IS THE APPROXIMATION RATE FOR CERTAIN STOCHASTIC INTEGRALS ALWAYS $1/\sqrt{n}$?

MIKA HUJO

Abstract. For a Borel-function $f : \mathbb{R} \to \mathbb{R}$ we consider the approximation of a random variable $f(W_1) \in L^2$ by stochastic integrals with respect to the Brownian motion $W = (W_t)_{t\in[0,1]}$ and the geometric Brownian motion, where the integrands are piece-wise constant within certain deterministic time intervals. In earlier papers it has been shown that under certain regularity conditions the optimal approximation rate is $\frac{1}{\sqrt{n}}$, if one optimizes over deterministic time-nets of cardinality n. We will show the existence of random variables $f(W_1) \in L^2$ such that the approximation error tends as slowly to zero as one wishes.

1. Introduction and result

In this article we consider the approximation of a random variable $f(W_1) \in L^2$ by stochastic integrals over piece-wise constant integrands, where $f : \mathbb{R} \to \mathbb{R}$ is a Borel-function, with respect to the standard Brownian motion $W = (W_t)_{t\in[0,1]}$ and the geometric Brownian motion $S = (S_t)_{t\in[0,1]}$. The approximation problem arises from Stochastic Finance where one is interested in variance optimal discrete time hedges in European type options in continuous time option pricing models because continuously adjusted portfolios have to be replaced by discretely adjusted portfolios in practice. As the option pricing model we take the discounted Black-Scholes model (where we may assume without loss of generality that the time horizon and the volatility are equal to one). The pay-off of an European type option we denote by $g(S_1)$ which we write in the following as $f(W_1)$ by $f(x) = g(e^{x-\frac{1}{2}})$. The approximation of the random variable $f(W_1)$ by a "discrete" stochastic integral with respect to the geometric Brownian motion corresponds to a discrete time hedge of the underlying option with pay-off $g(S_1)$.

To recall some earlier results from the literature and for later purpose we introduce the notation used in this paper. We will work with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is generated by W. It

Date: March 30, 2004.

2000 *Mathematics Subject Classification.* 60H05, 33C45.

Key words and phrases. stochastic integral, Hermite polynomial, approximation rate.

is also assumed that the filtration $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the natural filtration of W, all paths of W are continuous, and $W_0 \equiv 0$. The stochastic integrals we use in our approximation are driven by the process $X = (X_t)_{t\in[0,1]}$ which is either the standard Brownian motion W or the geometric Brownian motion S. The geometric Brownian motion is given by $S = (S_t)_{t\in[0,1]}$ with

$$S_t := e^{W_t - \frac{t}{2}}.$$

The random variable $f(W_1)$ has a representation as stochastic integral:

$$f(W_1) = \mathbb{E}f(W_1) + \int_0^1 \frac{\partial}{\partial x} F(s, X_s) dX_s \ a.s.$$

where

$$F(t,x) := \mathbb{E}_{X_t=x} f(W_1) \tag{1}$$

gives

$$F(t,x) \in C^\infty([0,1) \times E)$$

with

$$E := \begin{cases} \mathbb{R}, & X = W \\ (0,\infty), & X = S \end{cases}$$

(cf. [5] and [2]). The approximation that we use is of the form

$$f(W_1) \sim \mathbb{E}f(W_1) + \sum_{i=1}^n v_{i-1}(X_{t_i} - X_{t_{i-1}}),$$

where v_i is an $\mathcal{F}_{t_i}$-measurable step-function and $\tau = (t_i)_{i=0}^n$ is a deterministic, but not necessarily equidistant time-net, $0 = t_0 < t_1 < \cdots < t_n = 1$. The approximation error is measured with respect to L^2 via

$$a_X^{opt}(Z;\tau) := \inf_{v_0,\ldots,v_{n-1}} \left\| Z - \mathbb{E}Z - \sum_{i=1}^n v_{i-1}(X_{t_i} - X_{t_{i-1}}) \right\|_{L^2},$$

where $Z \in L^2$ and the infimum is taken over all $\mathcal{F}_{t_i}$-measurable step-functions v_i. The superscript *opt* refers to optimal, since we optimize over the random variables $(v_i)_{i=0}^{n-1}$.

We are interested in rates of convergence as the time-nets are optimized, i.e. in the asymptotics of

$$\inf_{\tau\in\mathcal{T}_n} a_X^{opt}(Z;\tau)$$

where

$$\mathcal{T}_n := \{(t_i)_{i=0}^m : 0 = t_0 < t_1 < \cdots < t_m = 1, m \le n\}.$$

Let us now recall some results from the literature. In the case of equidistant time-nets it follows from Zhang [6] that

$$C_g^{-1} n^{-1/2} \le a_S^{opt}\left(g(S_1); \left(\frac{i}{n}\right)_{i=0}^{n}\right) \le C_g n^{-1/2} \text{ for } n = 1, 2, \ldots$$

for certain absolutely continuous g that are not almost surely linear (cf. also [2]). Later on, Gobet and Temam [4] gave examples of functions g_η such that

$$C_\eta^{-1} n^{-\eta} \le a_S^{opt}\left(g_\eta(S_1); \left(\frac{i}{n}\right)_{i=0}^{n}\right) \le C_\eta n^{-\eta} \text{ for } n = 1, 2, \ldots$$

and $\eta \in [\frac{1}{4}, \frac{1}{2})$. Geiss [2] considered the problem with general deterministic time-nets. In [2] (see Lemma 2.1 below), it was shown that the approximation error can be completely controlled by the deterministic function

$$H(f(W_1); X)(u) := \left\| \left(\sigma^2 \frac{\partial^2}{\partial x^2} F\right)(u, X_u) \right\|_{L^2},$$

$u \in [0, 1)$, with

$$\sigma(y) := \begin{cases} 1, & X = W \\ y, & X = S \end{cases}$$

and F given as in (1). In particular it was shown that for a large class of random variables $f(W_1) \in L^2$ the optimal convergence rate is $n^{-1/2}$:

Theorem 1.1 ([1](Lemma 4.14, Proposition 4.16))**.** *Let $f(W_1) \in L^2$ and $X \in \{W, S\}$. Assume that $\sup_{u \in [0,1)} H(f(W_1); X)(u) > 0$ and*

$$H(f(W_1); X)(u) \le \frac{C}{\left[\alpha + \log\left(1 + \frac{1}{1-u}\right)\right]^{\alpha} (1-u)},$$

for all $u \in [0, 1)$, for some $C < \infty$ and $\alpha \in (1, \infty)$. Then

$$0 < \inf_n \sqrt{n} \left[\inf_{\tau \in \mathcal{T}_n} a_X^{opt}(f(W_1); \tau)\right] \le \sup_n \sqrt{n} \left[\inf_{\tau \in \mathcal{T}_n} a_X^{opt}(f(W_1); \tau)\right] < \infty.$$

Remark 1.2. (1) *The class of random variables satisfying the conditions of the above theorem is rather large. To see this we point out that for every $f(W_1) \in L^2$ one has (cf. for example [2]) that*

$$\int_0^1 (1-u) H(f(W_1); X)^2(u) du < \infty.$$

(2) *In [1] the above theorem is formulated in the case where X is the geometric Brownian motion, but by Lemma 2.1 below the theorem is true also for the Brownian motion.*

Using Theorem 1.1 it follows that in the case of Zhang [6] the equidistant time-nets give the optimal rate of approximation. By Theorem 1.1 it can be also proved that for the examples, given by Gobet and Temam [4], one has the rate $n^{-1/2}$, when the time-nets are optimized. Thus the approximation by equidistant time-nets does not necessarily give the optimal approximation rate.

The following question was left open: Are there random variables $f(W_1) \in L^2$ such that

$$\sup_n \sqrt{n} \left[\inf_{\tau \in \mathcal{T}_n} a_X^{opt}(f(W_1);\tau) \right] = \infty?$$

The difficulty in solving this problem arises from the fact that one has to optimize the approximation over all time-nets of a given cardinality. Here we solve this problem by a dynamic programming type argument. Our main theorem, Theorem 1.3, shows the existence of random variables $f(W_1) \in L^2$ such that the quantity $\inf_{\tau \in \mathcal{T}_n} a_X^{opt}(f(W_1);\tau)$ tends as slowly to zero as one wishes:

Theorem 1.3. *For every sequence $\beta = (\beta_n)_{n=1}^{\infty}$ of positive real numbers, $\beta_n \searrow 0$, there exists a random variable $f_\beta(W_1) \in L^2$ such that*

$$\inf_{\tau \in \mathcal{T}_n} a_X^{opt}(f_\beta(W_1);\tau) \geq \beta_n$$

for $n = 1, 2, \ldots$ and $X \in \{W, S\}$.

2. Proof

The normalized Hermite polynomials on $\mathbb{R}$ are given by

$$h_n(x) := \frac{1}{\sqrt{n!}} \frac{(-1)^n D^n e^{-\frac{x^2}{2}}}{e^{-\frac{x^2}{2}}}.$$

The family $\{h_n : n = 0, 1, 2, \ldots\}$ forms a complete orthonormal system in

$$L^2(\gamma) := \left\{ f : \mathbb{R} \to \mathbb{R} : f \text{ Borel-function }, \int_{-\infty}^{\infty} f^2(x) d\gamma(x) < \infty \right\}$$

where $d\gamma(x) = e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$ is the standard Gaussian measure on real line.

For the proof of our theorem we will need some preliminary results. First we recall that the approximation error can be strictly controlled by the deterministic function $H(f(W_1);X)$ defined above.

Lemma 2.1 ([2])**.** *For $f \in L^2(\gamma)$ and $\tau \in \cup_{n=1}^{\infty} \mathcal{T}_n$ one has*

$$\frac{1}{C} a_X(f(W_1);\tau) \le a_X^{opt}(f(W_1);\tau) \le C a_X(f(W_1);\tau)$$

where

$$a_X(f(W_1);\tau) := \left(\sum_{t_i \in \tau, t_i > 0} \int_{t_{i-1}}^{t_i} (t_i - u) H(f(W_1);X)^2(u) du \right)^{\frac{1}{2}}$$

and $C \ge 1$ is an absolute constant.

The next lemma recalls a representation of $H(f(W_1);X)$ used later on.

Lemma 2.2 ([3])**.** *Let $f = \sum_{k=0}^{\infty} \alpha_k h_k \in L^2(\gamma)$ and $u \in [0,1)$. Then*

$$H(f(W_1);W)^2(u) = \sum_{k=0}^{\infty} \alpha_{k+2}^2 (k+2)(k+1) u^k$$

and

$$H(f(W_1);S)^2(u) = \sum_{k=0}^{\infty} \left(\alpha_{k+2} - \frac{\alpha_{k+1}}{\sqrt{k+2}} \right)^2 (k+2)(k+1) u^k.$$

Consequently, for $u \in [0,1)$ one has

$$\begin{aligned} \frac{1}{12} H(f(W_1);W)^2(u) - \frac{2}{3} \left(\alpha_2^2 + \alpha_1^2 \right) &\le H(f(W_1);S)^2(u) \\ &\le 4 H(f(W_1);W)^2(u) + 2\alpha_1^2. \end{aligned}$$

To prove Theorem 1.3 we first consider the Brownian motion case. We show that there are functions $f_\theta \in L^2(\gamma)$ such that $H(f_\theta(W_1);W)(u) = (1-u)^{-\theta}$, for $\theta \in (0,1)$ and prove that the approximation error for these f_θ is vanishing "slowly" enough. Finally we construct a function H_β^2 as combination of the functions $H(f_\theta(W_1);W)^2$. From the function H_β^2 we get the desired function $f_{\beta,W}$ for the Brownian motion case such that $H(f_{\beta,W}(W_1);W)^2 = H_\beta^2$. So our first step is

Lemma 2.3. *For all $\theta \in (0,1)$ there exists a function $f_\theta \in L^2(\gamma)$ such that*

$$H(f_\theta(W_1);W)(u) = \frac{1}{(1-u)^\theta} \text{ for } u \in [0,1).$$

Proof. Because of Lemma 2.2 we have to find a function $f_\theta = \sum_{k=0}^{\infty} \alpha_{k,\theta} h_k \in L^2(\gamma)$ such that

$$\sum_{k=0}^{\infty} \left[\alpha_{k+2,\theta}^2 (k+2)(k+1)u^k\right] = \frac{1}{(1-u)^{2\theta}}.$$

Let $\alpha_{0,\theta} = \alpha_{1,\theta} = 0$. Using the Binomial series expansion

$$\frac{1}{(1-u)^{2\theta}} = \sum_{k=0}^{\infty} \binom{-2\theta}{k} (-u)^k = \sum_{k=0}^{\infty} \beta_k^{(\theta)} u^k \text{ for } u \in (-1,1),$$

with $\beta_0^{(\theta)} := 1$ and $\beta_k^{(\theta)} := \frac{2\theta(2\theta+1)\cdots(2\theta+(k-1))}{k!}$ for $k \geq 1$ we may set $\alpha_{k+2,\theta}^2 := \frac{\beta_k^{(\theta)}}{(k+2)(k+1)}$, $k \geq 0$, and have to show that $\sum_{k=0}^{\infty} \alpha_{k,\theta}^2 < \infty$ or

$$\sum_{k=1}^{\infty} \frac{2\theta(2\theta+1)\cdots(2\theta+(k-1))}{(k+2)!} < \infty$$

for every $\theta \in (0,1)$. The above sum can be written with Gamma-functions as

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{2\theta(2\theta+1)\cdots(2\theta+(k-1))}{(k+2)!} \\
&= \sum_{k=1}^{\infty} \frac{\Gamma(k+2\theta)/\Gamma(2\theta)}{(k+2)!} \\
&\leq \frac{1}{\Gamma(2\theta)} \sum_{k=0}^{\infty} \frac{\Gamma(k+2\theta)}{\Gamma(k+3)} \\
&= \frac{1}{2} \frac{\Gamma(3)}{\Gamma(2\theta)\Gamma(1)} \sum_{k=0}^{\infty} \frac{\Gamma(k+2\theta)\Gamma(k+1)}{\Gamma(k+3)} \frac{1}{k!} \\
&= \frac{1}{2} F(1, 2\theta; 3; 1)
\end{aligned}$$

where $F(a,b;c;d)$ is the Gauss hyper-geometric function which is finite in our case since $(c-a-b) > 0$ (see [7, p. 556]). □

Lemma 2.4. *For every $\theta \in (1,2)$ there exists a function $f_\theta \in L^2(\gamma)$ such that*

$$H(f_\theta(W_1); W)^2(u) = (2-\theta)(1-u)^{-\theta} \text{ on } [0,1)$$

and

$$\inf_{\tau \in \mathcal{T}_n} a_W(f_\theta(W_1); \tau) \geq (\theta-1)^{\frac{n-1}{2}} \text{ for all } n \in \{1,2,\ldots\}$$

where $a_W(\cdot;\cdot)$ was defined in Lemma 2.1.

Proof. From Lemma 2.3 it follows that there exists a function $f_\theta \in L^2(\gamma)$ such that

$$H(f_\theta(W_1);W)^2(u) = (2-\theta)(1-u)^{-\theta},$$

where $\theta \in (1,2)$. Let us define

$$b_n := (2-\theta) \inf_{0=t_0\le\cdots\le t_n=1} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i-u)(1-u)^{-\theta} du.$$

By a compactness argument there exists a partition $0 = t_0^0 \le \cdots \le t_n^0 = 1$ such that

$$\frac{b_n}{2-\theta} = \sum_{i=1}^n \int_{t_{i-1}^0}^{t_i^0} (t_i^0-u)(1-u)^{-\theta} du = \inf_{0=t_0\le\cdots\le t_n=1} \int_{t_{i-1}}^{t_i} (t_i-u)(1-u)^{-\theta} du.$$

By considering $\{t_0^0,\dots,t_n^0,t\}$ as a new partition, where $t \notin \{t_0^0,\dots,t_n^0\}$ we may conclude that $0 < b_{n+1} < b_n$ for $n \ge 1$. Moreover it is easy to see that $b_1 = 1$. Now we assume that $n \ge 2$. By a dynamic programming argument we can calculate a recursive representation for the expressions $\frac{b_n}{2-\theta}$: Fixing $s \in [0,1)$ and minimizing the error on $[s,1]$ by $n-1$ time-points and then minimizing the error as a function of s we get that

$$\begin{aligned}
\frac{b_n}{2-\theta} &= \inf_{0=t_0\le\cdots\le t_n=1} \Bigg[\int_0^{t_1} (t_1-u)(1-u)^{-\theta} du \\
&\qquad\qquad + \sum_{i=2}^n \int_{t_{i-1}}^{t_i} (t_i-u)(1-u)^{-\theta} du \Bigg] \\
&= \inf_{0\le s\le 1} \Bigg[\int_0^s (s-u)(1-u)^{-\theta} du \\
&\qquad + \inf_{s=y_0\le\cdots\le y_{n-1}=1} \sum_{i=1}^{n-1} \int_{y_{i-1}}^{y_i} (y_i-u)(1-u)^{-\theta} du \Bigg] \\
&\overset{(*)}{=} \inf_{0\le s\le 1} \left[\int_0^s (s-u)(1-u)^{-\theta} du + (1-s)^{2-\theta} \frac{b_{n-1}}{2-\theta} \right] \\
&= \inf_{0\le s\le 1} \left[\frac{1}{\theta-1} \left(\frac{(1-s)^{2-\theta}-1}{\theta-2} - s \right) + (1-s)^{2-\theta} \frac{b_{n-1}}{2-\theta} \right].
\end{aligned}$$

The equality $(*)$ can be seen by

$$
\begin{aligned}
&\inf_{s=y_0\leq\cdots\leq y_{n-1}=1}\sum_{i=1}^{n-1}\int_{y_{i-1}}^{y_i}(y_i-u)(1-u)^{-\theta}du\\
&=\inf_{s=y_0\leq\cdots\leq y_{n-1}=1}\sum_{i=1}^{n-1}\int_{1-y_i}^{1-y_{i-1}}(t-(1-y_i))t^{-\theta}dt\\
&=\inf_{0=y_0\leq\cdots\leq y_{n-1}=1}\sum_{i=1}^{n-1}\int_{(1-s)(1-y_i)}^{(1-s)(1-y_{i-1})}(t-(1-s)(1-y_i))t^{-\theta}dt\\
&=\inf_{0=y_0\leq\cdots\leq y_{n-1}=1}\sum_{i=1}^{n-1}\int_{1-y_i}^{1-y_{i-1}}((1-s)t-(1-s)(1-y_i))((1-s)t)^{-\theta}(1-s)dt\\
&=\inf_{0=y_0\leq\cdots\leq y_{n-1}=1}(1-s)^{2-\theta}\sum_{i=1}^{n-1}\int_{1-y_i}^{1-y_{i-1}}(t-(1-y_i))t^{-\theta}dt\\
&=(1-s)^{2-\theta}\inf_{0=y_0\leq\cdots\leq y_{n-1}=1}\sum_{i=1}^{n-1}\int_{y_{i-1}}^{y_i}(1-u-(1-y_i))(1-u)^{-\theta}du\\
&=(1-s)^{2-\theta}\frac{b_{n-1}}{2-\theta}.
\end{aligned}
$$

From above it follows that

$$b_n=\inf_{0\leq s\leq 1}\left[(1-s)^{2-\theta}\left(b_{n-1}-\frac{1}{\theta-1}\right)-\frac{2-\theta}{\theta-1}s+\frac{1}{\theta-1}\right].$$

Using the fact that $(b_n)_{n=1}^{\infty}$ is a strictly decreasing sequence and by minimizing

$$(1-s)^{2-\theta}\left(b_{n-1}-\frac{1}{\theta-1}\right)-\frac{2-\theta}{\theta-1}s+\frac{1}{\theta-1}$$

as a function of $s\in[0,1]$ we get that

$$b_n=1-(1-(\theta-1)b_{n-1})^{\frac{1}{\theta-1}}.$$

Let us assume that $b_n\geq(\theta-1)^{n-1}$, which is true for $n=1$. By induction we see that

$$
\begin{aligned}
b_{n+1}&=1-(1-(\theta-1)b_n)^{\frac{1}{\theta-1}}\\
&\geq 1-\left(1-(\theta-1)(\theta-1)^{n-1}\right)^{\frac{1}{\theta-1}}\\
&\geq(\theta-1)^n,
\end{aligned}
$$

which proves our assertion. $\square$

Lemma 2.5. *Let $(\beta_n)_{n=1}^\infty$ be a sequence such that $\beta_n \searrow 0$. Then there exists a function $f_{\beta,W} \in L^2(\gamma)$ with*

$$\inf_{\tau \in \mathcal{T}_n} a_W(f_{\beta,W}(W_1);\tau) \geq \beta_n \quad \textit{for all } n \in \{1,2,\ldots\}.$$

Proof. Fix $\varepsilon \in (0,1)$ and let $\sum_{n=1}^\infty p_n < \infty$, where $p_n \geq 0$ for all $n \in \{1,2,\ldots\}$. Assume a sequence $(a_n)_{n=1}^\infty$ such that

$$1 < a_n \nearrow 2 \text{ and } (a_n - 1)^{n-1} \geq (1-\varepsilon).$$

By Lemmas 2.2 and 2.3 we can choose coefficients α_{k,a_n} such that

$$(1-u)^{-a_n} = \sum_{k=0}^\infty \alpha_{k+2,a_n}^2 (k+2)(k+1)u^k \text{ and } \alpha_{0,a_n} = \alpha_{1,a_n} = 0.$$

Hence

$$\begin{aligned}\sum_{n=1}^\infty p_n(2-a_n)(1-u)^{-a_n} &= \sum_{n=1}^\infty p_n(2-a_n)\sum_{k=0}^\infty \alpha_{k+2,a_n}^2 (k+2)(k+1)u^k \\ &= \sum_{k=0}^\infty \left(\sum_{n=1}^\infty p_n(2-a_n)\alpha_{k+2,a_n}^2\right)(k+2)(k+1)u^k.\end{aligned} \tag{2}$$

Define

$$\alpha_{k+2} := \left(\sum_{n=1}^\infty p_n(2-a_n)\alpha_{k+2,a_n}^2\right)^{\frac{1}{2}} \quad \text{for } k \in \{0,1,\ldots\},$$

$\alpha_0 = \alpha_1 = 0$ and set $f_{p,\varepsilon} := \sum_{k=0}^\infty \alpha_k h_k$. To show that $f_{p,\varepsilon}$ is well defined we need to prove that

$$\sum_{k=0}^\infty \alpha_{k+2}^2 < \infty.$$

The definition of the coefficients α_{k,a_n} implies that

$$\begin{aligned}\sum_{k=0}^\infty \alpha_{k+2}^2 &= \sum_{k=0}^\infty \left(\sum_{n=1}^\infty p_n(2-a_n)\alpha_{k+2,a_n}^2\right) \\ &= \sum_{n=1}^\infty p_n(2-a_n)\sum_{k=0}^\infty \alpha_{k+2,a_n}^2\end{aligned}$$

$$= \sum_{n=1}^{\infty} p_n(2-a_n)\int_0^1 (1-u)(1-u)^{-a_n}du$$
$$= \sum_{n=1}^{\infty} p_n < \infty.$$

Now by equation (2) we have that

$$H(f_{p,\varepsilon}(W_1);W)^2(u) = \sum_{n=1}^{\infty} p_n(2-a_n)(1-u)^{-a_n}.$$

From this it follows that

$$\begin{aligned}
&\inf_{\tau\in\mathcal{T}_n} a_W(f_{p,\varepsilon}(W_1);\tau)^2\\
&= \inf_{\tau\in\mathcal{T}_n}\sum_{k=1}^{\infty} p_k \sum_{t_i\in\tau, t_i>0}\int_{t_{i-i}}^{t_i}(t_i-u)(2-a_k)(1-u)^{-a_k}du\\
&\geq \sum_{k=1}^{\infty} p_k \inf_{\tau\in\mathcal{T}_n}\sum_{t_i\in\tau, t_i>0}\int_{t_{i-i}}^{t_i}(t_i-u)(2-a_k)(1-u)^{-a_k}du.
\end{aligned}$$

Choosing $\theta = a_k$ in Lemma 2.4 we get that

$$\begin{aligned}
\inf_{\tau\in\mathcal{T}_n} a_W(f_{p,\varepsilon}(W_1);\tau)^2 &\geq \sum_{k=1}^{\infty} p_k(a_k-1)^{n-1}\\
&\geq (1-\varepsilon)\sum_{k=n}^{\infty} p_k.
\end{aligned}$$

Finally we set $f_{\beta,W} := \frac{f_{p,\varepsilon}}{\sqrt{1-\varepsilon}}$ and $p_k := \beta_k^2 - \beta_{k+1}^2$. □

Proof of Theorem 1.3. From Lemmas 2.1 and 2.5 it follows that the assertion is true in the case of the Brownian motion. Let $(\xi_n)_{n=1}^{\infty}$ be some sequence $\xi_n \searrow 0$. By virtue of Lemma 2.5 we can choose a function $f_{\xi,W} \in L^2(\gamma)$ such that $\xi_n \leq a_W(f_{\xi,W}(W_1);\tau)$ for all $\tau \in \mathcal{T}_n$ and $n \in \{1,2,\ldots\}$. From the proof of Lemma 2.5 we see that $H(f_{\xi,W}(W_1);W)^2(u) \nearrow \infty$ as $u \to 1$ and that we can also assume that $\alpha_0 = \alpha_1 = 0$. This means that we can find an $r \in (0,1)$ such that

$$\alpha_2^2 \leq H(f_{\xi,W}(W_1);S)^2(u)$$

for $u \in [r,1)$. By Lemma 2.2 this implies

$$H(f_{\xi,W}(W_1);W)^2(u) \leq 20H(f_{\xi,W}(W_1);S)^2(u)$$

for $u \in [r,1)$. Let $\tau_n \in \mathcal{T}_n$ and define $\tau_{2n} := \tau_n \cup \{\frac{ir}{n} : i = 0,1,\dots,n\} \in \mathcal{T}_{2n}$. Lemma 2.5 gives

$$
\begin{aligned}
\xi_{2n}^2 &\le \sum_{t_i \in \tau_{2n}, t_i > 0} \int_{t_{i-1}}^{t_i} (t_i - u) H(f_{\xi,W}(W_1); W)^2(u) du \\
&= \sum_{t_i \in \tau_{2n}, 0 < t_i \le r} \int_{t_{i-1}}^{t_i} (t_i - u) H(f_{\xi,W}(W_1); W)^2(u) du \\
&\quad + \sum_{t_i \in \tau_{2n}, t_i > r} \int_{t_{i-1}}^{t_i} (t_i - u) H(f_{\xi,W}(W_1); W)^2(u) du \\
&\le \frac{H(f_{\xi,W}(W_1); W)^2(r)}{2} \frac{r^2}{n} \\
&\quad + 20 \sum_{t_i \in \tau_{2n}, t_i > r} \int_{t_{i-1}}^{t_i} (t_i - u) H(f_{\xi,W}(W_1); S)^2(u) du \\
&\le \frac{H(f_{\xi,W}(W_1); W)^2(r)}{2} \frac{r^2}{n} + 20 a_S(f_{\xi,W}(W_1); \tau_n)^2.
\end{aligned}
$$

By choosing

$$\xi_{2n}^2 := 20\beta_n^2 + \frac{H(f_{\xi,W}(W_1); W)^2(r)}{2} \frac{r^2}{n}$$

and

$$f_\beta := 2 f_{\xi,W}$$

Lemma 2.2 gives that

$$\beta_n \le \frac{a_S(f_\beta(W_1); \tau_n)}{2} \le a_W(f_\beta(W_1); \tau_n).$$

□

3. Final remark

Theorem 1.3 yields to the following open problems which might be of interest for future work:

(i) As a consequence of Theorem 1.3 we get that Theorem 1.1 without a condition on the function $H(f(W_1); X)$ fails to be true. One might ask for weaker or even usable sufficient and necessary conditions on $H(f(W_1); X)$ such that

$$\sup_n \sqrt{n} \left[\inf_{\tau \in \mathcal{T}_n} a_X^{opt}(f(W_1); \tau) \right] < \infty. \tag{3}$$

Moreover it would be of interest to find a property, which is in a sense a property on f itself, equivalent to (3).

(ii) It seems that our techniques for deterministic time-nets do not apply for random time-nets. So one might ask whether it is possible to construct examples as in Theorem 1.3 if one is using random time-nets instead of deterministic time-nets.

Acknowledgments

The author would like to thank his advisor Stefan Geiss and Krzysztof Oleszkiewicz from the University of Warsaw for useful discussions.

References

[1] S. Geiss. On quantitative approximation of stochastic integrals with respect to the geometric brownian motion. *Report Series: SFB Adaptive Information Systems and Modelling in Economics and Management Science, Vienna University*, 43, 1999.

[2] S. Geiss. Quantitative approximation of certain stochastic integrals. *Stoch. Stoch. Rep.* 73 (2002), no. 3-4, 241–270.

[3] S. Geiss and M. Hujo. Interpolation and approximation in $L_2(\gamma)$. *Preprint 290, University of Jyväskylä*, 2004.

[4] E. Gobet and E. Temam. Discrete time hedging errors for options with irregular payoffs. *Finance Stoch.* 5 (2001), no. 3, 357–367.

[5] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer, 1988.

[6] R. Zhang. *Couverture approchée des options Européennes.* PhD thesis, Ecole Nationale des Ponts et Chaussées, Paris, 1998.

[7] *Handbook of mathematical functions (Ed. M. Abramowitz, I.A. Stegun).* Dover Publications, New York, 1970.

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland,

E-mail address: humika@maths.jyu.fi