_0)$. □

4 Proofs of Theorems 1.3 and 1.4

To prove Theorems 1.3 and 1.4, it suffices to choose φ and ψ such that the mapping $f : Q_0 \to Q_0$ constructed in Section 2 enjoys the desired properties. Theorems 1.3 and 1.4 then follow by scaling and shifting (note that f fixes the boundary ∂Q_0). Thus Theorem 1.3 follows from the following Lemma.

Lemma 4.1. *Let α be as in Theorem 1.3, and let $s \in (0,1)$. We can define*

$$\varphi(t) = \frac{1}{2} + \int_t^{\infty} \frac{dr}{(\alpha^{-1}(r^n))^{1/(n-1)}} \quad \textit{and} \quad \psi(t) = 2^{(1-n/s)t}$$

(for large t). Then E' is s-dimensional self-similar set, $f \in W^{1,1}(Q_0, \mathbb{R}^n)$, $J_f \in L^1(Q_0)$, $J_f = 0$ a.e. in E and

$$\int_{Q_0} \frac{(K_f)^{1/(n-1)}}{\alpha(K_f)} < \infty. \tag{4.1}$$

Proof. We have $\alpha(t) \le t$ and hence $\alpha^{-1}(t) \ge t$ for large t, whence

$$\int_{r_0}^{\infty} \frac{dr}{(\alpha^{-1}(r^n))^{1/(n-1)}} \le \int_{r_0}^{\infty} \frac{dr}{r^{n/(n-1)}} < \infty.$$

Thus φ is decreasing, $\lim_{t\to\infty} \varphi(t) = 1/2$ and

$$\varphi'(t) = -\frac{1}{(\alpha^{-1}(t^n))^{1/(n-1)}}$$

is increasing. (2.1) is easy to check. By Lemmas 3.2 and 3.3 it remains to show (4.1). Since

$$\frac{\psi(k-1) - \psi(k)}{\psi(k)} = 2^{n/s-1} - 1 > 1,$$

we have by (3.1) and (3.3) that

$$\begin{aligned}
\int_{Q_0} \frac{(K_f)^{1/(n-1)}}{\alpha(K_f)} &\lesssim \sum_k \frac{1}{\alpha\left(\left(\frac{1}{\varphi(k-1)-\varphi(k)}\right)^{n-1}\right)} \\
&\le \sum_k \frac{1}{\alpha\left(\left(\frac{1}{-\varphi'(k-1)}\right)^{n-1}\right)} \\
&\le \sum_k \frac{1}{(k-1)^n} < \infty.
\end{aligned}$$

$\square$

Theorem 1.4 follows from Lemma 4.2.

Lemma 4.2. *Let $s \in (0, n)$. We can define*

$$\varphi(t) = \frac{1}{2} + 2^{(1-n/s)t} \quad \textit{and} \quad \psi(t) = 2^{(1-n/s)t}$$

(for large t). Then E' is s-dimensional self-similar set, f is Lipschitz, $J_f \in L^1(Q_0)$, $J_f = 0$ a.e. in E and

$$\int_{Q_0} \frac{(K_f)^{1/(n-1)}}{\log^{1+\epsilon}(e + K_f)} < \infty$$

for all $\epsilon > 0$.

Proof. We have

$$\frac{\psi(k-1)-\psi(k)}{\varphi(k-1)-\varphi(k)} = 1 < \infty,$$

whence, by Proposition 3.1, f is Lipschitz, and for $x \in P_v \setminus Q_v$, $v \in V^k$,

$$K_f(x) \approx \left(\frac{\psi(k-1)-\psi(k)}{\varphi(k-1)-\varphi(k)}\,\frac{1}{\psi(k)}\right)^{n-1} = \left(\frac{1}{\psi(k)}\right)^{n-1}.$$

Therefore for $\epsilon > 0$

$$\begin{aligned}\int_{Q_0} \frac{(K_f)^{1/(n-1)}}{\log^{1+\epsilon}(e+K_f)}\,dx &\lesssim \sum_k \frac{\varphi(k-1)-\varphi(k)}{\psi(k)}\;\frac{1}{\log^{1+\epsilon}\big(e+1/\psi(k)^{n-1}\big)}\\ &\approx \sum_k \frac{1}{\log^{1+\epsilon}\big(2^{(n/s-1)(n-1)k}\big)}\\ &\approx \sum_k \frac{1}{k^{1+\epsilon}} < \infty.\end{aligned}$$

□

5 Remark

In Lemma 4.1 the set E' can be chosen to have any Hausdorff dimension $s \in (0,1)$. We can further guarantee that $\dim_{\mathcal{H}} E' = 0$ when $\alpha(t) = \log^{\epsilon}(e+t)$.

Lemma 5.1. *Let $p > 0$ and*

$$h(t) = \log^{-p}(1/t)$$

(for small t). We can define

$$\varphi(t) = \frac{1}{2} + \exp\big(-\exp\big(2^{2nt/p}\big)\big) \quad \text{and} \quad \psi(t) = \exp\big(-2^{nt/p}\big).$$

(for large t). Then $\mathcal{H}^h(E') < \infty$, $f \in W^{1,1}(Q_0,\mathbb{R}^n)$, $J_f \in L^1(Q_0)$, $J_f = 0$ a.e. in E and

$$\int_{Q_0} \frac{(K_f)^{1/(n-1)}}{\log^{\epsilon}(e+K_f)} < \infty$$

for all $\epsilon > 0$.

Proof. $\mathcal{H}^h(E) < \infty$ because for large k we have

$$\sum_{v\in V^k} h(\operatorname{diam}(Q_v)) = 2^{nk}h\big(c(n)2^{-k}\psi(k)\big) \le 2^{nk}h\big(\psi(k)\big) = 1.$$

Fix $\epsilon > 0$. Then we have

$$\begin{aligned}
\int_{Q_0} \frac{(K_f)^{1/(n-1)}}{\log^\epsilon(e+K_f)} &\lesssim \sum_k \frac{\frac{\psi(k-1)-\psi(k)}{\psi(k)}}{\log^\epsilon\left(\left(\frac{1}{\varphi(k-1)-\varphi(k)}\right)^{n-1}\right)} \\
&\lesssim \sum_k \frac{\frac{\psi(k-1)}{\psi(k)}}{\log^\epsilon\left(\frac{1}{\varphi(k-1)-1/2}\right)} \\
&\lesssim \sum_k \exp\left(\left(1-2^{-n/p}\right)2^{nk/p} - \epsilon 2^{2n(k-1)/p}\right) \\
&\lesssim \sum_k \exp\left(-2^{nk/p}\right) < \infty. \qquad \square
\end{aligned}$$

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Janne Kauhanen
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
FIN-40014 University of Jyväskylä
Finland

`janne.kauhanen@maths.jyu.fi`