Distortion of quasiconformal and quasiregular mappings at extremal points

Kai Rajala

Abstract

We prove local distortion results for quasiconformal and quasiregular mappings at points at which the local modulus of continuity of the mapping is in a sense extremal. These results extend to higher dimensions the results in [2], where the planar case is studied by using the Beltrami equation. Our proofs are based on the use of the conformal modulus of $n-1$-dimensional sets, and on the methods used in [3].

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. We call a mapping $f : \Omega \to \mathbb{R}^n$ quasiregular, if $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$, and if there exists $1 \leq K_O < \infty$ so that

$$|Df(x)|^n \leq K_O J_f(x)$$

for almost all $x \in \Omega$. Here $|Df(x)|$ is the operator norm of the differential matrix of f at x, and $J_f(x)$ is the Jacobian determinant of $Df(x)$. Quasiregular homeomorphisms are called quasiconformal maps. See [4], [5], [7] and [9] for the basic theory of quasiconformal and quasiregular mappings.

One of the properties of quasiregular mappings is the fact that they are locally Hölder continuous with exponent $\alpha(m, K_I) = (m/K_I)^{1/(n-1)}$. Here K_I is the inner distortion of f, and m is the local index of f at the point at which the local modulus of continuity is measured, see Section 2 for the definitions. Also, for non-constant quasiregular mappings it is true that the local modulus of continuity at a single point with local index m cannot be better than Hölder continuity with exponent $\beta(m, K_O) = (mK_O)^{1/(n-1)}$. See [5] III Theorem 4.7 for the proofs of these properties.

The purpose of this note is to study the asymptotic behavior of quasiconformal and quasiregular mappings at points at which the mappings behave in an extremal way, in the sense of the local modulus of continuity. That is, we look at the local distortion of a mapping f at points at which the local modulus of continuity is either no better than $\alpha(m, K_I)$, or no worse than

$\beta(m, K_O)$. This kind of study was initiated by Kovalev in [2], where the planar case was considered. For quasiconformal mappings with distortion K_O close to one, good estimates for the linear dilatation have been proved in [8] and [6].

The basic principle of the results in [2] is that, at the points that are extremal in the sense described above, the mapping distorts spheres somewhat like analytic functions do. Here we show that a similar phenomenom also occurs in higher dimensions. The results in [2] were proved by using estimates for solutions of the Beltrami equation, a method not available in higher dimensions. Here we will use methods similar to the ones established in [3]. Namely, we will use inequalities and estimates concerning the conformal modulus of families of $n-1$-dimensional spheres and their images and preimages under the mappings in question.

Our first result is stated for quasiconformal maps. As the result is local, it also holds true for quasiregular mappings at points at which the mapping is a local homeomorphism. The concepts used to state Theorems 1.1 and 1.4 below are defined in Section 2. In particular, we will use the notion of infinitesimal space, introduced in [1], in order to describe the asymptotic behavior.

Fix $K_I \geq 1$, and set

$$\omega_f(x_0, \delta) := \delta^{-K_I^{\frac{-1}{n-1}}} \max_{|x-x_0|\leq\delta} |f(x) - f(x_0)|,$$

and

$$\omega_f(x_0) := \limsup_{\delta\to 0} \omega_f(x_0, \delta).$$

We now have the following result that generalizes Theorem 2.1 of [2] to all dimensions $n \geq 2$.

Theorem 1.1. *Let $f : \Omega \to \mathbb{R}^n$ be a quasiconformal map, so that $K_I(x) \leq K_I$ for almost every $x \in \Omega$. Let $x_0 \in \Omega$, and suppose that $\omega_f(x_0) > 0$. Then $H_f(x_0) = 1$. Furthermore, for all $g : \mathbb{R}^n \to \mathbb{R}^n$ in the infinitesimal space of f at x_0,*

$$g(x) = |x|^{K_I^{\frac{-1}{n-1}}} h\Big(\frac{x}{|x|}\Big) \quad \text{for all } x \in \mathbb{R}^n, \tag{1.1}$$

where $h : S^{n-1}(0,1) \to S^{n-1}(0,1)$ is a homeomorphism that preserves surface measure. In particular, when $n = 2$, h is a rotation.

Remark 1.2. Although the mappings in the infinitesimal space satisfy the strong property (1.1), a mapping satisfying the assumptions of Theorem 1.1 does not have to be "differentiable" at x_0, meaning that the infinitesimal space of f at x_0 may include more than one mapping. See the proof of Theorem 2.2 in [2] for an example where "differentiability" fails.

Remark 1.3. For planar mappings f the mapping h in (1.1) is an isometry. In higher dimensions this does not have to be the case; for any $n \geq 3$ and any $K_I > 1$ there exists $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the assumptions of Theorem 1.1 for $x_0 = 0$ and K_I, so that the mapping h in (1.1) is not an isometry. To see this, fix $K_I > 1$, and take a quasiconformal mapping g of the form (1.1). Furthermore, choose h to be a sense-preserving $K_I^{1/(n-1)}$-bilipschitz homeomorphism that preserves surface measure but is not an isometry. Then g satisfies the assumptions of Theorem 1.1. In particular,

$$K_I(x, g) \leq K_I$$

almost everywhere: If we denote $v(t) = t^{K_I^{\frac{-1}{n-1}}}$, then, at a point of differentiability x,

$$l(Dg(x)) := \inf_{\{y \in \mathbb{R}^n : |y|=1\}} |Dg(x)y| = K_I^{\frac{-1}{n-1}} |x|^{K_I^{\frac{-1}{n-1}} - 1} = v'(|x|),$$

while

$$J_g(x) = l(Dg(x)) \cdot |x|^{(n-1)\left(K_I^{\frac{-1}{n-1}} - 1\right)} = K_I^{\frac{-1}{n-1}} |x|^{n\left(K_I^{\frac{-1}{n-1}} - 1\right)} = K_I l(Dg(x))^n.$$

Also, the infinitesimal space of g at the origin consists of one mapping, namely the mapping g itself. Mappings h as the one needed here do indeed exist. For instance, for the two-dimensional sphere $S^2(0,1) \subset \mathbb{R}^3$ such mappings can be constructed as follows. Take the caps $C_1 = S^2(0,1) \cap \{x_3 > 1/2\}$ and $C_2 = S^2(0,1) \cap \{x_3 < -1/2\}$. Now define h so that h restricted to $C_1 \cup C_2$ is the identity. For $|x_3| \leq 1/2$, set

$$h(r, \phi, x_3) = (r, \phi + \varphi(|x_3|), x_3)$$

in cylindrical coordinates, where $\varphi : [0,1] \to [0,1]$ is a nontrivial differentiable function so that $\varphi(1/2) = 0$ and $|\varphi'(t)| \leq K_I^{1/2} - 1$ for all $x \in [0, 1/2]$. Then h has the desired properties.

We do not know how to prove a result like Theorem 1.1 for quasiregular mappings at branch points, at which the local modulus of continuity is roughly $\alpha(m, K_I)$. The reason for this is that our methods (seem to) require that boundaries of certain balls get mapped into the boundaries of the images. In dimension two this problem does not exist, because, by the Stoilow factorization theorem, the result for quasiconformal maps implies that the corresponding result holds true also for quasiregular mappings.

For general points we have, however, the following result concerning the inverse distortion. Fix $K_O \geq 1$, and set

$$\sigma_f^m(x_0, \delta) := \delta^{-(mK_O)^{\frac{-1}{n-1}}} \max_{U(x_0, f, \delta)} |x - x_0|$$

(see Section 2 for the definition of $U(x, f, r)$), and

$$\sigma_f^m(x_0) := \limsup_{\delta \to 0} \sigma_f^m(x_0, \delta).$$

Theorem 1.4. *Let $f : \Omega \to \mathbb{R}^n$ be a quasiregular mapping, so that $K_O(x) \leq K_O$ for almost every $x \in \Omega$. Let $x_0 \in \Omega$, $i(x_0, f) = m$, and suppose that $\sigma_f^m(x_0) > 0$. Then $H_f^*(x_0) = 1$ and*

$$\sigma_f^m(x_0) = \lim_{\delta \to 0} \sigma_f^m(x_0, \delta).$$

In Section 2 we recall some definitions and the main tools that are needed to prove the results above. The material is mainly from [3] and [5]. Theorems 1.1 and 1.4 are proved in Section 3.

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2 Preliminaries

We will denote open euclidean balls with center x and radius r by $B(x, r)$, while the corresponding $(n-1)$-dimensional spheres are denoted by $S(x, r)$. In the case $x = 0$ the notations B_r and S_r are used. The boundary of a general set E is denoted by ∂E. Let f be a non-constant quasiregular mapping and $Df(x)$ the differential matrix of f at x. Set

$$|Df(x)| = \sup_{\{y \in \mathbb{R}^n : |y| = 1\}} |Df(x)y| \quad \text{and} \quad l(Df(x)) = \inf_{\{y \in \mathbb{R}^n : |y| = 1\}} |Df(x)y|,$$

where $J_f(x)$ is the Jacobian determinant of $Df(x)$. The inner and outer distortion functions of f are defined by

$$K_I(x) = K_I(x, f) = \frac{J_f(x)}{l(Df(x))^n}, \quad K_O(x) = K_O(x, f) = \frac{|Df(x)|^n}{J_f(x)},$$

respectively, whenever $J_f(x) \neq 0$, otherwise set $K_I(x) = K_O(x) = 0$. Recall that, for non-constant quasiregular mappings f, $J_f > 0$ almost everywhere, see [5] II Theorem 7.4. If f is quasiconformal and $K_I(x, f) \leq K_I$ almost everywhere, then f^{-1} is also quasiconformal and

$$K_O(x, f^{-1}) \leq K_I \tag{2.1}$$

almost everywhere, see [9] Corollary 13.3 and Theorem 34.4. The Lebesgue n-measure of a measurable set A is denoted by $|A|$. The Lebesgue measure of the unit n-ball is denoted by α_n. The $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ is normalized so that $\mathcal{H}^{n-1}(B_1) = \omega_{n-1}$, where ω_{n-1} denotes the surface measure of the unit sphere.

Let $f : \Omega \to \mathbb{R}^n$ be a non-constant quasiregular mapping. A domain $D \subset \Omega$ is called a normal domain (of f), if $f(\partial D) = \partial f(D)$. Furthermore, if D is a normal domain and $x \in D$ so that $f^{-1}(f(x)) \cap D = \{x\}$, then D is called a normal neighborhood of x. The x-component of the preimage of the ball $B(f(x), r)$ under f is denoted by $U(x, f, r)$. By [5] II Lemma 4.1, for each $x \in \Omega$ there exists $s_x > 0$ so that for each $s < s_x$ the following properties hold:

1. $U(x, f, s)$ is a normal neighborhood of x,
2. $U(x, f, s) = U(x, f, s_x) \cap f^{-1}(B(f(x), s))$,
3. $\partial U(x, f, s) = U(x, f, s_x) \cap f^{-1}(S(f(x), s))$,
4. $\mathbb{R}^n \setminus U(x, f, s)$ and $\mathbb{R}^n \setminus \overline{U}(x, f, s)$ are connected.

The local index $i(x, f)$ of a quasiregular mapping f at a point $x \in \Omega$ can be defined by

$$i(x, f) = \lim_{r \to 0} \sup_{y \in B(f(x), r)} \#\{f^{-1}(y) \cap U(x, f, r)\}.$$

The set of points in Ω at which $i(x, f) \geq 2$ is called the branch set of f and denoted by B_f.

We will use the following dilatation functions:

$$\begin{aligned}
L(x, f, r) &= \sup_{|x-y|=r} |f(y) - f(x)|, \quad l(x, f, r) = \inf_{|x-y|=r} |f(y) - f(x)|, \\
L^*(x, f, r) &= \sup_{z \in \partial U(x,f,r)} |x - z| \quad l^*(x, f, r) = \inf_{z \in \partial U(x,f,r)} |x - z|, \\
H(x, f, r) &= \frac{L(x, f, r)}{l(x, f, r)}, \quad H^*(x, f, r) = \frac{L^*(x, f, r)}{l^*(x, f, r)}, \\
H(x, f) &= \limsup_{r \to 0} H(x, f, r), \quad H^*(x, f) = \limsup_{r \to 0} H^*(x, f, r),
\end{aligned}$$

When U is a normal domain of f and $B_r \subset f(U)$, we denote $B'_r := f^{-1}(B_r) \cap U$. Also, a similar notation for components of preimages of spheres will be used. We define the surface modulus M_S of a family Λ of Borel-measurable subsets of $\mathbb{R}^n$ by setting

$$\begin{aligned}
M_S(\Lambda) &= \inf \Bigg\{ \int_{\mathbb{R}^n} \rho(x)^{\frac{n}{n-1}} \, dx : \rho : \mathbb{R}^n \to [0, \infty] \text{ is Borel measurable,} \\
&\qquad \int_S \rho(x) \, d\mathcal{H}^{n-1}(x) \geq 1 \, \forall S \in \Lambda \Bigg\}.
\end{aligned}$$

The surface modulus is a conformal invariant, and we have an inequality for quasiregular mappings as follows, see [3] Theorem 3.4: Suppose that $f : \Omega \to \mathbb{R}^n$ is a quasiregular mapping. Furthermore, assume that $f(0) = 0$,

$i(0,f) = m$ and that $U(0,f,1)$ is a normal neighborhood of 0. Let $I \subset (0,1)$ be a Borel measurable set. If $\Lambda := \{S_t : t \in I\}$ and $\Lambda' = \{S'_t : t \in I\}$, then

$$M_S\Lambda \leq (mK_O)^{\frac{1}{n-1}} M_S\Lambda'. \tag{2.2}$$

Also, by [5] III Lemma 4.1 there exists a constant d so that for all $r < d$ we have

$$H^*(0,f,r) \leq C, \tag{2.3}$$

where C is a constant only depending on n and K_O.

The isoperimetric defect $\delta(\Omega)$ of a bounded domain Ω is defined as

$$\delta(\Omega) = 1 - \frac{|\Omega|}{C_I \mathcal{H}^{n-1}(\partial\Omega)^{\frac{n}{n-1}}},$$

where C_I is the sharp constant in the isoperimetric inequality, so that $\delta(B) = 0$ for all balls $B \subset \mathbb{R}^n$. Hence $\delta(\Omega) \in [0,1)$ for every Ω. In addition to the isoperimetric defect, we will use the metric distortion. Let f be as in (2.2). For $t \in (0,1)$, set

$$\alpha(B'_t) = \inf\left\{\frac{R}{r} : S'_t \subset B(x,R) \setminus \overline{B}(x,r), x \in \mathbb{R}^n\right\}.$$

We then have the following connection between the isoperimetric defect and the metric distortion, see [3] Proposition 4.4: Let f be as in (2.2). Suppose that $\delta(B'_t) < a$. Then

$$\alpha(B'_t) \leq b(a), \tag{2.4}$$

where b is an increasing function depending only on n and K_O, and $b(a) \to 1$ as $a \to 0$.

Again, let f be as above. For each $t \in (0,1)$, the point symmetrizations of the set B'_t and its closure will be the open ball $B(0, \alpha^{\frac{-1}{n}}|B'_t|^{\frac{1}{n}})$ and its closure, respectively. Thus the symmetrization of each S'_t will be a sphere enclosing a ball with the same volume as the set enclosed by S'_t. We define a function $p : (0,1) \to (0,\infty)$ by setting $p(t) = \alpha^{\frac{-1}{n}}|B'_t|^{\frac{1}{n}}$. Thus the image of the set S'_t under point symmetrization is the sphere $S_{p(t)}$. Note that p is strictly increasing. We will use the following notation. If $s = p(t)$, we denote the isoperimetric defect and the linear distortion of B'_t by

$$\delta_s := \delta(B'_t) \quad \text{and} \quad \alpha_s := \alpha(B'_t),$$

respectively. The following estimate is proved in [3] Lemma 5.2: Suppose that $I \subset (0,1)$ is a Borel measurable set and

$$\Lambda = \{S'_t : t \in I\}.$$

Then

$$M_S(\Lambda) \le \omega_{n-1}^{\frac{-1}{n-1}} \int_{p(I)} \frac{1-\delta_s}{s}\, ds. \tag{2.5}$$

Finally, we will make use of the notion of infinitesimal space. Let $f : \Omega \to \mathbb{R}^n$ be a non-constant quasiregular mapping and $x_0 \in \Omega$. Set $\rho_0 = \operatorname{dist}(x_0, \partial\Omega)$ and $R(\rho) = \rho_0/\rho$ for all positive ρ:s. Define $F_\rho : B_{R(\rho)} \to \mathbb{R}^n$ by setting

$$F_\rho(x) = \frac{f(x_0 + \rho x) - f(x_0)}{r(x_0, f, \rho)}, \tag{2.6}$$

where

$$r(x_0, f, \rho) = \alpha_n^{\frac{-1}{n}} |f(B(x_0, \rho))|^{\frac{1}{n}}$$

Then the infinitesimal space of f at x_0 is defined as the set of all non-constant mappings $F : \mathbb{R}^n \to \mathbb{R}^n$ with the property that F is the limit under locally uniform convergence of some sequence (F_{ρ_j}), where $\rho_j \to 0$. By combining the Arzela-Ascoli theorem and Reshetnyak's compactness theorem, one can prove that the infinitesimal space is always non-empty, see [1]. Also,

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} K_I(x, F) \le \operatorname{ess\,sup}_{x\in\Omega} K_I(x, f)$$

and

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} K_O(x, F) \le \operatorname{ess\,sup}_{x\in\Omega} K_O(x, f).$$

3 Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.4. Without loss of generality, we may assume that $x_0 = 0 = f(x_0)$. Let (r_j) be a decreasing sequence converging to zero so that $r_1 < s_0$, where s_0 is as in Section 2, and so that the also the rest of the assumptions needed for the results in Section 2 to hold true are satisfied. Then, (2.3) holds in particular. Also, by the continuity estimate [5] III Theorem 4.7, the function $r \to \sigma_f^m(0, r)$ is bounded. Assume that

$$\frac{\sigma_f^m(0)}{\sigma_f^m(0, r_j)} \ge 1 - \frac{1}{j} \quad \forall j \in \mathbb{N} \tag{3.1}$$

and that $\sigma_f^m(0, r_j) \ge \sigma_f^m(0, r_k)$ for $j < k$. Notice that the assumption $\sigma_f^m(0) = 0$ is used here. For $r < r_1$, set $r' := \sup_{x\in B_r'} |x|$. For $j, k \in \mathbb{N}$, $j < k$, (3.1) yields

(3.2)

$$(mK_O)^{\frac{1}{n-1}} \log\Big(\frac{(1-\frac{1}{j})r_j'}{r_k'}\Big) \le (mK_O)^{\frac{1}{n-1}} \log\Big(\frac{\sigma_f^m(0, r_k) r_j'}{\sigma_f^m(0, r_j) r_k'}\Big) \le \log\frac{r_j}{r_k}.$$

Set $\Lambda_{jk} := \{S_t : r_k < t < r_j\}$, and $\Lambda'_{jk} := \{S'_t : r_k < t < r_j\}$. Then, by (2.2), $M_S\Lambda_{jk} \leq (mK_O)^{\frac{1}{n-1}} M_S\Lambda'_{jk}$. Furthermore, by (2.5),

$$M_S\Lambda'_{jk} \leq \omega_{n-1}^{\frac{-1}{n-1}} \int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds.$$

Recall that

$$p(r) = \alpha_n^{\frac{-1}{n}} |U(0,f,r)|^{\frac{1}{n}}.$$

Now

$$\begin{aligned} \omega_{n-1}^{\frac{-1}{n-1}} \log \frac{r_j}{r_k} &= M_S\Lambda_{jk} \leq (mK_O)^{\frac{1}{n-1}} M_S\Lambda'_{jk} \\ &\leq \Big(\frac{mK_O}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds, \end{aligned} \tag{3.3}$$

and combining (3.2) and (3.3) yields

$$\log\Big(\frac{(1-\frac{1}{j})r'_j}{r'_k}\Big) \leq \int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds. \tag{3.4}$$

Now fix $a > 0$ and denote $I = \{s \in (p(r_k), p(r_j)) : \delta_s > a\}$ and $J = (p(r_k), p(r_j)) \setminus I$. Then we have

$$\begin{aligned} \int_{p(r_k)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds &= \int_J \frac{1-\delta_s}{s}\, ds + \int_I \frac{1-\delta_s}{s}\, ds \leq \mu(J) + (1-a)\mu(I) \\ &= \log\frac{p(r_j)}{p(r_k)} - a\mu(I), \end{aligned} \tag{3.5}$$

where μ denotes the logarithmic measure. We have $p(r_j) \leq r'_j$ and $p(r_k) \geq \frac{r'_k}{C}$, where C is the constant in (2.3). Hence (3.4) and (3.5) give

$$\log\Big(\frac{(1-\frac{1}{j})r'_j}{r'_k}\Big) \leq \log\Big(\frac{Cr'_j}{r'_k}\Big) - a\mu(I),$$

i.e.

$$\mu(I) \leq \frac{1}{a}\log\frac{C}{1-\frac{1}{j}}. \tag{3.6}$$

Notice that this estimate does not depend on k, and thus (3.6) holds with

$$I = \{s \in (0, p(r_j)) : \delta_s > a)\}.$$

Next, fix $\epsilon > 0$, and consider the intervals $A_i := [(1+\epsilon)^{-i-1}, (1+\epsilon)^{-i})$ for $i \geq i_0$, where i_0 satisfies $(1+\epsilon)^{-i_0} < p(r_j)$. Since

$$\mu(A_i) = \log(1+\epsilon),$$

(3.6) implies that there exists $i(j)$ such that $A_i \cap J \neq \emptyset$ for all $i \geq i(j) - 1$. Now, for all $r < r_{i(j)}$, we find r^1 and r^2 so that if $p(r) \in A_L$, then $p(r^1) \in A_{L+1}$ and $p(r^2) \in A_{L-1}$, and $\delta_{r^1}, \delta_{r^2} < a$. By (2.4), $\alpha_{r^1}, \alpha_{r^2} < b(a)$, where $b(a) \to 1$ as $a \to 0$. Suppose that

$$S'_{r^2} \subset B(z, R_1) \setminus B(z, R_2) \quad \text{and} \quad S'_{r^1} \subset B(w, R_3) \setminus B(w, R_4),$$

so that $\alpha_{r^2} = R_1/R_2$ and $\alpha_{r^1} = R_3/R_4$. Then, since

$$|B'_{r^2}| = \alpha_n p(r^2)^n \leq \alpha_n (1+\epsilon)^{n(-L+2)}$$

and

$$|B'_{r^1}| = \alpha_n p(r^1)^n \geq \alpha_n (1+\epsilon)^{n(-L-1)},$$

we have

$$\begin{aligned} R_1 \leq b(a) R_2 \leq b(a) \alpha_n^{\frac{-1}{n}} |B'_{r^2}|^{\frac{1}{n}} &\leq (1+\epsilon^3) b(a) \alpha_n^{\frac{-1}{n}} |B'_{r^1}|^{\frac{1}{n}} \\ &\leq b(a)(1+\epsilon)^3 R_3 \leq b(a)^2 (1+\epsilon)^3 R_4. \end{aligned}$$

Notice that

$$B(w, R_4) \subset S'_{r^1} \subset S'_{r^2} \subset B(z, R_1).$$

Since there exists a ball $B(z, R_4 - 2(R_1 - R_4)) \subset B(w, R_4)$ (we may assume a and ϵ to be so small that $R_4 - 2(R_1 - R_4) > 0$), we further have

$$S'_r \subset B(z, R_1) \setminus B(w, R_4) \subset B(z, R_1) \setminus B(z, R_4 - 2(R_1 - R_4)),$$

and so

$$\alpha(B'_r) \leq \frac{R_1}{R_4 - 2(R_1 - R_4)} = \frac{R_1}{3R_4 - 2R_1} \leq \frac{1}{3C(a,\epsilon) - 2},$$

where $C(a, \epsilon) \to 1$ as $a, \epsilon \to 0$. We conclude that for all $a, \epsilon > 0$ there exists $r(a, \epsilon)$ so that for all $r < r(a, \epsilon)$,

$$\alpha(B'_r) \leq \frac{1}{3C(a,\epsilon) - 2},$$

and hence

$$\limsup_{r \to 0} \alpha(B'_r) = 1.$$

Let us next define a variant of σ_f^m; set

$$\phi_f^m(0, \delta) := \delta^{-(mK_O)^{\frac{-1}{n-1}}} p(\delta).$$

Then, by (2.3), $\phi_f^m(0, \delta) \geq \frac{1}{C} \sigma_f^m(0, \delta)$, and, in particular, we can choose a decreasing sequence (r_j), converging to zero, so that $\phi_f^m(0, r_j) > \phi_f^m(0, r_k)$ for $j < k$, and

$$\frac{\phi_f^m(0)}{\phi_f^m(0, r_j)} \geq 1 - \frac{1}{j} \quad \forall j \in \mathbb{N},$$

where

$$\phi_f^m(0) = \limsup_{\delta\to 0} \phi_f^m(0,\delta) \geq \frac{1}{C}\sigma_f^m(0) > 0.$$

We next show that actually

$$\phi_f^m(0) = \lim_{\delta\to 0} \phi_f^m(0,\delta). \tag{3.7}$$

Fix $j \in \mathbb{N}$, and consider $r < r_j$ and $r_k < r$. Denote $\Lambda_j = \{S_t : t \in (r, r_j)\}$, $\Lambda'_j = \{S'_t : t \in (r, r_j)\}$, $\Lambda_k = \{S_t : t \in (r_k, r)\}$ and $\Lambda'_k = \{S'_t : t \in (r_k, r)\}$. As before, we have

$$\begin{aligned}
\omega_{n-1}^{\frac{-1}{n-1}} \log \frac{r_j}{r} &= M_S\Lambda_j \leq (mK_O)^{\frac{1}{n-1}} M\Lambda'_j \leq \Big(\frac{mK_O}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \int_{p(r)}^{p(r_j)} \frac{1-\delta_s}{s}\, ds \\
&\leq \Big(\frac{mK_O}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \log \frac{p(r_j)}{p(r)} \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
\omega_{n-1}^{\frac{-1}{n-1}} \log \frac{r}{r_k} &= M_S\Lambda_k \leq (mK_O)^{\frac{1}{n-1}} M\Lambda'_k \leq \Big(\frac{mK_O}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \int_{p(r_k)}^{p(r)} \frac{1-\delta_s}{s}\, ds \\
&\leq \Big(\frac{mK_O}{\omega_{n-1}}\Big)^{\frac{1}{n-1}} \log \frac{p(r)}{p(r_k)}. \qquad (3.9)
\end{aligned}$$

By combining (3.8) and (3.9) with the definition of ϕ_f^m, we further have

$$\phi_f^m(0, r_k) \leq \phi_f^m(0, r) \leq \phi_f^m(0, r_j).$$

We conclude that the function $\delta \to \phi_f^m(0,\delta)$ is increasing, and (3.7) holds in particular.

In order to complete the proof of Theorem 1.4, we need to show that the balls defining $\alpha(B'_r)$:s have to be centerd sufficiently near to the origin when $r \to 0$. To this end, we use the following auxiliary result, cf. [3] Lemma 6.1: Let $B(x,r) \subset B(y,R)$, and denote by Λ the family of all sets separating $B(x,r)$ and $\mathbb{R}^n \setminus B(y,R)$. Then

$$M_S(\Lambda) \leq \Big(1 - A\Big(\frac{R}{r}, |x-y|\Big)\Big)\omega_{n-1}^{\frac{-1}{n-1}} \log \frac{R}{r}, \tag{3.10}$$

where $t \to A(\frac{R}{r}, t)$ is continuous and strictly increasing, and $A(\frac{R}{r}, 0) = 0$.

Now, fix R so that $\alpha(B'_r) < 1 + \frac{1}{k}$ for all $r < R$, and so that

$$\frac{\phi_f^m(0)}{\phi_f^m(0,r)} \geq 1 - \frac{1}{k} \quad \text{for all } r < R. \tag{3.11}$$

Denote $R_i := \frac{1}{i}R$, and set

$$\Lambda_i := \{S_t : t \in (R_i, R)\}, \quad \Lambda'_i := \{S'_t : t \in (R_i, R)\}.$$

Suppose that m_i is the radius of the largest ball $B(x_i, m_i)$ with the property $B(x_i, m_i) \subset B'_{R_i}$ for some $x_i \in \mathbb{R}^n$ which is the minimizing point in the definition of $\alpha(B'_{R_i})$. Also, let M_i be the smallest radius such that $B'_R \subset B(y_i, M_i)$ for some $y_i \in \mathbb{R}^n$ which is the minimizing point in the definition of $\alpha(B_R)$. Denote by Λ_i^* the family of all sets separating $B(x_i, m_i)$ and $\mathbb{R}^n \setminus B(y_i, M_i)$. We have

$$m_i \geq \frac{1}{\alpha(B'_{R_i})} p(R_i) \geq \frac{k}{k+1} p(R_i), \tag{3.12}$$

and similarly

$$M_i \leq \Big(1 + \frac{1}{k}\Big) p(R). \tag{3.13}$$

On the other hand, (3.11) gives

$$\frac{p(R_i)}{p(R)} = \frac{\phi_f^m(0, R_i)}{\phi_f^m(0, R)} \Big(\frac{R_i}{R}\Big)^{(mK_O)^{\frac{-1}{n-1}}} \geq \Big(1 - \frac{1}{k}\Big) i^{-(mK_O)^{\frac{-1}{n-1}}}. \tag{3.14}$$

Furthermore,

$$\omega_{n-1}^{\frac{-1}{n-1}} \log i = \omega_{n-1}^{\frac{-1}{n-1}} \log \frac{R}{R_i} = M_S \Lambda_i \leq (mK_O)^{\frac{1}{n-1}} M \Lambda'_i \leq (mK_O)^{\frac{1}{n-1}} M \Lambda_i^*, \tag{3.15}$$

and by (3.10),

$$M\Lambda_i^* \leq (1 - A(i, |x_i - y_i|)) \omega_{n-1}^{\frac{-1}{n-1}} \log \frac{M_i}{m_i}. \tag{3.16}$$

By (3.12), (3.13) and (3.14),

$$\log \frac{M_i}{m_i} \leq \log \Big(\frac{p(R)(k+1)^2}{p(R_i) k^2}\Big) \leq 3 \log \Big(\frac{k}{k-1}\Big) + (mK_O)^{\frac{-1}{n-1}} \log i. \tag{3.17}$$

By combining (3.15), (3.16) and (3.17), we have

$$\log i \leq (1 - A(i, |x_i - y_i|)) \Big(3 (mK_O)^{\frac{1}{n-1}} \log \Big(\frac{k}{k-1}\Big) + \log i\Big),$$

and so

$$A(i, |x_i - y_i|) \leq \frac{3 (mK_O)^{\frac{1}{n-1}} \log \Big(\frac{k}{k-1}\Big)}{\log i} =: \Phi(k, i).$$

If we denote the inverse of $t \to A(i, t)$ at a point T by $A^{-1}(i, T)$, we see that

$$\begin{aligned} H^*(0, f, R) &\leq \frac{|y_i + M_i|}{\frac{M_i}{\alpha(B'_R)} - |y_i|} \\ &\leq \frac{|x_i - y_i| + \alpha(B'_{R_i}) p(R_i) + (1 + \frac{1}{k}) p(R)}{\frac{k-1}{k} p(R) - |x_i - y_i| - \alpha(B'_{R_i}) p(R_i)} \\ &\leq \frac{A^{-1}(i, \Phi(k, i)) + (1 + \frac{1}{k})(p(R_i) + p(R))}{\frac{k-1}{k} p(R) - A^{-1}(i, \Phi(k, i)) - (1 + \frac{1}{k}) p(R_i)}. \end{aligned} \tag{3.18}$$

We may take $k \to \infty$ as $R \to 0$, and so (3.18) yields (notice that $A^{-1}(i, \Phi(k,i)) \to 0$ as $k \to \infty$)

$$H^*(0,f) \le \frac{p(R)+p(R_i)}{p(R)-p(R_i)}.$$

On the other hand, $p(R_i) \le \varphi(i)p(R)$, where φ depends only on i and $\varphi(i) \to 0$ as $i \to \infty$. So, letting $i \to \infty$ gives $H^*(0,f) = 1$, and hence

$$\lim_{\delta \to 0} \sigma_f^m(0,\delta) = \lim_{\delta \to 0} \phi_f^m(0,\delta) = \phi_f^m(0) = \sigma_f^m(0).$$

The proof of Theorem 1.4 is complete.

□

Proof of Theorem 1.1. We again assume that $x_0 = 0 = f(x_0)$. By applying Theorem 1.4 to f^{-1}, we see that $H(0,f) = 1$ and $\omega_f(0) = \lim_{\delta \to 0} \omega_f(0,\delta)$. Let g be the locally uniform limit of a converging sequence (F_ρ) as in (2.6). First, for each fixed r and each $x \in R(\rho)$ so that $|x| = r$ and $R(\rho) > r$,

$$|F_\rho(x)| = \frac{\alpha^{\frac{1}{n}}|f(\rho x)|}{|f(B_\rho)|^{\frac{1}{n}}} \le \frac{(r\rho)^{K_I^{\frac{-1}{n-1}}}\omega_f(0,r\rho)H(0,f,\rho)}{\rho^{K_I^{\frac{-1}{n-1}}}\omega_f(0,\rho)} \xrightarrow{\rho \to 0} r^{K_I^{\frac{-1}{n-1}}}$$

and

$$|F_\rho(x)| \ge \frac{(r\rho)^{K_I^{\frac{-1}{n-1}}}\omega_f(0,r\rho)}{\rho^{K_I^{\frac{-1}{n-1}}}\omega_f(0,\rho)H(0,f,r\rho)} \xrightarrow{\rho \to 0} r^{K_I^{\frac{-1}{n-1}}}.$$

Hence $|g(x)| = |x|^{K_I^{\frac{-1}{n-1}}}$ $\forall x \in \mathbb{R}^n$.

Recall that $K_I(x,g) \le K_I$ for almost every $x \in \mathbb{R}^n$. We next claim that

$$g(tx) = tg(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^n. \tag{3.19}$$

If this is not the case, then there exists a ray $L_y = \{ty : t > 0\}$ for some $y \in S_1$, such that $g^{-1}(L_y)$ is not a ray of the form $\{tx : t > 0\}$ for any $x \in S_1$. Furthermore, by continuity of g, there exist $\epsilon > 0$, a ball $B^\epsilon \subset S_1$ and $T > 0$ so that if we denote $A_T = B_{2T} \setminus B_T$, then for all $y \in B^\epsilon$ we have

$$\int_{g^{-1}(L_y)\cap A_T} G\, ds > 1 + \epsilon, \tag{3.20}$$

where $G(x) = \frac{1}{|x|}\log^{-1} 2$. Denote by Γ the family of all line segments of the form

$$\left\{ty : t \in \left(T^{K_I^{\frac{-1}{n-1}}}, (2T)^{K_I^{\frac{-1}{n-1}}}\right)\right\}.$$

Then, since for

$$F = \bigcup_{y \in B^\epsilon} \Big(g^{-1}(L_y) \cap A_T \Big)$$

we have $|F| > 0$, and since by (3.20) we can choose a test function G' for $M(g^{-1}(\Gamma))$ as

$$G'(x) = \begin{cases} \frac{1}{|x|} \log^{-1} 2, & x \in A_T \setminus F \\ \frac{1}{1+\epsilon} \frac{1}{|x|} \log^{-1} 2, & x \in F \end{cases}$$

we have

$$\begin{aligned} K_I \omega_{n-1} \log^{1-n} 2 &= \omega_{n-1} \log^{1-n} \left(\frac{2T}{T} \right)^{K_I^{\frac{-1}{n-1}}} = M\Gamma \leq K_I M(g^{-1}\Gamma) \\ &\leq K_I \int_{A_T} G'(x)^n \, dx < K_I \omega_{n-1} \log^{1-n} 2. \end{aligned}$$

This is a contradiction, and hence (3.19) holds true.

We have shown that g is of the form $g(x) = |x|^{K_I^{\frac{-1}{n-1}}} h(\frac{x}{|x|})$, where $h : S_1 \to S_1$ is a homeomorphism. To finish the proof, we need to show that $\mathcal{H}^{n-1}(h(V)) = \mathcal{H}^{n-1}(V)$ for all Borel sets $V \in S_1$. If this is not true, then there exists a Borel set $W \subset S_1$ so that

$$\mathcal{H}^{n-1}(h(W)) > \mathcal{H}^{n-1}(W). \tag{3.21}$$

Denote

$$\Gamma_W := \cup_{w \in W} \{tw : t \in (1,2)\}.$$

Then the paths in $g(\Gamma_W)$ are line segments of the form

$$\Big\{ th(w) : t \in \Big(1, 2^{K_I^{\frac{-1}{n-1}}}\Big) \Big\},$$

and we have

$$\begin{aligned} \omega_{n-1} \mathcal{H}^{n-1}(h(W)) K_I \log^{1-n} 2 &= M(g\Gamma_W) \leq K_I M(\Gamma_W) \\ &= \omega_{n-1} \mathcal{H}^{n-1}(W) K_I \log^{1-n} 2, \end{aligned}$$

which contradicts (3.21). The proof of Theorem 1.1 is complete. $\square$

References

[1] Gutlyanskii, V. Ya., Martio, O., Ryazanov, R. I., Vuorinen, M.: *Infinitesimal geometry of quasiregular mappings*, Ann. Acad. Sci. Fenn. Math., 25 (2000), no. 1, 101–130.

[2] Kovalev, L.: *Quasiregular mappings of maximal local modulus of continuity*, Ann. Acad. Sci. Fenn. Math., 29 (2004), 211–222.

[3] Rajala, K.: *The local homeomorphism property of spatial quasiregular mappings with distortion close to one*, Preprint 294, Department of Mathematics and Statistics, University of Jyväskylä, 2004.

[4] Reshetnyak, Yu. G.: Space Mappings with Bounded Distortion, Trans. of Mathematical Monographs, Amer. Math. Soc, vol. 73, 1989.

[5] Rickman, S.: Quasiregular mappings, Springer-Verlag, Berlin, 1993.

[6] Seittenranta, P.: *Linear dilatation of quasiconformal maps in space*, Duke Math. J., 91 (1998), no. 1, 1–16.

[7] Vuorinen, M.: Conformal geometry and quasiregular mappings, Springer-Verlag, Berlin, 1988.

[8] Vuorinen, M: *Quadruples and spatial quasiconformal mappings*, Math. Z., 205 (1990), 617–628.

[9] Väisälä, J.: Lectures on n-dimensional quasiconformal mappings, Springer-Verlag, Berlin - New York, 1971.

University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
FIN-40014 University of Jyväskylä
Finland
e-mail: kirajala@maths.jyu.fi