# NONSYMMETRIC CONICAL UPPER DENSITY AND $k$-POROSITY

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Abstract. We study how the Hausdorff measure is distributed in nonsymmetric narrow cones in $\mathbb{R}^n$. As an application, we find an upper bound close to $n-k$ for the Hausdorff dimension of strongly $k$-porous sets. With $k$-porous sets we mean sets which have holes in $k$ different directions on every small scale.

## 1. Introduction

It is a well known fact that for a set $A \subset \mathbb{R}^n$ with finite $s$-dimensional Hausdorff measure, $\mathcal{H}^s(A) < \infty$, we have

$$1 \le \limsup_{r \downarrow 0} \frac{\mathcal{H}^s\big(A \cap B(x,r)\big)}{r^s} \le 2^s \tag{1.1}$$

for $\mathcal{H}^s$ almost every $x \in A$. For a proof, see, for example, [9, Theorem 6.2(1)]. This is analogous to the classical Lebesgue Density Theorem. Using this fact, we know roughly how much of $A$ there is in small balls. Mattila [8] studied how $A$ is distributed in such balls. He was able to estimate how much of $A$ there is near $(n-m)$-planes. More precisely, assuming $0 \le m < s \le n$ and denoting

$$\begin{aligned} X(x,V,\alpha) &= \{y \in \mathbb{R}^n : \operatorname{dist}(y-x,V) < \alpha|y-x|\}, \\ X(x,r,V,\alpha) &= X(x,V,\alpha) \cap B(x,r), \end{aligned}$$

as $x \in \mathbb{R}^n$, $V \in G(n,m)$, $r > 0$, and $0 < \alpha \le 1$, he proved that there exists a constant $c = c(n,m,s,\alpha) > 0$ such that

$$\limsup_{r \downarrow 0} \inf_{V \in G(n,n-m)} \frac{\mathcal{H}^s\big(A \cap X(x,r,V,\alpha)\big)}{r^s} \ge c \tag{1.2}$$

for $\mathcal{H}^s$ almost every $x \in A$ whenever $A \subset \mathbb{R}^n$ is such that $\mathcal{H}^s(A) < \infty$. Here $G(n,m)$ denotes the collection of all $m$-dimensional linear subspaces of $\mathbb{R}^n$, see [9, §3.9]. Actually (1.2) is just a special case of Mattila's result, as his theorem can be applied also for more general cones, see [8, Theorem 3.3].

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In Theorem 2.5 we show that if $A$ is as above, then it cannot be concentrated in too small regions, not even inside the cones $X(x,r,V,\alpha)$. More precisely, denoting

$$H(x,\theta) = \{y \in \mathbb{R}^n : (y-x)\cdot\theta > 0\},$$
$$H(x,\theta,\eta) = \{y \in \mathbb{R}^n : (y-x)\cdot\theta > \eta|y-x|\},$$

for $x \in \mathbb{R}^n$, $\theta \in S^{n-1}$, and $0 < \eta \le 1$, we prove under the same assumptions as in (1.2) that there exists a constant $c = c(n,m,s,\alpha,\eta) > 0$ such that

$$\limsup_{r\downarrow 0} \inf_{\substack{\theta\in S^{n-1}\\ V\in G(n,n-m)}} \frac{\mathcal{H}^s\big(A\cap X(x,r,V,\alpha)\setminus H(x,\theta,\eta)\big)}{r^s} \ge c$$

for $\mathcal{H}^s$ almost every $x \in A$. Our method gives also a more elementary proof for (1.2) and it can also be used to obtain similar results for more general measures, see Theorem 2.6.

The nonsymmetric conical upper density theorem is essential in our application to $k$-porous sets, that is, the sets with $\mathrm{por}_k > 0$, see (1.5). The notation of porosity, or 1-porosity using our terminology, has aroused from the study of dimensional estimates related, for example, to the boundary behavior of quasiconformal mappings. See Koskela and Rohde [6], Martio and Vuorinen [7], Sarvas [12], Trocenko [14], and Väisälä [15]. The dimensional properties of 1-porous sets are well known. Using a version of (1.2), Mattila showed that

$$\sup\{s > 0 : \mathrm{por}_1(A) > \varrho \text{ and } \dim_H(A) > s \text{ for some } A \subset \mathbb{R}^n\} \longrightarrow n-1 \quad (1.3)$$

as $\varrho \to \frac{1}{2}$. Here $\dim_H$ refers to the Hausdorff dimension. Later Salli [11] generalized this result for the Minkowski dimension, and found the correct asymptotics. The concept of 1-porosity has also been generalized for measures, and it leads to similar kind of dimension bounds. See Järvenpää and Järvenpää [4] and references there.

Motivated by the fact that each $V \in G(n,n-1)$ has maximal 1-porosity, we introduce a porosity condition which describes also sets whose dimension is smaller than $n-1$. For $0 < k \le n$, $x \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$, and $r > 0$ we set

$$\begin{aligned}\mathrm{por}_k(A,x,r) = \sup\{\varrho :\ &\text{there are } z_1,\ldots,z_k \in \mathbb{R}^n \text{ such that}\\ &B(z_i,\varrho r) \subset B(x,r)\setminus A \text{ for every } i,\\ &\text{and } (z_i-x)\cdot(z_j-x) = 0 \text{ for } i \ne j\}.\end{aligned} \quad (1.4)$$

The *$k$-porosity* of $A$ at a point $x$ is defined to be

$$\mathrm{por}_k(A,x) = \liminf_{r\downarrow 0} \mathrm{por}_k(A,x,r),$$

and the $k$-porosity of $A$ is given by

$$\mathrm{por}_k(A) = \inf_{x\in A} \mathrm{por}_k(A,x). \quad (1.5)$$

This means that $k$-porous sets have holes in $k$ orthogonal directions near each of its points in every small scale. We shall now give a concrete example where $k$-porosity occurs naturally. Suppose $0 < \lambda < \frac{1}{2}$ and let $C_\lambda \subset \mathbb{R}$ be the usual $\lambda$-Cantor set, see [9, §4.10]. It is clearly a 1-porous set with $\mathrm{por}_1(C_\lambda) \approx \frac{1}{2} - \lambda$. Mattila's result (1.3) implies that $\dim_H(C_\lambda) \to 0$ as $\mathrm{por}_1(C_\lambda) \to \frac{1}{2}$. Of course, we could obtain the same information just by calculating the Hausdorff dimension of the self-similar set $C_\lambda$ and letting $\lambda \to 0$, but our aim was to provide the reader with an illustrative example. The sets $C_\lambda \times C_\lambda \subset \mathbb{R}^2$ and $C_\lambda \times C_\lambda \times [0,1] \subset \mathbb{R}^3$ are clearly 2-porous with $\mathrm{por}_2 \approx \frac{1}{2} - \lambda$. For these sets (1.3) does not give any reasonable dimension bound. However, it would be desirable to see, also in terms of porosity, that $\dim_H(C_\lambda \times C_\lambda) \to 0$ and $\dim_H(C_\lambda \times C_\lambda \times [0,1]) \to 1$ as $\lambda \to 0$. This follows as an immediate application of Theorem 3.2. Using our nonsymmetric conical upper density theorem, we show that

$$\sup\{s > 0 : \mathrm{por}_k(A) > \varrho \text{ and } \dim_H(A) > s \text{ for some } A \subset \mathbb{R}^n\} \longrightarrow n - k$$

as $\varrho \to \frac{1}{2}$. Observe also that in the proof of Theorem 3.2 the orthogonality in (1.4) plays no rôle and we may replace it by an assumption of a uniform lower bound for the angles between $z_i - x$ and the $(k-1)$-plane spanned by vectors $z_j - x$, $i \neq j$.

Let us now discuss the situation when porosity is small. It is well known (for example, see [7]) that if $A \subset \mathbb{R}^n$ with $\mathrm{por}_1(A, x, r) \geq \varrho > 0$ for all $x \in A$ and $0 < r < r_0$, then

$$\dim_M(A) < n - c\varrho^n, \tag{1.6}$$

where $c > 0$ depends only on $n$, and $\dim_M$ refers to the Minkowski dimension, see [9, §5.3]. It might be possible to get a better estimate if $\mathrm{por}_1$ is replaced by $\mathrm{por}_k$ for some $k > 1$, but this condition does not feel very natural if the size of the holes is small. However, if $V \in G(n,m)$ is fixed and the condition $\mathrm{por}_1(A, x, r) \geq \varrho$ is replaced by

$$\sup\big\{\varrho' : B(z, \varrho' r) \subset B(x, r) \setminus A \text{ for some } z \in V + \{x\}\big\} \geq \varrho,$$

then $n$ in (1.6) can be replaced by $m$, see Theorem 4.3. This is a rather immediate consequence of (1.6), but our main point is to give a simple proof for (1.6) using iterated function systems.

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## 2. Nonsymmetric conical upper density

We shall first prove a density theorem for nonsymmetric regions and then prove our main theorem by using a similar argument on $(n-m)$-planes. The proofs rely on the following geometric fact.

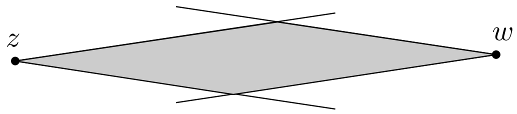


FIGURE A. All points lying on the gray region form a large angle with points $z$ and $w$.

**Lemma 2.1.** *For given $0 < \beta < \pi$, there is $q = q(n, \beta) \in \mathbb{N}$ such that in any set of $q$ points in $\mathbb{R}^n$, there are always three points which determine an angle between $\beta$ and $\pi$.*

*Remark* 2.2. Erdős and Füredi [1] have shown that for the smallest possible choice of $q$ it holds that

$$2^{(\pi/(\pi-\beta))^{n-1}} \leq q(n, \beta) \leq 2^{(4\pi/(\pi-\beta))^{n-1}} + 1.$$

For the convenience of the reader we shall give below a different proof which establishes the existence of some such $q$. The estimate that we get here for $q$ is, however, quite bad compared to the best possible one.

*Proof.* Let $A$ be a set of points in $\mathbb{R}^n$ so that all angles formed by its points are less than $\beta$. Let us fix a small number $\eta > 0$ and cover $\mathbb{R}^n \setminus \{0\}$ by cones $C_i = H(0, \theta_i, \eta)$, $i = 1, 2, \ldots, k$. Note that we can do this with a constant $k = k(n, \eta)$. For $y \in \mathbb{R}^n$ we use notation $C_{i,y} = C_i + \{y\}$.

For any index $i_1 i_2 \cdots i_j$, where $j \in \mathbb{N}$ and $i_m \in \{1, 2, \ldots, k\}$ for $1 \leq m \leq j$, we define sets $A_{i_1 i_2 \cdots i_j}$ in the following way: We begin by fixing $x \in A$ and setting $A_i = A \cap C_{i,x}$ for $1 \leq i \leq k$. If $A_{i_1 i_2 \cdots i_j}$ has been defined, we choose $y \in A_{i_1 i_2 \cdots i_j}$ and define $A_{i_1 i_2 \cdots i_j l} = A_{i_1 i_2 \cdots i_j} \cap C_{l,y}$ for $1 \leq l \leq k$ (if $A_{i_1 i_2 \cdots i_j}$ is empty, then so is $A_{i_1 i_2 \cdots i_j l}$). We refer to $y$ as the corner of $A_{i_1 i_2 \cdots i_j l}$. It follows directly from the definition of the sets $A_{i_1 i_2 \cdots i_j}$ that

$$\operatorname{card} A_{i_1 i_2 \cdots i_j} \leq 1 + \sum_{l=1}^{k} \operatorname{card} A_{i_1 i_2 \cdots i_j l}.$$

Iterating this, we get

$$\operatorname{card} A \leq \sum_{j=0}^{k} k^j + \sum_{i_1 i_2 \cdots i_k} \sum_{l=1}^{k} \operatorname{card} A_{i_1 i_2 \cdots i_k l}. \tag{2.1}$$

The main point of the proof is the observation that if $\eta = \eta(\beta)$ is chosen small enough in the beginning, then the following is true: If $z$ and $w$ are the corners of $A_{i_1 i_2 \cdots i_j}$ and $A_{i_1 i_2 \cdots i_j i_{j+1} \cdots i_m}$, respectively, and if $z \in C_{i_m,w}$, then $A \cap C_{i_j,z} \cap C_{i_m,w} = \emptyset$. See Figure A. It follows by induction from the above fact that for given $A_{i_1 i_2 \cdots i_j}$ we have

$$\operatorname{card}\{l : A_{i_1 i_2 \cdots i_j l} \neq \emptyset\} \leq k - j.$$

In particular, $A_{i_1 i_2 \cdots i_{k+1}} = \emptyset$ for any choice of $i_1 i_2 \dots i_{k+1}$. Combined with (2.1), this gives $\operatorname{card} A \le \sum_{j=0}^{k} k^j$. This number depends only on $k = k(n, \beta)$ and the claim follows. □

For $0 < \eta \le 1$ we define

$$t(\eta) = \sqrt{\frac{\eta^2 + 4}{\eta^2}},$$
$$\gamma(\eta) = \frac{1}{t(\eta)}.$$

Notice that $t(\eta) \ge 2$ and $\eta/\sqrt{5} \le \gamma(\eta) \le \eta/2$.

**Lemma 2.3.** *Suppose $y \in \mathbb{R}^n$, $\theta \in S^{n-1}$, $0 < \eta \le 1$, $t = t(\eta)$, and $\gamma = \gamma(\eta)$. If $z \in \mathbb{R}^n \setminus \big(B(y, tr) \cup H(y, \theta, \gamma)\big)$, then*

$$B(z, r) \cap H(y, \theta, \eta) = \emptyset.$$

*Proof.* Take $w \in \mathbb{R}^n$ such that it maximizes $(w - y) \cdot \theta / |w - y|$ in the closure of $B(z, r)$. It suffices to prove that $(w - y) \cdot \theta / |w - y| < \eta$, see Figure B. It is straightforward to check that $\eta\sqrt{s^2 - 1} \ge 1 + \gamma s$ when $s \ge t$. Denoting now $s = |y - z|/r$, we have $s \ge t > 1$ and thus

$$\begin{aligned} (w - y) \cdot \theta &< r + \gamma |y - z| = (1 + \gamma s) r \\ &\le \eta \sqrt{s^2 - 1} r = \eta |w - y|, \end{aligned}$$

which finishes the proof. □

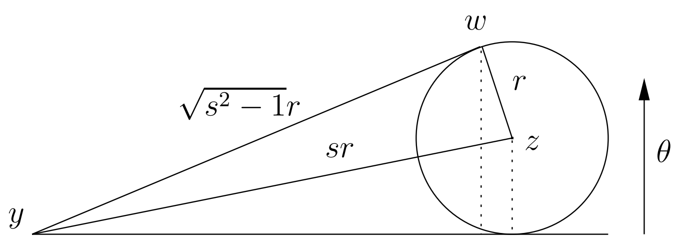


Figure B. Illustration for the proof of Lemma 2.3.

**Theorem 2.4.** *Suppose $0 < \eta \le 1$ and $0 < s \le n$. Then there is a constant $c = c(n, s, \eta) > 0$ such that*

$$\limsup_{r \downarrow 0} \inf_{\theta \in S^{n-1}} \frac{\mathcal{H}^s\big(A \cap B(x, r) \setminus H(x, \theta, \eta)\big)}{r^s} \ge c$$

*for $\mathcal{H}^s$ almost every $x \in A$ whenever $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$.*

*Proof.* Take $c > 0$ and assume there exists a set $B \subset \mathbb{R}^n$ with $\mathcal{H}^s(B) > 0$ such that for each $x \in B$ and $0 < r < r_0$ there is $\theta \in S^{n-1}$ for which

$$\mathcal{H}^s\big(B \cap B(x,r) \setminus H(x,\theta,\eta)\big) < cr^s. \tag{2.2}$$

According to (1.1), we may assume that

$$\mathcal{H}^s\big(B \cap B(x,r)\big) < 2^{s+1}r^s \tag{2.3}$$

for all $0 < r < r_0$ and $x \in B$. Using the same estimate, we find $0 < r < r_0/3$ and $x \in B$ such that

$$\mathcal{H}^s\big(B \cap B(x,r)\big) > \tfrac{1}{2}r^s. \tag{2.4}$$

Set $t = t(\eta)$, $\gamma = \gamma(\eta)$, and take $0 < \delta < 1$. Let us fix $\beta < \pi$ such that the opening angle of $H(x,\theta,\gamma)$ is smaller than $\beta$, and let $q = q(n,\beta)$ be as in Lemma 2.1. We can cover the set $B \cap B(x,r)$ by $(2n)^{n/2}\delta^{-n}$ balls of radius $\delta r$ with centers in $B$. Using now (2.4), we notice that there exists $x_1 \in B \cap B(x,r)$ such that

$$\mathcal{H}^s\big(B \cap B(x_1,\delta r)\big) > (2n)^{-n/2}\delta^n 2^{-1} r^s.$$

The set $B \cap B(x,r) \setminus B(x_1, t\delta r)$ can also be covered by $(2n)^{n/2}\delta^{-n}$ balls of radius $\delta r$ with centers in $B$. Since, using (2.3) and (2.4),

$$\mathcal{H}^s\big(B \cap B(x,r) \setminus B(x_1,t\delta r)\big) > (\tfrac{1}{2} - 2^{s+1}t^s\delta^s)r^s$$

and if $\frac{1}{2} - 2^{s+1}t^s\delta^s > 0$, we find $x_2 \in B \cap B(x,r) \setminus B(x_1,t\delta r)$ for which

$$\mathcal{H}^s\big(B \cap B(x_2,\delta r)\big) > (2n)^{-n/2}\delta^n(\tfrac{1}{2} - 2^{s+1}t^s\delta^s)r^s.$$

Choosing $\delta = \delta(n,s,\eta) > 0$ small enough and continuing in this manner, we find $q$ points $x_1,\ldots,x_q \in B \cap B(x,r)$ with $|x_i - x_j| > t\delta r$ for $i \neq j$, such that for each $i \in \{1,\ldots,q\}$ we have

$$\begin{aligned}\mathcal{H}^s\big(B \cap B(x_i,\delta r)\big) &> (2n)^{-n/2}\delta^n\big(\tfrac{1}{2} - (q-1)2^{s+1}t^s\delta^s\big)r^s \\ &=: c(n,s,\eta)(3r)^s,\end{aligned} \tag{2.5}$$

where $c(n,s,\eta) > 0$.

According to Lemma 2.1, we may choose three points $y, y_1, y_2$ from the set $\{x_1,\ldots,x_q\}$ such that for each $\theta \in S^{n-1}$ there is $i \in \{1,2\}$ for which $y_i \in \mathbb{R}^n \setminus \big(B(y,t\delta r) \cup H(y,\theta,\gamma)\big)$. We obtain, using Lemma 2.3, that for each $\theta \in S^{n-1}$ there is $i \in \{1,2\}$ such that

$$B(y_i,\delta r) \subset B\big(y, 2(1+\delta)r\big) \setminus H(y,\theta,\eta).$$

Thus, applying (2.5), we have

$$\mathcal{H}^s\big(B \cap B(y,3r) \setminus H(y,\theta,\eta)\big) > c(n,s,\eta)(3r)^s$$

for all $\theta \in S^{n-1}$. Recalling (2.2), we conclude that $c \geq c(n,s,\eta)$. The proof is finished. □

**Theorem 2.5.** *Suppose $0 < \alpha, \eta \le 1$ and $0 \le m < s \le n$. Then there is a constant $c = c(n,m,s,\alpha,\eta) > 0$ such that*

$$\limsup_{r\downarrow 0} \inf_{\substack{\theta\in S^{n-1}\\ V\in G(n,n-m)}} \frac{\mathcal{H}^s\big(A\cap X(x,r,V,\alpha)\setminus H(x,\theta,\eta)\big)}{r^s} \ge c$$

*for $\mathcal{H}^s$ almost every $x \in A$ whenever $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$.*

*Proof.* For any $V, W \in G(n,n-m)$, we set $d(V,W) = \sup_{x\in V\cap S^{n-1}} \operatorname{dist}(x,W)$. With this metric $G(n,n-m)$ is compact, see Salli [10]. Defining for each $V \in G(n,n-m)$ a set $\{W : d(V,W) < \alpha/2\}$ we notice that a finite number of these sets is still a cover. We assume that the sets assigned to the planes $V_1,\dots,V_l$, where $l = l(n,m,\alpha)$, cover $G(n,n-m)$. For any $W$, it holds that $d(V_i,W) < \alpha/2$ with some $i \in \{1,\dots,l\}$. This implies $X(0,V_i,\alpha/2) \subset X(0,W,\alpha)$. Thus, for each $W \in G(n,n-m)$, there is $i$ such that

$$X(x,r,W,\alpha) \supset X(x,r,V_i,\alpha/2) \tag{2.6}$$

when $r > 0$ and $x \in \mathbb{R}^n$. We shall prove that if $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$, then

$$\limsup_{r\downarrow 0} \inf_{\substack{\theta\in S^{n-1}\\ i=1,\dots,l}} \frac{\mathcal{H}^s\big(A\cap X(x,r,V_i,\alpha/2)\setminus H(x,\theta,\eta)\big)}{r^s} \ge c(n,m,s,\alpha,n)$$

for $\mathcal{H}^s$ almost every $x \in A$ from which the claim follows easily by using (2.6).

Take $c > 0$ and assume that there exists a set $B \subset \mathbb{R}^n$ with $\mathcal{H}^s(B) > 0$ such that for each $x \in B$ and $0 < r < r_0$ there are $i$ and $\theta \in S^{n-1}$ for which

$$\mathcal{H}^s\big(B\cap X(x,r,V_i,\alpha/2)\setminus H(x,\theta,\eta)\big) < cr^s.$$

According to (1.1) we may assume that

$$\mathcal{H}^s\big(B\cap B(x,r)\big) < 2^{s+1}r^s$$

for all $0 < r < r_0$ and $x \in B$. Using the same estimate, we find $0 < r < r_0/3$ and $x \in B$ such that

$$\mathcal{H}^s\big(B\cap B(x,r)\big) > \tfrac{1}{2}r^s. \tag{2.7}$$

Next we define

$$B_i = \big\{x\in B : \mathcal{H}^s\big(B\cap X(x,3r,V_i,\alpha/2)\setminus H(x,\theta,\eta)\big) < c(3r)^s \text{ for some } \theta\in S^{n-1}\big\}. \tag{2.8}$$

Since $\bigcup_{i=1}^l B_i = B$, we infer from (2.7) that there is $i_0 \in \{1,\dots,l\}$ for which

$$\mathcal{H}^s\big(B_{i_0}\cap B(x,r)\big) > 2^{-1}l^{-1}r^s.$$

Take $0 < \varepsilon < 1$. Since the set $(V_{i_0}^\perp + \{x\}) \cap B(x,r)$ can be covered with $(2m)^{m/2}\varepsilon^{-m}$ balls of radius $\varepsilon r$, there exists $y \in (V_{i_0}^\perp + \{x\}) \cap B(x,r)$ such that

$$\mathcal{H}^s\big(B_{i_0}\cap B(x,r)\cap P_{V_{i_0}^\perp}^{-1}(B(y,\varepsilon r))\big) > (2m)^{-m/2}2^{-1}l^{-1}\varepsilon^m r^s.$$

Choosing $\varepsilon = \varepsilon(n,m,s,\eta) > 0$ small enough, we find, as in the proof of Theorem 2.4, a point $z \in B_{i_0} \cap B(x,r) \cap P^{-1}_{V^{\perp}_{i_0}}\big(B(y,\varepsilon r)\big)$ such that for each $\theta \in S^{n-1}$ there is a ball

$$B(w,\varepsilon r) \subset B\big(x,(1+\varepsilon)r\big) \setminus \big(H(z,\theta,\eta) \cap B(z,4\varepsilon r/\alpha)\big)$$

with

$$\mathcal{H}^s\big(B \cap B(w,\varepsilon r) \cap P^{-1}_{V^{\perp}_{i_0}}(B(y,\varepsilon r))\big) > c(n,m,s,\alpha,\eta)(3r)^s > 0.$$

Since

$$P^{-1}_{V^{\perp}_{i_0}}\big(B(y,\varepsilon r)\big) \cap B(z,3r) \setminus B(z,4\varepsilon r/\alpha) \subset X(z,3r,V_{i_0},\alpha/2),$$

we get

$$\inf_{\theta \in S^{n-1}} \mathcal{H}^s\big(B \cap X(z,3r,V_{i_0},\alpha/2) \setminus H(z,\theta,\eta)\big) > c(n,m,s,\alpha,\eta)(3r)^s.$$

Figure C illustrates the situation. Since $z \in B_{i_0}$, we conclude, using (2.8), that $c \geq c(n,m,s,\alpha,\eta)$. The proof is finished. □

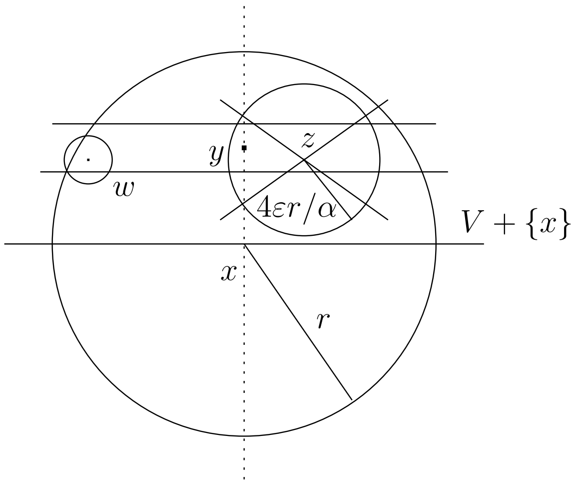


Figure C. Illustration for the proof of Theorem 2.5.

Our method can be applied also in a more general setting. A similar proof as above gives the following result. If $\mu$ is a measure on $\mathbb{R}^n$, $h\colon (0,r_0) \to (0,\infty)$, and $x \in \mathbb{R}^n$, we define $\overline{D}(\mu,x)$ and $\underline{D}(\mu,x)$ as the lower and upper limits, respectively, of the ratio $\mu(B(x,r))/h(r)$ as $r \downarrow 0$.

**Theorem 2.6.** *Suppose $0 \leq m < n$ and $h\colon (0,r_0) \to (0,\infty)$ is a function with*

$$\frac{h(\varepsilon r)}{\varepsilon^m h(r)} \longrightarrow 0 \qquad \textit{uniformly for all } 0 < r < r_0 \tag{2.9}$$

*as* $\varepsilon \downarrow 0$. *Let* $\mu$ *be a measure on* $\mathbb{R}^n$ *with* $\overline{D}(\mu, x) < \infty$ *for* $\mu$ *almost all* $x \in \mathbb{R}^n$. *For every* $0 < \alpha, \eta \leq 1$, *there is a constant* $c = c(n, m, h, \alpha, \eta) > 0$ *such that*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \eta)\big)}{h(r)} \geq c\overline{D}(\mu, x)$$

*for* $\mu$ *almost every* $x \in \mathbb{R}^n$.

Let us make few comments related to the above theorem. Suppose that $h$ fulfills condition (2.9). Let $\mathcal{H}_h$ be the generalized Hausdorff measure which is constructed using $h$ as a gauge function, see [9, §4.9]. If $\mu = \mathcal{H}_h|_A$, where $\mathcal{H}_h(A) < \infty$, then $\overline{D}(\mu, x) < \infty$ for $\mu$ almost every $x \in \mathbb{R}^n$, and thus Theorem 2.6 can be applied.

There are many natural gauge functions, such as $h(r) = r^s \log(1/r)$ where $m < s < n$, which satisfy (2.9). However, some interesting cases, such as $h(r) = r^m / \log(1/r)$, are not covered by this condition.

It seems to be unknown whether a similar result as Theorem 2.6 holds if one replaces the condition $\overline{D}(\mu, x) < \infty$ by $\underline{D}(\mu, x) < \infty$. The most interesting example falling into this category is obtained when $\mu = \mathcal{P}^s|_A$ and $h(r) = r^s$, where $\mathcal{P}^s(A) < \infty$ and $m < s < n$. See also Suomala [13] for related theorems. Here $\mathcal{P}^s$ denotes the $s$-dimensional packing measure, see [9, §5.10].

## 3. Sets with large $k$-porosity

Mattila [8] proved Theorem 2.5 in the case $m = n - 1$. Using this, he obtained the desired dimension bounds for 1-porous sets, see (1.3). Our result for $k$-porous sets follows applying a similar argument.

For $\sqrt{2} - 1 < \varrho < \frac{1}{2}$ we define

$$t(\varrho) = \frac{1}{\sqrt{1 - 2\varrho}},$$

$$\delta(\varrho) = \frac{1 - \varrho - \sqrt{\varrho^2 + 2\varrho - 1}}{\sqrt{1 - 2\varrho}}.$$

Notice that $\delta(\varrho) \to 0$ as $\varrho \to \frac{1}{2}$.

**Lemma 3.1.** *Suppose* $x \in \mathbb{R}^n$, $r > 0$, $\sqrt{2} - 1 < \varrho < \frac{1}{2}$, $t = t(\varrho)$, *and* $\delta = \delta(\varrho)$. *If* $z \in \mathbb{R}^n$ *is such that* $B(z, \varrho t r) \subset B(x, tr)$, *then*

$$H(x + \delta r \theta, \theta) \cap B(x, r) \subset B(z, \varrho t r),$$

*where* $\theta = (z - x)/|z - x|$.

*Proof.* To simplify the notation, we assume $r = 1$, $x = 0$, and $\theta = e_1 = (1, 0, \ldots, 0)$. This will not affect the generality. Let $y \in B(0, 1) \setminus B(z, \varrho t)$. We have to show that

$$y \notin H(x + \delta\theta, \theta). \tag{3.1}$$

By the Pythagorean Theorem we have

$$|z - y_1| = \sqrt{|z - y|^2 - |y - y_1|^2} \geq \sqrt{(\varrho t)^2 - 1}.$$

Using this, we obtain

$$y_1 = |z| - |z - y_1| \leq t - \varrho t - \sqrt{(\varrho t)^2 - 1} = \delta,$$

which implies (3.1). □

**Theorem 3.2.** *Suppose* $0 < k \leq n$. *Then*

$$\sup\{s > 0 : \mathrm{por}_k(A) > \varrho \text{ and } \dim_H(A) > s \text{ for some } A \subset \mathbb{R}^n\} \longrightarrow n - k$$

*as* $\varrho \to \frac{1}{2}$.

*Proof.* Assume on the contrary that there exists $s > n - k$ such that for each $\sqrt{2} - 1 < \varrho < \frac{1}{2}$ there is a set $A_\varrho$ for which $\dim_H(A_\varrho) > s$ and $\mathrm{por}_k(A_\varrho) > \varrho$. Take $\sqrt{2} - 1 < \varrho < \frac{1}{2}$ and such a set $A_\varrho$. Now $A_\varrho$ has a subset $B$ for which $\dim_H(B) > s$ and $\mathrm{por}_k(B, x, r) > \varrho$ for all $x \in B$ and $0 < r < r_0$ with some $r_0 > 0$. Clearly also the closure of $B$ satisfies these conditions. Thus there is a closed set $F \subset \overline{B}$ (for example, use [2, Theorem 5.4]) such that $0 < \mathcal{H}^s(F) < \infty$ and

$$\mathrm{por}_k(F, x, r) > \varrho \qquad \text{for all } x \in F \text{ and } 0 < r < r_0.$$

Therefore, for any $x \in F$ and $0 < r < r_0/t$, there are $z_1, \ldots, z_k \in \mathbb{R}^n$ such that $B(z_i, \varrho t r) \subset B(x, tr) \setminus F$ for $i = 1, \ldots, k$, and $(z_i - x) \cdot (z_j - x) = 0$ for $i \neq j$. Put $\theta_i = (z_i - x)/|z_i - x|$. Applying now Lemma 3.1 we have $H(x + \delta r \theta_i, \theta_i) \cap B(x, r) \subset B(z_i, \varrho t r)$ for every $i$. Here $t = t(\varrho)$ and $\delta = \delta(\varrho)$. Thus

$$F \cap B(x, r) \subset \bigcap_{i=1}^{k} B(x, r) \setminus H(x + \delta r \theta_i, \theta_i). \tag{3.2}$$

Put $\theta = -\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \theta_i$ and take $V \in G(n, k)$ such that $\theta_i \in V$ for every $i$. Now choosing $\alpha$ and $\eta$ small enough, we have, using (3.2), that

$$F \cap X(x, r, V, \alpha) \setminus H(x, \theta, \eta) \subset B(x, 2n^{1/2} \delta r). \tag{3.3}$$

Observe that the choice of $\alpha$ and $\eta$ does not depend on $\delta$ and hence not on $\varrho$ either. Figure D illustrates the situation. Using Theorem 2.5, we may fix $x \in F$ and $0 < r < r_0/t$ for which

$$\mathcal{H}^s\big(F \cap X(x, r, V, \alpha) \setminus H(x, \theta, \eta)\big) \geq c 2^{2s+1} n^{s/2} r^s, \tag{3.4}$$

where $c = c(n, k, s, \alpha, \eta) > 0$. By (1.1) we may assume that also

$$\mathcal{H}^s\big(F \cap B(x, 2n^{1/2} \delta r)\big) \leq 2^{2s+1} n^{s/2} \delta^s r^s. \tag{3.5}$$

Combining (3.3)–(3.5), we have $c 2^{2s+1} n^{s/2} r^s \leq 2^{2s+1} n^{s/2} \delta^s r^s$ and hence

$$s \leq \frac{\log c}{\log \delta(\varrho)}.$$

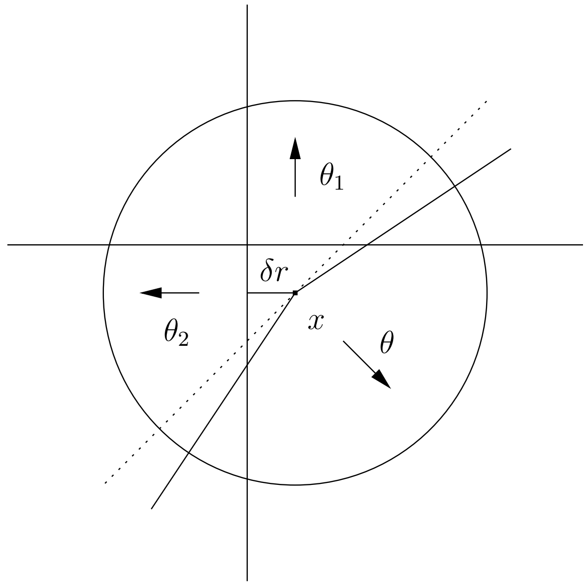


FIGURE D. The situation when $n = 2$, $k = 2$.

But the constant $c$ does not depend on $\varrho$, and thus $\log c/\log\delta(\varrho) \to 0$ as $\varrho \to \frac{1}{2}$ giving a contradiction. □

## 4. Sets with small porosity

Finally, let us briefly discuss the situation when porosity is small. The proof of the following theorem can be found for example in Martio and Vuorinen [7]. We shall give here a different proof, and then show how the theorem can be improved when more information on the location of the holes is given.

**Theorem 4.1.** *Let $A \subset \mathbb{R}^n$ be bounded and suppose that $\mathrm{por}_1(A, x, r) \geq \varrho$ for all $x \in A$ and $0 < r < r_0$. Then $\dim_M(A) < n - c\varrho^n$, where $c > 0$ depends only on $n$.*

*Proof.* We may assume that $r_0 = 1$ and $A \subset [0,1]^n$. Let us denote by $\mathcal{Q}_j$ the collection of all closed dyadic cubes $Q \subset [0,1]^n$ with side length $2^{-j}$. Let $l$ be the smallest integer with $2^{-l+2} < \varrho/\sqrt{n}$. It is easy to see that for any $Q \in \mathcal{Q}_j$ there is $Q' \in \mathcal{Q}_{j+l}$ such that $Q' \subset Q$ and $Q' \cap A = \emptyset$. Let us fix one such $Q'$ for each $Q \in \bigcup_{j=1}^{\infty} \mathcal{Q}_j$. Next we define a set $B \subset [0,1]^n$ by setting

$$B = [0,1]^n \setminus \bigcup_{j=0}^{\infty} \bigcup_{Q \in \mathcal{Q}_j} Q'. \tag{4.1}$$

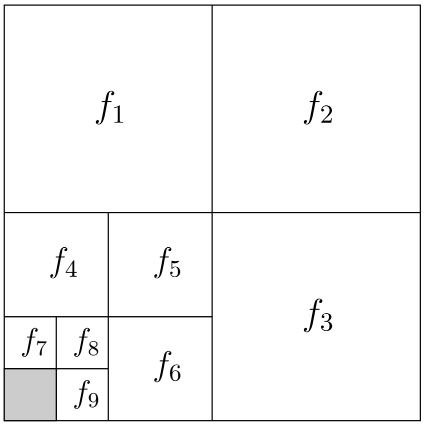


FIGURE E. Similitudes $f_k$, when $n = 2$, $l = 3$.

For any $Q \in \mathcal{Q}_j$, let $x_Q$ be the corner of $Q$ which is nearest to the origin, and let $\widetilde{Q} = \{x_Q\} + 2^{-j-l}[0,1]^n$. If we define $E \subset [0,1]^n$ by setting

$$E = [0,1]^n \setminus \bigcup_{j=0}^{\infty} \bigcup_{Q \in \mathcal{Q}_j} \mathrm{int}\widetilde{Q},$$

where int denotes the interior of a given set, then obviously $\dim_M(E) \geq \dim_M(B)$, see also Käenmäki [5, Example 3.14]. The set $E$ is the limit set of the iterated function system defined by the similitudes $f_k$, $k = 1, 2, \ldots, l(2^n - 1)$, see Figure E. For any $i \in \{1, \ldots, l\}$, there are $2^n - 1$ similitudes among $\{f_k\}_{k=1}^{l(2^n-1)}$ with contraction ratio $2^{-i}$. Since the open set condition is clearly satisfied, the dimension $s = \dim_M(E) = \dim_H(E)$ is given by

$$(2^n - 1) \sum_{i=1}^{l} 2^{-is} = 1,$$

see Hutchinson [3, §5]. This reduces to

$$2^{n-s} = 1 + (2^n - 1)2^{-(l+1)s}$$

and since $\log_2(1 + x) \geq x / \big((1 + x) \log 2\big)$ for $x \geq 0$, we have

$$\begin{aligned} s &= n - \log_2\big(1 + (2^n - 1)2^{-(l+1)s}\big) \\ &\leq n - \log_2\big(1 + (1 - 2^{-n})2^{-ln}\big) \\ &\leq n - \frac{2}{5 \log 2} 2^{-ln} \leq n - c\varrho^n, \end{aligned}$$

where $c = \big(2/(5 \log 2)\big)2^{-3n} n^{-n/2}$. Because $A \subset B$ and $\dim_M(B) \leq \dim_M(E) = s$, we conclude that also $\dim_M(A) \leq n - c\varrho^n$. $\square$

We believe it is known, that a straightforward box-counting argument can be used to prove Theorem 4.1. However, such a direct method seems to lead to quite complicated recursive formulae and to avoid this, we used the self-similar set $E$.

*Remark* 4.2. In a sense the above result is the best possible one. There is a constant $c' = c'(n) > 0$ and sets $A_\varrho$, $0 < \varrho < 1/2$, with $\dim_H(A_\varrho) > n - c'\varrho^n$, and $\mathrm{por}_1(A_\varrho, x, r) \geq \varrho$ for all $r > 0$ and $x \in \mathbb{R}^n$. See, for example, Koskela and Rohde [6], or estimate the Hausdorff dimension of the set $E$ from below.

**Theorem 4.3.** *Let $A \subset \mathbb{R}^n$ be bounded and suppose that there is $V \in G(n,m)$ such that for all $x \in A$ and $0 < r < r_0$ one has*

$$\sup\big\{\varrho' : B(z, \varrho' r) \subset B(x,r) \setminus A \text{ for some } z \in V + \{x\}\big\} \geq \varrho. \tag{4.2}$$

*Then $\dim_M(A) < n - c\varrho^m$, where $c > 0$ depends only on $n$ and $m$.*

*Proof.* Without losing the generality we may assume that $V = \mathbb{R}^m = \{x \in \mathbb{R}^n : x_{m+1} = x_{m+2} = \ldots = x_n = 0\}$, $r_0 = \sqrt{n}$, and $A \subset [0,1]^n$. Let $\mathcal{Q}_j$ be, as before, the collection of all closed dyadic cubes $Q \subset [0,1]^n$ with side length $2^{-j}$, and let $\widetilde{\mathcal{Q}}_j = \{P_V(Q) : Q \in \mathcal{Q}_j\}$ and $\mathcal{Q}'_j = \{P_{V^\perp}(Q) : Q \in \mathcal{Q}_j\}$. Here $P_V$ is the orthogonal projection onto $V$. Furthermore, let $l$ be the smallest integer with $2^{-l+2} < \varrho/\sqrt{n}$.

We define a set $E = E_{l,m} \subset V$ as in the proof of Theorem 4.1. For $j \in \mathbb{N}$ we let $a_j = a_{j,l,m}$ denote the minimum number of cubes from the collection $\widetilde{\mathcal{Q}}_j$ that are needed to cover $E$. The proof of Theorem 4.1 yields that

$$\lim_{j\to\infty} \frac{\log a_j}{\log(2^j)} \leq m - c2^{-ml}, \tag{4.3}$$

where $c > \frac{1}{2}$ is an absolute constant.

It is straightforward to convince oneself of the following fact: If $\widetilde{Q} \in \widetilde{\mathcal{Q}}_j$ and $Q' \in \mathcal{Q}'_{j+l}$, then there is $Q \in \mathcal{Q}_{j+l}$ such that $P_{V^\perp}(Q) = Q'$, $P_V(Q) \subset \widetilde{Q}$, and $A \cap Q = \emptyset$. From this observation it follows that given $Q' \in \mathcal{Q}'_j$, only $a_j$ cubes from the collection $\{Q \in \mathcal{Q}_j : P_{V^\perp}(Q) = Q'\}$ touch the set $A$. Thus only $2^{j(n-m)}a_j$ cubes from the collection $\mathcal{Q}_j$ are needed to cover $A$. Using (4.3), we calculate

$$\begin{aligned}\dim_M(A) &\leq \limsup_{j\downarrow 0} \frac{\log(2^{j(n-m)}a_j)}{\log(2^j)} = n - m + \limsup_{j\downarrow 0}\frac{\log a_j}{\log(2^j)}\\ &\leq n - c2^{-ml} \leq n - c2^{-3m}n^{-m/2}\varrho^m.\end{aligned}$$

□

*Remark* 4.4. Suppose that $V \in G(n,m)$ is fixed and $A \subset \mathbb{R}^n$ is such that (4.2) holds for every $x \in A$ and $0 < r < r_x$, where $r_x > 0$ depends on the point $x$. It follows immediately from Theorem 4.3 that $\dim_H(A) \leq \dim_P(A) \leq n - c\varrho^m$, where $c$ is as in Theorem 4.3 and $\dim_P$ denotes the packing dimension, see [9, §5.9]. The above dimension estimates are also sharp. Consider, for example, sets of the form $E \times \mathbb{R}^{n-m}$, where $E \subset \mathbb{R}^m$ is as in the proof of Theorem 4.1.

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