Quasiconformal removability and the quasihyperbolic metric

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Abstract

We establish an essentially sharp condition sufficient for the L^n-integrability of the quasihyperbolic metric in a domain $\Omega \subset \mathbf{R}^n$. As a corollary, we prove a result concerning (quasi)conformal and $W^{1,n}$-removability of the boundary $\partial\Omega$.

1 Introduction

Recall that a compact set $K \subset U \subset \mathbf{R}^n$ is called *(quasi)conformally removable inside a domain* U, if each homeomorphism of U, which is (quasi)conformal on $U \setminus K$, is (quasi)conformal on U. In the same way one defines a stronger property, Sobolev $W^{1,n}$-removability: K is $W^{1,n}$-removable if each continuous function in U that belongs to $W^{1,n}(U \setminus K)$ also belongs to $W^{1,n}(U)$. For the basics of removability and the motivation that partially arises from complex dynamics, see [Bi], [Jo], [JS].

In [JS], P. W. Jones and S. K. Smirnov established a connection between the L^n-integrability of the quasihyperbolic metric in a domain Ω and the removability of the boundary $\partial\Omega$ ([JS, Theorem 2]). Namely, if a domain Ω satisfies

$$k_\Omega(\cdot, z_0) \in L^n(\Omega), \tag{1}$$

then $\partial\Omega$ is quasiconformally and $W^{1,n}$-removable. Here the quasihyperbolic metric k_Ω is defined in the usual way: $k_\Omega(z, z') = \inf_\gamma \int_\gamma \frac{ds}{d(\omega,\partial\Omega)}$, where the infimum is taken over all rectifiable curves γ joining z to z' in Ω.

As a corollary, Jones and Smirnov proved that if a domain $\Omega \subset \mathbf{R}^2$ is simply connected and the Riemann uniformization map $f : B^2 \to \Omega$ has the modulus of continuity $\psi(t) < \exp(-\sqrt{\log\frac{1}{t}\log\log\frac{1}{t}}/o(1))$, then $\partial\Omega$ is conformally and $W^{1,2}$-removable. They reduced this result to the fact that the required modulus of continuity implies condition (1) for the image domain $f(B^2) = \Omega$. In this paper we show that the second order term $\log\log\frac{1}{t}$ in ψ can be disposed of in this conclusion.

Theorem 1.1. *For each $p \geq 1$ there exists a constant $C_p \in \mathbf{R}$ such that, if the conformal mapping $f : B^2 \longrightarrow \Omega \subset \mathbf{R}^2$ satisfies*

$$|f(x) - f(x')| \leq \exp(-(C_p \log \frac{1}{|x - x'|})^{1/2}) \tag{2}$$

for all $x, x' \in B^2$ sufficiently close to each other, then

$$\int_\Omega k_\Omega(y, y_0)^p dy < \infty$$

for $y_0 = f(0) \in \Omega$.

Combining Theorem 1.1 and [JS, Theorem 2], we obtain the following corollary concerning the removability of the boundary of a simply connected planar domain.

Corollary 1.2. *There is a constant C such that, if a planar domain Ω is simply connected and the Riemann uniformization map $f : B^2 \longrightarrow \Omega$ satisfies*

$$|f(x) - f(x')| \leq \exp(-(C \log \frac{1}{|x - x'|})^{1/2}) \tag{3}$$

for all $x, x' \in B^2$ sufficiently close to each other, then $\partial\Omega$ is conformally and $W^{1,2}$-removable.

Recall that the modulus of continuity $\varphi(t) = \exp(-\sqrt{\log\frac{1}{t} / \log\log\frac{1}{t}})$ guarantees that the area of $f(\partial B^2)$ is zero by [JM]. It remains open if this modulus of continuity could also be sufficient for removability. However, one cannot obtain such an improvement on Corollary 1.2 by techniques of this kind: in Section 3 below we describe an example of a situation where (3) holds for a small C but k_Ω fails to be in L^2 or even in L^1. It is known that $k_\Omega \in L^1$ is not sufficient for removability.

The results above can be extended to $\mathbf{R}^n$ with $n \geq 2$. We do this by establishing the critical quasihyperbolic growth condition that implies the L^p-integrability of the quasihyperbolic metric in $\Omega \subset \mathbf{R}^n$.

Theorem 1.3. *For each $p \geq 1$ there exists a constant $C_p = C_p(p, n)$ such that, if a domain $\Omega \subset \mathbf{R}^n$ satisfies*

$$k_\Omega(y, y_0) \leq \frac{1}{C_p}(\log \frac{d(y_0, \partial\Omega)}{d(y, \partial\Omega)})^{\frac{n}{n-1}} + C_0 \tag{4}$$

for all $y \in \Omega$ with some fixed point $y_0 \in \Omega$, then

$$\int_\Omega k_\Omega(y, y_0)^p dy < \infty.$$

Note that Theorem 1.3 is sharp in the following sense: the exponent $\frac{n}{n-1}$ is the best possible. In fact, for any $p \geq 1$ there even exists a constant C'_p and a domain $\Omega \subset \mathbf{R}^2$ such that it satisfies condition (4) with the constant C'_p, and that $\int_\Omega k_\Omega(y, y_0)^p dy = \infty$.

If we consider a simply connected planar domain, Theorem 1.3 gives us an essentially equivalent result with Theorem 1.1. Indeed, condition (4) is roughly equivalent with condition (2) when we are close to the boundary.

By combining Theorem 1.3 with [JS, Theorem 2], we arrive at the following result.

Corollary 1.4. *There is a constant $C = C(n)$ such that, if a domain $\Omega \subset \mathbf{R}^n$ satisfies*

$$k_\Omega(y, y_0) \leq \frac{1}{C}(\log \frac{d(y_0, \partial\Omega)}{d(y, \partial\Omega)})^{\frac{n}{n-1}} + C_0$$

for all $y \in \Omega$ with some fixed point $y_0 \in \Omega$, then $\partial\Omega$ is quasiconformally and $W^{1,n}$-removable.

2 Proofs of the theorems

For the proof of Theorem 1.1 we need the following well-known consequence of the Hardy-Littlewood maximal theorem (see [Bo]).

Lemma 2.1. *Let $\mathcal{Q}$ be a collection of cubes $Q \subset \boldsymbol{R}^n$ and let $p \geq 1$. Then*

$$\int_{\boldsymbol{R}^n} (\sum_{Q \in \mathcal{Q}} \chi_{pQ}(x))^k dx \leq (Ckp^n)^k \int_{\boldsymbol{R}^n} (\sum_{Q \in \mathcal{Q}} \chi_Q(x))^k dx$$

for all $k \geq 1$, where $C = C(n)$.

Proof of Theorem 1.1. Let $\omega \in \partial B^2$ and let $\gamma = f([0, \omega])$. Define for each $k \in \mathbf{Z}^+$ a function $\chi_k : \partial\Omega \longrightarrow \{0, 1\}$,

$$\chi_k(x) = \begin{cases} 1, \text{ if } \int_{A_k(x)\cap\gamma} \frac{dt}{d(t,\partial\Omega)} \leq \frac{k}{c} \\ 0, \text{ otherwise,} \end{cases}$$

where $0 < c < 1$ and $A_k(x) = \{y \in \mathbf{R}^2 : 2^{-k} \leq y < 2^{-k+1}\}$. Let $j \in \mathbf{Z}^+$ and let

$$S_j(x) = \sum_{k=1}^{j} \chi_k(x).$$

We show first that there is an absolute constant C such that, when we choose $c < CC_p$, then

$$\frac{S_j(f(\omega))}{j} > \frac{1}{2} \tag{5}$$

for all $j \geq j_0$, where the integer j_0 is independent of ω.

Consider an integer $k < j$ such that $\chi_k(f(\omega)) = 0$. The curve γ intersects the two boundary components of $A_k(f(\omega))$ in two points $a = f(t_a\omega)$ and $b = f(t_b\omega)$, say. The quasihyperbolic distance $k_\Omega(a,b)$ is at least $\frac{k}{c}$. Due to the invariance of the hyperbolic metric under conformal mappings, the quasihyperbolic distance $k_{B^2}(t_a\omega, t_b\omega)$ is at least $\frac{k}{4c}$.

Consider the largest $t < 1$ with

$$f(t\omega) - f(\omega) = 2^{-j}.$$

By assumption (2), there is an integer j_0 (independent of ω) such that

$$2^{-j} \leq \exp(-(C_p \log \frac{1}{1-t})^{1/2})$$

whenever $j \geq j_0$, and hence

$$\log \frac{1}{1-t} \leq \frac{j^2}{C_p}. \tag{6}$$

On the other hand

$$\log \frac{1}{1-t} = k_{B^2}(0, t\omega) \geq \sum k_{B^2}(t_a\omega, t_b\omega) \geq \sum \frac{k}{4c}, \tag{7}$$

where the summation is over all $k < j$ with $\chi_k(f(\omega)) = 0$.

Suppose the assertion (5) fails for some large j. Then, by combining (6) and (7), we obtain

$$\frac{j^2}{C_p} \geq \sum_{k=1}^{j/2} \frac{k}{4c} \geq C\frac{j^2}{c}$$

for each $j \geq j_0$ with some absolute constant C. This is a contradiction when we choose $c < CC_p$, and thus (5) is proved.

Next we define for each $j \in \mathbf{Z}^+$ a set

$$\Omega_j := \{x \in \Omega : d(x, \partial\Omega) < 2^{-j}\}$$

and we prove that there are constants M and $\tilde{C}$ such that

$$|\Omega_j| \leq M j^{-\tilde{C}C_p} \tag{8}$$

for all $j > j_0$.

Define a collection $\mathcal{Q}$ of pairwise disjoint squares in the domain Ω in the following way: Let $\mathcal{W}$ be the Whitney decomposition of Ω, and let $\mathcal{Q}$ consist of all the squares in the Whitney decompositions of the squares $Q \in \mathcal{W}$.

Let us now consider a dyadic annulus $A_k(f(\omega))$ such that $\chi_k(f(\omega)) = 1$. Recall that the quasihyperbolic distance between the points $a, b \in \Omega$ is comparable with the number of squares in the shortest chain of Whitney squares $Q \in \mathcal{W}$ connecting the points a and b (see e.g. [JS, p. 273]). Hence there is a chain of Whitney squares such that it connects two points a, b from the boundary components of the annulus and that there are at most $\frac{c_0 k}{c}$ squares in the chain. Here c_0 is some absolute constant. Since the width of the annulus is 2^{-k}, we find at least $\frac{c_0 k}{c}$ squares $Q_i^k \in \mathcal{Q}$ such that $Q_i^k \subset A_k(f(\omega))$ and the edge lengths of the squares Q_i^k are at least $\frac{c 2^{-k}}{c_1 k}$ with some absolute constant c_1.

Let $j > j_0$. For each $k \leq j$ let $N(k)$ be the smallest integer such that $N(k) \geq \frac{j}{k}$. We define a collection $\mathcal{Q}_j$ in the following way: If $Q \in \mathcal{Q}$ and there is $1 < k \leq j$ such that $\frac{c 2^{-k}}{c_1 k} \leq l(Q) < \frac{c 2^{-k+1}}{c_1 (k-1)}$, then each edge of the square Q is divided into $N(k)$ parts. As for a square Q with $l(Q) \geq \frac{c}{2 c_1}$, divide each edge into $N(1)$ parts. Hence Q is subdivided into $N(k)^2$ squares that have edge lengths of at least $\frac{c 2^{-k}}{2 c_1 j}$. Let $\mathcal{Q}_j$ be the collection of squares aqcuired in this manner from the squares $Q \in \mathcal{Q}$ with $l(Q) \geq \frac{c 2^{-j}}{c_1 j}$.

Let $y \in \Omega_j$. We choose $y' \in \partial\Omega$ such that $d(y, y') < 2^{-j}$. From now on, c_i, $i \geq 0$, denotes an absolute constant. Let $k \leq j$ satisfy $\chi_k(y') = 1$. By the calculations above we conclude that from the annulus $A_k(y')$ we find at least $\frac{c_0 k}{c} N(k)^2 = \frac{c_0 j^2}{ck}$ disjoint squares $Q_i^k(y') \in \mathcal{Q}_j$, such that $l(Q_i^k(y')) \geq \frac{c 2^{-k}}{2 c_1 j}$ for all $1 \leq i \leq \frac{c_0 j^2}{ck}$.

By enlarging the squares $Q \in \mathcal{Q}_j$ we have by (5) that

$$\sum_{Q \in \mathcal{Q}_j} \chi_{\frac{c_2 j}{c} Q}(y) \geq \sum_{k=j_0}^{j} \frac{c_0 j^2}{2ck} = \frac{c_0 j^2}{2c} \sum_{k=j_0}^{j} \frac{1}{k} \geq \frac{c_3 j^2 \log \frac{j}{j_0}}{c}$$

when the constant c_2 is chosen large enough. Hence we have the estimate

$$\frac{c^2}{c_3 j^2 c \log \frac{j}{j_0}} \sum_{Q \in \mathcal{Q}_j} \chi_{\frac{c_2 j}{c} Q}(y) \geq 1. \tag{9}$$

Next we use inequality (9) to estimate the Lebesgue measure of Ω_j. For all $\varepsilon > 0$ we have by (9) that

$$|\Omega_j| \exp(\varepsilon c \log \frac{j}{j_0}) \leq \int_{\Omega_j} \sum_{i \geq 0} \frac{1}{i!} (\varepsilon c \log \frac{j}{j_0})^i dy$$

$$\leq c_4 |\Omega| + \sum_{i \geq 2} \frac{1}{i!} (\varepsilon c \log \frac{j}{j_0})^i \int_{\mathbf{R}^2} \Big(\frac{c^2}{c_3 j^2 c \log \frac{j}{j_0}} \sum_{Q \in \mathcal{Q}_j} \chi_{\frac{c_2 j}{c} Q}(y) \Big)^{i/2} dy$$

By Lemma 2.1 we thus deduce that

$$|\Omega_j|\exp(\varepsilon c\log\frac{j}{j_0})$$

$$\leq c_4|\Omega|+\sum_{i\geq 2}\frac{\varepsilon^i}{i!}(\frac{c^2}{c_3j^2}c_5\frac{i}{2}\frac{c_2^2j^2}{c^2}c\log\frac{j}{j_0})^{i/2}\int_{\mathbf{R}^2}(\sum_{Q\in\mathcal{Q}_j}\chi_Q(y))^{i/2}dy$$

$$\leq|\Omega|(c_4+\sum_{i\geq 2}\frac{\varepsilon^i(c_6\frac{i}{2}c\log\frac{j}{j_0})^{i/2}}{i!}). \tag{10}$$

By the inequality $i^i\leq e^i i!$ we have that

$$|\Omega|(c_4+\sum_{i\geq 2}\frac{\varepsilon^i(c_6\frac{i}{2}c\log\frac{j}{j_0})^{i/2}}{i!})$$

$$\leq|\Omega|(c_4+\sum_{i\geq 2}\frac{(\frac{1}{2})^{i/2}\varepsilon^i(c_7c\log\frac{j}{j_0})^{i/2}}{i!^{1/2}}). \tag{11}$$

By Hölder's inequality we obtain

$$|\Omega|(c_4+\sum_{i\geq 2}\frac{(\frac{1}{2})^{i/2}\varepsilon^i(c_7c\log\frac{j}{j_0})^{i/2}}{i!^{1/2}})$$

$$\leq|\Omega|(c_4+(\sum_{i\geq 2}\frac{1}{2^i})^{1/2}(\sum_{i\geq 2}\frac{\varepsilon^{2i}(c_7c\log\frac{j}{j_0})^i}{i!})^{1/2})$$

$$\leq|\Omega|(c_4+\exp(\frac{1}{2}\varepsilon^2c_7c\log\frac{j}{j_0}). \tag{12}$$

Now

$$|\Omega|(c_4+\exp(\frac{1}{2}\varepsilon^2c_7c\log\frac{j}{j_0})\leq M_0\exp(\frac{1}{2}\varepsilon c\log\frac{j}{j_0}), \tag{13}$$

when we choose $\varepsilon=1/c_7$ and the constant M_0 suitably. Thus, by combining (10),(11),(12),(13), we arrive at

$$|\Omega_j|\leq M_0\exp(-\frac{1}{2}\varepsilon c\log\frac{j}{j_0}),$$

and hence

$$|\Omega_j|\leq M_0\exp(-\frac{1}{2}\varepsilon c\log j)\exp(\frac{1}{2}\varepsilon c\log j_0)\leq Mj^{-\tilde{C}C_p},$$

where the constant M depends on $|\Omega|$ and j_0, and $\tilde{C}$ is an absolute constant. This proves inequality (8).

It follows from (2) that

$$k_\Omega(f(x), f(0)) \lesssim \frac{1}{C_p}(\log \frac{d(f(0), \partial\Omega)}{d(f(x), \partial\Omega)})^2 + C_0 \tag{14}$$

for all points $x \in B^2$.

Now, by combining (8) and (14), we obtain for $y_0 = f(0) \in \Omega$ that

$$\int_\Omega k_\Omega(y, y_0)^p dy \lesssim \int_\Omega \frac{1}{C_p^p}(\log \frac{d(y_0, \partial\Omega)}{d(y, \partial\Omega)})^{2p} dy + \text{const}$$

$$\lesssim \sum_{j=j_0}^{\infty} \int_{\Omega_j \setminus \Omega_{j+1}} \frac{1}{C_p^p}(\log 2^{j+1})^{2p} dy + \text{const}$$

$$\lesssim \sum_{j=j_0}^{\infty} \frac{M}{C_p^p} j^{-\tilde{C}C_p + 2p} + \text{const}.$$

The sum above converges if $\tilde{C}C_p > 2p + 1$ or

$$C_p > \frac{2p+1}{\tilde{C}},$$

and hence the theorem is proved. □

Note that, by using the technique in the proof of Theorem 1.1, we obtain the estimate (corresponding to (8)) $|\{y : d(y, \partial\Omega) < r\}| \leq G(r)$ for all $r > 0$ with some increasing function G with $G(0) = 0$. This implies a generalized Hausdorff dimension estimate for the set $\partial\Omega$. This kind of estimates, that can be viewed of generalizations of the dimension bounds in [JM] and [KR], are considered further in [N].

Proof of Theorem 1.3. The proof is very similar to the proof of Theorem 1.1, and thus we only indicate the important modifications needed. Instead of using the curves $\gamma = f([0, \omega])$, we directly work with quasihyperbolic geodesics. In the definition of χ_k, the bound $\frac{k}{c}$ is replaced with $\frac{k^{\frac{1}{n-1}}}{c}$. The crucial estimate (5) is now obtained by applying [KOT, Lemma 4.6]. The rest of the proof then follows essentially as in the proof of Theorem 1.1. □

3 Sharpness of the results

We show the essential sharpness of Theorem 1.3 by sketching an example of a domain $\Omega \subset \mathbf{R}^2$ which satisfies condition (4) with some constant, but $\int_\Omega k_\Omega(y, y_0)^p dy = \infty$ for all $p \geq 1$. A similar construction can be carried out in $\mathbf{R}^n$, $n > 2$, to show that the exponent $\frac{n}{n-1}$ in Theorem 1.3 is critical for the volume of $\partial\Omega$ to be zero, see [N]. Because $k_\Omega \in L^n$ can be shown to guarantee that the volume of $\partial\Omega$ is zero, it follows that the exponent $\frac{n}{n-1}$ in Theorem 1.3 is the best possible.

Let $\alpha :]0,1[\to]0,\infty[$,

$$\alpha(t) = \frac{\varepsilon t}{\log \frac{1}{t}},$$

where $0 < \varepsilon < \frac{1}{2}$. Let $Q_1 = \{x \in \mathbf{R}^2 : |x_i| < \frac{1}{4} \text{ for } i = 1,2\}$, and denote the side length of Q_1 by r_1. Let Ω^1 be the $\alpha(r_1)$-neighborhood of the coordinate axes in Q_1. Let $Q_2 = Q_1 \setminus \Omega^1$. Now Q_2 consists of 4 squares with side lengths $r_2 = \frac{1}{2} r_1 (1 - \frac{2\varepsilon}{\log \frac{1}{r_1}})$. Denote the components of Q_2 by Q_2^l. Let Ω^2 be the union of the $\alpha(r_2)$-neighborhoods of the centered coordinate axes in the squares Q_2^l. Then let $Q_3 = Q_2 \setminus \Omega^2$. Now Q_3 consists of 4^2 squares with side lengths $r_3 = (\frac{1}{2})^2 r_1 (1 - \frac{2\varepsilon}{\log \frac{1}{r_1}})(1 - \frac{2\varepsilon}{\log \frac{1}{r_2}})$. Define for every $k \geq 3$ the sets Ω^k and Q_k accordingly. Now the set Ω^k consists of the $\alpha(r_k)$-neighborhoods of the centered coordinate axes in the squares $Q_k^l, l = 1, 2, ..., 4^{k-1}$, with side lenghts

$$r_k = r_1 (\frac{1}{2})^{k-1} \prod_{i=1}^{k-1} (1 - \frac{2\varepsilon}{\log r_i}) \geq r_1 (\frac{1}{2})^{k-1} \prod_{i=1}^{k-1} (1 - \frac{2\varepsilon}{i}).$$

Define the domain Ω by

$$\Omega = \bigcup_{k=1}^{\infty} \Omega^k.$$

Now an easy calculation shows that Ω satisfies condition (4) with some constants $C(\varepsilon)$ and C_0. Moreover, choosing the constant ε small enough implies $\int_\Omega k_\Omega(y, y_0)^p dy = \infty$ for any $p \geq 1$. Indeed, notice that $r_k \asymp 2^{-k} k^{-2\varepsilon}$, and

$$\int_\Omega k(y, y_0)^p dy \gtrsim \sum_{k=1}^{\infty} 4^{k-1} r_k \alpha(r_k) (\frac{r_k}{\alpha(r_k)})^p$$

$$= \sum_{k=1}^{\infty} 4^{k-1} r_k^2 (\frac{\log \frac{1}{r_k}}{\varepsilon})^{p-1}$$

$$\gtrsim \sum_{k=1}^{\infty} k^{-4\varepsilon} (\frac{k}{\varepsilon})^{p-1}.$$

This sum diverges if $\varepsilon < \frac{p}{4}$. Notice that Ω is not simply connected because of loops. This can be easily fixed by closing certain gates in the construction. We leave the details to the reader. Moreover, condition (4) implies condition (3) with a constant $\tilde{C}(\varepsilon)$.

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