NATURAL ERGODIC MEASURES ARE NOT ALWAYS OBSERVABLE

ESA JÄRVENPÄÄ[1] AND TAPANI TOLONEN[2]

Abstract. We construct a continuous dynamical system on a compact connected metric space which has a unique ergodic invariant natural measure but has no observable measures.

1. Introduction

One of the most important properties of axiom A attractors is the existence of an SRB-measure. If the system is topologically mixing the SRB-measure is unique. These results are due to Sinai [S], Ruelle [R], and Bowen [B]. In this setting the unique SRB-measure μ has many well-known characterizations. First of all, it is observable, that is, the Birkhoff averages of Lebesgue almost all initial points can be calculated using μ (for more precise definitions see section 2). Secondly, the SRB-measure is natural meaning that any measure which is absolutely continuous with respect to the Lebesgue measure converges to μ under the iteration of the dynamics. Thirdly, in the expanding case μ is equivalent to the Lebesgue measure, and in the genuinely hyperbolic case the conditional distributions of μ along the unstable manifolds are absolutely continuous with respect to the Lebesgue measure on unstable leaves. Finally, μ is an equilibrium state for a potential function constructed using the derivative of the map.

Since the original works of Sinai, Ruelle, and Bowen there has been an extensive study of SRB-measures. We refer to the nice survey of Young [Y] for more information. Despite of many important results, the understanding of systems having or having not SRB-measures is far from being satisfactory. Therefore it is important to find out relations between the above mentioned properties in general dynamical systems. The third and fourth properties are closely tied to hyperbolic systems since one needs the existence of unstable leaves or symbolic dynamics to even define them. Thus from the general point of view the first to properties are more interesting. Blank and Bunimovich have studied the relations between observable and natural measures in [BB]. They prove (see [BB, Theorem 2.1]) that if an observable measure exists then it is also natural. On the other hand, natural measures are not

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necessarily observable (see [BB, Theorem 2.1]). However, Blank and Bunimovich proved that any ergodic invariant natural measure which is equivalent to the Lebesgue measure is also observable. They also show that ergodicity or equivalence to the Lebesgue measure is necessary in this implication: using a result of Inoue [I, Corollary 2.2] (see also [K]), they verify the existence of a non-ergodic invariant natural measure which is not equivalent to the Lebesgue measure and not observable.

In this paper we prove that ergodicity does not guarantee the equivalence of natural and observable measures. Namely, we construct a continuous dynamical system on a compact connected metric space which has a unique ergodic invariant natural measure but no observable measures.

The paper is organized as follows. The notation is introduced in section 2. In section 3 we construct the compact metric space and the mapping defined on it. The aforementioned properties of this dynamical system are proved in section 4. Finally, in section 5 we give some modifications of our system.

2. Basic definitions and notation

In this section we give the basic definitions and introduce the notation used throughout this paper.

Definition 2.1. Let X be a topological space equipped with a locally finite Borel regular measure $\mathcal{L}$, and let $T : X \to X$ be Borel measurable. A Borel regular probability measure μ is *natural* (with respect to $\mathcal{L}$), if there exists an open set $U \subset X$ such that for all Borel regular probability measures ν, which are absolutely continuous with respect to $\mathcal{L}$ (denoted by $\nu \ll \mathcal{L}$) and supported on U, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (T_*)^j \nu = \mu$$

where the limit is taken in the weak* topology and $T_*\nu$ is the image measure of ν under the map T. A Borel regular probability measure μ is *observable* (with respect to $\mathcal{L}$), if there exists an open set $U \subset X$ such that for $\mathcal{L}$-almost all $x \in U$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{T^j(x)} = \mu$$

where δ_x is the normalized point mass at x. (Here again the weak* topology is used.)

We will represent our construction using a modification of the stacking method originally due to von Neumann and Kakutani (see [F, chapter 6]). For this we need some notation. Let $C_i = [c_i, c_i^*)$, $i = 0, 1, \ldots, n$, be non-empty intervals with disjoint closures. Denoting $C' = \cup_{i=0}^{n} C_i$, we can define a one-to-one mapping $\tau_C : C' \setminus C_n \to C' \setminus C_0$ in the following manner: when $1 \le i < n$, set $\tau_i : C_i \to C_{i+1}$ by

$$\tau_i(x) = \frac{c^*_{i+1} - c_{i+1}}{c^*_i - c_i}(x - c_i) + c_{i+1}, \tag{2.1}$$

and define $\tau_C(x) = \tau_i(x)$, when $x \in C_i$. We represent the *tower* associated to the function τ_C as an ordered product $C = \prod_{i=0}^n C_i = C_0 \cdots C_n$. Clearly τ_i is the unique affine order-preserving function of C_i onto C_{i+1}. Therefore the intervals C_i define the function τ_C uniquely and one can equivalently speak of the tower C and the function τ_C. The tower C has C_0 as its *base* and C_n as its *top*. We say that a tower *starts* from its base. Every x in C' has some preimage $\tau_C^{-k}(x)$ in C_0, admitting $k = 0$. This k, which is unique for x, is called the *height* of x and denoted by $h(x)$. The points with the same height are said to be on the same *level*. (The concepts "above", "below", and "vertical" make thus sense due to the correspondence between iteration and ascending height. They only refer to the location of points in the tower, not their actual location on the real line.) The height of a tower C is

$$h(C) = \max\{h(x) \mid x \in C'\} - \min\{h(x) \mid x \in C'\} + 1.$$

For the tower C defined above we have $h(C) = n + 1$. Later we will use towers starting from different levels.

It is obvious that also infinitely high towers $\prod_{i=0}^\infty C_i$ can be defined. Two towers C and D are disjoint, if $C' \cap D' = \emptyset$. If c'_i divides the interval C_i in some proportion, say in half, then $\tau_C(c'_i) = c'_{i+1}$ divides C_{i+1} in the same proportion. We see that τ_C maps interval $[c_i, c'_i) =: C^1_i$ onto $[c_{i+1}, c'_{i+1}) =: C^1_{i+1}$ and $[c'_i, c^*_i) =: C^2_i$ onto $[c'_{i+1}, c^*_{i+1}) =: C^2_{i+1}$. Thus we can think the tower $C = \prod_{i=0}^n C_i$ as consisting of two (parallel) subtowers $\prod_{i=0}^n C^1_i$ and $\prod_{i=0}^n C^2_i$ as well as the (successive) subtowers $\prod_{i=0}^m C_i$ and $\prod_{i=m+1}^n C_i$. (The term subtower will be used in both senses.) Say we have two disjoint towers C and D with heights $n = h(C) > h(D) = m$. We can *replace* the subtower $\prod_{i=j}^{j+m-1} C^2_i$ $(0 \le j \le n + 1 - m)$ by tower $\prod_{i=0}^{m-1} D_i$ defining the function τ from the interval C^2_{j-1} onto D_0 and from D_{m-1} onto C^2_{j+m} as in (2.1) and elsewhere as a restriction of τ_C and τ_D in an obvious manner. Since the closures of the intervals are disjoint, the resulting function is continuous everywhere in its domain excluding the point c'_{j-1}.

3. Construction of the mapping

In this section we will construct our mapping. The basic idea of the construction is as follows: The mapping is defined on a subset of the interval $[-1, 1]$. We start with a short interval near zero. It is mapped linearly onto an interval which is left to the starting interval. After continuing in this manner for a while, the interval is divided into two parts. One part continues to move towards -1 while the other one starts to move back to 0 and further to 1. After spending a long time (compared to its history) close to 1, it returns back to 0 and further to -1. Then a half of those points which have been close to -1 will start to move towards 1, spend there a relatively long time, and come back close to -1 again. In this way every point will spend arbitrarily long periods of time near -1 and near 1, which guarantees that there will be no observable measures. On the other hand, the proportion of points being near -1 at fixed time will tend to 1 as time tends to infinity implying that the point mass at -1 will be the unique natural measure. To make the mapping continuous, we need to remove the neighbourhoods of the splitting points of the intervals, and their preimages. Finally, adding an extra dimension to our system makes the space connected.

We start by constructing a compact metric space $Z \subset [-1, 1]$ and an auxiliary mapping $\widetilde{T} : Z \to Z$. We will denote the Lebesgue measure on $[-1, 1]$ by $\mathcal{L}$.

Definition 3.1. For $A, B \subset [-1, 1]$ we write $A < B$, if $x < y$ for every $x \in \overline{A}$ and $y \in \overline{B}$. Here the closure of a set A is denoted by $\overline{A}$. We use also the notation $x < A$ if $\{x\} < A$, and $x \leq A$ if $x < A$ or $x \in \overline{A}$. The distance of a point x to a set A is denoted by $d(x, A) = \inf\{d(x, y) \mid y \in A\}$, where $d(x, y)$ is the usual Euclidean distance between two points.

For $i = 0, 1, \ldots$, let $C_i = [c_i, c_i^*)$ and $D_i = [d_i, d_i^*)$ be half-open subintervals of the open interval $(-1, 0)$, and $E_i = [e_i, e_i^*)$ and $F_i = [f_i, f_i^*)$ subintervals of $(0, 1)$, such that $C_0 > D_0 > C_1 > D_1 > \ldots$ and $E_0 < F_0 < E_1 < F_1 < \ldots$ with $\lim_{i\to\infty} d(-1, C_i) = \lim_{i\to\infty} d(1, E_i) = 0$. For $j = 1, 2, \ldots$, let $D_i^j = [d_i^j, d_i^{j*})$ and $F_i^j = [f_i^j, f_i^{j*})$ be subintervals of D_i and F_i, respectively, so that for every $i = 0, 1, \ldots$ we have

(1) $D_i^1 < D_i^2 < \ldots$
(2) $\lim_{j\to\infty} d(d_i^*, D_i^j) = 0$,
(3) $\mathcal{L}(D_i^1) < \frac{1}{2}\mathcal{L}(D_i)$ and $\mathcal{L}(D_i^{j+1}) < \frac{1}{2}\mathcal{L}(D_i^j)$,

and similarly for F_i^j. Now if H stands for C, D, E or F, then $\mathcal{L}(\cup_{i:n\leq i} H_i)$ tends to zero as n tends to infinity. If H is either D or F, then $\mathcal{L}(\cup_{j:n\leq j} H_i^j) < \frac{1}{2^{n-1}}\mathcal{L}(H_i) < \frac{1}{2^{n-1}}$ for every i and therefore

$$\mathcal{L}\Big(\bigcup_{\substack{i,j \\ i\leq n\leq j}} H_i^j\Big) \leq \frac{n+1}{2^{n-1}} \xrightarrow[n\to\infty]{} 0. \tag{3.1}$$

The union of the intervals H_i will be denoted by H', where $H = C$, D, E, or F.

The construction of $\widetilde{T}$ is based on the tower $C = \prod_{i=0}^{\infty} C_i$ and towers B_i with $h(B_i) = h_i$ to be defined precisely in (3.2). All the towers

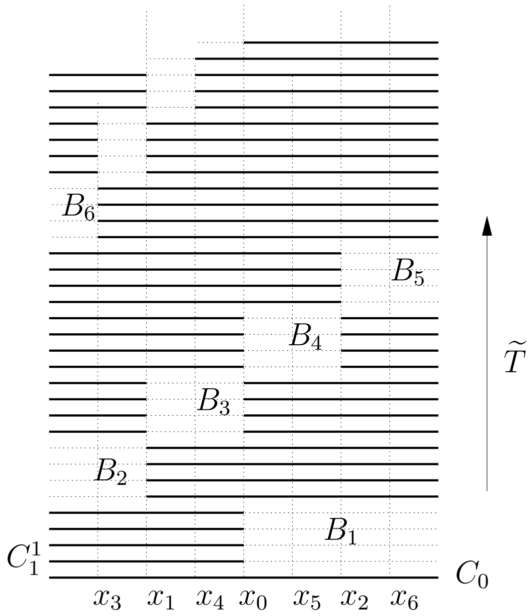

Figure 1. Schematic picture of the modified tower corresponding to $\widetilde{T}$. In the picture the intervals are scaled equally long. We have curtailed the heights of subtowers B_i so that they fit nicely in the picture. The points x_i are dyadic on the base interval C_0.

are mutually disjoint. Dividing C (vertically) in half gives two parallel subtowers with bases C_1^1 and C_1^2. Then the subtower $\prod_{i=1}^{h_1} C_i^2$ is replaced by B_1 in the manner described in section 2. Setting $n = h_1 + 1$, divide the subtowers starting from intervals C_n^1 and C_n^2 in half and re-enumerate them such that $C_n^1 < C_n^2 < C_n^3 < C_n^4$. Replace the subtower starting from C_n^1 with height h_2 by B_2, the subtower starting from $C_{n+h_2}^2$ with height h_3 by B_3, and the subtowers starting from $C_{n+h_2+h_3}^3$ and $C_{n+h_2+h_3+h_4}^4$ with heights h_4 and h_5 by B_4 and B_5, respectively. Then continue by dividing the subtowers in half, re-enumerating them, and replacing subtowers with corresponding lengths by $B_6, \dots, B_{13}$. This procedure will be carried on and on always inserting the subtowers B_j in the tower so that the base of B_{j+1} is one level above the top of B_j (see Figure 1).

The function defined above is denoted by $\widetilde{T}$ and its domain by Z. Finally, we define the towers B_i explicitly. The structure of B_i depends on the heights of $B_1, \dots, B_{i-1}$. Namely, if B_i is inserted in C so that it starts from the nth level, it is of the form

$$B_i = D_n^n \cdot D_{n-1}^n \cdots D_0^n \cdot \prod_{j=n}^{4n} E_j \cdot F_{4n}^{4n} \cdot F_{4n-1}^{4n} \cdots F_0^{4n}. \tag{3.2}$$

Set $B' := \cup_i B'_i$ where the union of intervals in B_i is denoted by B'_i. Every $x \in Z$ has some nth preimage in C_0, so the height of points will be understood with respect to $\widetilde{T}$ and the base C_0. The points which divide C_0 into 2^n equally long intervals for some n, i.e., the dividing points due to bisecting C_0 over and over again, are called *dyadic* (see Figure 1).

Note that in the definition of $\widetilde{T}$ we did not use all the intervals defined above. The only use for the unused intervals is that they make the indexation simpler.

Lemma 3.2. *The mapping $\widetilde{T} : Z \to Z$ has the following properties:*

(1) Its points of discontinuity are some iterates of the dyadic points in C_0.
(2) On every level (except C_0) of $\widetilde{T}$ there is exactly one interval in B'. We call such interval $\mathcal{B}_n$.
(3) For every $x \in Z$ there are infinitely many positive integers n and m such that $\widetilde{T}^n(x) \in C'$ and $\widetilde{T}^m(x) \in E'$.
(4) If $x \in C'$ (respectively E') and $n > 0$ is the smallest integer such that $\widetilde{T}^n(x) \in E'$ ($\in C'$) then $\widetilde{T}^i(x) \in E'$ ($\in C'$) for all $n \leq i \leq 2n$.
(5) If $G_n = \{x \in Z \mid h(x) \geq n\}$, then $\lim_{n\to\infty} \mathcal{L}(G_n) = 0$.
(6) $\lim_{n\to\infty} \mathcal{L}(\widetilde{T}^{-n}(\mathcal{B}_n)) = 0$.
(7) If $Q_n^k = \{x \in Z \mid -1 < \widetilde{T}^n(x) < \frac{1}{k} - 1\}$, then

$$\lim_{n\to\infty} \mathcal{L}(Z \setminus Q_n^k) = 0$$

for every fixed $k > 0$.

Proof. (1) As mentioned before, the only points where $\widetilde{T}$ is discontinuous arise in the construction when dividing the subtowers in half and replacing them. The preimages of such points in C_0 are dyadic.

(2) This is obvious considering the construction of $\widetilde{T}$.

(3) In view of the construction, it is easy to see that every $x \in Z$ has some iterate $\widetilde{T}^n(x)$ in B'_i for some i. Since B_i is of the form (3.2), it is obvious that some iterate will also be in E'. The corresponding claim for C' instead of E' is similar.

(4) Let $n > 0$ be the smallest integer such that $\widetilde{T}^n(x) \in E'$ for $x \in C'$. Then $\widetilde{T}^n(x) \in E_m \subset B'_k$ for some m and k, and moreover

$$B_k = D_m^m \cdot D_{m-1}^m \cdots D_0^m \cdot \prod_{j=m}^{4m} E_j \cdot F_{4m}^{4m} \cdot F_{4m-1}^{4m} \cdots F_0^{4m}$$

(recall (3.2)). Therefore there are $m-1$ levels below B_k (that is, below the interval D_m^m) and m levels between D_m^m and E_m, altogether $2m$ levels below E_m. Thus $n < 2m$. For $x \in C'$ the claim then follows from the fact that $\prod_{j=m}^{4m} E_j$ is a subtower of B_k with height $3m+1 > 2m > n$.

Let $x \in E'$ and n be the smallest integer such that $\widetilde{T}^n(x) \in C'$. Say $x \in E_m \subset B'_k$. Then $n \leq h_k$. It is easy to see in the light of (3.2) that $h_{k+1} > 2h_k \geq 2n$ and also that there are at least h_{k+1} intervals in C' above the subtower B_k, from which the corresponding claim for $x \in E'$ follows.

(5) Every $x \in G_n$ belongs to some C_m or $\mathcal{B}_m$ with $m \geq n$. Moreover, according to (3.2), $\mathcal{B}_m$ is either of the form E_i with $i \geq n$, or of the form D_i^j or F_i^j with $j \geq n$ and $i \leq j$. Since $\lim_{n\to\infty} \mathcal{L}(\cup_{i:i\geq n} C_i \cup D_i \cup E_i \cup F_i) = 0$, it suffices to prove that $\lim_{n\to\infty} \mathcal{L}(\cup_{i,j:j\geq n\geq i} D_i^j \cup F_i^j) = 0$, but this has already been stated in (3.1).

(6) As n tends to infinity, smaller and smaller fractions of the points on level $n-1$ are mapped onto $\mathcal{B}_n$ (see Figure 1). The inverse function of $\widetilde{T}$ preserves the proportion of such points. Therefore smaller and smaller subintervals of C_0 are mapped onto $\mathcal{B}_n$ by $\widetilde{T}^n$ as n increases.

(7) Let $\varepsilon > 0$. We know that when j is large then $d(-1, C_i) < \frac{1}{k}$ for all $i \geq j$. Thus for $n > j$ every $x \in Z \setminus Q_n^k$ is such that $\widetilde{T}^n(x) \in \mathcal{B}_i$ for some $i \geq j$. By (5) we may take m so large that $\mathcal{L}(G_m) < \frac{\varepsilon}{2}$. Then

$$\mathcal{L}(Z \setminus Q_n^k) \leq \mathcal{L}(G_m) + \mathcal{L}(H_m),$$

where

$$H_m = \{x \in Z \mid h(x) \leq m \text{ and } \widetilde{T}^n(x) \in B'\} \subset \cup_{j=0}^m \widetilde{T}^j(\widetilde{T}^{-(n+j)}(\mathcal{B}_{n+j})).$$

Choosing $n > j$ large enough, we have by (6) that

$$\mathcal{L}(\cup_{j=0}^m \widetilde{T}^j(\widetilde{T}^{-(n+j)}(\mathcal{B}_{n+j}))) < \frac{\varepsilon}{2}.$$

Observe that $|\widetilde{T}'| < K = K(m)$ here, where the notation $K(m)$ means that the constant K depends on m. In fact, setting some lower bound on the decrease rate of the lengths of the intervals of the construction, one has $|\widetilde{T}'(x)| < K$ for all $x \in Z$, that is, one obtains a constant independent of m. □

Definition 3.3. Consider an increasing sequence (m_i) of positive integers and a point $x \in Z$. Define

$$N_i(x) = \{1 \leq k \leq m_i \mid \widetilde{T}^k(x) < 0\} \text{ and}$$
$$P_i(x) = \{1 \leq k \leq m_i \mid \widetilde{T}^k(x) > 0\}.$$

From now on we will write N_i instead of $N_i(x)$ as the point x will be clear from the context. Define $n_i = \#N_i$ and $p_i = \#P_i$, where the number of the elements in a set A is denoted by $\#A$.

Lemma 3.4. *For every $x \in Z$ there is an increasing sequence (m_i) of positive integers such that $p_i = n_i = \frac{m_i}{2}$.*

Proof. We may assume that $x < 0$ because the case $x > 0$ is similar. According to Lemma 3.2.(3), there is n such that $\widetilde{T}^n(x) \in E'$. Take the smallest such n which is easily seen to be the smallest integer such that $\widetilde{T}^n(x) > 0$. Using Lemma 3.2.(4), we know that the points $\widetilde{T}^n(x), \dots, \widetilde{T}^{2n}(x)$ are also in E', and thus positive. We set $m_1 = 2n-2$, whence $n_1 = n-1 = p_1$. The rest of the sequence is defined recursively. Assume $z := \widetilde{T}^{m_i}(x) > 0$. (The case < 0 is treated similarly by inverting all the inequalities.) Then there is again the smallest n such that $\widetilde{T}^n(z), \dots, \widetilde{T}^{2n}(z) < 0$. We define $m_{i+1} = m_i + 2n - 2$ and get $n_{i+1} = n_i + n - 1 = \frac{m_i}{2} + n - 1 = p_i + n - 1 = p_{i+1}$. □

The set Z is not compact since it lacks the right endpoints of its intervals and the accumulation points -1, 1, d_i^*, and f_i^* for $i = 0, 1, \dots$ (We denote the accumulation points outside Z by $\overline{Z} \setminus Z =: Y$.) However, the mapping $\widetilde{T}$ can be easily extended to Y. The right endpoints of intervals in Z are mapped to each other in an obvious manner just expanding the affine function to a left-continuous one on the closed interval in accordance with (2.1). Define also $\widetilde{T}(-1) = -1$, $\widetilde{T}(1) = 1$, $\widetilde{T}(d_i^*) = d_{i-1}^*$ and, $\widetilde{T}(f_i^*) = f_{i-1}^*$ for $i = 1, 2, \dots$, $\widetilde{T}(d_0^*) = 1$ and $\widetilde{T}(f_0^*) = -1$.

Lemma 3.5. *The mapping $\widetilde{T} : \overline{Z} \to \overline{Z}$ on the compact set $\overline{Z} = Z \cup Y$ is continuous on Y.*

Proof. It suffices to prove the continuity at 1, -1, and the points d_i^*, f_i^*, $i = 0, 1, \dots$ (Recall that the left-continuity at the right endpoints of the intervals is enough, since the closures of the intervals are disjoint.) First we note that if $-1 < x < C_i$, then x is either in C' or in D'. In the former case $\widetilde{T}(x) < C_{i+1} < C_{i-1}$, and in the latter one $\widetilde{T}(x) \leq D_{i-1} < C_{i-1}$. For arbitrary $\varepsilon > 0$ choose i so large that $d(-1, C_j) < \varepsilon$ for every $j \geq i-1$, and let $\delta < d(-1, C_i)$. Now if $d(-1, x) < \delta$, then $x < C_i$ and $d(\widetilde{T}(x), -1) < d(-1, C_{i-1}) < \varepsilon$. Thus $\widetilde{T}$ is continuous at -1. The proof of the continuity at 1 and d_i^*, f_i^*, $i = 0, 1, \dots$ is similar. □

Although $\widetilde{T}$ is continuous at the new points in Y which are added to Z to make it compact, it is not continuous originally on Z. However, according to Lemma 3.2.(1) the points of discontinuity in Z are only some iterates of the dyadic points in C_0. In order to make $\widetilde{T}$ continuous, we modify the set Z by omitting all the dyadic points and their images in the following manner. Denote the middle point of C_0 by x_0 and the middle points of the halves of C_0 by x_1 and x_2, respectively. Enumerate all the dyadic points like this (see Figure 1). Remove some small open interval including x_0. Then remove much smaller open intervals containing x_1 and x_2. Carry this procedure on so that the remaining set, (which will be still called C_0), is of positive measure, compact and free of the dyadic points. Note that the new C_0 is not a standard Cantor type set since the intervals are not removed from the middle of the previous level intervals. In particular, one may remove an interval inside an interval which has already been removed. Nevertheless, the resulting set is compact and it has positive Lebesgue measure if the lengths of the removed intervals tend to zero rapidly enough. Since $\widetilde{T}$ is continuous on (the new) C_0 and on all of its iterates, the set

$$X = \cup_{n=0}^{\infty} \widetilde{T}^n(C_0)$$

is compact and of positive Lebesgue measure. The restriction $T = \widetilde{T}|_X$ is therefore a continuous function on the compact set X. With obvious modifications it still has the properties described in Lemma 3.2.

4. Natural and observable measures of T

In this section we show that the mapping T has a unique natural measure which is ergodic and invariant, but it has no observable measures.

Theorem 4.1. *For any Borel regular probability measure $\mu \ll \mathcal{L}$ with support in X we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (T_*)^j \mu = \delta_{-1}$$

where the convergence is in the weak topology. Furthermore, the measure δ_{-1} is ergodic and invariant with respect to T.*

Proof. Let $\varphi : X \to \mathbb{R}$ be a continuous function and μ a Borel regular probability measure supported in X such that $\mu \ll \mathcal{L}$. By the definition of the image measure

$$\int \varphi d(\frac{1}{n} \sum_{j=1}^{n} (T_*)^j \mu) = \int \frac{1}{n} \sum_{j=1}^{n} \varphi \circ T^j d\mu.$$

For $Q_n^k = \{x \in Z \mid -1 < T^n(x) < \frac{1}{k} - 1\}$ we know from Lemma 3.2.(7) that $\lim_{n\to\infty} \mathcal{L}(X \setminus Q_n^k) = 0$ for every fixed k. Since $\mu \ll \mathcal{L}$ also $\lim_{n\to\infty} \mu(Q_n^k) = 1$. We fix a positive integer l and let $R_n = \{x \in X \mid |\varphi(T^n(x)) - \varphi(-1)| < \frac{1}{l}\}$. Since φ is continuous, for all positive integers n we have $Q_n^k \subset R_n$ for all large k depending on l. This gives that $\lim_{n\to\infty} \mu(R_n) = 1$. Thus, letting $\varepsilon > 0$, there is a positive integer n_ε so that if $x \in R_n$ with $n > n_\varepsilon$, then $|\varphi(T^n(x)) - \frac{\varphi(-1)}{\mu(R_n)}| < \frac{1}{l} + \varepsilon$ and $\mu(X \setminus R_n) < \varepsilon$. Now

$$\begin{aligned}
\left| \int \frac{1}{n} \sum_{j=1}^{n} \varphi \circ T^j d\mu - \int \varphi d\delta_{-1} \right| &\leq \frac{1}{n} \sum_{j=1}^{n} \int_{R_j} |\varphi(T^j(x)) - \tfrac{\varphi(-1)}{\mu(R_j)}| d\mu(x) \\
&\quad + \frac{1}{n} \sum_{j=1}^{n} \int_{X \setminus R_j} |\varphi(T^j(x))| d\mu(x) \\
&= A + B.
\end{aligned}$$

Fixing $\varepsilon > 0$, noting that $\mu(R_j) > 0$ for all j, and writing $M = \max_{x\in X} |\varphi(x)|$ we have

$$
\begin{aligned}
A &= \frac{1}{n}\sum_{j=1}^{n_\varepsilon}\int_{R_j} |\varphi(T^j(x)) - \tfrac{\varphi(-1)}{\mu(R_j)}| d\mu(x) \\
&\quad + \frac{1}{n}\sum_{j=n_\varepsilon+1}^{n}\int_{R_j} |\varphi(T^j(x)) - \tfrac{\varphi(-1)}{\mu(R_j)}| d\mu(x) \\
&\le \frac{MK(n_\varepsilon)}{n} + \frac{1}{l} + \varepsilon
\end{aligned}
$$

Letting $n \to \infty$, $\varepsilon \to 0$, and $l \to \infty$ this term tends to zero. To estimate B, choose n_ε so big that $\mu(X \setminus R_j) < \frac{\varepsilon}{M}$ whenever $j > n_\varepsilon$. Then

$$
\begin{aligned}
B &= \frac{1}{n}\sum_{j=1}^{n_\varepsilon}\int_{X\setminus R_j} |\varphi(T^j(x))| d\mu(x) + \frac{1}{n}\sum_{j=n_\varepsilon+1}^{n}\int_{X\setminus R_j} |\varphi(T^j(x))| d\mu(x) \\
&\le \frac{n_\varepsilon M}{n} + \frac{M\varepsilon}{M} \xrightarrow[n\to\infty]{} \varepsilon.
\end{aligned}
$$

Therefore

$$
\lim_{n\to\infty}\int \frac{1}{n}\sum_{j=1}^{n} \varphi \circ T^j d\mu = \varphi(-1),
$$

that is, δ_{-1} is the unique natural measure. Obviously, δ_{-1} is invariant and ergodic. □

Theorem 4.2. *For $\mathcal{L}$-almost every $x \in X$ we have that*

$$
\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \delta_{T^j(x)} \neq \delta_{-1}.
$$

Proof. Let $\varphi : X \to \mathbb{R}$ be a continuous function with $\varphi(x) = 0$ for $x < C_0$ and $\varphi(x) = 1$ for $x > 0$. Take $x \in Z \cap X$ recalling that $\mathcal{L}(X \setminus Z) = 0$. According to Lemma 3.4, there is an increasing sequence (m_i) such that $p_i = n_i = \frac{m_i}{2}$ (see Definition 3.3). Since the preimage $T^{-1}(C_0)$ is empty, $\varphi(T^j(x)) = 0$, if $T^j(x) < 0$. Therefore

$$
\begin{aligned}
\frac{1}{m_i}\sum_{j=1}^{m_i} \varphi(T^j(x)) &= \frac{1}{m_i}\sum_{j\in N_i} \varphi(T^j(x)) + \frac{1}{m_i}\sum_{j\in P_i} \varphi(T^j(x)) \\
&= \frac{1}{2p_i}\sum_{P_i} 1 = \frac{1}{2}
\end{aligned}
$$

for all i. Hence

$$
\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \varphi(T^j(x)) \neq 0 = \varphi(-1).
$$

□

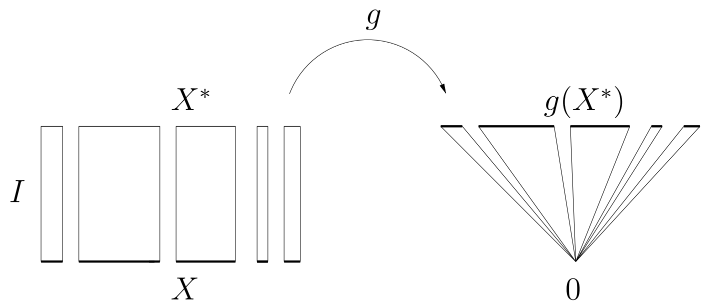

Figure 2. The image $g(X^*)$ is connected through the origin.

Corollary 4.3. *The mapping $T : X \to X$ has no observable measures.*

Proof. If T had an observable measure μ then [BB, Theorem 2.1] would imply that μ is also natural. Thus $\mu = \delta_{-1}$ by Theorem 4.1 which is a contradiction with Theorem 4.2. □

5. Modifications of the mapping T

We have constructed a continuous function on a compact metric space X with a unique natural measure but without observable measures. The set X is not connected but it can be modified by increasing the dimension. First expand X into $\mathbb{R}^2$ by taking $X^* = X \times [0, 1]$. Let $S(x) = \sqrt{2x - x^2}$. Notice that S has 0 as a repelling and 1 as an attracting fixed point. Define T^* on X^* by $T^*(x, y) = (T(x), S(y))$, which makes T^* continuous on its domain. It is obvious that under iteration T^* will eventually behave like T on the set $X \times \{1\}$. The natural measure of T^* is $\delta_{(-1,1)}$ but it does not have observable measures. Now to make X^* connected, we define a continuous function $g : X^* \to \mathbb{R}^2$ by $g(x, y) = (yx, y)$ which is invertible except at the origin. It is easy to see that $X' = g(X^*)$ is connected (see Figure 2). Define finally function $\overline{T} : X' \to X'$ by making the origin its fixed point and defining it as $g \circ T^* \circ g^{-1}$ everywhere else. Obviously this function is continuous and it has the same properties as T^*.

It is also easy to modify T such that it has nontrivial limit dynamics. Namely, let $\widehat{X} = X' \times S^1$ and define $\widehat{T} : \widehat{X} \to \widehat{X}$ by $\widehat{T}(x, z) = (\overline{T}(x), z^2)$. Then $\delta_{(-1,1)} \times \mathcal{L}$ is the unique natural measure and $\widehat{T}$ has no observable measures.

Remark 5.1. In the view of Corollary 4.3 we know that the essential assumption in [BB, Theorem 2.1] guaranteeing that a natural measure is observable is its equivalence to the Lebesgue measure (restricted to some open set). It would be nice to know if there is any other condition implying this property.

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland[1,2]

E-mail address: `esaj@maths.jyu.fi`[1]

E-mail address: `taptolo@maths.jyu.fi`[2]