Interpolation and approximation in $L_2(\gamma)$

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February 20, 2004

Abstract

Assume a standard Brownian motion $W = (W_t)_{t\in[0,1]}$ and a Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $Z = f(W_1) \in L_2$. We show that certain approximation properties of Z with respect to the Brownian motion and the geometric Brownian motion are equivalent to the fact that f belongs to some fractional Sobolev space obtained by the real interpolation method from the couple $(\mathbb{D}_{1,2}(\gamma), L_2(\gamma))$, where γ is the standard Gaussian measure on the real line.

Keywords: Approximation of stochastic integrals, interpolation
Mathematics Subject classification: 46B70, 46B28, 46E35, 60G44, 60H05

1 Introduction

The recent study of quantitative approximation problems for stochastic integrals has been motivated by problems in Stochastic Finance, where one replaces continuously adjusted portfolios by discretely adjusted ones and is interested in the occurring approximation error, which can be interpreted as risk. After considering special pay-off functions the investigations were indicating that there are close relations between stochastic approximation properties and interpolation properties. A first connection in this direction was given in [3]. The aim of this paper is to extend these results in the case that the underlying diffusion, which acts as integrator of our stochastic integrals, is the Brownian motion or the geometric Brownian motion. The particular choice of the diffusion allows us to exploit Hermite polynomial expansions to understand better the interplay between interpolation and stochastic approximation.

We let $W = (W_t)_{t\in[0,1]}$ be a standard Brownian motion with $W_0 \equiv 0$ and having continuous paths for all $\omega \in \Omega$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where we assume that $\mathcal{F}$ is the completion of $\sigma(W_t : t \in [0,1])$. We let $(\mathcal{F}_t)_{t\in[0,1]}$ be the augmentation of the natural filtration of W, the process $S = (S_t)_{t\in[0,1]}$ be the geometric Brownian motion

$$S_t := e^{W_t - \frac{t}{2}}, \quad t \in [0,1],$$

and γ be the standard Gaussian measure on the real line

$$d\gamma(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

The complete orthonormal system of Hermite polynomials $(h_k)_{k=0}^{\infty} \subset L_2(\gamma)$ is obtained by

$$\exp\left(\lambda x - \frac{\lambda^2}{2}\right) = \sum_{k=0}^{\infty} \lambda^k \frac{h_k(x)}{\sqrt{k!}} \quad \text{for} \quad \lambda \in \mathbb{R}$$

and the Sobolev space $\mathbb{D}_{1,2}(\gamma)$ as the Banach space of all $f = \sum_{k=0}^{\infty} \alpha_k h_k$ such that

$$\|f\|_{\mathbb{D}_{1,2}} := \left(\sum_{k=0}^{\infty}(k+1)\alpha_k^2\right)^{\frac{1}{2}} < \infty.$$

Given $f \in L_2(\gamma)$ we want to approximate $f(W_1)$ as follows.

Definition 1.1 *Let X be the Brownian motion or the geometric Brownian motion and $\mathcal{T}$ be the set of all deterministic time-nets $0 = t_0 < \cdots < t_n = 1$. Given $\tau = (t_i)_{i=0}^{n} \in \mathcal{T}$ and $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$, we let*

$$a_X(Z;\tau) := \inf \left\| [Z - \mathbb{E}Z] - \sum_{i=1}^{n} v_{i-1}(X_{t_i} - X_{t_{i-1}}) \right\|_{L_2}$$

where the infimum is taken over all $\mathcal{F}_{t_{i-1}}$-measurable step-functions v_{i-1}.

To introduce the fractional Sobolev spaces we use the real interpolation method described for example in [1] and [2].

Definition 1.2 *Let (X_0, X_1) be a compatible couple of Banach spaces. Given $x \in X_0 + X_1$ and $\lambda \geq 0$, the K-functional is defined by*

$$K(x, \lambda; X_0, X_1) := \inf \{\|x_0\|_{X_0} + \lambda \|x_1\|_{X_1} : x = x_0 + x_1\}.$$

Moreover, given $\eta \in (0,1)$ and $q \in [1, \infty]$, we let $(X_0, X_1)_{\eta,q}$ be the space of all $x \in X_0 + X_1$ such that

$$\|x\|_{(X_0,X_1)_{\eta,q}} := \left\| \lambda^{-\eta} K(x, \lambda; X_0, X_1) \right\|_{L_q((0,\infty), \frac{d\lambda}{\lambda})} < \infty.$$

The following result was, in a sense, announced in [3] in the case, X being the Brownian motion, and gives a first connection between real interpolation and stochastic approximation. To shorten the notation, given $A, B \geq 0$ and $c \geq 1$, the expression $A \sim_c B$ stands for $A/c \leq B \leq cA$.

Theorem 1.3 *Let $\eta \in (0,1)$, $f \in L_2(\gamma)$, and X be either the Brownian motion or the geometric Brownian motion. Then the following assertions are equivalent:*

(i) $f \in (\mathbb{D}_{1,2}(\gamma), L_2(\gamma))_{\eta,\infty}$.

(ii) *There is a $c > 0$ such that, for all $\tau = (t_i)_{i=0}^n \in \mathcal{T}$,*

$$a_X(f(W_1); \tau) \leq c \sup_{i=1,\ldots,n} |t_i - t_{i-1}|^{\frac{1-\eta}{2}}.$$

Moreover, if $|f(W_1)|_{\eta,X}$ is the smallest possible $c > 0$ in (ii)*, then*

$$\|f\|_{(\mathbb{D}_{1,2}(\gamma), L_2(\gamma))_{\eta,\infty}} \sim_d \|f\|_{L_2(\gamma)} + |f(W_1)|_{\eta,X},$$

where $d > 0$ depends on η only.

As a byproduct of the paper we prove this theorem as well. This might be of interest since the method in this paper is completely different from the method used in [3]. The theorem has a couple of 'drawbacks': firstly, it is not visible at all that

$$\sup_{n\geq 1}\sqrt{n}\inf_{(t_i)_{i=0}^n\in\mathcal{T}} a_X\big(f(W_1);(t_i)_{i=0}^n\big)<\infty$$

under the conditions of Theorem 1.3 (this estimate and that the rate $1/\sqrt{n}$ is optimal follow from [3]). Secondly, the usage of the interpolation spaces with the parameters $(\theta,2)$ seems to be more favorable because of the already existing very intuitive characterizations (see, for example, [5]). Finally, we would like to connect path-properties of the martingale $\big(\mathbb{E}(f(W_1)|\mathcal{F}_t)\big)_{t\in[0,1]}$ and approximation properties of $f(W_1)$. A first attempt in this direction was done in [3]: letting X be the Brownian motion or the geometric Brownian motion and

$$I_\theta(f)(\omega):=\left(\int_0^1(1-t)^{-2\theta-1}|f(W_1(\omega))-\mathbb{E}(f(W_1)|\mathcal{F}_t)(\omega)|^2dt\right)^{\frac{1}{2}} \tag{1}$$

for $\theta\in(0,1/2)$, it was shown for certain f that $I_\theta(f)\in L_2$ implies

$$a_X(f(W_1);\tau)\leq c\sup_{i=1,\ldots,n}|t_i-t_{i-1}|^\theta, \tag{2}$$

and that the latter inequality implies in turn that $I_{\widetilde{\theta}}(f)\in L_2$ for $\widetilde{\theta}<\theta$. The question about an equivalence was raised. Here we answer this question by modifying the right-hand side of Formula (2) or, equivalently, by passing from the interpolation spaces with the parameters (η,∞) to the spaces with the parameters $(\eta,2)$.

2 Results

Our first result characterizes $f\in(\mathbb{D}_{1,2}(\gamma),L_2(\gamma))_{\eta,q}$ by the martingale obtained from $f(W_1)$. This is slightly different from the semi-group approach used for example in [5], where the corresponding Ornstein-Uhlenbeck semi-group is exploited. Given $f\in L_2(\gamma)$, we define

$$F(t,x):=\mathbb{E}f(x+W_{1-t})$$

for $(t,x)\in[0,1)\times\mathbb{R}$. The function F is smooth. In particular, one can define F on $(-\varepsilon,1)\times\mathbb{R}$ for some $\varepsilon>0$ (cf. [4] (Lemma A.2)) and

$$\frac{\partial F}{\partial t}+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}=0.$$

By Itô's formula we obtain $\mathbb{E}(f(W_1)|\mathcal{F}_t) = F(t, W_t)$ a.s. for $t \in [0,1)$ and

$$f(W_1) - \mathbb{E}f(W_1) = \int_0^1 \frac{\partial F}{\partial x}(t, W_t) dW_t \text{ a.s.}$$

with $\mathbb{E}\sup_{t\in[0,b]} |(\partial F/\partial x)(t, W_t)|^2 < \infty$ for $b \in [0,1)$ (cf. [4] (Corollary 4.1)).

Theorem 2.1 *For $\eta \in (0,1)$, $q \in [2,\infty]$, and $f \in L_2(\gamma)$ one has that*

$$\begin{aligned} \|f\|_{(\mathbb{D}_{1,2}(\gamma), L_2(\gamma))_{\eta,q}} &\sim_d \|f\|_{L_2(\gamma)} + \left\| (1-t)^{\frac{\eta}{2}} \left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_2} \right\|_{L_q(\mu)} \\ &\sim_d \|f\|_{L_2(\gamma)} + \left\| (1-t)^{\frac{\eta}{2}-\frac{1}{2}} \|f(W_1) - F(t, W_t)\|_{L_2} \right\|_{L_q(\mu)}, \end{aligned}$$

where $d > 0$ depends on (η, q) only and $d\mu(t) := \frac{dt}{1-t}$ on $[0,1)$.

Now let us turn to the approximation problem we are interested in. As already pointed out in [4] and [3], equidistant time nets are not optimal in general when one minimizes $a_X(f(W_1);\tau)$ while τ is restricted to a certain cardinality. Instead of equidistant nets, the nets $\tau_n^\eta = (t_i^{(n,\eta)})_{i=0}^n$ with

$$t_i^{(n,\eta)} := 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{1-\eta}} \tag{3}$$

and $\eta \in (0,1)$ were used there. The following theorem shows that $f \in (\mathbb{D}_{1,2}(\gamma), L_2(\gamma))_{\eta,2}$ is exactly characterized by the fact that the nets τ_n^η give already the optimal approximation rate of $1/\sqrt{n}$, which was slightly surprising to us. Moreover it should be noted, that the following characterization is independent from the choice of the diffusion W or S.

Theorem 2.2 *Let $\eta \in (0,1)$, $f \in L_2(\gamma)$, and X be either the Brownian motion or the geometric Brownian motion. Then the following assertions are equivalent:*

(i) $f \in (\mathbb{D}_{1,2}(\gamma), L_2(\gamma))_{\eta,2}$.

(ii) *There is a $c_2 > 0$ such that, for all $\tau = (t_i)_{i=0}^n \in \mathcal{T}$,*

$$a_X(f(W_1);\tau) \leq c_2 \sup_{i=1,\dots,n} \frac{(t_i - t_{i-1})^{\frac{1}{2}}}{(1-t_{i-1})^{\frac{\eta}{2}}}.$$

(iii) *There is a $c_3 > 0$ such that, for all $n = 1, 2, \dots$,*

$$a_X(f(W_1);\tau_n^\eta) \leq \frac{c_3}{\sqrt{n}}.$$

Moreover, if $|f(W_1)|_{\eta,X}$ is the smallest possible $c_2 > 0$ in (ii) and $\|f(W_1)\|_{\eta,X}$ is the smallest possible $c_3 > 0$ in (iii), then

$$\begin{aligned} \|f\|_{(\mathbb{D}_{1,2}(\gamma),L_2(\gamma))_{\eta,2}} &\sim_d \|f\|_{L_2(\gamma)} + |f(W_1)|_{\eta,X} \\ &\sim_d \|f\|_{L_2(\gamma)} + \|f(W_1)\|_{\eta,X}, \end{aligned}$$

where $d > 0$ depends on η only.

Combining Theorems 2.1 and 2.2 gives immediately the

Corollary 2.3 *Assume that $\theta \in (0,1/2)$ and $\eta \in (0,1)$ with $2\theta = 1-\eta$ and that $I_\theta(f)$ is given by* Formula (1) *for $f \in L_2(\gamma)$. Let X be the Brownian motion or the geometric Brownian motion. Then $I_\theta(f) \in L_2$ if and only if there is a constant $c > 0$ such that, for all $\tau = (t_i)_{i=0}^n \in \mathcal{T}$,*

$$a_X(f(W_1);\tau) \leq c \sup_{i=1,\dots,n} \frac{(t_i - t_{i-1})^{\frac{1}{2}}}{(1-t_{i-1})^{\frac{\eta}{2}}}.$$

3 Proof of the theorems

In the following we let $d_{1,2}$ and ℓ_2 be the spaces of all $a = (\alpha_k)_{k=0}^\infty \subset \mathbb{R}$ such that

$$\|a\|_{d_{1,2}} := \left(\sum_{k=0}^\infty (k+1)\alpha_k^2 \right)^{\frac{1}{2}} < \infty$$

and $\|a\|_{\ell_2} := (\sum_{k=0}^\infty \alpha_k^2)^{\frac{1}{2}} < \infty$, respectively. First we compute the K-functional of the couple $(d_{1,2}, \ell_2)$ (we believe that this should be folklore).

Lemma 3.1 *For $a = (\alpha_k)_{k=0}^\infty \in \ell_2$ one has $K(a, \lambda; d_{1,2}, \ell_2) = \lambda \|a\|_{\ell_2}$ for $\lambda \in [0,1]$ and*

$$K(a, \lambda; d_{1,2}, \ell_2) \sim_{\sqrt{2}} \left(\sum_{k=0}^\infty \min\{k+1, \lambda^2\}\alpha_k^2 \right)^{\frac{1}{2}} \quad \textit{for} \quad \lambda > 1.$$

Proof. Since the case $\lambda \in [0,1]$ is obvious, we suppose $\lambda \in (\sqrt{n}, \sqrt{n+1}]$ for some $n \in \{1, 2, \ldots\}$. Letting $a_0 = (\alpha_{k,0})_{k=0}^\infty \in d_{1,2}$ and $a_1 = (\alpha_{k,1})_{k=0}^\infty \in \ell_2$ such that $a = a_0 + a_1$ we get

$$\begin{aligned} \sum_{k=0}^\infty (k+1)\alpha_{k,0}^2 + \lambda^2 \sum_{k=0}^\infty \alpha_{k,1}^2 &\geq \sum_{k=0}^\infty \min\left\{k+1, \lambda^2\right\}\left(\alpha_{k,0}^2 + \alpha_{k,1}^2\right) \\ &\geq \frac{1}{2} \sum_{k=0}^\infty \min\left\{k+1, \lambda^2\right\} \alpha_k^2 \end{aligned}$$

so that $\sum_{k=0}^\infty \min\left\{k+1, \lambda^2\right\} \alpha_k^2 \leq 2K(a, \lambda; d_{1,2}, \ell_2)^2$. On the other hand,

$$\begin{aligned} K(a, \lambda; d_{1,2}, \ell_2) &\leq \left(\sum_{k=0}^{n-1} (k+1)\alpha_k^2 \right)^{\frac{1}{2}} + \lambda \left(\sum_{k=n}^\infty \alpha_k^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{2} \left(\sum_{k=0}^\infty \min\left\{k+1, \lambda^2\right\} \alpha_k^2 \right)^{\frac{1}{2}}. \end{aligned}$$

□

Now we use a transformation, in a sense, similar to the Hermite transform: given $a = (\alpha_k)_{k=0}^\infty \in \ell_2$, we let

$$(Ta)(t) := \sum_{k=0}^\infty \alpha_k^2 t^k \quad \text{for} \quad t \in [0,1)$$

and obtain

Proposition 3.2 *For $\eta \in (0,1)$, $q \in [2,\infty]$, and $a = (\alpha_k)_{k=0}^\infty \in \ell_2$ one has*

$$\begin{aligned} \|a\|_{(d_{1,2},\ell_2)_{\eta,q}} &\sim_c \|a\|_{\ell_2} + \left\| (1-t)^{\frac{\eta}{2}} \sqrt{(Ta)'(t)} \right\|_{L_q([0,1),\frac{dt}{1-t})} \\ &\sim_c \|a\|_{\ell_2} + \left\| (1-t)^{\frac{\eta}{2}-\frac{1}{2}} \sqrt{(Ta)(1)-(Ta)(t)} \right\|_{L_q([0,1),\frac{dt}{1-t})} \end{aligned}$$

where $c > 0$ depends on (η, q) only and the expressions may be infinite.

Proof. (a) We start with the first equivalence. For $t \in [0,1)$ take $n_t \in \{1,2,...\}$ such that $1 - \frac{1}{n_t} \le t < 1 - \frac{1}{n_t+1}$. Then we get

$$\begin{aligned} & \left\| (1-t)^{\frac{\eta}{2}} \sqrt{(Ta)'(t)} \right\|_{L_q([0,1),\frac{dt}{1-t})} \\ &= \left\| (1-t)^{\frac{\eta}{2}} \left(\sum_{k=1}^\infty \alpha_k^2 k t^{k-1} \right)^{\frac{1}{2}} \right\|_{L_q([0,1),\frac{dt}{1-t})} \\ &\le \left\| (1-t)^{\frac{\eta}{2}} \left(\sum_{k=1}^{n_t-1} \alpha_k^2 k t^{k-1} \right)^{\frac{1}{2}} \right\|_{L_q([0,1),\frac{dt}{1-t})} \\ &\quad + \left\| (1-t)^{\frac{\eta}{2}} \left(\sum_{k=n_t}^\infty \alpha_k^2 k t^{k-1} \right)^{\frac{1}{2}} \right\|_{L_q([0,1),\frac{dt}{1-t})} \\ &\le \left\| (1-t)^{\frac{\eta}{2}} \left(\sum_{k=1}^{n_t-1} k \alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([0,1),\frac{dt}{1-t})} \\ &\quad + c \left\| (1-t)^{\frac{\eta}{2}} \left(\frac{1}{1-t} \sum_{k=n_t}^\infty \alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([0,1),\frac{dt}{1-t})} \end{aligned}$$

since $kt^{k-1} \leq c^2/(1-t)$ for $k \geq n_t$ and some absolute $c > 0$ and where 'empty' sums are treated as zero. Using the transformation $\lambda^2 = 1/(1-t)$ and observing that $n_t \leq \lambda^2 < n_t + 1$ we can continue from the last sum to

$$
\begin{aligned}
& 2^{\frac{1}{q}} \left\| \lambda^{-\eta} \left(\sum_{k=1}^{[\lambda^2]-1} k\alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} + c2^{\frac{1}{q}} \left\| \lambda^{-\eta} \left(\lambda^2 \sum_{k=[\lambda^2]}^{\infty} \alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\
\leq \; & (1+c)2^{\frac{1}{q}} \left\| \lambda^{-\eta} \left(\sum_{k=1}^{\infty} \min\{k+1,\lambda^2\}\alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\
\leq \; & (1+c)2^{\frac{1}{q}}\sqrt{2} \left\| \lambda^{-\eta} K(a,\lambda;d_{1,2},\ell_2) \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\
\leq \; & (1+c)2^{\frac{1}{q}}\sqrt{2} \|a\|_{(d_{1,2},\ell_2)_{\eta,q}}
\end{aligned}
$$

where we have used Lemma 3.1. Moreover,

$$
\|a\|_{\ell_2} \leq c_{\eta,q} \|a\|_{(d_{1,2},\ell_2)_{\eta,q}}
$$

so that one direction of the equivalence is verified. To get the remaining estimate we again use Lemma 3.1 to obtain

$$
\begin{aligned}
& \|a\|_{(d_{1,2},\ell_2)_{\eta,q}} \\
= \; & \left\| \lambda^{-\eta} K(a,\lambda;d_{1,2},\ell_2) \right\|_{L_q((0,\infty),\frac{d\lambda}{\lambda})} \\
\leq \; & c'_{\eta,q}\|a\|_{\ell_2} + \left\| \lambda^{-\eta} K(a,\lambda;d_{1,2},\ell_2) \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\
\leq \; & c'_{\eta,q}\|a\|_{\ell_2} + \sqrt{2} \left\| \lambda^{-\eta} \left(\sum_{k=0}^{\infty} \min\{k+1,\lambda^2\}\alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\
\leq \; & c''_{\eta,q}\|a\|_{\ell_2} + \sqrt{2} \left\| \lambda^{-\eta} \left(\sum_{k=1}^{\infty} \min\{k+1,\lambda^2\}\alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\
= \; & c''_{\eta,q}\|a\|_{\ell_2} \\
& \quad + \sqrt{2} \left\| \lambda^{-\eta} \left(\sum_{k=1}^{[\lambda^2]-1} (k+1)\alpha_k^2 + \lambda^2 \sum_{k=[\lambda^2]}^{\infty} \alpha_k^2 \right)^{\frac{1}{2}} \right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} .
\end{aligned}
$$

Moreover, for $t_n := 1 - \frac{1}{n}$, $n \geq 1$, one has

$$\sum_{k=1}^{n-1}(k+1)\alpha_k^2 + n\sum_{k=n}^{\infty}\alpha_k^2 \leq 5\sum_{k=1}^{\infty}\alpha_k^2\frac{1-t_n^k}{1-t_n} = 5n\int_{t_n}^{1}(Ta)'(s)ds$$

so that

$$\begin{aligned} & \left\| \lambda^{-\eta}\left(\sum_{k=1}^{[\lambda^2]-1}(k+1)\alpha_k^2 + \lambda^2\sum_{k=[\lambda^2]}^{\infty}\alpha_k^2\right)^{\frac{1}{2}}\right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\ \leq \quad & \sqrt{10}\left\|\lambda^{-\eta}\left([\lambda^2]\int_{1-\frac{1}{[\lambda^2]}}^{1}(Ta)'(s)ds\right)^{\frac{1}{2}}\right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})}. \end{aligned}$$

Letting $Z(t) := (Ta)'(1-t)$, $t \in (0,1]$, we have that

$$[\lambda^2]\int_{1-\frac{1}{[\lambda^2]}}^{1}(Ta)'(s)ds = [\lambda^2]\int_0^{\frac{1}{[\lambda^2]}}Z(t)dt \leq \lambda^2\int_0^{\frac{1}{\lambda^2}}Z(t)dt$$

and

$$\begin{aligned} & \left\|\lambda^{-\eta}\left([\lambda^2]\int_{1-\frac{1}{[\lambda^2]}}^{1}(Ta)'(s)ds\right)^{\frac{1}{2}}\right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\ \leq \quad & \left\|\lambda^{-\eta}\left(\lambda^2\int_0^{\frac{1}{\lambda^2}}Z(t)dt\right)^{\frac{1}{2}}\right\|_{L_q([1,\infty),\frac{d\lambda}{\lambda})} \\ = \quad & 2^{-\frac{1}{q}}\left\|r^{\frac{\eta}{2}}\left(\frac{1}{r}\int_0^r Z(t)dt\right)^{\frac{1}{2}}\right\|_{L_q((0,1],\frac{dr}{r})}. \end{aligned}$$

Applying one of Hardy's inequalities (see [1] (Lemma 3.9, p. 124)) (note that $q/2 \geq 1$ and $\eta \in (0,1)$) implies that the last term can be estimated from above by

$$\frac{1}{\sqrt{1-\eta}}\left\|t^{\frac{\eta}{2}}\sqrt{Z(t)}\right\|_{L_q((0,1],\frac{dt}{t})} = \frac{1}{\sqrt{1-\eta}}\left\|t^{\frac{\eta}{2}}\sqrt{(Ta)'(1-t)}\right\|_{L_q((0,1],\frac{dt}{t})}$$

so that the remaining inequality follows.

(b) The second equivalence follows from step (a) and

$$r^{\frac{\eta}{2}}\left(\frac{1}{r}\int_0^r Z(t)dt\right)^{\frac{1}{2}} = (1-s)^{\frac{\eta}{2}-\frac{1}{2}}\sqrt{(Ta)(1)-(Ta)(s)}$$

for $r \in (0,1]$ and $r+s=1$. $\square$

Lemma 3.3 *Let $\eta \in (0,1)$, τ_n^η be the nets defined in* Formula (3)*, and $\varphi : [0,1) \longrightarrow [0,\infty)$ be a continuous and increasing function. Then the following assertions are equivalent:*

(i) *There is a constant $c_1 > 0$ such that*

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u)\varphi(u)du \le c_1 \sup_{i=1,\dots,n} \frac{t_i - t_{i-1}}{(1-t_{i-1})^\eta}$$

for all $0 = t_0 < t_1 < \cdots < t_n = 1$.

(ii) *There is a constant $c_2 > 0$ such that*

$$\sum_{i=1}^n \int_{t_{i-1}^{(n,\eta)}}^{t_i^{(n,\eta)}} (t_i^{(n,\eta)} - u)\varphi(u)du \le \frac{c_2}{n}$$

for all $n = 1, 2, \dots$

(iii) *There is a constant $c_3 > 0$ such that*

$$\int_0^1 (1-u)^\eta \varphi(u)du \le c_3.$$

For (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) one can take $c_2 = dc_1$ and $c_3 = dc_2$, where $d > 0$ is a constant depending at most on η, for (iii) $\Rightarrow$ (i) one can take $c_1 = c_3$.

Proof. *(i) $\Rightarrow$ (ii)* is trivial, so let us check first *(ii) $\Rightarrow$ (iii)*: given time-nets $(t_i^{(n)})_{i=0}^n$ with $0 = t_0^{(n)} < \cdots < t_n^{(n)} = 1$ and $\lim_n \sup_{i=1,\dots,n} |t_i^{(n)} - t_{i-1}^{(n)}| = 0$ we have that

$$\liminf_n \sum_{i=1}^n (1-t_{i-1}^{(n)})^\eta \varphi(t_{i-1}^{(n)})(t_i^{(n)} - t_{i-1}^{(n)}) \ge \int_0^1 (1-u)^\eta \varphi(u)du.$$

Furthermore,

$$
\begin{aligned}
&\sum_{i=1}^{n}(1-t_{i-1}^{(n)})^{\eta}\varphi(t_{i-1}^{(n)})(t_i^{(n)}-t_{i-1}^{(n)}) \\
&= \sum_{i=1}^{n}\frac{(1-t_{i-1}^{(n)})^{\eta}}{t_i^{(n)}-t_{i-1}^{(n)}}(t_i^{(n)}-t_{i-1}^{(n)})^2\varphi(t_{i-1}^{(n)}) \\
&\le \left(\sup_{i=1,..,n}\frac{(1-t_{i-1}^{(n)})^{\eta}}{t_i^{(n)}-t_{i-1}^{(n)}}\right)\sum_{i=1}^{n}(t_i^{(n)}-t_{i-1}^{(n)})^2\varphi(t_{i-1}^{(n)}) \\
&\le 2\left(\sup_{i=1,..,n}\frac{(1-t_{i-1}^{(n)})^{\eta}}{t_i^{(n)}-t_{i-1}^{(n)}}\right)\sum_{i=1}^{n}\int_{t_{i-1}^{(n)}}^{t_i^{(n)}}(t_i^{(n)}-u)\varphi(u)du.
\end{aligned}
$$

Applying this to $(t_i^{(n,\eta)})_{i=0}^n$ gives

$$
\int_0^1(1-u)^{\eta}\varphi(u)du \le 2\liminf_n\left(\sup_{i=1,..,n}\frac{(1-t_{i-1}^{(n,\eta)})^{\eta}}{t_i^{(n,\eta)}-t_{i-1}^{(n,\eta)}}\right)\frac{c_2}{n}\le dc_2
$$

for some $d=d(\eta)>0$.

$(iii)\Rightarrow(i)$ Here one gets that

$$
\begin{aligned}
&\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-u)\varphi(u)du \\
&= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{t_i-u}{(1-u)^{\eta}}(1-u)^{\eta}\varphi(u)du \\
&\le \left(\sup_{i=1,...,n}\sup_{u\in[t_{i-1},t_i)}\frac{t_i-u}{(1-u)^{\eta}}\right)\int_0^1(1-u)^{\eta}\varphi(u)du \\
&= \left(\sup_{i=1,...,n}\frac{t_i-t_{i-1}}{(1-t_{i-1})^{\eta}}\right)\int_0^1(1-u)^{\eta}\varphi(u)du.
\end{aligned}
$$

□

Given $f\in L_2(\gamma)$ and $F(t,x)=\mathbb{E}f(x+W_{1-t})$ we let

$$
G(t,y):=F\left(t,\frac{t}{2}+\log y\right)
$$

for $(t,y) \in (-\varepsilon,1)\times(0,\infty)$, where $\varepsilon>0$ is taken from the definition of F, so that $G \in C^\infty((-\varepsilon,1)\times(0,\infty))$ and

$$\frac{\partial G}{\partial t} + \frac{y^2}{2}\frac{\partial^2 G}{\partial y^2} = 0.$$

As function φ in Lemma 3.3 we shall take

$$H_W(f)(t) := \left\| \frac{\partial^2 F}{\partial x^2}(t,W_t) \right\|_{L_2} \quad \text{and} \quad H_S(f)(t) := \left\| S_t^2 \frac{\partial^2 G}{\partial y^2}(t,S_t) \right\|_{L_2}.$$

The functions $H_W(f)$ and $H_S(f)$ are the main tool in order to estimate $a_X(f(W_1);\tau)$ and can be computed by

Lemma 3.4 *For $f=\sum_{k=0}^\infty \alpha_k h_k \in L_2(\gamma)$ and $t\in[0,1)$ one has that*

$$\begin{aligned} H_W(f)^2(t) &= \sum_{k=0}^\infty \alpha_{k+2}^2 (k+2)(k+1)t^k, \\ H_S(f)^2(t) &= \sum_{k=0}^\infty \left(\alpha_{k+2} - \frac{\alpha_{k+1}}{\sqrt{k+2}}\right)^2 (k+2)(k+1)t^k. \end{aligned}$$

Proof. For $f\in L_2(\gamma)$, $t\in[0,1)$, and $x\in\mathbb{R}$ we use the Ornstein-Uhlenbeck type operators

$$\begin{aligned} S_t^1(f)(x) &:= \int_{\mathbb{R}} f(\sqrt{t}\,x + \sqrt{1-t}\,\eta)\,\eta\,d\gamma(\eta), \\ S_t^2(f)(x) &:= \int_{\mathbb{R}} f(\sqrt{t}\,x + \sqrt{1-t}\,\eta)\,(\eta^2-1)\,d\gamma(\eta). \end{aligned}$$

The operators S_t^1 and S_t^2 are bounded on $L_2(\gamma)$. In particular, $S_t^1(h_0) = S_t^2(h_0) = S_t^2(h_1) = 0$ and

$$\begin{aligned} S_t^1(h_k) &= \sqrt{1-t}\,\sqrt{k}\,t^{\frac{k-1}{2}} h_{k-1}, \\ S_t^2(h_l) &= (1-t)\,\sqrt{l(l-1)}\,t^{\frac{l-2}{2}} h_{l-2} \end{aligned}$$

for $k\geq 1$ and $l\geq 2$. We only check the formula for $H_S(f)$. It is known (cf., for example, [4] (Lemma A.2)) that

$$y^2\frac{\partial^2 G}{\partial y^2}(t,y) = \mathbb{E}\left(g(yS_{1-t})\left(\frac{W_{1-t}^2}{(1-t)^2} - \frac{W_{1-t}+1}{1-t}\right)\right)$$

and

$$
\begin{aligned}
& H_S(f)^2(t) \\
= & \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(\sqrt{t}\, x + \sqrt{1-t}\eta) \left(\frac{\eta^2 - 1}{1-t} - \frac{\eta}{\sqrt{1-t}} \right) e^{-\frac{\eta^2}{2}} \frac{d\eta}{\sqrt{2\pi}} \right]^2 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
= & \left\| \frac{1}{1-t} S_t^2(f) - \frac{1}{\sqrt{1-t}} S_t^1(f) \right\|_{L^2(\gamma)}^2 \\
= & \sum_{k=0}^{\infty} \left(\alpha_{k+2} - \frac{\alpha_{k+1}}{\sqrt{k+2}} \right)^2 (k+2)(k+1) t^k.
\end{aligned}
$$

□

One gets from the above lemma that the behavior of $H_W(f)$ and $H_S(f)$ is not far from each other since a simple computation implies

Lemma 3.5 *For $f = \sum_{k=0}^{\infty} \alpha_k h_k \in L_2(\gamma)$ and $t \in [0,1)$ one has*

$$
\frac{1}{12} H_W(f)^2(t) - \frac{2}{3} \left(\alpha_2^2 + \alpha_1^2 \right) \leq H_S(f)^2(t) \leq 4 H_W(f)^2(t) + 2\alpha_1^2.
$$

The following lemma is taken from [4] where it is formulated for the geometric Brownian motion. The case of the Brownian motion can be obtained by the results of [4] as well.

Lemma 3.6 *For $f \in L_2(\gamma)$, $Z = f(W_1)$, and $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ one has that*

$$
a_X(Z; \tau) \sim_c \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u) H_X(f)^2(u) du \right)^{\frac{1}{2}}
$$

where X is either the Brownian motion or the geometric Brownian motion and $c \geq 1$ is an absolute constant.

Proof of Theorem 2.1. Assuming $f = \sum_{k=0}^{\infty} \alpha_k h_k$ we get by the properties of the Ornstein-Uhlenbeck semi-group that $\mathbb{E}|\mathbb{E}(f(W_1)|\mathcal{F}_t)|^2 = \sum_{k=0}^{\infty} \alpha_k^2 t^k$ and $\mathbb{E}|(\partial F/\partial x)(t,W_t)|^2 = \sum_{k=1}^{\infty} \alpha_k^2 k t^{k-1} = (Ta)'(t)$ for $t \in [0,1)$. Now our assertion follows from Proposition 3.2 and

$$\begin{aligned}\mathbb{E}|f(W_1) - \mathbb{E}(f(W_1)|\mathcal{F}_t)|^2 &= \mathbb{E}|f(W_1)|^2 - \mathbb{E}|\mathbb{E}(f(W_1)|\mathcal{F}_t)|^2 \\ &= (Ta)(1) - (Ta)(t).\end{aligned}$$

□

Proof of Theorem 1.3. Applying Proposition 3.2 to $f = \sum_{k=0}^{\infty} \alpha_k h_k$ and $q = \infty$ gives

$$\begin{aligned}\|f\|_{(\mathbb{D}_{1,2}(\gamma),L_2(\gamma))_{\eta,\infty}} &\sim_c \|f\|_{L_2(\gamma)} + \sup_{t\in[0,1)} (1-t)^{\frac{\eta}{2}} \sqrt{(Ta)'(t)} \\ &\sim_{c'} \|f\|_{L_2(\gamma)} + \sup_{t\in[0,1)} (1-t)^{\frac{\eta}{2}} \sqrt{(Ta)'(t) - \alpha_1^2} \\ &= \|f\|_{L_2(\gamma)} + \sup_{t\in[0,1)} (1-t)^{\frac{\eta}{2}} \sqrt{\int_0^t H_W(f)^2(u)du} \\ &\sim_{c''} \|f\|_{L_2(\gamma)} + \sup_{t\in[0,1)} (1-t)^{\frac{\eta}{2}} \sqrt{\int_0^t H_S(f)^2(u)du}\end{aligned}$$

where we used Lemmas 3.4 and 3.5, and where $c, c', c'' > 0$ depend at most on η. Now the assertion follows from Lemma 3.6 and [3] (Lemma 3.1). □

Proof of Theorem 2.2. Again, applying Proposition 3.2 to the function $f = \sum_{k=0}^{\infty} \alpha_k h_k$ and, now, $q = 2$ gives

$$\begin{aligned}\|f\|_{(\mathbb{D}_{1,2}(\gamma),L_2(\gamma))_{\eta,2}} &\sim_c \|f\|_{L_2(\gamma)} + \sqrt{\int_0^1 (1-t)^{\eta-1}(Ta)'(t)dt} \\ &\sim_{c'} \|f\|_{L_2(\gamma)} + \sqrt{\int_0^1 (1-t)^{\eta-1}[(Ta)'(t) - \alpha_1^2]dt} \\ &= \|f\|_{L_2(\gamma)} + \sqrt{\int_0^1 (1-t)^{\eta-1} \int_0^t H_W(f)^2(u)dudt}\end{aligned}$$

$$
\begin{aligned}
&= \quad \|f\|_{L_2(\gamma)} + \sqrt{\frac{1}{\eta}\int_0^1 (1-t)^\eta H_W(f)^2(t)dt} \\
&\sim_{c''} \quad \|f\|_{L_2(\gamma)} + \sqrt{\int_0^1 (1-t)^\eta H_S(f)^2(t)dt}
\end{aligned}
$$

by Lemmas 3.4 and 3.5, where $c, c', c'' > 0$ depend at most on η. Finally, the assertion follows from Lemmas 3.6 and 3.3. $\square$

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