Menger curvature and Lipschitz parametrizations in metric spaces

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Abstract. We show that pointwise bounds on the Menger curvature imply Lipschitz parametrization for general compact metric spaces. We also give some estimates on the optimal Lipschitz constants of the parametrizing maps for the metric spaces in $\Omega(\varepsilon)$, which is the class of bounded metric spaces E such that the maximum angle for every triple in E is at least $\pi/2 + \arcsin \varepsilon$. Finally we in a certain way extend Peter Jones' travelling salesman theorem to general metric spaces.

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1 Introduction

In this paper E is always a metric space and $d : E \times E \to \mathbb{R}$ is a metric on E. We denote

$$d(E) = \sup\{\, d(x,y) \,:\, x, y \in E \,\},$$

and for $x \in E$ and $r > 0$

$$B(x,r) = \{\, y \in E \,:\, d(y,x) \le r \,\}.$$

Let $\{x,y,z\}$ be a metric triple and i an isometry from $\{x,y,z\}$ to $\mathbb{R}^2$. For $\{x,y,z\}$ the angle at x, denoted by $\sphericalangle yxz$, is the angle at vertex $i(x)$ of the planar triangle whose other vertices are $i(y)$ and $i(z)$. Using the cosine formula we can write

$$\sphericalangle yxz = \arccos \frac{d(x,y)^2 + d(x,z)^2 - d(y,z)^2}{2d(x,y)d(x,z)}.$$

We also denote the maximum angle of $\{x,y,z\}$ by $\max \sphericalangle\{x,y,z\}$. The Menger curvature of the triple $\{x,y,z\}$, denoted by $c(x,y,z)$, is the inverse of the radius of the circle passing through $i(x)$, $i(y)$ and $i(z)$. By elementary plane geometry

$$c(x,y,z) = \frac{2 \sin \sphericalangle xyz}{d(x,z)}, \tag{1}$$

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from which we easily get

$$c(x,y,z)=\frac{\sqrt{(d_1+d_2+d_3)(d_1+d_2-d_3)(d_1-d_2+d_3)(-d_1+d_2+d_3)}}{d_1d_2d_3},$$

where $d_1 = d(x,y)$, $d_2 = d(y,z)$ and $d_3 = d(x,z)$. The condition $c(x,y,z) = 0$ means that the maximum distance in $\{x,y,z\}$ is the same as the sum of the other two distances.

Karl Menger introduced this definition of curvature in [10]. In his terminology a metric space E has at a point p the curvature $K_M(p)$ if $c(x,y,z) \to K_M(p)$ as the distinct points x, y and z converge independently and simultaneously to p. He proved that a simple metric arc Γ such that $K_M(p) = 0$ for all $p \in \Gamma$ and that each subset of four points of Γ is isometric with a subset of $\mathbb{R}^3$ is isometric with a segment of $\mathbb{R}$. Schoenberg showed in [12] that the latter condition in this statement can be replaced by the weaker condition, that for each four points of Γ so called ptolemaic inequality is satisfied.

Menger curvature has turned out to be a useful tool for studying relations between rectifiability, Cauchy integral and analytic capacity. For $z_1, z_2, z_3 \in \mathbb{C}$ we have

$$c(z_1,z_2,z_3)^2=\sum_{\sigma}\frac{1}{(z_{\sigma(1)}-z_{\sigma(3)})\overline{(z_{\sigma(2)}-z_{\sigma(3)})}}, \tag{2}$$

where σ runs through all six permutations of $\{1,2,3\}$. This relation between Menger curvature and the Cauchy kernel $1/z$, $z \in \mathbb{C}$, was found by Melnikov in [8]. We say that $F \subset \mathbb{C}$ is 1-regular if there exists $C < \infty$ such that $C^{-1}r \le \mathcal{H}^1(F \cap B(x,r)) \le Cr$ whenever $x \in F$ and $r \in]0, d(F)[$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. In [7] Mattila, Melnikov and Verdera proved that for a compact 1-regular set $F \subset \mathbb{C}$ the Cauchy singular integral operator is bounded in $L^2(F)$ with respect to the restriction of $\mathcal{H}^1$ to F if and only if F is contained in a 1-regular curve. They first proved, by using earlier work of David and Semmes (see [4]) that the latter condition is satisfied if and only if there exists $M < \infty$ such that

$$\iiint_{(F\cap B)^3} c(z_1,z_2,z_3)^2\, d\mathcal{H}^1 z_1\, d\mathcal{H}^1 z_2\, d\mathcal{H}^1 z_3 \le M d(B)$$

for every ball B in $\mathbb{C}$. Using the indentity (2) they received the final conclusion.

David and Léger have proved that if $F \subset \mathbb{C}$ with $\mathcal{H}^1(F) < \infty$ and

$$\int_F\int_F\int_F c(z_1,z_2,z_3)^2\, d\mathcal{H}^1 z_1\, d\mathcal{H}^1 z_2\, d\mathcal{H}^1 z_3 < \infty,$$

then there are rectifiable curves $\Gamma_1, \Gamma_2, \ldots$ such that

$$\mathcal{H}^1\left(F \backslash \bigcup_{i=1}^{\infty}\Gamma_i\right) = 0.$$

We say that a set is a rectifiable curve if it is the image of a bounded interval under a Lipschitz map. Léger's proof can be found in [6]. David used this theorem when he proved in [3] that if $F \subset \mathbb{C}$ is compact with $\mathcal{H}^1(F) < \infty$ and $\mathcal{H}^1(F \cap \Gamma) = 0$ for every rectifiable curve Γ, then F is removable for bounded analytic functions. The previous conclusion means that for every open set U containing F every bounded analytic function in $U \setminus F$ has an analytic extension to U or, equivalently, every bounded analytic function in $\mathbb{C} \setminus F$ is constant. In [13] Tolsa proved that a compact set $F \subset \mathbb{C}$ is not removable for bounded analytic functions if and only if F supports a positive Radon measure μ such that $\mu(B) \leq d(B)$ for every ball B in $\mathbb{C}$ and

$$\iiint c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3 < \infty.$$

We say that E has the complete property Ω if $\max \sphericalangle\{x, y, z\} > \pi/2$ for every triple $\{x, y, z\} \subset E$. If there is $\alpha > 0$ such that $\max \sphericalangle\{x, y, z\} \geq \pi/2 + \alpha$ for every triple $\{x, y, z\} \subset E$, we say that E has the complete property Ω^* (with a constant α). This means that

$$d(x, z)^2 \geq d(x, y)^2 + d(y, z)^2 + 2d(x, y)d(y, z) \sin \alpha \tag{3}$$

for $\{x, y, z\} \subset E$ whenever $d(x, z) = d(\{x, y, z\})$. We also denote by $\Omega(\varepsilon)$, $0 < \varepsilon \leq 1$, the set of the bounded metric spaces which have the complete property Ω^* with the constant $\arcsin \varepsilon$. We say that E has the property Ω^* at a point $x \in E$, if there exists $\delta_x > 0$ such that $B(x, \delta_x)$ has the complete property Ω^*. If E has the property Ω^* at each of its points, we say that E has the property Ω^*.

Compact connected metric spaces with properties Ω and Ω^* have been studied in [2]. In this paper we prove that pointwise bounds on the Menger curvature imply Lipschitz parametrization for general compact metric spaces. We also give rather sharp estimates on the Lipschitz constants of the parametrizing maps. In Theorem 3.7 we show that for $E \in \Omega(\varepsilon)$ there exist $A \subset [0, 1]$ and a surjective map $f : A \to E$ such that

$$d(E)\frac{\varepsilon}{2}|s - t| \leq d(f(s), f(t)) \leq d(E)\frac{9}{2\varepsilon}|s - t|$$

for all $s, t \in A$.

For $F \subset \mathbb{R}^n$ and a cube $Q \subset \mathbb{R}^n$ set

$$\beta_F(Q) = \inf_L d(Q)^{-1} \sup\{\, d(y, L) \, : \, y \in F \cap 3Q \,\},$$

where the infimum is taken over all lines in $\mathbb{R}^n$ and $3Q$ is the cube with the same center as Q and sides parallel to to the sides of Q, but whose diameter is $3d(Q)$. A cube $Q \subset \mathbb{R}^n$ is a dyadic cube if $Q = \prod_{i=1}^n [k_i 2^{-k}, (k_i + 1)2^{-k}]$, where $k \in \mathbb{Z}$ and $k_i \in \mathbb{Z}$ for $i = 1, \ldots n$. P. W. Jones proved in [5] that a compact $F \subset \mathbb{R}^n$ is contained in a rectifiable curve if

$$\sum_Q \beta_F(Q)^2 d(Q) < \infty, \tag{4}$$

where the sum is taken over all dyadic cubes in $\mathbb{R}^n$. F. Ferrari, B. Franchi and H. Pajot have extended this result to geodesic metric spaces of a certain type. Theorem 5.2 is some kind of an analog in the setting of general metric spaces.

In fact, Jones proved in the case $n = 2$ that for a compact $F \subset \mathbb{R}^n$ the condition (4) is satisfied if and only if F lies in a rectifiable curve. In [11] K. Okikiolu extended this result to general $n \in \mathbb{N}$.

2 Order

We say that an injective map $j : E \to \mathbb{R}$ is an order on E, if for all $x, y, z \in E$ the condition $j(x) < j(y) < j(z)$ implies that $d(x, z) > \max\{d(x, y), d(y, z)\}$.

If $j : E \to \mathbb{R}$ is an order on E and $E' \subset E$, clearly the restriction $j|_{E'} : E' \to \mathbb{R}$ is an order on E'. For $A \subset \mathbb{R}$ a function $j : A \to \mathbb{R}$ is an order if and only if j is strictly increasing or decreasing. If j_1 and j_2 are orders on E then $j_2 = s \circ j_1$, where $s = j_2 \circ j_1^{-1} : j_1(E) \to \mathbb{R}$ is an order on $j_1(E)$. On the other hand, if j is an order on E and $s : j(E) \to \mathbb{R}$ is strictly increasing or decreasing, then $s \circ j$ also is an order on E. If E has an order, we can by the next proof construct one in the following way: Choose $a, b \in E$, $a \neq b$, and set for all $x \in E$

$$j(x) = \begin{cases} -d(x,a) & \text{if } d(x,b) > \max\{d(x,a), d(a,b)\}, \\ d(x,a) & \text{elsewhere.} \end{cases} \tag{5}$$

Then the order j is 2-Lipschitz. If E is compact, we can choose $a, b \in E$ such that $d(a, b) = d(E)$ and we get $j : x \mapsto d(x, a)$. At least a compact metric space, which has an order, is homeomorphic with a subset of $\mathbb{R}$.

For $\{x_1, \ldots, x_n\} \subset E$, $n \in \mathbb{N}$, we will use a notation $x_1 x_2 \ldots x_n$ if there is an order j on $\{x_1, \ldots, x_n\}$ such that $j(x_i) < j(x_{i+1})$ for $i = 1, \ldots, n-1$. Especially a notation xyz will symbolize the relation $d(x, z) > \max\{d(x, y), d(y, z)\}$.

Proposition 2.1. *Let E be a metric space such that each subset of E, which consists of at most four points, has an order. Then the whole space E has an order.*

Proof. Choose $a, b \in E$, $a \neq b$, and define $j : E \to \mathbb{R}$ by formula (5). We check first that j is injective by verifying that $j(x) \neq j(y)$ for all $x, y \in E$ with $x \neq y$. Clearly $j(x) \neq 0 = j(a)$ for $x \neq a$ and $j(x) \neq j(y)$ when $d(x, a) \neq d(y, a)$. Hence we can assume that $x, y \neq a$ and $d(x, a) = d(y, a)$. Let i be an order on $\{a, b, x, y\} \subset E$. Since $d(x, a) = d(y, a)$, we have either $i(x) < i(a) < i(y)$ or $i(y) < i(a) < i(x)$. We can assume that $i(x) < i(a) < i(y)$ is true. If now $i(b) < i(a)$, then $d(x, b) < d(\{x, a, b\})$ and yab. Thus $j(x) = d(x, a) \neq -d(x, a) = -d(y, a) = j(y)$. If $i(b) > i(a)$, we get similarly that $j(x) < 0$ and $j(y) > 0$.

We next show that every subset of E, which consists of five points, has an order. Let $\{x_1, x_2, x_3, x_4, x_5\} \subset E$ be such a set and let $i : \{x_1, x_2, x_3, x_4\} \to \mathbb{R}$ be an order such that $i(x_k) = k$ for $k = 1, 2, 3, 4$. Choose $l, m \in \{1, 2, 3, 4\}$, $l < m$, such that $d(x_5, x_l) \leq d(x_5, x_n)$ and $d(x_5, x_m) \leq d(x_5, x_n)$ for $n \in \{1, 2, 3, 4\} \setminus \{l, m\}$. Now $m = l + 1$. Suppose this is false. Then we have $l < n < m$ with some integer n and $x_l x_n x_m$. Therefore for any order i' on $\{x_l, x_n, x_m, x_5\}$ either $i'(x_l) < i'(x_n) < i'(x_m)$ or $i'(x_m) < i'(x_n) < i'(x_l)$. Thus we have $d(x_5, x_n) < \max\{d(x_5, x_l), d(x_5, x_m)\}$, which contradicts the choice of l and m.

If $x_5 x_l x_{l+1}$, set $p = l - 1/2$. If $x_l x_5 x_{l+1}$, set $p = l + 1/2$. Finally, if $x_5 x_{l+1} x_l$, we set $p = l + 3/2$. Define a function $h : \{x_1, x_2, x_3, x_4, x_5\} \to \mathbb{R}$ by setting $h(x_k) = i(x_k) = k$ for $k = 1, 2, 3, 4$ and $h(x_5) = p$. We claim that h is an order on $\{x_1, x_2, x_3, x_4, x_5\}$. Clearly h is injective. We have to show that for every triple $\{k, m, n\} \subset \{1, 2, 3, 4, 5\}$ the condition $h(x_k) < h(x_m) < h(x_n)$ implies that $x_k x_m x_n$. For $\{l, l+1, 5\}$ this is true by the definition of h. Naturally it suffices to check the triples of indices which contain 5.

If $l = 1$, then in any case $h(x_5) \leq l + 3/2 < h(x_3) < h(x_4)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_1x_3x_4$. Thus for any order i' on $\{x_1, x_3, x_4, x_5\}$ either $i'(x_1) < i'(x_3) < i'(x_4)$ or $i'(x_4) < i'(x_3) < i'(x_1)$. Since $d(x_5, x_1) \leq d(x_5, x_3)$ and i' is an order, necessarily $x_5x_3x_4$.

If $l = 2$, then in any case $h(x_1) < l - 1/2 \leq h(x_5) \leq l + 3/2 < h(x_4)$ Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_1x_2x_4$. Thus for any order i' on $\{x_1, x_2, x_4, x_5\}$ either $i'(x_1) < i'(x_2) < i'(x_4)$ or $i'(x_4) < i'(x_2) < i'(x_1)$. Since $d(x_5, x_2) \leq d(x_5, x_1)$, $d(x_5, x_2) \leq d(x_5, x_4)$ and i' is an order, necessarily $x_1x_5x_4$.

If $l = 3$, then in any case $h(x_5) \geq l - 1/2 > h(x_2) > h(x_1)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_1x_2x_3$. Thus for any order i' on $\{x_1, x_2, x_3, x_5\}$ either $i'(x_1) < i'(x_2) < i'(x_3)$ or $i'(x_3) < i'(x_2) < i'(x_1)$. Since $d(x_5, x_3) \leq d(x_5, x_2)$ and i' is an order, necessarily $x_5x_2x_1$.

Suppose that $l \leq 2$, $k \in \{l+2, 4\}$ and we have $x_5x_lx_{l+1}$ or $x_lx_5x_{l+1}$. Then $h(x_5) \leq l+1/2 < h(x_{l+1}) < h(x_k)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_lx_{l+1}x_k$. Thus for any order i' on $\{x_l, x_{l+1}, x_k, x_5\}$ either $i'(x_l) < i'(x_{l+1}) < i'(x_k)$ or $i'(x_k) < i'(x_{l+1}) < i'(x_l)$. Since $d(x_5, x_l) < d(\{x_5, x_{l+1}, x_l\})$ and i' is an order, necessarily $x_5x_{l+1}x_k$.

Suppose that $l \leq 2$, $k \in \{l+2, 4\}$ and we have $x_lx_5x_{l+1}$ or $x_5x_{l+1}x_l$. Then $h(x_l) < l + 1/2 \leq h(x_5) \leq l + 3/2 < h(x_k)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_lx_{l+1}x_k$. Thus for any order i' on $\{x_l, x_{l+1}, x_k, x_5\}$ either $i'(x_l) < i'(x_{l+1}) < i'(x_k)$ or $i'(x_k) < i'(x_{l+1}) < i'(x_l)$. Since $d(x_5, x_{l+1}) < d(\{x_5, x_l, x_{l+1}\})$, $d(x_5, x_{l+1}) \leq d(x_5, x_k)$ and i' is an order, necessarily $x_lx_5x_k$.

Suppose that $l \leq 2$, $k \in \{l+2, 4\}$ and $x_5x_lx_{l+1}$. Then $h(x_5) < h(x_l) < h(x_k)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_lx_{l+1}x_k$. Thus for any order i' on $\{x_l, x_{l+1}, x_k, x_5\}$ either $i'(x_l) < i'(x_{l+1}) < i'(x_k)$ or $i'(x_k) < i'(x_{l+1}) < i'(x_l)$. Since $x_5x_lx_{l+1}$ and i' is an order, necessarily $x_5x_lx_k$.

Suppose that $l \leq 2$, $k \in \{l+2, 4\}$ and $x_5x_{l+1}x_l$. Then $h(x_{l+1}) < h(x_5) = l + 3/2 < h(x_k)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_lx_{l+1}x_k$. Thus for any order i' on $\{x_l, x_{l+1}, x_k, x_5\}$ either $i'(x_l) < i'(x_{l+1}) < i'(x_k)$ or $i'(x_k) < i'(x_{l+1}) < i'(x_l)$. Since $d(x_5, x_{l+1}) \leq d(x_5, x_k)$, $x_5x_{l+1}x_l$ and i' is an order, necessarily $x_{l+1}x_5x_k$.

Suppose that $l \geq 2$, $k \in \{1, l-1\}$ and we have $x_lx_5x_{l+1}$ or $x_5x_{l+1}x_l$. Then $h(x_5) \geq l+1/2 > h(x_l) > h(x_k)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_kx_lx_{l+1}$. Thus for any order i' on $\{x_k, x_l, x_{l+1}, x_5\}$ either $i'(x_k) < i'(x_l) < i'(x_{l+1})$ or $i'(x_{l+1}) < i'(x_l) < i'(x_k)$. Since $d(x_5, x_{l+1}) < d(\{x_5, x_l, x_{l+1}\})$ and i' is an order, necessarily $x_5x_lx_k$.

Suppose that $l \geq 2$, $k \in \{1, l-1\}$ and we have $x_5x_lx_{l+1}$ or $x_lx_5x_{l+1}$. Then $h(x_{l+1}) > l + 1/2 \geq h(x_5) \geq l - 1/2 > h(x_k)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_kx_lx_{l+1}$. Thus for any order i' on $\{x_k, x_l, x_{l+1}, x_5\}$ either $i'(x_k) < i'(x_l) < i'(x_{l+1})$ or $i'(x_{l+1}) < i'(x_l) < i'(x_k)$. Since $d(x_5, x_l) < d(\{x_5, x_{l+1}, x_l\})$, $d(x_5, x_l) \leq d(x_5, x_k)$ and i' is an order, necessarily $x_kx_5x_{l+1}$.

Suppose that $l \geq 2$, $k \in \{1, l-1\}$ and $x_5x_{l+1}x_l$. Then $h(x_k) < h(x_{l+1}) < h(x_5)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_kx_lx_{l+1}$. Thus for any order i' on $\{x_k, x_l, x_{l+1}, x_5\}$ either $i'(x_k) < i'(x_l) < i'(x_{l+1})$ or $i'(x_{l+1}) < i'(x_l) < i'(x_k)$. Since $x_5x_{l+1}x_l$ and i' is an order, necessarily $x_5x_{l+1}x_k$.

Suppose that $l \geq 2$, $k \in \{1, l-1\}$ and $x_5x_lx_{l+1}$. Then $h(x_k) < h(x_5) < h(x_l)$. Since i is an order on $\{x_1, x_2, x_3, x_4\}$, we have $x_kx_lx_{l+1}$. Thus for any order i' on $\{x_k, x_l, x_{l+1}, x_5\}$ either $i'(x_k) < i'(x_l) < i'(x_{l+1})$ or $i'(x_{l+1}) < i'(x_l) < i'(x_k)$. Since $x_5x_lx_{l+1}$, $d(x_5, x_l) \leq d(x_5, x_k)$ and i' is an order, necessarily $x_kx_5x_l$.

So we have shown that every subset of E, which consists of five points, has an order. Let now $x_1, x_2, x_3 \in E$ and $0 \leq j(x_1) < j(x_2) < j(x_3)$. Let i be an order on $\{a, b, x_1, x_2, x_3\}$ such that $i(a) < i(b)$. Since $d(x_k, b) \leq \max\{d(x_k, a), d(a, b)\}$ for $k = 1, 2, 3$, necessarily $i(x_k) \geq i(a)$ for $k = 1, 2, 3$. Since $d(x_1, a) < d(x_2, a) < d(x_3, a)$, we further have $i(x_1) < i(x_2) < i(x_3)$. This implies that $x_1x_2x_3$.

Let next $x_1, x_2, x_3 \in E$ and $j(x_1) < 0 \leq j(x_2) < j(x_3)$ and let i be an order on $\{a, b, x_1, x_2, x_3\}$ such that $i(a) < i(b)$. Since $d(x_k, b) \leq \max\{d(x_k, a), d(a, b)\}$ for $k = 2, 3$, necessarily $i(x_k) \geq i(a)$ for $k = 2, 3$. Since $d(x_2, a) < d(x_3, a)$, we have $i(x_2) < i(x_3)$. Moreover, $i(x_1) < i(a)$, because x_1ab. So $i(x_1) < i(a) \leq i(x_2) < i(x_3)$, which implies $x_1x_2x_3$.

Let next $x_1, x_2, x_3 \in E$ and $j(x_1) < j(x_2) < 0 \leq j(x_3)$ and let i be an order on $\{a, b, x_1, x_2, x_3\}$ such that $i(a) < i(b)$. Since x_kab for $k = 1, 2$, necessarily $i(x_k) < i(a)$ for $k = 1, 2$. Since $d(x_2, a) < d(x_1, a)$, we have $i(x_1) < i(x_2)$. Moreover $i(x_3) \geq i(a)$, because $d(x_3, b) \leq \max\{d(x_3, a), d(a, b)\}$. So $i(x_1) < i(x_2) < i(a) \leq i(x_3)$, which implies $x_1x_2x_3$.

Finally, let $x_1, x_2, x_3 \in E$ and $j(x_1) < j(x_2) < j(x_3) < 0$ and let i be an order on $\{a, b, x_1, x_2, x_3\}$ such that $i(a) < i(b)$. Since x_kab for $k = 1, 2, 3$, necessarily $i(x_k) < i(a)$ for $k = 1, 2, 3$. Since $d(x_3, a) < d(x_2, a) < d(x_1, a)$, we have $i(x_1) < i(x_2) < i(x_3)$. This implies $x_1x_2x_3$. $\square$

The following two lemmas we are going to use in the chapter 5.

Lemma 2.2. *Let $K \geq 1$ and $\varepsilon \geq K/(K+1)$. Suppose that E is a metric space of four points such that $d(x, y) < Kd(z, w)$ for all $x, y, z, w \in E$, $z \neq w$, and $d(x, z) \geq d(x, y) + \varepsilon d(y, z)$ whenever $x, y, z \in E$ such that $d(x, z) = d(\{x, y, z\})$. Then E has an order or $E = \{x_1, x_2, x_3, x_4\}$ such that $x_1x_2x_3$, $x_2x_1x_4$, $x_2x_3x_4$, $x_1x_4x_3$, $\varepsilon d(x_1, x_2) \leq d(x_3, x_4) \leq \varepsilon^{-1}d(x_1, x_2)$ and $\varepsilon d(x_1, x_4) \leq d(x_2, x_3) \leq \varepsilon^{-1}d(x_1, x_4)$.*

Proof. Let $E = \{x_1, x_2, x_3, x_4\}$ such that $x_1x_2x_3$. We denote $\delta = d(E)/K$ and $d_{ij} = d(x_i, x_j)$ for $i, j = 1, \ldots, 4$. If now $x_ix_jx_k$, we have $d_{ij} \leq d_{ik} - \varepsilon d_{jk} < \delta(K - \varepsilon)$.

Suppose first that $x_1x_2x_4$ and $x_1x_3x_4$. Then $d_{24} - d_{23} \geq d_{14} - d_{12} - d_{23} \geq d_{13} + \varepsilon d_{34} - d_{12} - d_{23} \geq d_{12} + \varepsilon d_{23} + \varepsilon d_{34} - d_{12} - d_{23} > \delta((\varepsilon - 1)K + \varepsilon) \geq 0$ and $d_{24} - d_{34} \geq d_{14} - d_{12} - (d_{14} - \varepsilon d_{13}) \geq \varepsilon(d_{12} + \varepsilon d_{23}) - d_{12} > \delta((\varepsilon - 1)(K - \varepsilon) + \varepsilon^2) = \delta((\varepsilon - 1)K + \varepsilon) \geq 0$. Thus we have $x_1x_2x_3x_4$.

If $x_1x_2x_4$ and $x_1x_4x_3$, then $d_{23} - d_{24} \geq d_{23} - (d_{14} - \varepsilon d_{12}) \geq d_{23} - (d_{12} + d_{23} - \varepsilon d_{34} - \varepsilon d_{12}) > \delta((\varepsilon - 1)K + \varepsilon) \geq 0$ and $d_{23} - d_{34} \geq d_{23} - (d_{12} + d_{23} - \varepsilon d_{14}) \geq \varepsilon(d_{12} + \varepsilon d_{24}) - d_{12} > \delta((\varepsilon - 1)(K - \varepsilon) + \varepsilon^2) \geq 0$, which implies $x_1x_2x_4x_3$.

If $x_1x_4x_2$ and $x_1x_4x_3$, then $d_{34} - d_{24} \geq d_{12} + \varepsilon d_{23} - d_{14} - (d_{12} - \varepsilon d_{14}) > \delta((\varepsilon - 1)K + \varepsilon) \geq 0$ and $d_{34} - d_{23} \geq d_{12} + \varepsilon d_{23} - d_{14} - d_{23} \geq d_{14} + \varepsilon d_{24} + \varepsilon d_{23} - d_{14} - d_{23} > \delta((\varepsilon - 1)K + \varepsilon) \geq 0$, which implies $x_1x_4x_2x_3$.

If $x_2x_1x_4$ and $x_3x_1x_4$, then $d_{34} - d_{24} \geq d_{13} + \varepsilon d_{14} - (d_{12} + d_{14}) \geq d_{12} + \varepsilon d_{23} + \varepsilon d_{14} - (d_{12} + d_{14}) > \delta((\varepsilon - 1)K + \varepsilon) \geq 0$ and $d_{34} - d_{23} \geq d_{12} + \varepsilon d_{23} + \varepsilon d_{14} - d_{23} > \delta((\varepsilon - 1)K + 1 + \varepsilon) > 0$, which implies $x_4x_1x_2x_3$.

Assume now that $x_2x_1x_4$ and $x_1x_4x_3$. Since $d_{24} + \varepsilon d_{34} - d_{23} \geq d_{12} + \varepsilon d_{14} + \varepsilon(d_{12} + \varepsilon d_{23} - d_{14}) - d_{23} > \delta((\varepsilon^2 - 1)K + 1 + \varepsilon) = \delta((\varepsilon - 1)K + 1)(1 + \varepsilon) > 0$ and $d_{24} + \varepsilon d_{23} - d_{34} \geq d_{12} + \varepsilon d_{14} + \varepsilon d_{23} - (d_{12} + d_{23} - \varepsilon d_{14}) > \delta((\varepsilon - 1)K + 2\varepsilon) > 0$, we must have $x_2x_3x_4$. Now $d_{24} = d_{34} + \varepsilon_1 d_{23} = d_{12} + \varepsilon_2 d_{14}$ and $d_{34} = d_{12} + \varepsilon_4 d_{23} - \varepsilon_3 d_{14}$ for some $\varepsilon \leq \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \leq 1$. This gives $(\varepsilon_2 + \varepsilon_3)d_{14} = (\varepsilon_1 + \varepsilon_4)d_{23}$, from which we get $\varepsilon d_{14} \leq d_{23} \leq \varepsilon^{-1}d_{14}$.

The rest of the alternatives are not possible because of the triangle inequality. Namely, $x_1x_2x_4$ and $x_3x_1x_4$ would imply $d_{34} - d_{24} - d_{23} \geq d_{12} + \varepsilon d_{23} + \varepsilon d_{14} - (d_{14} - \varepsilon d_{12}) - d_{23} > \delta(2(\varepsilon - 1)K + 1 + \varepsilon) \geq 0$. If $x_1x_4x_2$ and $x_1x_3x_4$, then $d_{23} - d_{24} - d_{34} \geq d_{23} - (d_{12} - \varepsilon d_{14}) - (d_{14} - \varepsilon(d_{12} + \varepsilon d_{23})) > \delta(2(\varepsilon - 1)K + 1 + \varepsilon^2) \geq 0$. If $x_1x_4x_2$ and $x_3x_1x_4$, then $d_{34} - d_{24} - d_{23} \geq d_{12} + \varepsilon d_{23} + \varepsilon d_{14} - (d_{12} - \varepsilon d_{14}) - d_{23} > \delta((\varepsilon - 1)K + 2\varepsilon) > 0$. If $x_2x_1x_4$ and $x_1x_3x_4$, then $d_{24} - d_{34} - d_{23} \geq d_{14} + \varepsilon d_{12} - (d_{14} - \varepsilon(d_{12} + \varepsilon d_{23})) - d_{23} > \delta((\varepsilon^2 - 1)K + 2\varepsilon) \geq 0$. □

Lemma 2.3. *Let $K \geq 1$ and $\varepsilon^3 \geq (4K-1)/(4K+1)$. Suppose that E is a metric space such that $\#E \neq 4$, $d(x,y) < Kd(z,w)$ for all $x, y, z, w \in E$, $z \neq w$, and $d(x,z) \geq d(x,y) + \varepsilon d(y,z)$ whenever $x, y, z \in E$ such that $d(x,z) = d(\{x,y,z\})$. Then E has an order.*

Proof. We assume that there are at least five points in E. We need to show that every quadruple of E has an order. Suppose that this is not true and let $\{x_1, x_2, x_3, x_4, x_5\} \subset E$ be a subset of five points such that $\{x_1, x_2, x_3, x_4\}$ has no order. By the previous lemma we can assume that $x_1x_2x_3$, $x_2x_1x_4$, $x_2x_3x_4$ and $x_1x_4x_3$. We denote $\delta = d(E)/K$ and $d_{ij} = d(x_i, x_j)$ for $i, j = 1, \ldots, 4$. If now $x_ix_jx_k$, we have $d_{ij} \leq d_{ik} - \varepsilon d_{jk} < \delta(K - \varepsilon)$.

Applying the proof of Lemma 2.2 to quadruples $\{x_1, x_2, x_3, x_5\}$ and $\{x_1, x_4, x_3, x_5\}$, we see that the next eight cases are not possible:

$$\begin{array}{ccc}
x_1x_2x_5 & \text{and} & x_3x_1x_5, \\
x_1x_5x_2 & \text{and} & x_1x_3x_5, \\
x_1x_5x_2 & \text{and} & x_3x_1x_5, \\
x_2x_1x_5 & \text{and} & x_1x_3x_5, \\
x_1x_4x_5 & \text{and} & x_3x_1x_5, \\
x_1x_5x_4 & \text{and} & x_1x_3x_5, \\
x_1x_5x_4 & \text{and} & x_3x_1x_5, \\
x_4x_1x_5 & \text{and} & x_1x_3x_5.
\end{array}$$

Furthermore, $x_2x_1x_5$ and $x_1x_5x_3$ implies $\varepsilon d_{15} \leq d_{23} \leq \varepsilon^{-1}d_{15}$. Similarly, if $x_4x_1x_5$ and $x_1x_5x_3$, we have $\varepsilon d_{15} \leq d_{34} \leq \varepsilon^{-1}d_{15}$.

The next three alternatives are not possible by the triangle inequality: If $x_1x_5x_2$ and $x_1x_5x_4$, then $d_{24} - d_{25} - d_{45} \geq d_{12} + \varepsilon d_{14} - (d_{12} - \varepsilon d_{15}) - (d_{14} - \varepsilon d_{15}) > \delta((\varepsilon - 1)K + 2\varepsilon) > 0$. Similarly, if $x_1x_5x_2$ and $x_1x_4x_5$, then $d_{24} - d_{25} - d_{45} \geq d_{12} + \varepsilon d_{14} - (d_{12} - \varepsilon d_{15}) - (d_{15} - \varepsilon d_{14}) > \delta((\varepsilon - 1)K + 2\varepsilon) > 0$ and if $x_1x_2x_5$ and $x_1x_5x_4$, we have $d_{24} - d_{25} - d_{45} \geq d_{14} + \varepsilon d_{12} - (d_{15} - \varepsilon d_{12}) - (d_{14} - \varepsilon d_{15}) > \delta((\varepsilon - 1)K + 2\varepsilon) > 0$. The alternative $x_2x_1x_5$ and $x_4x_1x_5$ is impossible, because in that case $d_{24} + \varepsilon d_{25} - d_{45} \geq d_{14} + \varepsilon d_{12} + \varepsilon(d_{15} + \varepsilon d_{12}) - (d_{14} + d_{15}) > \delta((\varepsilon - 1)K + \varepsilon + \varepsilon^2) > 0$, $d_{24} + \varepsilon d_{45} - d_{25} \geq d_{12} + \varepsilon d_{14} + \varepsilon(d_{15} + \varepsilon d_{14}) - (d_{12} + d_{15}) > \delta((\varepsilon - 1)K + \varepsilon + \varepsilon^2) > 0$ and $d_{25} + \varepsilon d_{45} - d_{24} \geq d_{12} + \varepsilon d_{15} + \varepsilon(d_{14} + \varepsilon d_{15}) - (d_{12} + d_{14}) > \delta((\varepsilon - 1)K + \varepsilon + \varepsilon^2) > 0$.

By the above examination not more than the next six cases are possible:

$$x_2x_1x_5, \quad x_1x_5x_3, \quad x_1x_4x_5, \tag{6}$$
$$x_2x_1x_5, \quad x_1x_5x_3, \quad x_1x_5x_4, \tag{7}$$
$$x_1x_2x_5, \quad x_1x_5x_3, \quad x_4x_1x_5, \tag{8}$$
$$x_1x_5x_2, \quad x_1x_5x_3, \quad x_4x_1x_5, \tag{9}$$
$$x_1x_2x_5, \quad x_1x_3x_5, \quad x_1x_4x_5, \tag{10}$$
$$x_1x_2x_5, \quad x_1x_5x_3, \quad x_1x_4x_5. \tag{11}$$

From (6) it follows that $d_{45} \leq d_{15} - \varepsilon d_{14} \leq d_{15} - \varepsilon^2 d_{23} \leq d_{15} - \varepsilon^3 d_{15} < (1-\varepsilon^3)(K-\varepsilon)\delta \leq \delta$, which is a contradiction. In the case (7) we would have $d_{45} \leq d_{14} - \varepsilon d_{15} \leq d_{14} - \varepsilon^2 d_{23} \leq d_{14} - \varepsilon^3 d_{14} < \delta$. Similarly, in the case (8) $d_{25} \leq d_{15} - \varepsilon d_{12} \leq d_{15} - \varepsilon^2 d_{34} \leq d_{15} - \varepsilon^3 d_{15} < \delta$ and (9) would imply $d_{25} \leq d_{12} - \varepsilon d_{15} \leq d_{12} - \varepsilon^2 d_{34} \leq d_{12} - \varepsilon^3 d_{12} < \delta$. Thus we must have (10) or (11). Since $d_{24} + \varepsilon d_{25} - d_{45} \geq d_{12} + \varepsilon d_{14} + \varepsilon(d_{15} - d_{12}) - (d_{15} - \varepsilon d_{14}) > \delta((\varepsilon - 1)K + 1 + \varepsilon) > 0$ and $d_{24} + \varepsilon d_{45} - d_{25} \geq d_{12} + \varepsilon d_{14} + \varepsilon(d_{15} - d_{14}) - (d_{15} - \varepsilon d_{12}) > \delta((\varepsilon - 1)K + 1 + \varepsilon) > 0$, we further have $x_2x_5x_4$. Thus $d_{12} + \varepsilon_1 d_{14} = d_{24} = d_{45} + \varepsilon_2 d_{25} = d_{15} - \varepsilon_3 d_{14} + \varepsilon_2(d_{15} - \varepsilon_4 d_{12})$ for some $\varepsilon \leq \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \leq 1$, from which we get

$$\varepsilon(d_{12} + d_{14}) \leq d_{15} \leq \frac{2(d_{12} + d_{14})}{1 + \varepsilon}.$$

Now it follows from (10) that $d_{35} \leq d_{15} - \varepsilon d_{13} \leq 2(1+\varepsilon)^{-1}(d_{12} + d_{14}) - \varepsilon(d_{12} + \varepsilon d_{23}) \leq 2(1+\varepsilon)^{-1}(d_{12} + d_{14}) - \varepsilon(d_{12} + \varepsilon^2 d_{14}) < (4(1+\varepsilon)^{-1} - \varepsilon - \varepsilon^3)(K-\varepsilon)\delta \leq (4(1+\varepsilon)^{-1} - 2\varepsilon^2)(K-\varepsilon)\delta \leq \delta$ and (11) yields $d_{35} \leq d_{13} - \varepsilon d_{15} \leq d_{12} + d_{23} - \varepsilon^2(d_{12} + d_{14}) \leq d_{12} + d_{23} - \varepsilon^2(d_{12} + \varepsilon d_{23}) < (2 - \varepsilon^2 - \varepsilon^3)(K-\varepsilon)\delta \leq \delta$. □

From the previous lemmas we easily get the following result.

Proposition 2.4. *Let E be a metric space such that $c(x,y,z) = 0$ for all $x,y,z \in E$. Then E is isometric with a subset of $\mathbb{R}$ or, alternatively, with some positive numbers a and b, isometric with a set $\{(0,0),(a,0),(0,b),(a,b)\} \subset \mathbb{R}^2$ equipped with a metric d_1, where $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$ for $x = (x_1,x_2), y = (y_1,y_2)$.*

In fact, Menger proved in [9] that a metric space of more than $n+3$ points, for which each of its subsets of $n+2$ points is isometric with a subset of $\mathbb{R}^n$, is isometric with a subset of $\mathbb{R}^n$. He also showed that for each n there is a metric space of $n+3$ points, for which each of its subsets of $n+2$ points is isometric with a subset of $\mathbb{R}^n$, but which is not isometric with a subset of $\mathbb{R}^n$. For the proof see also [1].

3 Metric spaces with property Ω^*

We will first show that a compact metric space with the property Ω^* is a Lipschitz image of a compact set of real numbers.

Lemma 3.1. *Let E be a metric space, which has the complete property Ω^* with a constant $\alpha > 0$, and let*

$$R = \frac{d(E)\sqrt{1+\sin\alpha}}{2\left(\sqrt{2} + \sqrt{1+\sin\alpha}\right)}.$$

Then for all $a \in E$ and $r < R$ there exist $A \subset [0,1]$ and a bijection $f : A \to B(a,r)$ such that

$$d(B(a,r))|s-t| \leq d\left(f(s), f(t)\right) \leq \frac{d(B(a,r))}{\sin\alpha}|s-t|.$$

for all $s, t \in A$.

Proof. Let $a \in E$, $r < R$ and $\varepsilon = \sin\alpha$ (> 0). For every triple $\{x, y, z\} \subset E$ we have by the assumption $d(x,y)^2 \geq d(x,z)^2 + d(y,z)^2 + 2\varepsilon d(x,z)d(y,z)$ whenever $d(x,y) = d(\{x,z,y\})$. Set

$$d' = \left(\frac{\sqrt{2}}{\sqrt{1+\varepsilon}} + 1\right) r.$$

Since $d' < d(E)/2$, there exists $b \in E$ such that $d(a,b) > d'$. Define a function

$$g : B(a,r) \to [d(a,b) - r, d(a,b) + r]$$

by setting $g(x) = d(x,b)$. Let $x, y \in B(a,r)$. Now

$$\begin{aligned} d(x,b)^2 + d(y,b)^2 + 2\varepsilon d(x,b)d(y,b) &\geq 2\left(d(a,b) - r\right)^2 + 2\varepsilon\left(d(a,b) - r\right)^2 \\ &> 2(1+\varepsilon)(d' - r)^2 = 4r^2 \geq d(x,y)^2, \end{aligned}$$

and thus $d(x,y) < d(\{x,b,y\})$. Suppose $d(x,b) \geq d(y,b)$. Since

$$d(x,b)^2 \geq d(y,b)^2 + d(x,y)^2 + 2\varepsilon d(y,b)d(x,y) \geq \left(d(y,b) + \varepsilon d(x,y)\right)^2,$$

we get

$$\varepsilon d(x,y) \leq d(x,b) - d(y,b) = |g(x) - g(y)| \leq d(x,y)$$

and further for $s, t \in g(B(a,r))$

$$|s-t| \leq d\left(g^{-1}(s), g^{-1}(t)\right) \leq \frac{1}{\varepsilon}|s-t|,$$

where $g^{-1} : g(E) \to E$ is the inverse of g. If $B(a,r)$ contains at least two points, we take $A = h^{-1}\left(g(B(a,r))\right) \subset [0,1]$ and $f = g^{-1} \circ h : A \to B(a,r)$, where $h(s) = d(B(a,r))s + \inf g(B(a,r))$. □

Now we get immediately the following result.

Proposition 3.2. *Let E be a compact metric space with the property Ω^*. Then there exist $A \subset [0,1]$ and a Lipschitz surjection $f : A \to E$. (Moreover, f and A can be chosen such that $A = \bigcup_{i=1}^n A_i$, where $n \in \mathbb{N}$, sets A_i are compact and restrictions $f|_{A_i}$ are bi-Lipschitz maps.)*

By the proof of Lemma 3.1 it is clear that we do not have to suppose so much in the previous proposition.

Proposition 3.3. *Let E be a compact metric space and suppose that for all $a \in E$ there are $r > 0$, $\alpha > 0$ and $b \in E \backslash \{a\}$ such that $\max \sphericalangle\{b,c,d\} \geq \pi/2 + \alpha$ for all $c, d \in B(a,r)$. Then there exist $A \subset [0,1]$ and a Lipschitz surjection $f : A \to E$.*

Further we get the following corollaries.

Corollary 3.4. *Let E be a compact metric space and suppose that for all $a \in E$ there is $r > 0$ such that $c(x,y,z)d(x,y) < \sqrt{3}$ for $x,y,z \in B(a,r)$. Then there exist $A \subset [0,1]$ and a Lipschitz surjection $f : A \to E$.*

Proof. By (1), the condition $c(x,y,z)d(x,y) < \sqrt{3}$ implies that $\sin\alpha < \sqrt{3}/2$, where α is the angle at z for the triple $\{x,y,z\}$. So by the assumption, for every $a \in E$ there is $r > 0$ such that $\max \sphericalangle\{x,y,z\} > 2\pi/3$ whenever $x,y,z \in B(a,r)$. Since E has the property Ω^*, the corollary follows from Proposition 3.2. □

Corollary 3.5. *Let E be a compact metric space and suppose that there is $M \in \mathbb{R}$ such that $c(x,y,z) \le M$ for all $x,y,z \in E$. Then there exist $A \subset [0,1]$ and a Lipschitz surjection $f : A \to E$.*

Now we are going to show that every bounded metric space with the complete property Ω^* is a Lipschitz image of a bounded set of real numbers. We also try to estimate the optimal Lipschitz constant. For that reason we use the following lemma.

Lemma 3.6. *Let E be a bounded metric space, which has the complete property Ω^* with a constant $\alpha > 0$, and let*

$$R = \frac{d(E)}{\sqrt{2}\sqrt{1+\sin\alpha}}$$

and $a \in E$. Then $d(x,y) < R$ for all $x,y \in E \backslash B(a,R)$.

Proof. Let $a \in E$ and $x,y \in E\backslash B(a,R)$. Suppose $d(x,y) \ge R$. Then

$$\max\{d(x,a), d(y,a), d(x,y)\} > \sqrt{2}\sqrt{1+\sin\alpha}R = d(E),$$

which is a contradiction. □

Theorem 3.7. *If a bounded metric space E has the complete property Ω^* with a constant $\alpha > 0$, then there exist $A \subset [0,1]$ and a bijective map $f : A \to E$ such that*

$$d(E)\frac{\sin\alpha}{2}|s-t| \le d(f(s), f(t)) \le d(E)\frac{9}{2\sin\alpha}|s-t|$$

for all $s,t \in A$.

Proof. Let $a \in E$, $d = d(E)$, $\varepsilon = \sin\alpha$ and

$$\lambda = \max\left\{ \frac{1}{\sqrt{1+4\varepsilon+2\varepsilon^2}}, \frac{\sqrt{2}}{\sqrt{3+\varepsilon+2\sqrt{2}\varepsilon\sqrt{1+\varepsilon}}} \right\},$$

$$R = \frac{d}{\sqrt{2}\sqrt{1+\varepsilon}}.$$

For $r > 0$ denote

$$E_r = \{\, x \in E \,:\, \lambda r < d(x,a) \le r \,\}$$

and

$$\delta_r = \frac{\lambda r\sqrt{1+\varepsilon}}{\sqrt{2}}.$$

Let $r > 0$ and $x_1, x_2, x_3 \in E_r$. We shall first show that one of the distances $d(x_1, x_2)$, $d(x_1, x_3)$ and $d(x_2, x_3)$ must be less than δ_r. Denote $d_i = d(x_i, a)$ and $d_{ij} = d(x_i, x_j)$ for $i, j = 1, 2, 3$, and suppose $d_1 \le d_2 \le d_3$. Then one or the other inequality is true in the following three pairs of expressions:

$$d_2^2 \ge d_1^2 + d_{12}^2 + 2\varepsilon d_1 d_{12} \tag{12}$$
$$d_{12}^2 \ge d_1^2 + d_2^2 + 2\varepsilon d_1 d_2 \tag{13}$$

$$d_3^2 \ge d_1^2 + d_{13}^2 + 2\varepsilon d_1 d_{13} \tag{14}$$
$$d_{13}^2 \ge d_1^2 + d_3^2 + 2\varepsilon d_1 d_3 \tag{15}$$

$$d_3^2 \ge d_2^2 + d_{23}^2 + 2\varepsilon d_2 d_{23} \tag{16}$$
$$d_{23}^2 \ge d_2^2 + d_3^2 + 2\varepsilon d_2 d_3 \tag{17}$$

At least one of inequalities (12), (14) and (16) must be true. Otherwise, we would have (13), (15) and (17). In that case, the smallest distance in $\{x_1, x_2, x_3\}$ would be at least $\sqrt{d_1^2 + d_2^2 + 2\varepsilon d_1 d_2}$ and another one at least $\sqrt{d_1^2 + d_3^2 + 2\varepsilon d_1 d_3}$. The third distance is, of course, not more than $d_2 + d_3$. Now we have

$$\begin{aligned}
& d_1^2 + d_2^2 + 2\varepsilon d_1 d_2 + d_1^2 + d_3^2 + 2\varepsilon d_1 d_3 \\
& + 2\varepsilon\sqrt{d_1^2 + d_2^2 + 2\varepsilon d_1 d_2}\sqrt{d_1^2 + d_3^2 + 2\varepsilon d_1 d_3} - (d_2 + d_3)^2 \\
& = d_1^2 + 2\varepsilon d_1 d_2 + d_1^2 + 2\varepsilon d_1 d_3 + 2\varepsilon\sqrt{d_1^2 + d_2^2 + 2\varepsilon d_1 d_2}\sqrt{d_1^2 + d_3^2 + 2\varepsilon d_1 d_3} - 2d_2 d_3 \\
& > 2\left((\lambda r)^2 + 2\varepsilon(\lambda r)^2 + 2\varepsilon(1+\varepsilon)(\lambda r)^2 - r^2\right) \\
& = 2r^2\left((1 + 4\varepsilon + 2\varepsilon^2)\lambda^2 - 1\right) \ge 0,
\end{aligned}$$

and thus we would have $\max \sphericalangle\{x_1, x_2, x_3\} < \pi/2 + \alpha$, which is a contradiction. If $x, y \in E_r$ such that $d(x, y) \le \max\{d(x, a), d(a, y)\}$, then

$$d(x, y) < r\left(\sqrt{(\varepsilon^2 - 1)\lambda^2 + 1} - \varepsilon\lambda\right) \le \frac{\lambda r\sqrt{1+\varepsilon}}{\sqrt{2}}$$

by (3) and the choice of λ. Thus $\min\{d_{12}, d_{13}, d_{23}\} < \delta_r$. If $x, y \in E_r$ such that $d(x, y) = d(\{x, a, y\})$, then

$$d(x, y) \ge \sqrt{d(x, a)^2 + d(y, a)^2 + 2\varepsilon d(x, a)d(y, a)} > \sqrt{2(\lambda r)^2 + 2\varepsilon(\lambda r)^2} = \lambda r\sqrt{2}\sqrt{1+\varepsilon}.$$

This means that for all $r > 0$ and $x, y \in E_r$ either $d(x, y) > 2\delta_r$ or $d(x, y) < \delta_r$ and in the latter case $d(x, y) < \min\{d(x, a), d(y, a)\}$. Further E_r has a unique decomposition into two sets A_r and B_r such that $d(A_r) \le \delta_r$, $d(B_r) \le \delta_r$ and $d(A_r, B_r) \ge 2\delta_r$. (If $E_r \ne \emptyset$ we can choose $z \in E_r$ and take $A_r = \{x \in E_r : d(x, z) < \delta_r\}$ and $B_r = E_r \backslash A_r$.)

Set $F_{-1} = E\backslash B(a, R)$ and $G_{-1} = \emptyset$. Define for all k sets F_k and G_k inductively as follows. Let $k \in \mathbb{N}$ and suppose that we have defined the sets F_{k-1} and G_{k-1} such that $E_{\lambda^{k-1}R} = F_{k-1} \cup G_{k-1}$. Suppose first that there exists $\lambda^k R < r_k < \lambda^{k-1} R$ such that $E_{r_k} \cap E_{\lambda^{k-1}R} \neq \emptyset$ and $E_{r_k} \cap E_{\lambda^k R} \neq \emptyset$. Choose points $z_{k-1} \in E_{r_k} \cap E_{\lambda^{k-1}R}$ and $w_k \in E_{r_k} \cap E_{\lambda^k R}$. Then we have the following alternatives:

$$z_{k-1} \in F_{k-1} \quad \textit{and} \quad d(z_{k-1}, w_k) < \delta_{r_k}, \tag{18}$$

$$z_{k-1} \in F_{k-1} \quad \textit{and} \quad d(z_{k-1}, w_k) > 2\delta_{r_k}, \tag{19}$$

$$z_{k-1} \in G_{k-1} \quad \textit{and} \quad d(z_{k-1}, w_k) < \delta_{r_k}, \tag{20}$$

$$z_{k-1} \in G_{k-1} \quad \textit{and} \quad d(z_{k-1}, w_k) > 2\delta_{r_k}. \tag{21}$$

If (18) or (21) is true, set $F_k = \{x \in E_{\lambda^k R} : d(x, w_k) < \delta_{\lambda^k R}\}$ and $G_k = E_{\lambda^k R} \setminus F_k$. Else we put $G_k = \{x \in E_{\lambda^k R} : d(x, w_k) < \delta_{\lambda^k R}\}$ and $F_k = E_{\lambda^k R} \setminus G_k$. If $E_r \cap E_{\lambda^{k-1}R} = \emptyset$ or $E_r \cap E_{\lambda^k R} = \emptyset$ for all $\lambda^k R < r < \lambda^{k-1} R$, we define F_k and G_k arbitrarily such that $E_{\lambda^k R} = F_k \cup G_k$, $d(F_k) \leq \delta_{\lambda^k R}$, $d(G_k) \leq \delta_{\lambda^k R}$ and $d(F_k, G_k) \geq 2\delta_{\lambda^k R}$.

Set $F = \bigcup_{k=-1}^{\infty} F_k$ and $G = \bigcup_{k=0}^{\infty} G_k$. Then $E = F \cup G \cup \{a\}$. Define a function $g : E \to [-R, d]$ by setting

$$g(x) = \begin{cases} -d(x,a) & \text{for } x \in G, \\ d(x,a) & \text{for } x \in F \cup \{a\}. \end{cases}$$

We now show that g is bi-Lipschitz.

If $x, y \in F_k$ or $x, y \in G_k$ for some $k \in \mathbb{N}$, we have by (3)

$$\varepsilon d(x,y) \leq |d(x,a) - d(y,a)| = |g(x) - g(y)| \leq d(x,y), \tag{22}$$

because $d(x,y) \leq \delta_{\lambda^k R}$ implies $d(x,y) < d(\{x,a,y\})$. The same is true for $x, y \in F_{-1}$ by Lemma 3.6. If $x \in F_k$ and $y \in G_k$ for some $k \in \mathbb{N}$, we have

$$1 \leq \frac{d(x,a) + d(y,a)}{d(x,y)} = \frac{|g(x) - g(y)|}{d(x,y)} \leq \frac{2\lambda^k R}{2\delta_{\lambda^k R}} = \frac{\sqrt{2}}{\lambda\sqrt{1+\varepsilon}}.$$

From now on we suppose that $x, y \in E$ such that $d(y,a) \leq d(x,a)$ and $x \neq a$.

If $d(y,a) \leq \lambda d(x,a)$, then

$$\frac{1-\lambda}{1+\lambda} \leq \frac{d(x,a) - d(y,a)}{d(x,a) + d(y,a)} \leq \frac{|g(x) - g(y)|}{d(x,y)} \leq \frac{d(x,a) + d(y,a)}{d(x,a) - d(y,a)} \leq \frac{1+\lambda}{1-\lambda}.$$

Suppose $d(y,a) > \lambda d(x,a)$. Then either $x, y \in F_{-1}$ or $x, y \in E_{\lambda^{k-1}R} \cup E_{\lambda^k R}$ for some $k \in \mathbb{N}$. We have to check the case $x \in E_{\lambda^{k-1}R}$ and $y \in E_{\lambda^k R}$ for some k. Since $d(y,a) \leq d(x,a) \leq d(y,a)/\lambda$, we have $x \in E_{r_k}$ or $y \in E_{r_k}$. We may assume that $x \in E_{r_k}$. Then $x, y, w_k \in E_{d(x,a)}$.

Suppose first that $d(x, z_{k-1}) < \delta_{\lambda^{k-1}R}$, $d(y, w_k) < \delta_{\lambda^k R}$ and $d(z_{k-1}, w_k) < \delta_{r_k}$. Then we have either $x \in F_{k-1}$, $y \in F_k$ and (18) or $x \in G_{k-1}$, $y \in G_k$ and (20). Since $\delta_{\lambda^{k-1}R} < 2\delta_{r_k}$, we have $d(x, z_{k-1}) < \delta_{r_k}$. Thus $d(x, w_k) \leq d(x, z_{k-1}) + d(z_{k-1}, w_k) < 2\delta_{r_k}$ and so $d(x, w_k) < \delta_{r_k}$, because $x, w_k \in E_{r_k}$. Since $\delta_{r_k} < 2\delta_{d(x.a)}$ and $x, w_k \in E_{d(x,a)}$ we further have $d(x, w_k) < \delta_{d(x,a)}$. Therefore $d(x,y) \leq d(x, w_k) + d(y, w_k) < \delta_{d(x,a)} + \delta_{\lambda^k R} < 2\delta_{d(x,a)}$. Since $x, y \in E_{d(x,a)}$, we have $d(x,y) < \delta_{d(x,a)}$ and (22).

Suppose now that $d(x, z_{k-1}) < \delta_{\lambda^{k-1}R}$, $d(y, w_k) < \delta_{\lambda^k R}$ and $d(z_{k-1}, w_k) > 2\delta_{r_k}$. Then we have either $x \in F_{k-1}$, $y \in G_k$ and (19) or $x \in G_{k-1}$, $y \in F_k$ and (21). Since

$\delta_{\lambda^{k-1}R} < 2\delta_{r_k}$, we have $d(x, z_{k-1}) < \delta_{r_k}$. Thus $d(x, w_k) \geq d(w_k, z_{k-1}) - d(x, z_{k-1}) > \delta_{r_k}$ and so $d(x, w_k) > 2\delta_{r_k}$, because $x, w_k \in E_{r_k}$. Therefore $d(x, y) \geq d(x, w_k) - d(y, w_k) > 2\delta_{r_k} - \delta_{\lambda^k R} > \delta_{d(x,a)}$. Since $x, y \in E_{d(x,a)}$, we have $d(x, y) > 2\delta_{d(x,a)}$ and

$$1 \leq \frac{d(x,a) + d(y,a)}{d(x,y)} = \frac{|g(x) - g(y)|}{d(x,y)} \leq \frac{2d(x,a)}{2\delta_{d(x,a)}} = \frac{\sqrt{2}}{\lambda\sqrt{1+\varepsilon}}.$$

The other cases can be treated similarly. The inequality

$$\frac{1-\lambda}{1+\lambda} \leq \frac{|g(x) - g(y)|}{d(x,y)} \leq \frac{1+\lambda}{1-\lambda}$$

holds for all $x, y \in E$. Thus we get $A \subset [0,1]$ and a surjection $f : A \to E$ such that

$$d(E)\frac{1+\sqrt{2}\sqrt{1+\varepsilon}}{\sqrt{2}\sqrt{1+\varepsilon}}\frac{1-\lambda}{1+\lambda}|s-t| \leq d(f(s), f(t)) \leq d(E)\frac{1+\sqrt{2}\sqrt{1+\varepsilon}}{\sqrt{2}\sqrt{1+\varepsilon}}\frac{1+\lambda}{1-\lambda}|s-t|$$

and further the estimate

$$d(E)\frac{\varepsilon}{2}|s-t| \leq d(f(s), f(t)) \leq d(E)\frac{9}{2\varepsilon}|s-t|$$

for all $s, t \in A$. □

We now give an example of a compact and connected metric space, which has an order and the complete property Ω, but which is not a Lipschitz image of a bounded set of real numbers.

Example 3.8. Let $1 < p < 2$ and $x \in \ell^p \backslash \ell^1$, $x = (x_k)_{k=1}^{\infty}$, where $x_k \geq 0$ for every k. Set

$$E = \left\{ \sum_{k=1}^{n} x_k e_k + t e_{n+1} \ : \ n \in \mathbb{N}, t \in [0, x_{n+1}] \right\} \cup \{ x \} \subset \ell^p,$$

where $\{e_k,\ k \in \mathbb{N}\}$ is the standard base of ℓ^p. Now $j : y \mapsto \|y\|_p$ is an order on E and maps E onto $[x_1, \|x\|_p]$. We check that every triple in E contains an obtuse angle. Let

$$a_1 = \sum_{k=1}^{n_1} x_k e_k \quad a_2 = \sum_{k=1}^{n_2} x_k e_k + t e_{n_2+1} \quad \text{and} \quad a_3 = \sum_{k=1}^{n_3} x_k e_k,$$

where $\|a_1\|_p < \|a_2\|_p < \|a_3\|_p$, $n_1 \leq n_2 < n_3 \leq \infty$, $t \in [0, x_{n_2+1}]$. Set

$$A = \sum_{k=n_1+1}^{n_2} x_k^p \quad \text{and} \quad B = \sum_{k=n_2+2}^{n_3} x_k^p.$$

Now

$$\begin{aligned} d(a_1, a_3)^2 - d(a_1, a_2)^2 - d(a_2, a_3)^2 &= \|a_1 - a_3\|_p^2 - \|a_1 - a_2\|_p^2 - \|a_2 - a_3\|_p^2 \\ &= (A + x_{n_2+1}^p + B)^{2/p} - (A + t^p)^{2/p} - ((x_{n_2+1} - t)^p + B)^{2/p} \\ &\geq (A + x_{n_2+1}^p + B)^{2/p} - (A + t^p)^{2/p} - (x_{n_2+1}^p - t^p + B)^{2/p} > 0, \end{aligned}$$

because $(a+b)^s > a^s + b^s$ for $a, b > 0$ and $s > 1$. So for $\{a_1, a_2, a_3\}$ the angle at a_2 is obtuse. However, even

$$E' = \left\{ \sum_{k=1}^{n} x_k e_k \; : \; n \in \mathbb{N} \right\} \subset E$$

can not be a Lipschitz image of a bounded set of real numbers when $x \notin \ell^1$. Namely, if $A \subset [0,1]$ and $f : A \to E'$ is a Lipshitz map such that

$$\left\{ \sum_{k=1}^{n} x_k e_k \; : \; n = 1, \ldots, n_0 \right\} \subset f(A),$$

the Lipschitz constant of f must be at least $\sum_{k=2}^{n_0} x_k$.

4 Connected, ordered and ptolemaic spaces

Let $\mathcal{M}$ be the collection of all the bounded metric spaces. For $E \in \mathcal{M}$ we denote

$$l(E) = \inf\{ \operatorname{Lip}(f) \; : \; f : A \to E \text{ is a surjection and } A \subset [0,1] \},$$

where $\operatorname{Lip}(f) \in [0, \infty]$ is the Lipschitz constant of f. For $\mathcal{A} \subset \mathcal{M}$ we set

$$\tilde{L}(\mathcal{A}) = \sup\{ l(E)/d(E) \; : \; E \in \mathcal{A} \}.$$

Further for $0 < \varepsilon \leq 1$ and $\mathcal{A} \subset \mathcal{M}$ we put $L(\varepsilon, \mathcal{A}) = \tilde{L}(\Omega(\varepsilon) \cap \mathcal{A})$. Clearly $\varepsilon \mapsto L(\varepsilon, \mathcal{A})$ is a decreasing function on $]0,1]$ for fixed $\mathcal{A} \subset \mathcal{M}$ and $L(\varepsilon, \mathcal{A}) \leq L(\varepsilon, \mathcal{B})$ if $\mathcal{A} \subset \mathcal{B} \subset \mathcal{M}$. By Proposition 2.4 we have $L(1, \mathcal{M}) = 3/2$ and $L(\varepsilon, \mathcal{M}) \leq C/\varepsilon$ with some constant $C > 0$ by Theorem 3.7.

We denote by $\mathcal{C}$ the collection of the connected metric spaces and let $\mathcal{O}$ be the collection of metric spaces which have an order. We next show that $L(\varepsilon, \mathcal{C}) = L(\varepsilon, \mathcal{O}) = 1/\varepsilon$ for $0 < \varepsilon \leq 1$.

Lemma 4.1. *Let E be a connected metric space such that $\max \sphericalangle\{x, y, z\} \geq \pi/2$ for every triple $\{x, y, z\} \subset E$. If $f : E \to \mathbb{R}$ is a homeomorphism to its image, then f is an order.*

Proof. We can of course assume that E contains more than one point. Suppose that f is not an order. Then there exists $\{x, y, z\} \subset E$ such that

$$f(x) < f(y) < f(z) \tag{23}$$

and $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. We can assume xzy, because $\max \sphericalangle\{x, y, z\} \geq \pi/2$. Define $g : f(E) \to \mathbb{R}$ by setting $g(a) = d(x, f^{-1}(a))$, where f^{-1} is the inverse of f. Now g is continuous. Since $f(E)$ is connected, $[f(x), f(y)] \subset f(E)$. Let $b \in [f(x), f(y)]$ such that $g(b) = \max\{g(a) : a \in [f(x), f(y)]\}$. Now $b \in]f(x), f(y)[$ because of (23). Now

$$d(x, f^{-1}(c)) = g(c) \uparrow g(b) = d(x, f^{-1}(b))$$

as $c \uparrow b$ and

$$d(x, f^{-1}(e)) = g(e) \uparrow g(b) = d(x, f^{-1}(b))$$

as $e \downarrow b$, where $d(x, f^{-1}(b)) > 0$. Further by the continuity of f^{-1} we simultaneously have $d(f^{-1}(c), f^{-1}(e)) \downarrow 0$. From this we conclude that E contain a triple whose maximum angle is less than $\pi/2$, which is a contradiction. $\square$

Proposition 4.2. $L(\varepsilon, \mathcal{C}) = L(\varepsilon, \mathcal{O}) = 1/\varepsilon$ *for all* $\varepsilon \in]0,1]$.

Proof. Let $0 < \varepsilon \leq 1$. By Theorem 3.7 and Lemma 4.1 we have $L(\varepsilon, \mathcal{C}) \leq L(\varepsilon, \mathcal{O})$. Clearly $L(\varepsilon, \mathcal{O}) \leq 1/\varepsilon$. Namely, if $E \in \Omega(\varepsilon) \cap \mathcal{O}$, it follows from Theorem 3.7 that the completion of E is compact. Since clearly also the completion of E is in $\Omega(\varepsilon) \cap \mathcal{O}$, we may assume that E is compact. Take $a, b \in E$ such that $d(a,b) = d(E)$ and define a function $g : E \to [0, d(E)]$ by setting $g(x) = d(x,a)$. As before, we see by (3) that the inverse of g is $1/\varepsilon$-Lipshitz from $d(E)$ to E.

We are left to show $L(\varepsilon, \mathcal{C}) \geq 1/\varepsilon$. We define a metric d on an interval $[0, N]$, $N \in \mathbb{N}$, as follows: Define real numbers r_k, $k = 0, 1, \dots$ by setting $r_0 = 0$ and

$$r_{k+1} = \sqrt{{r_k}^2 + 1 + 2\varepsilon r_k}.$$

Let $x, y \in [0, N]$ with $x < y$. If $\mathbb{N} \cap [x,y] = \emptyset$, we set $d(x,y) = |x - y|$. Else we put $m = \inf(\mathbb{N} \cap [x,y])$, $M = \sup(\mathbb{N} \cap [x,y])$, $s = \min\{y - M, m - x\}$ and $t = \max\{y - M, m - x\}$. Then we set

$$d(x,y) = \sqrt{u^2 + s^2 + 2\varepsilon us},$$

where

$$u = \sqrt{{r_{M-m}}^2 + t^2 + 2\varepsilon r_{M-m} t}.$$

Denote this metric space by E_N. Now $E_N \in \Omega(\varepsilon) \cap \mathcal{C}$. Since $r_{k+1} - r_k \to \varepsilon$ as $k \to \infty$, we have

$$\frac{l(E_N)}{d(E_N)} = \frac{N}{r_N} \to \frac{1}{\varepsilon}$$

as $N \to \infty$. $\square$

We say that a metric space E has the four-point property if any subset of four points of E is isometric with some subset of $\mathbb{R}^3$. E is called ptolemaic provided for all $x, y, z, w \in E$ the inequality $d(x,y)d(z,w) + d(x,z)d(y,w) \geq d(x,w)d(y,z)$ is true. Denote the set of the metric spaces with four-point property by $\mathcal{F}$ and the set of the ptolemaic metric spaces by $\mathcal{P}$. Since $\mathbb{R}^3$ is ptolemaic, we have $\mathcal{F} \subset \mathcal{P}$. It is easy to construct metric spaces, which are ptolemaic but which do not have the four-point property. For example we can take a quadruple such that one distance between points equals 2 while the other five distances are 1. Since $\mathcal{F} \cap \Omega(\sqrt{3}/2) \subset \mathcal{O}$, we have at least $L(\varepsilon, \mathcal{F}) \leq 1/\varepsilon$ for $\sqrt{3}/2 \leq \varepsilon \leq 1$. We now show that $L(\varepsilon, \mathcal{P}) \downarrow 1$ as $\varepsilon \uparrow 1$.

Lemma 4.3. *Let E be a ptolemaic metric space with the complete property Ω. Then* $\min\{d(x,y), d(z,w)\} < \max\{d(x,z), d(x,w), d(y,z), d(y,w)\}$ *for each four pairwise distinct points* $x, y, z, w \in E$.

Proof. If opposite is true, then

$$\begin{aligned} d(x,y)^2 d(z,w)^2 &> \frac{(d(x,z)^2 + d(y,w)^2 + d(x,w)^2 + d(y,z)^2)^2}{4} \\ &\geq \frac{(2d(x,z)d(y,w) + 2d(x,w)d(y,z))^2}{4} \\ &= (d(x,z)d(y,w) + d(x,w)d(y,z))^2, \end{aligned}$$

which means that E is not ptolemaic. $\square$

Proposition 4.4. *$L(\varepsilon, \mathcal{P}) \downarrow 1$ as $\varepsilon \uparrow 1$.*

Proof. Let now $E \in \Omega(\varepsilon) \cap \mathcal{P}$. It follows from Theorem 3.7 that the completion of E is compact. Since clearly also the completion of E is in $\Omega(\varepsilon) \cap \mathcal{P}$, we can assume that E is compact. Let $a, b \in E$ such that $d(a, b) = d(E)$. We define a function $g : E \to [0, d(E)]$ by setting $g(x) = d(x, a)$. Let $x, y \in E \backslash \{a, b\}$. If $d(x, y) \leq \max\{d(x, a), d(a, y)\}$, then we have $|g(x) - g(y)| \geq \varepsilon d(x, y)$ by (3). Suppose xay. By the previous lemma $d(x, y) \leq \max\{d(x, b), d(b, y)\}$. We may assume that $d(y, b) > d(x, b)$. Denote $d = d(E)$, $p = d(x, a)$, $q = d(y, a)$, $r = d(x, y)$, $s = d(y, b)$ and $t = d(x, b)$. Now $d^2 \geq q^2 + s^2 + 2\varepsilon qs$, which gives

$$s \leq \sqrt{d^2 + (\varepsilon^2 - 1)q^2} - \varepsilon q.$$

We also have $t \geq d - p$ and $r^2 \geq q^2 + p^2 + 2\varepsilon qp$. Thus we get

$$\begin{aligned}
&s^2 - t^2 - r^2 \\
&\leq d^2 + (2\varepsilon^2 - 1)q^2 - 2\varepsilon q\sqrt{d^2 + (\varepsilon^2 - 1)q^2} - d^2 - p^2 + 2dp - q^2 - p^2 - 2\varepsilon qp \\
&= 2\left[(\varepsilon^2 - 1)q^2 - p^2 + dp - \varepsilon qp - \varepsilon q\sqrt{d^2 + (\varepsilon^2 - 1)q^2}\right] \leq 2p(d - p)
\end{aligned}$$

and further

$$\varepsilon \leq \frac{s^2 - t^2 - r^2}{2rt} \leq \frac{p}{\sqrt{q^2 + p^2 + 2\varepsilon qp}}.$$

This yields $p \geq \varepsilon(\varepsilon q + p)$, which gives $q \leq p(1 - \varepsilon)/\varepsilon^2$. Thus

$$\frac{|g(x) - g(y)|}{d(x, y)} = \frac{p - q}{r} \geq \frac{p - q}{p + q} \geq \frac{\varepsilon^2 + \varepsilon - 1}{\varepsilon^2 - \varepsilon + 1},$$

and we get

$$L(\varepsilon, \mathcal{P}) \leq \max\left\{\frac{\varepsilon^2 - \varepsilon + 1}{\varepsilon^2 + \varepsilon - 1}, \frac{1}{\varepsilon}\right\} = \frac{\varepsilon^2 - \varepsilon + 1}{\varepsilon^2 + \varepsilon - 1},$$

when $(\sqrt{5} - 1)/2 < \varepsilon \leq 1$. Therefore $L(\varepsilon, \mathcal{P}) \downarrow 1$ as $\varepsilon \uparrow 1$. □

5 Travelling salesman theorem

Let E be a bounded metric space and let $C_1 \geq C_2 > 115680$. For any $x \in E$ and $t > 0$ we set

$$\beta(x, t) = \sup\{\, c(z_1, z_2, z_3) \;:\; z_1, z_2, z_3 \in B(x, t),\; d(z_i, z_j) \geq C_1^{-1}t \;\forall i \neq j \,\}.$$

We say that an increasing sequence $(\Delta_k)_{k \in \mathbb{Z}}$ of subsets of E is a net of E if for all $k \in \mathbb{Z}$

(i) for any $x, y \in \Delta_k$, $x \neq y$, $d(x, y) > 2^{-k}$,

(ii) for any $x \in E$ there exists $y \in \Delta_k$ such that $d(x, y) \leq 2^{-k}$.

Now we define

$$\beta(E) = \inf \left\{ \sum_{k\in\mathbb{Z}} \sum_{x\in\Delta_k} \beta(x, C_2 2^{-k})^2 (2^{-k})^3 \ : \ (\Delta_k)_k \text{ is a net of } E \right\}.$$

For $F \subset \mathbb{R}^n$ the conditions $\beta(F) < \infty$ and (4) are equivalent. We are going to show that for any bounded metric space E the condition $\beta(E) < \infty$ implies that E is a Lipschitz image of a bounded set of real numbers.

Lemma 5.1. *Let $x, y, z \in \mathbb{R}^2$ be distinct points, L_{yz} the line passing through y and z and $P : \mathbb{R}^2 \to L_{yz}$ the orthogonal projection to L_{yz}. Denote $d_{yx} = |y - x|$, $d_{xz} = |x - z|$ and $d_{yz} = |y - z|$. If $d_{yz} = |y - P(x)| + |P(x) - z|$, then*

$$\frac{c(x,y,z)^2}{8} \le \frac{d_{yx} + d_{xz} - d_{yz}}{d_{yx} d_{xz}(d_{yx} + d_{xz})} \le \frac{c(x,y,z)^2}{4}.$$

Proof. Denote $s = |y - P(x)|$, $t = |P(x) - z|$ and $h = |x - P(x)|$. By the Pythagorean Theorem

$$d_{yx} + d_{xz} - d_{yz} = d_{yx} - s + d_{xz} - t = \frac{d_{yx}^2 - s^2}{d_{yx} + s} + \frac{d_{xz}^2 - t^2}{d_{xz} + t} = h^2 \left(\frac{1}{d_{yx} + s} + \frac{1}{d_{xz} + t} \right)$$

and by (1)

$$c(x,y,z) = \frac{2h}{d_{yx} d_{xz}}.$$

Hence

$$d_{yx} + d_{xz} - d_{yz} = \frac{c(x,y,z)^2 d_{yx}^2 d_{xz}^2}{4} \left(\frac{1}{d_{yx} + s} + \frac{1}{d_{xz} + t} \right),$$

from which we get the conclusion. $\square$

Theorem 5.2. *Let E be a bounded metric space such that $\beta(E) < \infty$. Then there exist $A \subset [0,1]$ and a Lipschitz surjection $f : A \to E$. Moreover, f can be chosen such that* $\mathrm{Lip}(f) \le C(\beta(E) + d(E))$, *where $C > 0$ is an absolute constant.*

Proof. Let $(\Delta_k)_{k\in\mathbb{Z}}$ be a net of E such that $\sum_k \sum_{x\in\Delta_k} \beta(x, C_2 2^{-k})^2 (2^{-k})^3 < \infty$ and let C_3, C_4 and ε_0 be positive constants such that $C_3 \ge 9$, $C_4 > 24(1+C_3)$, $C_2 \ge 2C_4(1+C_4)$ and $C_4(1+C_4)\left(1 + C_4(1+C_4)\right)\varepsilon_0 \le \sqrt{2C_4(1+C_4)+1}$.

Suppose that $\beta(x, C_2 2^{-k}) 2^{-k} < \varepsilon \le \sqrt{3}/4$ for some $x \in D_{y,k}$, where $y \in E$, $k \in \mathbb{Z}$ and $D_{y,k} = B(y, 2^{-k+1}) \cap \Delta_k$. Since $C_1 \ge C_2 \ge 4$, we have $d(z_1, z_2) \ge C_1^{-1} C_2 2^{-k}$ for all $z_1, z_2 \in D_{y,k} \subset B(x, C_2 2^{-k})$. By (1)

$$\sin \sphericalangle z_1 z_2 z_3 \le \frac{d(D_{y,k}) \sin \sphericalangle z_1 z_2 z_3}{d(z_1, z_3)} \le \frac{d(D_{y,k})}{2} c(z_1, z_2, z_3) < 2^{k-1} d(D_{y,k}) \varepsilon \le 2\varepsilon \le \frac{\sqrt{3}}{2}$$

for any triple $\{z_1, z_2, z_3\} \subset D_{y,k}$. So $D_{y,k} \in \Omega(\sqrt{1 - 4\varepsilon^2})$ and further $\# D_{y,k} \le 18/\sqrt{1-4\varepsilon^2} + 1$ by Theorem 3.7. Moreover, if $289(1 - 4\varepsilon^2)^3 \ge 225$, we have by Lemma 2.3 and Proposition 4.2 that $\# D_{y,k} \le 4/\sqrt{1 - 4\varepsilon^2} + 1 < 6$. Especially Δ_k is finite for each k. Let

$$\bigcup_{k\in\mathbb{Z}} \Delta_k = \{ x_1, x_2, x_3, \dots \}$$

such that for all $k \in \mathbb{Z}$

$$D_{\#\Delta_k} = \Delta_k,$$
$$d\left(x_{j+1}, D_j\right) = \max\left\{ d\left(x, D_j\right) \,:\, x \in \Delta_k \right\} \text{ for } j = 1, \ldots, \#\Delta_k - 1,$$

where $D_j = \{x_1, \ldots, x_j\}$ for $j \in \mathbb{N}$.

We are going to construct a sequence (G_j) of connected weighted graphs with no cycles. For each j we denote by V_j and E_j the sets of the vertices and the edges of G_j. For each j we have an injection $g_j : D_j \to V_j$. For all $x, y \in D_j$ such that $\{g_j(x), g_j(y)\} \in E_j$ we will have $w_j(\{g_j(x), g_j(y)\}) = d(x,y)$, where $w_j : E_j \to]0, \infty[$ is the weight function on the graph G_j. We denote $l(G_j) = \sum_{e \in E_j} w_j(e)$ and for $y \in D_j$ we will use the notation

$$N_j(y) = \{ z \in D_j \,:\, \{g_j(y), g_j(z)\} \in E_j \}.$$

Each vertex in $V_j \backslash g_j(D_j)$ will have only one neighbour. Thus the subgraph of G_j induced by $g_j(D_j)$ will also be connected. We will denote this graph and the set of its edges by G_j^* and E_j^*. In our construction the number $l(G_j^*) = \sum_{e \in E_j^*} w_j(e)$ will remain bounded, from which we get the final conclusion.

We define a graph G_2 with 4 vertices and 3 edges as follows. Put $V_2 = \{a_1, a_2, b_1, b_2\}$ and define $g_2 : D_2 \to V_2$ by setting $g_2(x_i) = a_i$ for $i = 1, 2$. Further we set $E_2 = \{\{a_1, a_2\}, \{a_1, b_1\}, \{a_2, b_2\}\}$, $w_2(\{a_1, a_2\}) = d(x_1, x_2)$ and $w_2(\{a_i, b_i\}) = C_3 d(x_1, x_2)$ for $i = 1, 2$. Now

$$l(G_2) \leq (1 + 2C_3) d(E). \tag{24}$$

We set $I_2 = \{0, d(x_1, x_2)\}$ and define $h_2 : I_2 \to D_2$ by setting $h_2(0) = x_1$ and $h_2(d(x_1, x_2)) = x_2$.

Let now $j \geq 2$ and assume by induction that we have constructed a graph $G_j = (V_j, E_j)$, $w_j : E_j \to]0, \infty[$, an injection $g_j : D_j \to V_j$ and a 1-Lipschitz surjection $h_j : I_j \to D_j$, where $I_j \subset [0, 2l(G_j^*)]$. We also assume that G_j and h_j satisfy the following properties:

(*) Let $y \in D_j$. If $d(y, z_1) < C_4 d(y, z_2)$ and $d(z_1, z_2) < d(\{z_1, y, z_2\})$ for all $z_1, z_2 \in N_j(y)$, then there exists $b \in V_j \backslash g_j(D_j)$ such that $\{g_j(y), b\} \in E_j$.

(**) If $z_1, z_2 \in D_j$ such that $\{g_j(z_1), g_j(z_2)\} \in E_j$, then there exist $s_1, s_2 \in I_j$, such that $s_2 - s_1 = d(z_1, z_2)$, $h_j(\{s_1, s_2\}) = \{z_1, z_2\}$ and $I_j \cap]s_1, s_2[= \emptyset$.

We denote $x = x_{j+1}$. Let y be a nearest neighbour of x in D_j and let k be the smallest integer such that $x \in \Delta_k$. In other words $\#\Delta_{k-1} \leq j < \#\Delta_k$.

Case 1. $\beta\left(x, C_2 2^{-k}\right) 2^{-k} \geq \varepsilon_0$.

We set $V_{j+1} = V_j \cup \{a, b\}$, where $a \neq b$, $V_j \cap \{a, b\} = \emptyset$, and define $g_{j+1} : D_{j+1} \to V_{j+1}$ by setting $g_{j+1}(x) = a$ and $g_{j+1}(v) = g_j(v)$ for $v \in D_j$. Further we define

$$E_{j+1} = E_j \cup \{\{g_j(y), a\}, \{a, b\}\}$$

and $w_{j+1} : E_{j+1} \to]0, \infty[$ by setting

$$w_{j+1}(e) = \begin{cases} d(y, x) & \text{for } e = \{g_j(y), a\}, \\ C_3 d(y, x) & \text{for } e = \{a, b\}, \\ w_j(e) & \text{for } e \in E_j. \end{cases}$$

Now G_{j+1} satisfies the property (*) and

$$l(G_{j+1}) - l(G_j) = (1+C_3)d(y,x) \le (1+C_3)2^{-(k-1)} \le \frac{2(1+C_3)}{\varepsilon_0^2}\beta\left(x, C_2 2^{-k}\right)^2 \left(2^{-k}\right)^3. \tag{25}$$

Let $t \in I_j$ such that $h_j(t) = y$. We set

$$I_{j+1} = J_1 \cup \{t + d(y,x)\} \cup J_2,$$

where $J_1 = I_j \cap [0,t]$ and $J_2 = (I_j \cap [t,\infty[) + 2d(y,x)$, and define $h_{j+1} : I_{j+1} \to D_{j+1}$ by setting

$$h_{j+1}(s) = \begin{cases} h_j(s) & \text{for } s \in J_1, \\ x & \text{for } s = t + d(y,x), \\ h_j(s - 2d(y,x)) & \text{for } s \in J_2. \end{cases}$$

Now (**) is satisfied, $I_{j+1} \subset [0, 2l(G^*_{j+1})]$ and h_{j+1} is surjective and 1-Lipschitz.

For the rest of the cases we assume that $\beta\left(x, C_2 2^{-k}\right) 2^{-k} < \varepsilon_0$.

Case 2. There exists $z \in N_j(y)$ such that $C_4 d(y,x) \le d(y,z)$.

We define G_{j+1}, g_{j+1}, I_{j+1} and h_{j+1} as in the case 1. Now

$$l(G_{j+1}) - l(G_j) = (1+C_3)d(y,x) \le \frac{1+C_3}{C_4}d(y,z). \tag{26}$$

By the construction $\{g_j(y), g_j(z)\} \in E^*_m$ for all $m \ge j$.

For the rest of the cases we assume that $d(y,z) < C_4 d(y,x)$ for all $z \in N_j(y)$.

Case 3. There exists $z \in N_j(y)$ such that $d(x,z) \le d(y,z)$.

We set $V_{j+1} = V_j \cup \{a\}$, where $a \notin V_j$, and define $g_{j+1} : D_{j+1} \to V_{j+1}$ by setting $g_{j+1}(x) = a$ and $g_{j+1}(v) = g_j(v)$ for $v \in D_j$. Further we define

$$E_{j+1} = (E_j \setminus \{\{g_j(y), g_j(z)\}\}) \cup \{\{g_j(y), a\}, \{a, g_j(z)\}\}$$

and $w_{j+1} : E_{j+1} \to]0,\infty[$ by setting

$$w_{j+1}(e) = \begin{cases} d(y,x) & \text{for } e = \{g_j(y), a\}, \\ d(x,z) & \text{for } e = \{a, g_j(z)\}, \\ w_j(e) & \text{for } e \in E_j \setminus \{\{g_j(y), g_j(z)\}\}. \end{cases}$$

By Lemma 5.1

$$\begin{aligned} l(G_{j+1}) - l(G_j) &= d(y,x) + d(x,z) - d(y,z) \\ &\le \frac{c(y,x,z)^2}{4} d(y,x)d(x,z)\left(d(y,x) + d(x,z)\right) \\ &\le \frac{C_4(1+C_4)}{4} c(y,x,z)^2 (2^{-(k-1)})^3 \\ &\le 2C_4(1+C_4)\beta\left(x, C_2 2^{-k}\right)^2 \left(2^{-k}\right)^3. \end{aligned} \tag{27}$$

The last inequality holds, because $C_1 \geq C_2 \geq 2C_4$.

We next show that G_{j+1} satisfies the property (*) at z. Suppose that $\{g_{j+1}(z), b\} \notin E_{j+1}$ for all $b \in V_{j+1} \backslash g_{j+1}(D_{j+1})$, which implies that $\{g_j(z), b\} \notin E_j$ for all $b \in V_j \backslash g_j(D_j)$. By (*) $N_j(z) \backslash \{y\} \neq \emptyset$. Suppose that $d(z,v) < C_4 d(x,z)$ for all $v \in N_j(z)$. Then $C_4^{-1} d(y,z) < d(y,x) \leq d(z,v) < C_4 d(x,z) \leq C_4 d(y,z)$ for all $v \in N_j(z)$. If now $d(z,v_1) < C_4 d(z,v_2)$ for all $v_1, v_2 \in N_j(z)$, there exist by (*) $y', z' \in N_j(z)$ for which $d(y',z') = d(\{y',z,z'\})$. If $y \notin \{y',z'\}$, then $y', z' \in N_{j+1}(z)$ and the property (*) is satisfied at z. Thus we may assume $y' = y$. Now $2^{-k} < d(y,x) \leq 2^{-(k-1)}$, $d(x,z) \leq d(y,z) < C_4 d(y,x) \leq C_4 2^{-(k-1)}$, $d(y,z') \leq d(y,z) + d(z,z') < C_4(1+C_4) d(y,x) \leq C_4(1+C_4)2^{-(k-1)}$ and $d(x,z') \leq d(x,z) + d(z,z') < C_4(1+C_4) d(y,x) \leq C_4(1+C_4)2^{-(k-1)}$. Since $C_1 \geq C_2 \geq 2C_4(1+C_4)$ and $2C_4(1+C_4)\varepsilon_0 \leq \sqrt{3}$, we have $\{y,x,z,z'\} \in \Omega(\delta)$, where

$$\delta \geq \sqrt{1 - C_4^2(1+C_4)^2\varepsilon_0^2} \geq \frac{C_4(1+C_4)}{C_4(1+C_4)+1}.$$

Since now yxz and yzz', $\{y,x,z,z'\}$ has an order by Lemma 2.2. Thus $d(x,z') = d(\{x,z,z'\})$ and the property (*) is satisfied at z. In the similar way we see that (*) is satisfied at y.

Let $t, u \in I_j$ such that $u - t = d(y,z)$, $h_j(\{t,u\}) = \{y,z\}$ and $I_j \cap]t,u[= \emptyset$. We set

$$I_{j+1} = J_1 \cup \{t + d(h_j(t), x)\} \cup J_2,$$

where $J_1 = I_j \cap [0,t]$ and $J_2 = (I_j \cap [u,\infty[) + d(y,x) + d(x,z) - d(y,z)$, and define $h_{j+1} : I_{j+1} \to D_{j+1}$ by setting

$$h_{j+1}(s) = \begin{cases} h_j(s) & \text{for } s \in J_1, \\ x & \text{for } s = t + d(h_j(t),x), \\ h_j(s - d(y,x) - d(x,z) + d(y,z)) & \text{for } s \in J_2. \end{cases}$$

Now (**) is satisfied, $I_{j+1} \subset [0, 2l(G_{j+1}^*)]$ and h_{j+1} is surjective and 1-Lipschitz.

Case 4. $d(y,z) < d(x,z)$ for all $z \in N_j(y)$.

We first show that there exists $b \in V_j \backslash g_j(D_j)$ such that $\{g_j(y), b\} \in E_j$. Suppose this fails. Now $d(y,v_1) < C_4 d(y,x) \leq C_4 d(y,v_2)$ for all $v_1, v_2 \in N_j(y)$. Thus by (*) there are $z_1, z_2 \in N_j(y)$ such that $d(z_1,z_2) = d(\{z_1,y,z_2\})$. Since $C_1 \geq C_2 \geq 2(1+C_4)$ and $4C_4\varepsilon_0 \leq \sqrt{3}$, we have $\{z_1,x,y,z_2\} \in \Omega(\delta)$, where

$$\delta \geq \sqrt{1 - 4C_4^2\varepsilon_0^2} \geq \frac{2C_4}{2C_4+1}.$$

Since now xyz_1 and xyz_2, it follows from Lemma 2.2 that yz_1z_2 or yz_2z_1, which is a contradiction.

We set $V_{j+1} = V_j \cup \{a\}$, where $a \notin V_j$, and define $g_{j+1} : D_{j+1} \to V_{j+1}$ by setting $g_{j+1}(x) = a$ and $g_{j+1}(v) = g_j(v)$ for $v \in D_j$. Further we define

$$E_{j+1} = (E_j \backslash \{\{g_j(y), b\}\}) \cup \{\{g_j(y), a\}, \{a, b\}\}$$

and $w_{j+1}: E_{j+1} \to]0,\infty[$ by setting

$$w_{j+1}(e) = \begin{cases} d(y,x) & \text{for } e = \{g_j(y), a\}, \\ w_j(\{g_j(y), b\}) & \text{for } e = \{a, b\}, \\ w_j(e) & \text{for } e \in E_j \backslash \{g_j(y), b\}. \end{cases}$$

Now

$$l(G_{j+1}) - l(G_j) = d(y,x) \le \frac{w_j(\{g_j(y), b\})}{C_3}. \tag{28}$$

Since $d(x,z) = d(\{x,y,z\})$, the property (*) is satisfied at y. For all $m \ge j$ there is $z \in D_m$ such that $\{g_m(z), b\} \in E_m$ and $w_m(\{g_m(z), b\}) = w_j(\{g_j(y), b\})$ by the construction. We define I_{j+1} and h_{j+1} as in the cases 1 and 2.

By iterating the above algorithm, we construct a sequence (G_j) of graphs and a sequence $h_j : I_j \to D_j$ of 1-Lipschitz surjections such that $I_j \subset [0, 2l(G_j^*)]$ for all $j \ge 2$. Let n_0 be the smallest integer such that $\#\Delta_{n_0} \ge 2$. For all $n \ge n_0$ we denote $T_n = G^*_{\#\Delta_n}$, $A_n = I_{\#\Delta_n}$ and $f_n = h_{\#\Delta_n}$.

Since $289(1-4\varepsilon_0^2)^3 \ge 225$, for any $y \in E$ and k the case 2 is applied at most to four points in $B(y, 2^{-k+1}) \cap \Delta_k$ by the calculation at the beginning of the proof. Thus by (26) and the remark after it

$$\sum_{j \in Y_m} \sum_{e \in E_j \backslash E_{j-1}} w_j(e) \le 2\left(1 + \sum_{i=0}^{\infty} 2^{-i}\right) \frac{4(1+C_3)}{C_4} l(G_m^*) = \frac{24(1+C_3)}{C_4} l(G_m^*) \tag{29}$$

for all $m \ge 3$, where $Y_m = \{j \in \{3, \ldots, m\} :$ The case 2 is applied to $x_j\}$.

We now show that for any fixed $b \in \bigcup_j (V_j \backslash g_j(D_j))$ for all k the case 4 can occur at most for three points in Δ_k. Suppose this fails for some k and let $\#\Delta_{k-1} < i_1 < i_2 < i_3 < i_4 \le \#\Delta_k$, $i_0 < i_1$ such that $\{g_{i_l}(x_{i_l}), b\} \in E_{i_l}$ for $l = 0, \ldots, 4$. Then, since $\max\{d(x, D_{i_1-1}) : x \in \Delta_k\} = d(x_{i_1}, x_{i_0}) < d(x_{i_2}, x_{i_0})$, there is $z \in D_{i_1-1} \backslash \{x_{i_0}\}$ such that $d(x_{i_2}, z) \le d(x_{i_1}, x_{i_0})$. Since $2^{-k} < d(z_1, z_2) \le 2^{-k+3}$ for all $z_1, z_2 \in \{x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, z\} \subset B(x_{i_2}, 2^{-k+2})$, $\beta\left(x_{i_2}, C_2 2^{-k}\right) 2^{-k} < \varepsilon_0$, $C_1 \ge C_2 \ge 4$ and $8\varepsilon_0 \le \sqrt{3}$, we have $\{x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, z\} \in \Omega(\delta)$, where $\delta \ge \sqrt{1 - 16\varepsilon_0^2}$. Since $\delta^3 \ge 31/33$, $\{x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, z\}$ has an order by Lemma 2.3. Since $d(x_{i_l}, x_{i_{l-1}}) = d(x_{i_l}, D_{i_l-1})$ for $l = 1, \ldots, 4$, we must have $x_{i_0} x_{i_1} x_{i_2} x_{i_3} x_{i_4} z$. From this we get $d(x_{i_2}, z) \ge d(x_{i_2}, x_{i_3}) + \delta d(x_{i_3}, x_{i_4}) + \delta d(x_{i_4}, z) > (1+2\delta)2^{-k} > 2^{-(k-1)} \ge d(x_{i_1}, x_{i_0})$, which is a contradiction. Thus by (28) and the remark after it

$$\sum_{j \in Z_m} \sum_{e \in E_j \backslash E_{j-1}} w_j(e) \le \left(1 + \sum_{i=0}^{\infty} 2^{-i}\right) \frac{3}{C_3} \left[l(G_m) - l(G_m^*)\right] = \frac{9}{C_3} \left[l(G_m) - l(G_m^*)\right] \tag{30}$$

for all $m \ge 3$, where $Z_m = \{j \in \{3, \ldots, m\} :$ The case 4 is applied to $x_j\}$.

Using the estimates (24), (25), (27), (29) and (30), we get for all $n \ge n_0$

$$\begin{aligned} l(T_n) &\le (1 + 2C_3) d(E) \\ &+ \max\left\{ \frac{2(1+C_3)}{\varepsilon_0^2}, 2C_4(1+C_4) \right\} \sum_{k=n_0}^{n} \sum_{x \in \Delta_k \backslash \Delta_{k-1}} \beta\left(x, C_2 2^{-k}\right)^2 \left(2^{-k}\right)^3 \\ &+ \frac{24(1+C_3)}{C_4} l(T_n) + \frac{9}{C_3}\left[l(G_{\#\Delta_n}) - l(T_n)\right] - \left[l(G_{\#\Delta_n}) - l(T_n)\right]. \end{aligned}$$

Since $C_3 \geq 9$, $C_4 > 24(1+C_3)$ and the net $(\Delta_k)_k$ is arbitrary, we have an absolute constant C such that $2l(G_j^*) \leq C\,(\beta(E)+d(E))$ for all $j \geq 2$. Now there exists a compact $A \subset [0, C(\beta(E)+d(E))]$ such that $A_n \to A$ in the Kuratowski sense:

(i) If $a = \lim_{k\to\infty} a_{n_k}$ for some subsequence (a_{n_k}) of a sequence (a_n) such that $a_n \in A_n$ for any n, then $a \in A$.

(ii) If $a \in A$, then there exists a sequence (a_n) such that $a_n \in A_n$ for any n and $a = \lim_{n\to\infty} a_n$.

Let $a \in A$ and let (a_n) be a sequence such that $a_n \in A_n$ for any n and $a_n \to a$ as $n \to \infty$. Let $m \geq n \geq n_0$. By the construction there is $b \in A_m$ such that $|a_n - b| \leq 2\,(l(T_m) - l(T_n))$ and $f_n(a_n) = f_m(b)$. Using this we get

$$\begin{aligned} d(f_m(a_m), f_n(a_n)) &= d(f_m(a_m), f_m(b)) \leq |a_m - b| \leq |a_m - a_n| + |a_n - b| \\ &\leq |a_m - a_n| + 2\,(l(T_m) - l(T_n))\,. \end{aligned}$$

So $(f_n(a_n))$ is a Cauchy sequence in E. Thus we can define $f : A \to \overline{E}$, where $\overline{E}$ is the completion of E, by setting for $a \in A$

$$f(a) = \lim_{n\to\infty} f_n(a_n),$$

where (a_n) is a sequence such that $a_n \in A_n$ for any n and $a_n \to a$ as $n \to \infty$. Clearly $f(a)$ does not depend on the choice of the sequence (a_n). Let $a, b \in A$ and let $a_n \to a$ and $b_n \to b$ such that $a_n, b_n \in A_n$ for any n. Now, since f_n is 1-Lipschitz for any n,

$$\begin{aligned} d(f(a), f(b)) &\leq d(f(a), f_n(a_n)) + d(f_n(a_n), f_n(b_n)) + d(f_n(b_n), f(b)) \\ &\leq d(f(a), f_n(a_n)) + |a_n - b_n| + d(f_n(b_n), f(b)) \to |a - b| \end{aligned}$$

as $n \to \infty$. So f is 1-Lipschitz. It is also surjective. To check this let $x \in \Delta_k$ for some k. Then there is $c_k \in A_k$ such that $x = f_k(c_k)$. By the construction we have a sequence $(c_n)_{n\geq k}$ such that $c_n \in A_n$, $f_n(c_n) = x$ and $|c_{n+1} - c_n| \leq 2\,(l(T_{n+1}) - l(T_n))$ for any $n \geq k$. From this we see that (c_n) is a Cauchy sequence and thus there is $c \in [0, C(\beta(E)+d(E))]$ such that $c_n \to c$. Now $c \in A$ by (i) and $x = \lim_{n\to\infty} f_n(c_n) = f(c)$. Thus $\bigcup_{n\in\mathbb{Z}} \Delta_n \subset f(A)$. Since $\bigcup_{n\in\mathbb{Z}} \Delta_n \subset \overline{E}$ is dense and $f(A)$ is compact, we have $E \subset f(A) = \overline{E}$. Finally, we restrict f to $f^{-1}(E)$. $\square$

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