CONTINUITY OF THE MAXIMAL OPERATOR IN SOBOLEV SPACES

HANNES LUIRO

Abstract. We establish the continuity of the Hardy-Littlewood maximal operator on Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. As an auxiliary tool we prove an explicit formula for the derivative of the maximal function.

1. Introduction

The classical Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on $L^1_{loc}(\mathbb{R}^n)$ by setting for all $f \in L^1_{loc}(\mathbb{R}^n)$

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy = \sup_{r>0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y)|\,dy\,, \tag{1}$$

for every $x \in \mathbb{R}^n$; here m denotes the Lebesgue measure in $\mathbb{R}^n$ and $B_r = B(0,r)$.

The theorem of Hardy, Littlewood and Wiener asserts that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. This theorem is one of the cornerstones of harmonic analysis. Applications e.g. to the study of Sobolev-functions indicate that it is also useful to know how it preserves differentiability properties of functions. Quite recently, Kinnunen observed [K] that $\mathcal{M}$ is bounded on the Sobolev-space $W^{1,p}(\mathbb{R}^n)$, for $1 < p \leq \infty$. Extensions and related results can be found from e.g. [KL], [Ko], [KS], [HO].

Continuity of the maximal operator in $L^p(\mathbb{R}^n)$ follows from its sublinearity and boundedness. Because of boundedness in $W^{1,p}(\mathbb{R}^n)$, it is very natural to ask whether the maximal operator is continuous in $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, or not. This question was posed in [HO, Question 3] where it was attributed to T. Iwaniec. In general, bounded non-sublinear operators need not to be continuous. An important example of this kind of phenomenon is the result of Almgren and Lieb [AL] who proved that the (known to be bounded) symmetric rearrangement $\mathcal{R} : W^{1,p}(\mathbb{R}^n) \mapsto W^{1,p}(\mathbb{R}^n)$ is not continuous when $1 < p < n$ and $n > 1$. On Sobolev-spaces, $\mathcal{M}$ is not sublinear and the issue of the continuity of $\mathcal{M}$ is not trivial even though we know the boundedness.

Our main result (Thm. 4.1 below) is the positive answer to the question of Iwaniec. A central role in our proof is played by a careful analysis of the set $\mathcal{R}f(x)$ (see 2.1 below), which consists of the radii r for which equality is achieved in (1). As a useful auxiliary tool we establish in Thm. 3.1 an explicit formula for the derivative of the maximal function.

2. Definitions and auxiliary results

Let us first introduce some notation. If $A \subset \mathbb{R}^n$ and $r \in \mathbb{R}^n$, we define

$$d(r,A) := \inf_{a\in A} |r-a|\,, \text{ and } A_{(\lambda)} := \{x \in \mathbb{R}^n : d(x,A) \leq \lambda\} \text{ for } \lambda \geq 0\,.$$

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We endow $W^{1,p}(\mathbb{R}^n)$ with the norm

$$\|f\|_{1,p} = \|f\|_p + \|\nabla f\|_p ,$$

where ∇f is the weak gradient of f. Let us also denote by $\|f\|_{p,A}$ the L^p-norm of $\chi_A f$ for all measurable sets $A \subset \mathbb{R}^n$.

The following new concept will be central in this work.

Definition 2.1. *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. The set $\mathcal{R}f(x)$ is defined as*

$$\mathcal{R}f(x) = \{r \geq 0 : Mf(x) = \limsup_{r_k \to r} \fint_{B(x,r_k)} |f(y)|\,dy\,,\ \text{for some } r_k > 0\}.$$

Remarks. We comment on the above definition and the properties of the sets $\mathcal{R}f(x)$. Firstly, the definition clearly implies that $\mathcal{R}f(x)$ is always closed. Moreover, for fixed $x \in \mathbb{R}^n$ define $u_x : [0,\infty) \mapsto \mathbb{R}$ by

$$u_x(0) = |f(x)| \text{ and } u_x(r) = \fint_{B(x,r)} |f(y)|\,dy \quad \text{when } r \in (0,\infty).$$

First of all, the functions u_x are continuous for almost all x. The continuity on $(0,\infty)$ is clearly true for all x and at 0 it follows a.e., because almost every point $x \in \mathbb{R}^n$ is a Lebesgue's point for f. Moreover, by Hölder's inequality we have

$$u_x(r) \leq \|f\|_p (m(B_r))^{\frac{1}{q}-1}, \tag{2}$$

where q is the conjugate exponent of p, and hence $\lim_{r\to\infty} u_x(r) = 0$.

These facts together imply that, for almost all x, the function u_x has at least one maximum point in $[0,\infty)$. Furthermore, they guarantee that for all $x \in \mathbb{R}^n$ the set $\mathcal{R}f(x)$ is nonempty and

$$Mf(x) = \fint_{B(x,r)} |f(y)|\,dy \text{ if } r \in \mathcal{R}f(x) \text{ and } r > 0,\ \forall x \in \mathbb{R}^n, \text{ and}$$
$$Mf(x) = |f(x)| \text{ for almost every } x \text{ such that } 0 \in \mathcal{R}f(x).$$

Also, it is useful to observe that for every $R > 0$ (assuming $f \not\equiv 0$) it is true that

$$\sup\{r : r \in \mathcal{R}f(x), x \in B(0,R)\} < \infty.$$

The following lemma tells us how the sets $\mathcal{R}f(x)$ and $\mathcal{R}g(x)$ are related to each other, especially when $\|f-g\|_p$ is small.

Lemma 2.2. *Let $1 \leq p < \infty$ and suppose $f_j \to f$ in $L^p(\mathbb{R}^n)$ when $j \to \infty$. Then for all $R > 0$ and $\lambda > 0$ it holds that*

$$m(\{x \in B(0,R) : \mathcal{R}f_j(x) \not\subset \mathcal{R}f(x)_{(\lambda)}\}) \to 0 \quad \text{if } j \to \infty.$$

Proof. First we remark that one can show that the above set is Lebesgue-measurable always when f_j and f are in $L^p(\mathbb{R}^n)$. It is sufficient to prove the claim in the case where both f and f_j are non-negative, because $\mathcal{R}f(x) = \mathcal{R}|f|(x)$. Observe that $\mathcal{R}f(x)$ is $[0,\infty)$ for all x if $f \equiv 0$ a.e., whence this case is trivial. Let $\lambda > 0, R > 0$ and $\varepsilon > 0$. For almost every $x \in B(0,R)$ there exists a natural number $i(x) \in \mathbb{N}$ so that

$$\fint_{B(x,r)} f(y)\,dy < Mf(x) - \frac{1}{i(x)}, \text{ when } d(r, \mathcal{R}f(x)) > \lambda. \tag{3}$$

This can be seen in the following way: If the claim is not true there is a sequence of radii $(r_k)_{k=1}^{\infty}$ so that

$$\fint_{B(x,r_k)} f(y)\,dy \to Mf(x) \text{ and } d(r_k, \mathcal{R}f(x)) > \lambda\,.$$

By moving to a subsequence, if needed, we may assume that $r_k \to r$ as $k \to \infty$, because (2) implies that the sequence $(r_k)_{k=1}^{\infty}$ must be bounded. It follows that $r \in \mathcal{R}f(x)$. This is a contradiction, since, obviously r satisfies $d(r, \mathcal{R}f(x)) \geq \lambda$.

From (3) we conclude that there exists $i \in \mathbb{N}$ so that

$$B(0,R) \subset \{x : \fint_{B(x,r)} f(y)\,dy < Mf(x) - \frac{1}{i}, \text{ if } d(r, \mathcal{R}f(x)) > \lambda\} \bigcup E =: A \bigcup E\,,$$

where E is a measurable set with $m(E) < \varepsilon$. The weak type (1,1)-estimate for the maximal operator implies that there exists $j_0 \in \mathbb{N}$ so that

$$m\big(\{x \in B(0,R) : |M(f - f_j)(x)| \geq \frac{1}{4i}\}\big) < \varepsilon \text{ when } j \geq j_0\,.$$

For all j we observe that

$$\begin{aligned} A \subset &\{x : \fint_{B(x,r)} f_j(y)\,dy < Mf(x) - \frac{1}{2i}, \text{ if } d(r, \mathcal{R}f(x)) > \lambda\} \\ &\bigcup \{x : \Big| \fint_{B(x,r)} f(y)\,dy - \fint_{B(x,r)} f_j(y)\,dy \Big| \geq \frac{1}{2i}, \text{ for some } r, d(r, \mathcal{R}f(x)) > \lambda\} \\ =:& A_j \bigcup B_j\,. \end{aligned}$$

Continuing the same reasoning, and using the fact $|Mf(x) - Mf_j(x)| \leq |M(f - f_j)(x)|$, we get

$$\begin{aligned} A_j \subset &\{x : \fint_{B(x,r)} f_j(y)\,dy < Mf_j(x) - \frac{1}{4i}, \text{ if } d(r, \mathcal{R}f(x)) > \lambda\} \\ &\bigcup \{x : |M(f - f_j)(x)| \geq \frac{1}{4i}\} \\ =:& C_j \bigcup D_j\,. \end{aligned}$$

It holds that

$$C_j \subset \{x : \mathcal{R}f_j(x) \subset \mathcal{R}f(x)_{(\lambda)}\}\,.$$

By combining the above observations, we conclude that for all j

$$B(0,R) \subset \{x : \mathcal{R}f_j(x) \subset \mathcal{R}f(x)_{(\lambda)}\} \bigcup E \bigcup D_j \bigcup B_j\,.$$

Observe finally that $B_j \subset D_j$ and, by our choice of j_0 we have $m(D_j) < \varepsilon$ if $j \geq j_0$, and therefore

$$m(\{x \in B(0,R) : \mathcal{R}f_j(x) \not\subset \mathcal{R}f(x)_{(\lambda)}\}) < 2\varepsilon\,,$$

if $j \geq j_0$. □

Let us introduce more notation. Assume that $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Let e_i be one of the standard basevectors of $\mathbb{R}^n$. For all $h \in \mathbb{R}$, $|h| > 0$ we define the functions f_h^i and $f_{\tau(h)}^i$ by setting

$$f_h^i(x) = \frac{f(x + he_i) - f(x)}{|h|} \quad \text{and} \quad f_{\tau(h)}^i(x) = f(x + he_i)\,.$$

Now we know that $f^i_{\tau(h)} \to f$ in $L^p(\mathbb{R}^n)$ when $|h| \to 0$ and, if $p > 1$, for functions $f \in W^{1,p}(\mathbb{R}^n)$ we have (see [GT, 7.11]) that $f^i_h \to D_i f$ in $L^p(\mathbb{R}^n)$ when $|h| \to 0$.

Corollary 2.3. *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Then for all i, $1 \le i \le n$, $R > 0$, $\lambda > 0$ one has*

$$m(\{x \in B(0,R) : \mathcal{R}f(x) \not\subset \mathcal{R}f^i_{\tau(h)}(x)_{(\lambda)} \text{ or } \mathcal{R}f^i_{\tau(h)}(x) \not\subset \mathcal{R}f(x)_{(\lambda)}\}) \overset{h\to 0}{\longrightarrow} 0\,.$$

Proof. Now $f^i_{\tau(h)} \to f$ and as a consequence of Lemma 2.2 it is clearly sufficient to prove that

$$m(\{x \in B_R : \mathcal{R}f(x) \not\subset \mathcal{R}f^i_{\tau(h)}(x)_{(\lambda)}\}) \to 0 \text{ as } h \to 0\,.$$

But this also follows easily from Lemma 2.2, because for $|h| < 1$ one has that

$$\begin{aligned}&\{x \in B_R : \mathcal{R}f(x) \not\subset \mathcal{R}f^i_{\tau(h)}(x)_{(\lambda)}\}\\ =&\{x \in B_R : \mathcal{R}f^i_{\tau(-h)}(x+he_i) \not\subset \mathcal{R}f(x+he_i)_{(\lambda)}\}\\ \subset&\{y \in B_{R+1} : \mathcal{R}f^i_{\tau(-h)}(y) \not\subset \mathcal{R}f(y)_{(\lambda)}\} - he_i\,.\end{aligned}$$

□

Remark. The previous corollary will became useful after the following observation. Let us denote by

$$\pi(A,B) := \inf\{\delta > 0 : A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}$$

the Hausdorff distance of the sets A and B. Let f be in $L^p(\mathbb{R}^n)$. With the new notation, the corollary says that

$$m(\{x \in B_R : \pi(\mathcal{R}f(x), \mathcal{R}f(x+he_i)) > \lambda\}) \to 0 \text{ when } h \to 0\,.$$

Therefore we easily infer that there is a sequence $(h_k)_{k=1}^\infty$, $h_k > 0$ with $h_k \to 0$, and such that $\pi(\mathcal{R}f(x), \mathcal{R}f(x+h_k e_i)) \to 0$ as $k \to \infty$ for almost every $x \in B_R$. This is the decisive fact needed in the following section.

3. A formula for the derivative of the maximal function

Let us denote by $D_i f(x)$ the partial derivative $\frac{\partial f}{\partial x_i}$.

Theorem 3.1. *Let $f \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. Then we have for almost all $x \in \mathbb{R}^n$ that*

(1) $D_i Mf(x) = \displaystyle⨍_{B(x,r)} D_i|f|(y)\,dy$ *for all* $r \in \mathcal{R}f(x)$, $r > 0$ *and*

(2) $D_i Mf(x) = D_i|f|(x)$ *if* $0 \in \mathcal{R}f(x)$.

Proof. It is sufficient to prove the claim for non-negative functions, because $Mf = M|f|$ and $|f| \in W^{1,p}(\mathbb{R}^n)$ if $f \in W^{1,p}(\mathbb{R}^n)$. Let $R > 0$. We start by choosing a sequence $(h_k)_{k=1}^\infty$, $h_k > 0$ and $h_k \to 0$ so that $\pi(\mathcal{R}f(x), \mathcal{R}f(x+h_k e_i)) \to 0$ as $k \to \infty$ for almost all $x \in B_R$ (see the Remark after Corollary 2.3). Then we have

(i) $\|D_i Mf - (Mf)^i_{h_k}\|_{p,B_R} \to 0$ as $k \to \infty$,

(ii) $\|D_i f - f^i_{h_k}\|_{p,B_R} \to 0$ as $k \to \infty$,

(iii) $\|M(D_i f - f^i_{h_k})\|_{p,B_R} \to 0$ as $k \to \infty$.

Now, by extracting a subsequence if needed, we may assume that the convergences above are true pointwise almost everywhere as well. Moreover, we recall that the set

$$\{x \in \mathbb{R}^n : \exists\, k \in \mathbb{N} \text{ s.t. } 0 \in \mathcal{R}f(x+h_k e_i) \text{ with } Mf(x+h_k e_i) \ne f(x+h_k e_i)\}$$

has measure zero as an countable union of the sets having measure zero. Let $x \in B_R$ be a point outside the union of all these unwanted sets of measure zero (in particular, the pointwise analogies of (i–iii) hold at x) and let $r \in \mathcal{R}f(x)$.

Now, because $\pi(\mathcal{R}f(x), \mathcal{R}f(x+h_k e_i)) \to 0$, we find radii $r_k \in \mathcal{R}f(x+h_k e_i)$ so that $r_k \to r$ when $k \to \infty$. If $r > 0$ we can estimate:

$$\begin{aligned} D_i Mf(x) &= \lim_{k\to\infty} \frac{1}{h_k}(Mf(x+h_k e_i) - Mf(x)) \\ &\le \lim_{k\to\infty} \frac{1}{h_k}\Big(\fint_{B(x+h_k e_i, r_k)} f(y)\,dy - \fint_{B(x,r_k)} f(y)\,dy \Big) \\ &= \lim_{k\to\infty} \frac{1}{m(B(x,r_k))} \int_{B(x,r_k)} \frac{f(y+h_k e_i) - f(y)}{h_k}\,dy \\ &= \fint_{B(x,r)} D_i f(y)\,dy\,. \end{aligned}$$

The last equation holds, because $m(B_{r_k}) \to m(B_r)$ and

$$\chi_{B(x,r_k)} f^i_{h_k} \to \chi_{B(x,r)} D_i f \ \text{ in } L^1(\mathbb{R}^n) \text{ as } k \to \infty\,.$$

On the other hand, we get that

$$\begin{aligned} D_i Mf(x) &\ge \lim_{k\to\infty} \frac{1}{h_k}\Big(\fint_{B(x+h_k e_i, r)} f(y)\,dy - \fint_{B(x,r)} f(y)\,dy \Big) \\ &= \lim_{k\to\infty} \frac{1}{m(B(x,r))} \int_{B(x,r)} \frac{f(y+h_k e_i) - f(y)}{h_k}\,dy = \fint_{B(x,r)} D_i f(y)\,dy\,. \end{aligned}$$

Let then be $r = 0$. The proof of the lower bound of $D_i Mf(x)$ applies now, too, and we get that $D_i Mf(x) \ge D_i f(x)$. If we have $r_k = 0$ for infinite many k, we can decide straightforwardly that $D_i Mf(x) = D_i f(x)$. If $r_k > 0$ starting from some k_0, we get by the same way as when studying the upper bound of $D_i Mf(x)$ in the case $r > 0$ that

$$D_i Mf(x) \le \lim_{k\to\infty} \fint_{B(x,r_k)} f^i_{h_k}(y)\,dy = D_i f(x)\,,$$

because

$$\begin{aligned} &\lim_{k\to\infty} \Big| \fint_{B(x,r_k)} f^i_{h_k}(y)\,dy - D_i f(x) \Big| = \lim_{k\to\infty} \Big| \fint_{B(x,r_k)} \big(f^i_{h_k}(y) - D_i f(y) \big)\,dy \Big| \\ &\le \lim_{k\to\infty} M(f^i_{h_k} - D_i f)(x) = 0\,. \end{aligned}$$

Now we have showed the claim in the ball $B(0,R)$. Since R was arbitrary, this completes the proof. □

4. Continuity of the maximal operator in $W^{1,p}(\mathbb{R}^n)$

By using Theorem 3.1 and Lemma 2.2, we can establish quite easily our main result which verifies the continuity of the maximal operator in $W^{1,p}(\mathbb{R}^n)$.

Theorem 4.1. *$M : W^{1,p}(\mathbb{R}^n) \mapsto W^{1,p}(\mathbb{R}^n)$ is continuous for all $1 < p < \infty$.*

Proof. Let $f_j \to f$ in $W^{1,p}(\mathbb{R}^n)$ when $j \to \infty$. We have to show that $\|Mf_j - Mf\|_{1,p} \to 0$. Because we know the continuity of M in $L^p(\mathbb{R}^n)$, it is sufficient to prove that $\|D_i Mf_j - D_i Mf\|_p \to 0$ for all i, $1 \le i \le n$. Also it is clear that we may assume the functions f_j and f to be non-negative.

We start by choosing $R>0$ so that $\|2MD_if\|_{p,C_1}<\varepsilon$, where $C_1=\mathbb{R}^n\setminus B(0,R)$. By absolute continuity we choose $\alpha>0$ so that $\|2MD_if\|_{p,A}<\varepsilon$ always when $m(A)<\alpha$ and A is measurable subset of $B(0,R)$.

We let (compare with the remark after Definition 2.1) $u_x(r)$ stand for the average of D_if in the ball $B(x,r)$ and $u_x(0)=D_if(x)$. As already observed, for almost every $x\in\mathbb{R}^n$ functions u_x are continuous on $[0,\infty)$. Moreover they converge to 0 when $r\to\infty$. Therefore there exists $\delta>0$ such that

$$m(\{x\in B_R:|u_x(r_1)-u_x(r_2)|>\frac{\varepsilon}{(m(B_R))^{\frac{1}{p}}}\text{ for some }r_1,r_2,|r_1-r_2|<\delta\})$$
$$=:m(C_2)<\frac{\alpha}{2}.$$

The set C_2 is easily shown to be measurable. Furthermore, Lemma 2.2 says that we can find j_0 so that

$$m(\{x:\mathcal{R}f_j(x)\not\subset\mathcal{R}f(x)_{(\delta)}\})=:m(C^j)<\frac{\alpha}{2}\text{ when }j\geq j_0.$$

Then, let $j\geq j_0$ be fixed. It follows from Theorem 3.1 that almost everywhere in $\mathbb{R}^n$

$$\begin{aligned}|D_iMf_j(x)-D_iMf(x)|&=\Big|\fint_{B(x,r_1)}D_if_j(y)\,dy-\fint_{B(x,r_2)}D_if(y)\,dy\Big|\\
&\leq\Big|\fint_{B(x,r_1)}D_if_j(y)\,dy-\fint_{B(x,r_1)}D_if(y)\,dy\Big|\\
&\quad+\Big|\fint_{B(x,r_1)}D_if(y)\,dy-\fint_{B(x,r_2)}D_if(y)\,dy\Big|\\
&\leq M(D_if_j-D_if)(x)+\Big|\fint_{B(x,r_1)}D_if(y)\,dy-\fint_{B(x,r_2)}D_if(y)\,dy\Big|\end{aligned}$$

for all $r_1\in\mathcal{R}f_j(x),r_2\in\mathcal{R}f(x)$. This inequation applies also to the cases $r_1=0$ or $r_2=0$ when we agree that

$$\fint_{B(x,0)}D_if(y)\,dy:=D_if(x).$$

This is obvious because for almost every x it is true that $Mf(x)\geq f(x)$, and by Theorem 3.1 $D_iMf(x)=D_if(x)$ if $0\in\mathcal{R}f(x)$.

Now, if $x\notin C_1\bigcup C_2\bigcup C^j$, we can pick $r_1\in\mathcal{R}f_j(x)$ and $r_2\in\mathcal{R}f(x)$ so that $|r_1-r_2|<\delta$. Our choice of δ implies that

$$s:=\Big|\fint_{B(x,r_1)}D_if(y)\,dy-\fint_{B(x,r_2)}D_if(y)\,dy\Big|<\frac{\varepsilon}{(m(B_R))^{\frac{1}{p}}}.$$

If $x\in C_1\bigcup C_2\bigcup C^j$, we estimate that $s\leq 2MD_if(x)$. Observe also that $m(C_2\bigcup C^j)<\alpha$.

Combining the above estimates it follows that

$$\begin{aligned}\|D_iMf_j-D_iMf\|_{p,\mathbb{R}^n}\leq&\|M(D_if_j-D_if)\|_{p,\mathbb{R}^n}+\Big\|\frac{\varepsilon}{(m(B_R))^{\frac{1}{p}}}\Big\|_{p,B_R}\\
&+\|2MD_if\|_{p,C_1}+\|2MD_if\|_{p,C_2\cup C_j}.\end{aligned}$$

The first term in the righthandside of the inequation converges to zero when $j\to\infty$. The rest of the terms are less than ε, because of the choices of R and α. As ε was arbitrary we conclude that $\|D_iMf_j-D_iMf\|_p\to 0$ as $j\to\infty$. The proof is complete. □

Question. How good estimates one can find for the modulus of continuity of M?

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Department of Mathematics and Statistics, University of Jyväskylä, P.O.Box 35 (MaD), 40014 University of Jyväskylä, Finland

E-mail address: haluiro@maths.jyu.fi