INVERSE SCHRÖDINGER SCATTERING PROBLEM FOR A RANDOM POTENTIAL IN THE PLANE

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Abstract. We study an inverse problem for the two-dimensional random Schrödinger equation $(\Delta+q+k^2)u=0$. The potential $q(x)$ is assumed to be a Gaussian random function that defines a Markov field. We show that the backscattered field, obtained from a single realization of the random potential q, determines uniquely the principal symbol of the covariance operator of q. The analysis is carried out by combining methods of harmonic and microlocal analysis with stochastic methods.

1. Introduction

Consider the Schrödinger equation with outgoing radiation condition

$$\begin{aligned} &(\Delta - q + k^2)u = \delta_y, && \text{in } \mathbb{R}^2 \\ &\left(\frac{\partial}{\partial r} - ik\right) u(x) = o(|x|^{-1/2}) && \text{as } |x| \to \infty \end{aligned} \tag{1.1}$$

where the potential q is a random generalized function supported in a compact domain D. The wave u is decomposed into two parts

$$u = u_0(x,y,k) + u_s(x,y,k),$$

where $u_s(x,y,k)$ is the scattered field and

$$u_0(x,y,k) = \Phi_k(x-y) = -\frac{i}{4}H_0^{(1)}(k|x-y|)$$

is the incident field corresponding to a point source at y and $H_0^{(1)}(\cdot\,)$ is the Hankel function of the first kind. We shall assume that $y \in U$, where the domain $U \subset \mathbb{R}^2 \setminus \overline{D}$, called the *measurement domain*, is bounded and convex. In the measurement domain U it is possible to make measurements of the scattered wave.

The aim of inverse scattering theory is usually to determine the scatterer q from appropriate measurements. On the other hand, in many applications the scatterer can be very complicated and non-smooth. For such scatterers, the inverse problem is not so much to recover the exact micro-structure of an object but merely to determine the parameters or functions describing the properties of the micro-structure. One example of such a parameter is the correlation length of the medium which is related to the typical size of "particles" inside of the scatterer. In mathematical terms,

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assume that the potential q has been created by a random process, i.e., $q = q(x, \omega_0)$ is a realization of a random function $q(x, \omega)$. Here ω_0 denotes an element of the probability space Ω. As $q(x, \omega)$ is random, the scattered field is also random and we sometimes emphasize this by writing $u_s(x, y, k) = u_s(x, y, k, \omega)$. The inverse problem is to determine the parameters describing the random process $q(x, \omega)$, e.g., from the energy of the scattered wave $|u_s(x, y, k, \omega_0)|^2$.

In applied literature the measured data is often assumed to coincide with the averaged data $\mathbb{E}\,|u_s(x, y, k, \omega)|^2$. This corresponds to the case when the measurements could be made from many independent samples of the scatterer and these measurements could be averaged. This appears to not always be a well justified assumption since often the scatterer does not change during the period of measurements. Also, in applications the multiple scattering is often omitted. This leads to a linearization of the inverse problem which can be justified only when q is small.

A related approach for the scattering from a random medium is the study of the multi-scale asymptotics of the scattered field. In this case the approximations made can be justified when the frequency k and the spatial frequency of the scatterer have appropriate magnitudes. This type of asymptotic analysis has been studied by Papanicolaou and others in various cases, cf. e.g. [40], [41], [6], [9]. Random Schrödinger operators have also been studied from the point of view of spectral theory in celebrated papers of Kotani, [26],[27] and extensively generalized by B. Simon and others (cf. [42],[16],[46],[45],[28]).

To avoid any approximations such as linearization we apply to the above stochastic inverse problem techniques that are developed for similar deterministic problems. Actually, our stochastic setup leads to new type of analysis problems. Our tools for treating them include harmonic and microlocal analysis, which techniques are also often used in the deterministic case, cf. [11], [31], [32], [48]. An extensive review for this is given in [50]. For inverse problems involving non-smooth deterministic structures see e.g. [10], [37], [38], [18], and [5]. From probability theory we apply e.g. the theory of Gaussian fields and ergodicity of random processes.

The main result of this paper (Theorem 1.3 below) shows that the mean values over the frequency k of the backscattered energy $|u_s(x, x, k, \omega_0)|^2$, obtained from a single realization $q(z, \omega_0)$, almost surely determine the micro-structure of the random potential, or more exactly the principal symbol of the covariance operator of the random function $q(x, \omega)$. We stress that, after the model for the random potential is fixed, no approximations are made. In particular, we study the full non-linear problem. We next describe our results in detail in Subsections 1.1 and 1.2 below.

1.1. **The model for the random potential.** Fix a bounded simply connected domain $D \subset \mathbb{R}^2$. We assume that the potential q is a generalized Gaussian field supported in D. Recall, that this means that q is a measurable map from the probability space Ω to the space of distributions $\mathcal{D}'(\mathbb{R}^2)$ such that for all $\phi_1, \dots, \phi_m \in C_0^\infty(\mathbb{R}^2)$ the mapping $\Omega \ni \omega \mapsto (\langle q(\omega), \phi_j \rangle)_{j=1}^m$ is a Gaussian random variable.

We will assume that the probability measure space $(\Omega, \mathcal{F}, \mathcal{P})$ is complete. The distribution of q is determined by the expectation $\mathbb{E}\, q$ and the covariance operator $C_q : C_0^\infty(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ defined by

$$\langle \psi_1, C_q \psi_2 \rangle = \mathbb{E}\,(\langle q - \mathbb{E}\, q, \psi_1 \rangle \langle q - \mathbb{E}\, q, \psi_2 \rangle). \tag{1.2}$$

Let $k_q(x, y)$ be the Schwartz kernel of the covariance operator C_q. We call $k_q(x, y)$ the covariance function of q. Then, in the sense of generalized functions, (1.2) reads as

$$k_q(x, y) = \mathbb{E}\,((q(x) - \mathbb{E}\, q(x))(q(y) - \mathbb{E}\, q(y))).$$

It is often assumed that the covariance function $k_q(x, y)$ is singular only on the diagonal since it is natural to assume that the long range interactions depend smoothly on location. Also the basic stochastic processes like the Brownian bridge or the free Gaussian field possess this property. As the above property is characteristic for Schwartz kernels of pseudodifferential operators, we introduce the following definition.

Definition 1.1. Let $\mu \in C_0^\infty(D)$, $\mu(x) \geq 0$. A generalized Gaussian random field q on $\mathbb{R}^2$ is said to be microlocally isotropic (of order $\tau \geq 0$) in D, if the realizations of q are almost surely supported in the domain D and its covariance operator C_q is a classical pseudodifferential operator having the principal symbol $\mu(x)|\xi|^{-\tau}$.

In particular, we are interested in the case $\tau = 2$, cf. subsection 1.1 where examples of the micro-locally isotropic fields of order two are considered. In Section 3 we will show that the Schrödinger equation has a.s. a unique solution for such potentials.

We call μ the *micro-correlation strength* of q. In our situation the covariance function $k_q(z_1, z_2)$ is locally integrable for fixed z_2 and has the asymptotics

$$k_q(z_1, z_2) = -\mu(z_2) \log |z_1 - z_2| + f(z_1, z_2)$$

where f is locally bounded. Hence the function $\mu(z)$ describes the strength of the singularity of k_q near the diagonal, and it determines approximately the radius of the set $\{z_1 : \ k_q(z_1, z_2) > M\}$ with a given large bound M. Thus micro-correlation strength function μ is closely related to the local correlation length of the random field.

We are ready to formulate the measurement configuration. Recall that u_s is the scattered field corresponding to problem (1.1).

Definition 1.2. Given $\omega \in \Omega$, the measurement $m(x, y, \omega)$, $x, y \in U$ is the pointwise limit

$$m(x, y, \omega) = \lim_{K \to \infty} \frac{1}{K - 1} \int_1^K k^4 |u_s(x, y, k, \omega)|^2 dk. \tag{1.3}$$

An important special case is the backscattering measurement $m(x, x, \omega)$.

The measurement in the above definition is an average over all frequencies whence it is not sensitive to measurement errors. For example, the white noise error in the measurement is filtered out by frequency averaging. Note also that the measurement uses information only from the amplitude (not the phase) of the scattered field. It is truly a non-trivial fact that the above definition gives a well-defined, finite and non-zero quantity. That this is so, is part of Theorem 1.3 below.

1.2. **The result.** The main result of this paper is

Theorem 1.3. *Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain, $U \subset \mathbb{R}^2 \setminus \overline{D}$ be a bounded and convex domain, and let q be a microlocally isotropic Gaussian random field of order two in D, as described in Definition* 1.1. *Then*

- **(i)** *For any $x, y \in U$ the measurement $m(x,y,\omega)$ is well-defined* (*that is, the limit in* (1.3) *exists almost surely*).
- **(ii)** *There exists a continuous deterministic function $m_0(x,y)$ such that for any $x,y \in U$ the equality $m(x,y,\omega) = m_0(x,y)$ holds almost surely. In particular, the function $m_0(x,x)$ is almost surely determined by the backscattering data $m(x,x,\omega)$.*
- **(iii)** *The backscattering data $n_0(x) := m_0(x,x), \quad x \in U$ uniquely determines the micro-correlation strength function μ in the domain Ω. Moreover, there is a linear operator T such that*

$$T(n_0) = \mu.$$

We point out that by the above result the principal structure of the covariance is determined by measurements from only a single realization of the potential only! Observe that the needed data is the energy averages of the back-scattered field. However, one should note that the result uses backscattering data from sending locations that are at a finite distance from the potential – often one considers far-field data in connection with the deterministic backscattering problem (cf. [15], [14], [47] and [35]). We refer to the Remark 3 at the end of Section 7 for a more thorough discussion of the relation of the above result to its deterministic counterparts. Property (ii) in Theorem 1.3 is sometimes called statistical stability, c.f. [9].

Finally, we note that using the fact that the measurements $m(x,y,\omega)$ exist, our analysis could be generalized to other measurements. For instance one can analyze fixed source point measurements $x \mapsto m(x,y_0,\omega)$ where $y_0 \in U$ is fixed and $x \in U$. This and other physically important measurements will be considered elsewhere in detail.

Because of notational simplicity, we will assume in the proof of Theorem 1.3 that $\mathbb{E}\, q = 0$; one can easily dispense with this assumption (see Remark 1 at the end of Section 7).

The rest of the paper is organized as follows: Subsection 1.3 describes some natural examples of random potentials. The smoothness of the random potential is considered in Section 2. It turns out that q is a proper distribution almost surely.

Section 3 treats the scattering problem for a class of distributional potentials. Especially, one obtains that (1.1) has a unique solution in our case. Section 4 considers oscillatory integrals in order to establish the asymptotic independence of the solutions $u(x, y, k_1)$ and $u(x, y, k_2)$ for large values of $|k_1 - k_2|$ in the Born approximation. The validity of this approximation in the context of our measurements is verified in Section 5. The results of the previous sections are combined in Section 6, where it is shown that the measurements can be expressed as a deterministic weighted avarage over the unknown parameter μ. Finally, Section 7 verifies that this data allows us to recover μ almost surely.

1.3. **Markov fields.** Here we introduce the notion of Markov fields which provide natural examples of microlocally isotropic fields. Assume that the potential q is a localization of a generalized Gaussian field, that is, $q = \chi Q$, where $\chi \in C_0^\infty(D)$ and Q is a centered (i.e., $\mathbb{E}\, Q = 0$) generalized Gaussian field on $\mathbb{R}^2$. The reason for introducing the cutoff χ is to avoid the possible effects arising from discontinuity at the boundary ∂D.

To obtain a more concrete structure we will assume further that Q has additionally a Markov structure. Below we will recall the definition of such fields. The basic properties of generalized Markov fields can be found in [44]. The definition of Markov fields mimics the situation where physical particles in a lattice have no long-term interaction, i.e., only neighboring particles have direct interaction. Assume that $S_1 \subset D$ is an open set with $\overline{S_1} \subset D$. We set $S_2 = D \setminus \overline{S_1}$ and $S_\epsilon = \{x \in D : d(x, \partial S_1) \leq \varepsilon\}$, $\varepsilon > 0$, a collar neighborhood of the boundary ∂S_1. Intuitively the Markov property means that the influence from the inside to the outside must pass through the collar.

Definition 1.4. A generalized random field Q on $\mathbb{R}^2$ satisfies the Markov property if for any S_1, S_2 and S_ε as described above, and $\varepsilon > 0$ small enough, the conditional expectations satisfy

$$\mathbb{E}\,(h \circ Q(\psi)|\mathcal{B}(S_\varepsilon)) = \mathbb{E}\,(h \circ Q(\psi)|\mathcal{B}(S_\varepsilon \cup S_1))$$

for any complex polynomial h and for any test function $\psi \in C_0^\infty(S_2)$.

Here $\mathcal{B}(S_j)$ is the σ-algebra generated by the random variables $Q(\phi)$, $\phi \in C_0^\infty(S_j)$, $j = 1, 2$, and $\mathcal{B}(S_\varepsilon)$ is defined respectively.

The Markov property has dramatic implications to the structure of the field Q and especially to its covariance operator C_Q. Under minor additional conditions (cf. [44]), we may define the inverse operator $(C_Q)^{-1}$ which turns out to be a local operator: it cannot increase the support of a test function. By a well-known theorem of J. Peetre [43] $(C_Q)^{-1}$ must be a linear partial differential operator. As C_Q is non-negative operator, $(C_Q)^{-1}$ has to be of even order. To obtain an isotropic situation we finally assume that $(C_Q)^{-1}$ is a non-degenerate elliptic operator, is of 2nd order, has smooth coefficients, and finally its principal part is positive and homogeneous.

This implies that

$$(C_Q)^{-1} = P(z, D_z) = -\sum_{j,k=1}^{2} \frac{\partial}{\partial z^j} a(z) \frac{\partial}{\partial z^k} + b(z), \tag{1.4}$$

where $a(z) > 0$ and $b(z)$ are smooth real functions in $\mathbb{R}^2$. Then the field Q is microlocally isotropic of order two as C_Q is a pseudodifferential operator with an isotropic principal symbol.

To motivate the assumption that the order of $(C_Q)^{-1}$ is two, let us consider the case where $(C_Q)^{-1}$ would be of fourth order or higher, with smooth coefficients. Then one could easily verify (cf. the proof of Theorem 2.3) that the realizations of q are in the Sobolev class $H^{s,p}_{comp}(\mathbb{R}^2)$ for all $s < 1$ and $1 < p < \infty$. As our aim is to consider the case of non-smooth potentials, the second order case is the most interesting in view of many applications. An important example of such random fields of this type is obtained by the free Gaussian fields, which appear in two dimension quantum field theory (c.f. e.g. [19]). The free Gaussian field on the bounded domain D, corresponding to Dirichlet boundary values, has the (Dirichlet-)Green's function as the kernel of its covariance operator. This corresponds to choices $a(z) = 1$, $b(z) = 0$. Examples with variable $a(z)$ can be constructed easily.

Finally, the covariance operator C_q of the potential q has the kernel

$$k_q(z_1, z_2) = \chi(z_1) k_Q(z_1, z_2) \chi(z_2).$$

This implies that q is microlocally isotropic of order two in D and has the microcorrelation strength function $\mu(z) = \chi(z)^2\, a(z)^{-1}$.

2. Regularity of the stochastic potential

We will study what kind of regularity (or irregularity) is implied for the potential by Definition 1.2. Actually it turns out that $q(\omega)$ is not a function (or even a measure); almost surely it is a proper distribution. This is not so surprising since similar phenomena are well known in the case of a free Gaussian field. However, the potential just barely fails to be a function: almost every realization of the potential satisfies

$$q(\omega) \in H_0^{-\epsilon,p}(D) \quad \text{for all} \quad \varepsilon > 0 \quad \text{and} \quad 1 < p < \infty. \tag{2.1}$$

Here, $H^{s,p}(\mathbb{R}^2) = \mathcal{F}^{-1}((1+|\xi|^2)^{-s/2}\mathcal{F}L^p(\mathbb{R}^2))$ is the standard Sobolev space, defined with the Fourier transform $\mathcal{F}$ and $H_0^{s,p}(D)$ is the closure of $C_0^\infty(D)$ in $H^{s,p}(\mathbb{R}^2)$. In this section we verify (2.1), which is crucial for the success of the subsequent analysis of our problem. For example, it enables us to prove in the following section the uniqueness for the corresponding scattering problem, even though the uniqueness is known to fail for certain integrable potentials.

We start by recording a result which yields a criterion for realizations of a random field to lie in $\bigcap_{p>1} L^p(D)$. Here and below c denotes a generic constant the value of which may change even inside a formula.

Lemma 2.1. *Assume that the covariance operator K of a random field F on the open bounded set $D \subset \mathbb{R}^n$ has a regular kernel (denoted also by $K(x,y)$) satisfying*

$$|K(x,y)| \leq c < \infty \quad \text{for every} \quad x, y \in D.$$

Then the realizations of F belong almost surely to $\bigcap_{p>1} L^p(D)$.

Proof. This is an immediate consequence of [8, Prop. 3.11.15]. To sketch a direct proof of this result, one may first mollify F and observe that in the smooth case $\mathbb{E}\,(\|F\|_p)^p = c_p \int_D |K(x,x)|^{p/2} dx$. $\square$

We next analyze the singularity of the covariance operator of q which we denote by $C = C_q$.

Proposition 2.2. *The Schwartz kernel of the covariance operator C may be decomposed as*

$$C(x,y) = c_0(x,y) \log|x-y| + r_1(x,y),$$

where $c_0 \in C_0^\infty(D \times D)$ and the term r_1 satisfies $\widehat{r}_1 \in L^1(\mathbb{R}^4)$, and, consequently

$$\sup_{x,y \in D} |r_1(x,y)| < \infty. \tag{2.2}$$

Here $\widehat{r}_1$ denotes the Fourier transform of r_1 with respect to both variables x, y.

Proof. By definition, $C(x,y)$ is a kernel of a (compactly supported) classical pseudodifferential operator with symbol $a(x,\xi) = \mu(x)(1-\psi(\xi))|\xi|^{-2} + b(x,\xi)$ in the class $S^{-2}(\mathbb{R}^2 \times \mathbb{R}^2)$ (c.f. [22]), where the smooth cutoff $\psi \in C_0^\infty(\mathbb{R}^2)$ equals 1 near the origin, and $b \in S^{-3}(\mathbb{R}^2 \times \mathbb{R}^2)$. We obtain

$$\begin{aligned} 2\pi^2 C(x,y) &= \mu(x) \int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi} (1-\psi(\xi))|\xi|^{-2} d\xi + \int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi} b(x,\xi) d\xi \\ &= I(x,y) + r_2(x,y), \end{aligned}$$

where $I(x,y)$ may clearly be written in the desired form $c_0(x,y)\log|x-y| + r(x,y)$. In order to analyze the residual term $r_2(x,y)$, write $R(x,y) = r_2(x, x-y)$, and observe that $\widehat{r}_1 \in L^1(\mathbb{R}^4)$ follows as soon as we show that $\widehat{R}$ is integrable. A simple computation shows that $\widehat{R}$ is the Fourier transform of the symbol b with respect to x. Since the support of b is compact with respect to x, and we have the uniform estimates $|\partial_x^\alpha b(x,\xi)| \leq c_\alpha (1+|\xi|)^{-3}$, $\alpha \geq 0$, it follows that $|\mathcal{F}_x b(\eta,\xi)| \leq c(1+|\eta|)^{-3}(1+|\xi|)^{-3}$. This verifies that $\widehat{R} \in L^1$. The proof is complete. $\square$

The following implication is needed for realizations of q.

Theorem 2.3. *Almost surely $q(\omega) \in H^{-\epsilon,p}(D)$ for all $\varepsilon > 0$ and $1 < p < \infty$.*

Proof. Recall that for given $s \in \mathbb{R}$ the Bessel potential J^s provides an isomorphism $J^s : H^{t,p}(\mathbb{R}^2) \to H^{t+s,p}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$ and $1 < p < \infty$. Moreover, J^s is a pseudodifferential operator, whence it preserves singular supports. Thus it is enough to verify that locally the covariance of $J^\varepsilon q$ has a uniformly bounded kernel for any small $\varepsilon > 0$. That is, by letting J^ε_{loc} stand for a suitable localization of J^ε we have to study the kernel $J^\varepsilon_{loc} c J^\varepsilon_{loc}$. It is well known that for small $\varepsilon > 0$ the kernel has form

$$J^\varepsilon(x,y) = \frac{c}{|x-y|^{2-\varepsilon}} + S(x,y),$$

where S has a lower order singularity. Now the claim follows by combining Proposition 2.2 and the fact

$$\int_{B(0,R)} \frac{|\log|x||}{|x|^{2-\varepsilon}} dx < \infty$$

for any radius $R > 0$. $\square$

3. Direct scattering problem for a distributional potential.

3.1. **Unique continuation.** We showed above that the random potential $q(\omega)$ belongs with probability one to the Sobolev space $H^{-\epsilon,p}(D)$ for all $1 \le p < \infty$ and $\epsilon > 0$. Consequently, we need to study the existence and properties of the solution for the Schrödinger equation for such irregular potentials. In this section we accomplish this by considering scattering from a deterministic non-smooth potential $q_0 \in H^{-\epsilon,p}(D)$, and the obtained results have independent interest.

The direct scattering theory from a potential that is in a weighted L^2 space is classical (c.f. [7],[3]). For the L^p scattering theory the key tool is the unique continuation of the solution. Jerison and Kenig showed in [23] that the strong unique continuation principle for L^p-potentials in $\mathbb{R}^n$ holds for $p \ge n/2$ and fails for $p < n/2$ in dimensions $n > 2$. In dimension two the unique continuation holds in a space of functions that is close to L^1 [23]. For Sobolev space potentials, the selfadjointness of the operator has been studied in [34]. Below in Lemma 3.2 we show a positive result for negative index Sobolev spaces.

More precisely, we study the scattering problem

$$\begin{cases} (\Delta - q_0 + k^2)u = \delta_y \\ \left(\frac{\partial}{\partial r} - ik\right) u(x) = o(|x|^{-1/2}) \end{cases} \tag{3.1}$$

where the potential $q_0 \in H^{-\epsilon,p'}_{\text{comp}}(\mathbb{R}^2)$, $p^{-1} + (p')^{-1} = 1$, $1 < p < 2$. We claim that the problem (3.1) is equivalent to the Lippmann-Schwinger equation

$$u(x) = u_0(x) - \int_{\mathbb{R}^2} \Phi_k(x-y) q_0(y) u(y) dy. \tag{3.2}$$

In the proof we show that the pointwise product $q_0 u$ in the integrand of (3.2) is well defined and that the integral exists in the sense of distributions. We will then show that (3.2) has a unique solution $u \in H^{2p,\epsilon}_{loc}(\mathbb{R}^n)$. The starting point is the unique continuation principle. Roughly speaking, it says that if u is a compactly supported solution of the Schrödinger equation with $q_0 \in H^{-\epsilon,r}, r > n/2$ and if ϵ is small then u must vanish identically. It appears to the authors that this result could also be obtained as a special case of D. Tataru's and H. Koch's recent unique continuation results based on L^p Carleman estimates [25]. In our case, we present a direct and simple proof for unique continuation. We start by observing that known pointwise multiplication results allow us to define the product distribution $q_0 u$.

Lemma 3.1. *Assume that $u \in H^{\epsilon,2p}_{loc}(\mathbb{R}^n)$, $q_0 \in H_0^{-\epsilon,p'}(\mathbb{R}^n)$, $1 < p < \infty$, $\epsilon > 0$. Then the product $q_0 u$ is well-defined as an element of $H_0^{-\epsilon,\widetilde{p}}(\mathbb{R}^n)$, where $\widetilde{p} = \frac{2p}{2p-1}$ and*

$$||q_0 u||_{H_0^{-\epsilon,\widetilde{p}}(\mathbb{R}^n)} \leq c||q_0||_{H_0^{-\epsilon,p'}(\mathbb{R}^n)}||u||_{H^{\epsilon,2p}(\mathbb{R}^n)}. \tag{3.3}$$

Proof: Take $\phi \in C_0^\infty(\mathbb{R}^n)$ to be a test function. By duality, the product $q_0 u \in \mathcal{D}'(\mathbb{R}^n)$ is a well defined through

$$\langle q_0 u, \phi\rangle = \langle q_0, \phi u\rangle \tag{3.4}$$

when $q_0 \in H^{-\epsilon,p'}_{\rm comp}(\mathbb{R}^n)$ and $u \in H^{\epsilon,p}_{\rm loc}(\mathbb{R}^n)$. By using Bony's paraproducts one can verify the following pointwise multiplier estimate in Sobolev spaces ([49, pp. 105])

$$\|\phi u\|_{H^{\epsilon,p}(\mathbb{R}^n)} \leq c\left(\|\phi\|_{L^{r_1}(\mathbb{R}^n)}\|u\|_{H^{\epsilon,r_2}(\mathbb{R}^n)} + \|u\|_{L^{r_1}(\mathbb{R}^n)}\|\phi\|_{H^{\epsilon,r_2}(\mathbb{R}^n)}\right) \tag{3.5}$$

for $1/p = 1/r_1 + 1/r_2$. From (3.4) and (3.5) with $r_1 = r_2 = 2p$ it readily follows by duality that $q_0 u \in H_0^{-\epsilon,\widetilde{p}}(\mathbb{R}^n)$ where $\widetilde{p} = \frac{2p}{2p-1}$. □

Proposition 3.2 (Unique continuation principle into an interior domain)**.** *Assume that $p' \in (n/2, \infty)$, together with $0 < \epsilon < \frac{n}{4}(2/n - 1/p')$. Let $q_0 \in H^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$. If $u \in H^{\epsilon,2p}_{loc}(\mathbb{R}^n)$ is compactly supported and satisfies the Schrödinger equation*

$$(\Delta - q_0 + k^2)u = 0$$

in the weak sense, then $u = 0$ identically.

Proof: Assume that the support of q is contained in the bounded domain $D \subset \mathbb{R}^n$. To prove the unique continuation we use the well-known techniques of exponentially growing solutions for the Schrödinger equation, cf. [20], [24]. To this end we write the equation $(\Delta + k^2)u = q_0 u$ as

$$(\Delta + 2i\zeta \cdot \nabla)e^{-i\zeta\cdot x}u = e^{-i\zeta\cdot x}q_0 u,$$

where $\zeta \in \mathbf{C}^n$ is such that $\zeta \cdot \zeta = k^2$. Since u is supposed to have compact support we have $v := e^{-i\zeta\cdot x}u \in H^{\epsilon,2p}_{\rm comp}(\mathbb{R}^2)$. For v we obtain the equation

$$v = \mathcal{G}_\zeta(q_0 v) \tag{3.6}$$

where the Faddeev operator $\mathcal{G}_\zeta$ is defined as the Fourier multiplier

$$\mathcal{G}_\zeta(f)(x) = \mathcal{F}^{-1}\Big(\frac{-1}{\xi^2 + 2\zeta\cdot\xi}\hat{f}\Big)(x).$$

It is well known (see for example the proof of Theorem 4.1 in [36]) that for $0 \leq s \leq \frac{1}{2}$

$$\|\mathcal{G}_\zeta\|_{H_0^{-s}(D)\to H^s(D)} \leq \frac{c}{|\zeta|^{1-2s}} \tag{3.7}$$

where $H^s(D) = H^{s,2}(D)$ and $H_0^s(D) = H_0^{s,2}(D)$ are L^2-based Sobolev spaces. By [24], G_ζ is a bounded operator

$$G_\zeta : L^r(D) \to L^{r'}(D), \tag{3.8}$$

for $r = \frac{2n}{n+2}$ if $n \geq 3$ and for $r > 1$ for $n = 2$. We continue first in the case $n \geq 3$. Interpolation of (3.7) and (3.8) yields

$$||G_\zeta||_{H_0^{-\epsilon,\widetilde{p}}(D)\to H^{\epsilon,2p}(D)} \leq c|\zeta|^{-(1-2s)\theta} \tag{3.9}$$

where $\epsilon = \theta s$ and $\theta = 1 - \frac{n}{2p'}$. Finally, (3.3), (3.6), and (3.9) show that

$$||v||_{H^{\epsilon,2p}(D)} \leq \frac{c}{|\zeta|^{(1-2s)\theta}} ||v||_{H^{\epsilon,2p}(D)} \tag{3.10}$$

Choosing $0 < s < \frac{1}{2}$ and ζ large enough, we conclude that v and hence u must vanish identically. Finally in the case $n = 2$ we interpolate (3.7) and (3.8) for $r > 1$ and by letting $r \to 1$ the same conclusion follows. □

Remark. Note that for $n = 2$ the uniqueness follows for $u \in H^{\varepsilon,r}_{\mathrm{loc}}(\mathbb{R}^2)$ when $r > 2$, $0 < \varepsilon < \frac{1}{r}$, and $q_0 \in H^{-\varepsilon,r'}_{\mathrm{comp}}(\mathbb{R}^2)$.

3.2. Existence and uniqueness for solutions of the scattering problem.

After having proven the unique continuation principle, the proofs of Theorems 3.3 and 3.4 below are relatively straightforward extensions of classical proofs for regular potentials. For the convenience of the reader, we include the details.

Theorem 3.3. *For $q_0 \in H^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$, with $n \geq 2$, $p' \in (n/2, \infty)$, and $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p'})$, the Lippmann-Schwinger equation* (3.2) *has a unique solution $u \in H^{\epsilon,2p}_{loc}(\mathbb{R}^n)$.*

Proof: Let D be a bounded domain such that $\mathrm{supp}\,(q_0) \subset D$. Consider the equation (3.2) in $H^{\epsilon,2p}(D)$. Since the operator H_k,

$$H_k f = \Phi_k * f, \tag{3.11}$$

defines a bounded operator $H_k : H_0^{-s}(D) \to H^s(D)$ for $s \leq 1$ we see from Sobolev embedding and Rellich's compact embedding theorem that $H_k : H_0^{-\epsilon,\widetilde{p}}(D) \to H^{\epsilon,2p}(D)$ compactly. This and Proposition 3.2 give that the operator $K_k : H^{\epsilon,2p}(D) \to H^{\epsilon,2p}(D)$, $K_k f = H_k q_0 f$ is compact.

Thus by Fredholm's alternative it is enough to show that in $H^{\epsilon,2p}(D)$ the homogeneous equation

$$u = H_k q_0 u \tag{3.12}$$

has only the trivial solution $u = 0$. If $u \in H^{\epsilon,2p}(D)$ satisfies (3.12) then u belongs to the Schwartz class $\mathcal{S}'$ and by taking the Fourier transform we obtain in the sense of distributions that $(\Delta + k^2)u(x) = q_0 u$. In particular u must be smooth in $\mathbb{R}^n \setminus \overline{D}$ and satisfy $(\Delta + k^2)u = 0$ there. Note that by (3.12) the values of u in D define u in all of $\mathbb{R}^n$.

As the fundamental solution and its derivatives satisfy the radiation condition, we see from (3.12) that u also satisfies the radiation condition in (1.1). Thus, as u is a classical solution in $\mathbb{R}^n \setminus D$ satisfying the radiation condition, it has a far field expansion (cf. [12, Thm. 2.14]). By Rellich's lemma (cf. [12, Lem. 2.11]) and the unique continuation principle it is enough to show that the far field u_∞ of u, defined by

$$u(x) = \frac{e^{ik|x|}}{4\pi|x|^{(n-1)/2}} u_\infty\left(\frac{x}{|x|}\right) + o\left(|x|^{-(n-1)/2}\right)$$

as $|x| \to \infty$, vanishes for u.

Note that

$$\Delta u = (q_0 - k^2)u \in H^{-\epsilon,\widetilde{p}}(\mathbb{R}^2) + H^{\epsilon,2p}_{loc}(\mathbb{R}^n).$$

This implies that $\nabla u \in L^2_{loc}$ and that u and Δu belong locally to spaces that are dual to each other. Take $r > 0$ so large that $\overline{D} \subset B(0,r)$. Thus by approximating u by smooth functions we get from Green's formula

$$\mathrm{Im} \int_{|x|=r} u \frac{\partial}{\partial\nu}\overline{u}\, ds = \mathrm{Im} \int_{|x|\le r} \left(|\nabla u|^2 + (q_0 - k^2)|u|^2\right)\, dx = 0.$$

Thus

$$\int_{|x|=r} \left(\left|\frac{\partial}{\partial\nu} u\right|^2 + k^2|u|^2\right)\, ds = \int_{|x|=r} \left|\frac{\partial}{\partial\nu} u - iku\right|^2\, ds \to 0$$

as $r \to \infty$. Especially, this implies that $||u||_{L^2(\{|x|=r\})} \to 0$ as $r \to \infty$. This is possible only if $u_\infty \equiv 0$. Thus the assertion is proven. $\square$

Theorem 3.4. *For $q_0 \in H^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$, with $n \ge 2$, $p' \in (n/2, \infty)$, and $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p'})$, the scattering problem (3.1) is equivalent to the Lippmann-Schwinger equation and thus has a unique solution $u \in H^{\epsilon,2p}_{loc}(\mathbb{R}^n)$.*

Proof: As reasoned in the proof of the previous theorem a solution of the Lippmann-Schwinger equation satisfies (3.1). Suppose $u \in H^{\epsilon,2p}_{loc}(\mathbb{R}^n) \cap \mathcal{S}'$ is a solution of (3.1). We need to show that

$$u_s(x) = \int \Phi_k(x-y) q_0(y) u(y)\, dy. \tag{3.13}$$

Since $(\Delta + k^2)u_s = q_0 u \in H^{-\epsilon,\widetilde{p}}_{\mathrm{comp}}(\mathbb{R}^n)$ and $\Phi_k(x - \cdot) \in H^{\epsilon,2p}_{\mathrm{loc}}(\mathbb{R}^n)$ and both functions are real-analytic outside a large ball we have from (3.1) in the sense of distributions that

$$\int_{|y|\le r} \Phi_k(x-y)(\Delta + k^2)u_s(y)\, dy = H_k(q_0 u). \tag{3.14}$$

Denote the operator that operates to u_s in the left hand side of (3.14) by T. Now for $\phi \in C^\infty(\mathbb{R}^n)$,

$$T\phi = \phi + \int_{|y|=r} \left(\Phi_k(\cdot - y) \frac{\partial}{\partial r(y)} \phi(y) - \frac{\partial}{\partial r(y)} \Phi_k(\cdot - y)\phi(y) \right) ds(y).$$

Thus, approximating u_s with smooth functions we obtain

$$u_s(x) + \int_{|y|=r} (\Phi_k(x-y) \frac{\partial}{\partial r(y)} u_s(y) - \frac{\partial}{\partial r(y)} \Phi_k(x-y) u_s(y))\, ds(y) = H_k(q_0 u).$$

From the radiation condition it follows that the boundary integral in the above formula approaches zero as $r \to \infty$, cf. [12, Thm. 2.4]. This proves (3.13) and hence the theorem. □

Note that, in view of Theorem 2.3, Theorem 3.4 implies that the original stochastic scattering problem (1.1) has a unique solution almost surely.

4. The asymptotic independence of the first order Born term

By iterating the Lippmann-Schwinger equation, one can formally represent u as the Born series,

$$u(x,y,k) = u_0(x,y,k) + u_1(x,y,k) + u_2(x,y,k) + \dots \tag{4.1}$$

where $u_0(x,y,k) = \Phi_k(x-y)$ and $u_{n+1} = (\Delta + k^2 + i0)^{-1}(qu_n)$. A considerable part of our work consist of analyzing the different terms in this development. We will later prove in subsection 5.2 that the series (4.1) converges for large enough values of k. In the proof of our main result we need to establish asymptotic independence for the first terms in the Born series, corresponding to different values of k. The verification of this fact leads to estimation of certain oscillatory integrals, and needs a fairly involved computation. As a useful tool we apply the calculus of conormal distributions. The results of this section will be applied later in section 6.

As the first term in the Born series is

$$u_1(x,y,k) = \int_D \Phi_k(x-z) q(z) \Phi_k(z-y)\, dz,$$

we start with the asymptotics of $\Phi_k(z) = -\frac{i}{4} H_0^{(1)}(k|z|)$, when $k \to \infty$. These are given by

$$\Phi_k(z) = \sqrt{\frac{1}{k|z|}} e^{i(k|z| - \pi/4)} F(\frac{1}{k|z|}), \quad F(t) = \sum_{j=0}^\infty d_j t^j, \quad t > 0, \tag{4.2}$$

where $d_0 = -\frac{i}{\sqrt{8\pi}}$ and d_j are constants whose actual values are not important for us in the sequel. The series (4.2) and its derivative have the property that for $N > 1$ (c.f. [1, formulae 9.1.27, 9.2.7–9.2.10])

$$|F(t) - \sum_{j=0}^N d_j t^j| \le c t^{N+1}, \quad |\frac{d}{dt}(F(t) - \sum_{j=0}^N d_j t^j)| \le c t^N, \quad 0 < t < 1. \tag{4.3}$$

Using first three terms in the asymptotics of Φ_k, we write

$$u_1(x,y,k) = \widetilde{u}_1(x,y,k) + b(x,y,k) \tag{4.4}$$

where, for $k \geq 1$

$$\widetilde{u}_1(x,y,k) = \int_D \Phi_k^{(3)}(x-z)q(z)\Phi_k^{(3)}(z-y)\,dz,$$

$$\Phi_k^{(3)}(z) = (k|z|)^{-\frac{1}{2}} e^{i(k|z|-\pi/4)} \sum_{j=0}^{3} d_j\,(k|z|)^{-j}.$$

Let us denote by $\mathcal{O}(k_1^{-n_1}k_2^{-n_2})$ functions $h(x,y,k_1,k_2)$ which satisfy an estimate $|h(x,y,k_1,k_2)| \leq ck_1^{-n_1}k_2^{-n_2}$ for $x,y \in U$ and $k_1,k_2 \geq 1$ where c is independent of x,y,k_1, and k_2. Next we compute the asymptotic expansion for the covariance of $\widetilde{u}_1$ thus showing that the fields $\widetilde{u}_1$ with different frequencies are asymptotically independent. We emphasize that formula (4.8) below is crucial for the construction of $\mu(z)$ in Section 7.

Proposition 4.1. *For $k_1, k_2 \geq 1$ the random variable $\widetilde{u}_1$ satisfies uniformly for $x, y \in U$ the estimates*

$$|\mathbb{E}\,(\widetilde{u}_1(x,y,k_1)\overline{\widetilde{u}_1(x,y,k_2)})| \leq \frac{c_n}{(k_1+k_2)^4(1+|k_1-k_2|)^n}, \tag{4.5}$$

$$|\mathbb{E}\,(\widetilde{u}_1(x,y,k_1)\widetilde{u}_1(x,y,k_2))| \leq c_n'(k_1+k_2)^{-n}, \tag{4.6}$$

where n is arbitrary. Moreover, for $k_1 = k_2 = k$ we have the asymptotics

$$\mathbb{E}\,(\widetilde{u}_1(x,y,k)\overline{\widetilde{u}_1(x,y,k)}) = R(x,y)k^{-4} + \mathcal{O}(k^{-5}) \tag{4.7}$$

where $R \in C^\infty(U \times U)$. Especially, it holds that

$$R(x,x) = \frac{1}{2^8\pi^2}\int_{\mathbb{R}^2} \frac{\mu(z)}{|z-x|^2}\,dz \quad \text{for } x \in U. \tag{4.8}$$

Proof. Denote $\phi(z,x,y) = |x-z| + |z-y|$. As the covariance operator C_q has a weakly singular kernel $C(z_1,z_2) = k_q(z_1,z_2)$ with asymptotics given as in Proposition 2.2, we see that

$$\mathbb{E}\,(\widetilde{u}_1(x,y,k_1)\overline{\widetilde{u}_1(x,y,k_2)}) = \sum_{j_1,j_2,l_1,l_2=0}^{3} I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y) \tag{4.9}$$

where

$$I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y) = \tag{4.10}$$
$$\frac{d_{j_1}d_{j_2}\overline{d}_{l_1}\overline{d}_{l_2}}{k_1^{1+j_1+j_2}k_2^{1+l_1+l_2}}\int_{\mathbb{R}^4} \frac{\exp\big(ik_1\phi(z_1,x,y) - ik_2\phi(z_2,x,y)\big)\,\mathbb{E}\,(q(z_1)q(z_2))}{|x-z_1|^{j_1+\frac{1}{2}}|z_1-y|^{j_2+\frac{1}{2}}|x-z_2|^{l_1+\frac{1}{2}}|z_2-y|^{l_2+\frac{1}{2}}}\,dz_1dz_2.$$

Assumption 1.1 with $\tau = 2$ states that $C(z_1,z_2)$ is the Schwartz kernel of a pseudodifferential operator C_q with a classical symbol $c(x,\xi) \in S_{1,0}^{-2}(\mathbb{R}^2 \times \mathbb{R}^2)$, and the principal symbol of C_q is given by $c^p(z,\xi) = \mu(z)(1+|\xi|^2)^{-1}$. The support of $C_q(z_1,z_2)$ is contained in $D \times D$. We may write (c.f. [22])

$$C_q(z_1,z_2) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(z_1-z_2)\cdot\xi} c(z_1,\xi)\, d\xi. \tag{4.11}$$

All symbols appearing below will be classical symbols [22].

In order to obtain uniform estimates with respect to variables x and y we shall introduce them as variables in the covariance in the following way. Let us define the function $C_1(z_1,z_2,x,y) = C_q(z_1,z_2)\theta(x)\theta(y)$ where $\theta \in C_0^\infty(\mathbb{R}^2)$ equals one in the domain U and has its support outside $\overline{D}$.

The formula (4.11) now takes the form

$$C_1(z_1,z_2,x,y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(z_1-z_2)\cdot\xi} c_1(z_1,x,y,\xi)\, d\xi \tag{4.12}$$

where $c_1(z_1,x,y,\xi) \in S^{-2}_{1,0}(\mathbb{R}^6\times\mathbb{R}^2)$. In fact, $c_1 \in S^{-2}_{1,0}((D\times\mathbb{R}^4)\times\mathbb{R}^2)$, but we consider it extended by zero to values $z_1 \notin D$. By definition, (4.12) means that $C_1(z_1,z_2,x,y)$ is a conormal distribution in $\mathbb{R}^8$ of Hörmander type having conormal singularity on the surface $S_1 = \{(z_1,z_2,x,y) \in \mathbb{R}^8 :\ z_1 - z_2 = 0\}$. Using notations of [22], if $X \subset \mathbb{R}^n$ is an open set and $S \subset X$ is a smooth submanifold of X, we denote by $I(X;S)$ the distributions in $\mathcal{D}'(X)$ that are smooth in $X \setminus S$ and have a conormal singularity at S. The set of distributions in $I(X;S)$ supported in a compact subset of X is denoted by $I_{comp}(X;S)$. Let $\mathbf{D} \subset \mathbb{R}^8$ be an open set containing $D \times D \times \text{supp}\ (\theta) \times \text{supp}\ (\theta)$ so that $C_1 \in I_{comp}(\mathbf{D}; S_1 \cap \mathbf{D})$.

We employ the fact that conormal distributions are invariant under a change of coordinates. Actually, our plan is to consider several different coordinates systems.

The first set of coordinates that we consider are (V,W,x,y), defined as $V = z_1 - z_2$ and $W = z_1 + z_2$. Denote by η the change of coordinates $\eta : (V,W,x,y) \mapsto (z_1,z_2,x,y)$ and consider the pull-back $C_2 = \eta^*(C_1)$. Then a direct substitution shows that

$$C_2(V,W,x,y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{iV\cdot\xi} c_2(V,W,x,y,\xi)\, d\xi,$$
$$c_2(V,W,x,y,\xi) = c_1(z_1(V,W,x,y),x,y,\xi)$$

which means that $C_2 \in I(\mathbb{R}^8; S_2)$ where $S_2 = \{(V,W,x,y) :\ V = 0\}$.

To find out how the symbol transforms in the change of coordinates, we have to represent $C_2(V,W,x,y)$ with a symbol that does not depend on V. We can achieve this by way of the representation theorem for conormal distributions [22, Lemma 18.2.1] because of the special form of the surface $S_2 = \{V = 0\}$. We have:

$$C_2(V,W,x,y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{iV\cdot\xi} c_3(W,x,y,\xi)\, d\xi, \tag{4.13}$$
$$c_3(W,x,y,\xi) \sim \sum_{l=0}^{\infty} \langle -iD_V, D_\xi\rangle^l c_2(V,W,x,y,\xi)|_{V=0} \in S^{-2}_{1,0}(\mathbb{R}^6 \times \mathbb{R}^2).$$

In particular, we see that $c_3(W,x,y,\xi)$ has the principal symbol

$$c_3^p(W,x,y,\xi) = \mu(z_1(V,W,x,y))(1+|\xi|^2)^{-1}\theta(x)\theta(y)\big|_{V=0}. \tag{4.14}$$

The second set of coordinates that we consider are (v,w,x,y) defined below. For this, to consider the oscillatory integrals (4.10) we change the coordinates so that $\phi(z_1,x,y)-\phi(z_2,x,y)$ will be a coordinate. We will do this change of coordinates in two steps. First we change the coordinates (z_1,z_2,x,y) to (Z_1,Z_2,x,y), where $Z_j = Z_j(x,y,z_j) \in \mathbb{R}^2$, $j=1,2$ are related to ellipses having focal points in x and y. More precisely, we write

$$\begin{aligned} &Z_j = (t_j,s_j) \in \mathbb{R}^2,\\ &t_j = \frac{1}{2}\phi(z_j,x,y), \quad s_j = \frac{1}{2}\phi(z_j,x,y)\cdot \arcsin(e_1\cdot \frac{\nabla_{z_j}\phi(z_j,x,y)}{||\nabla_{z_j}\phi(z_j,x,y)||}), \end{aligned}$$

where $e_1 = (1,0)$. In other words, here t_j corresponds to the semi-major axis of the ellipse having focal points x and y and containing the point z_j. The variable s_j specifies a 'normalized' angle of the normal vector of the ellipse with the x-axis at the point z_j. Since the domain U is convex and D is simply connected, our definition of the new coordinates is well-posed in a neighborhood of the domain D.

Secondly, we change from (Z_1,Z_2,x,y) to coordinates (v,w,x,y) where $v = Z_1 - Z_2$, $w = Z_1+Z_2$. Together, the above steps define the coordinates (v,w,x,y) and the map $\tau : (v,w,x,y) \mapsto (z_1,z_2,x,y)$. Note that the first component of $v(z_1,z_2,x,y)$ equals $(\phi(z_1,x,y)-\phi(z_2,x,y))/2$.

To simplify the notation, we denote $X_1 = \mathbf{D}$, $X_2 = \eta^{-1}(\mathbf{D})$ and $X_3 = \tau^{-1}(\mathbf{D})$ so that $\tau : X_3 \to X_1$ and $\eta : X_2 \to X_1$. We are ready to represent the conormal distribution $C_1(z_1,z_2,x,y)$ in coordinates (v,w,x,y) as the pull-back distribution $C_4 = \tau^*(C_1) \in I(X_3; S_3\cap X_3)$, $S_3 = \{(v,w,x,y) :\ v=0\}$. By the invariance of conormal distributions under the change of variables we may write

$$\begin{tikzcd} & I_{comp}(X_1;S_1\cap X_1) \arrow[dl, "\eta^*"'] \arrow[dr, "\tau^*"] & \\ I_{comp}(X_2;S_2\cap X_2) & & I_{comp}(X_3;S_3\cap X_3) \end{tikzcd}$$

To apply this diagram and the integral representation (4.13) of $C_2 \in I_{comp}(X_2;S_2\cap X_2)$, consider the transformation $\kappa = \eta^{-1}\circ\tau$. We will below use [22, Theorem 18.2.9], to provide a representation for the pull-back $C_4 = \kappa^* C_2$. Since surfaces S_2 and S_3 have the special form $S_2 = \{V=0\}$ and $S_3 = \{v=0\}$, and κ maps $S_3\cap X_3$ onto $S_2\cap X_2$, we obtain

$$C_4(v,w,x,y) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{iv\cdot\xi} c_4(w,x,y,\xi)\,d\xi,$$

where $c_4(w,x,y,\xi)\in S^{-2}_{1,0}(\mathbb{R}^6\times\mathbb{R}^2)$ is a symbol satisfying

$$
\begin{aligned}
(4.15)\qquad & c_4(w,x,y,\xi)= \\
& c_3(\kappa_2(v,w,x,y),((\kappa_{11}'(v,w,x,y))^{-1})^t\xi)|\det\kappa_{11}'(v,w,x,y)|^{-1}\big|_{v=0}+r(w,x,y,\xi).
\end{aligned}
$$

Here, $r(w,x,y,\xi)\in S^{-3}_{1,0}(\mathbb{R}^6\times\mathbb{R}^2)$ and the coordinate transform κ is decomposed into two parts, the $\mathbb{R}^2$-valued function $\kappa_1(v,w,x,y)=V(v,w,x,y)$ and the $\mathbb{R}^6$-valued function $\kappa_2(v,w,x,y)=(W(v,w,x,y),x,y)$. This yields for the differential κ' of κ the corresponding representation

$$\kappa'=\begin{pmatrix}\kappa_{11}' & \kappa_{12}'\\ \kappa_{21}' & \kappa_{22}'\end{pmatrix}.$$

We note that the transformation rule in κ^* in [22, Theorem 18.2.9] is presented for half-densities. The proof of the analogous result for distributions, however, is immediate.

Plugging the principal symbol of $c_3(x,\xi)$ given in (4.14) to formula (4.15), we see that the principal symbol of $c_4(w,x,y,\xi)$ is

$$
\begin{aligned}
c_4^p(w,x,y,\xi) &= \mu(z_1(v,w,x,y))(1+|((\kappa_{11}'(v,w,x,y))^{-1})^t\xi|^2)^{-1}\big|_{v=0}\\
&\quad\cdot\theta(x)\theta(y)J(w,x,y)
\end{aligned}
$$

where $J(w,x,y)=|\det\kappa_{11}'(0,w,x,y)|^{-1}$.

We are ready to compute the asymptotics of $I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y)$. We denote $\vec{j}=(j_1,j_2,l_1,l_2)$. By writing the integral $I_{\vec{j}}=I_{j_1,j_2,l_1,l_2}(k_1,k_2,x,y)$ in coordinates (v,w,x,y) we obtain

$$
\begin{aligned}
I_{\vec{j}} &= k_1^{-(1+j_1+j_2)}k_2^{-(1+l_1+l_2)}\int_{\mathbb{R}^4}\exp(i\big((k_1+k_2)e_1\cdot v+(k_1-k_2)e_1\cdot w\big))\cdot\\
&\qquad\cdot C_4(v,w,x,y)H^{\vec{j}}(v,w,x,y)\,dvdw \qquad (4.16)
\end{aligned}
$$

where $e_1=(1,0)$ is the unit vector and $H^{\vec{j}}=H^{\vec{j}}(v,w,x,y)$ is

$$H^{\vec{j}}=\frac{d_{j_1}d_{j_2}\overline{d}_{l_1}\overline{d}_{l_2}}{|x-z_1|^{j_1+\frac12}|z_1-y|^{j_2+\frac12}|x-z_2|^{l_1+\frac12}|z_2-y|^{l_2+\frac12}}\det(\tau'(v,w,x,y))$$

where $z_1=z_1(v,w,x,y)$ and $z_2=z_2(v,w,x,y)$.

Since $H^{\vec{j}}$ is smooth in X_3 in all variables and the class $I(\mathbb{R}^8;S_3)$ is closed under multiplication with a smooth function, we have $C_4(v,w,x,y)\,H^{\vec{j}}(v,w,x,y)\in I(\mathbb{R}^8;S_3)$. To evaluate the oscillatory integrals (4.16) in a convenient way, we need to represent this conormal distribution with a symbol that does not depend on v. Again, by using the representation theorem for conormal distributions [22, Lemma

18.2.1], we obtain

$$C_4(v,w,x,y)\,H^{\vec{j}}(v,w,x,y) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{iv\cdot\xi} c_5^{\vec{j}}(w,x,y,\xi)\,d\xi, \tag{4.17}$$

$$c_5^{\vec{j}}(w,x,y,\xi) \sim \sum_{l=0}^{\infty}\langle -iD_v, D_\xi\rangle^l (c_4(w,x,y,\xi)H^{\vec{j}}(v,w,x,y))|_{v=0}.$$

In particular, we see that $c_5^{\vec{j}}(w,x,y,\xi)$ has the principal symbol

$$\begin{aligned} c_5^{\vec{j}p}(w,x,y,\xi) &= \mu(z_1(v,w,x,y))(1+|((\kappa_{11}'(v,w,x,y))^{-1})^t\xi|^2)^{-1}\cdot \\ &\cdot \theta(x)\theta(y)\,J(w,x,y)H^{\vec{j}}(v,w,x,y)\Big|_{v=0}. \end{aligned} \tag{4.18}$$

By substituting (4.17) in to (4.16) and using the Fourier inversion rule we obtain the important formula

$$I_{\vec{j}} = k_1^{-(1+j_1+j_2)}k_2^{-(1+l_1+l_2)}(\mathcal{F}_w c_5^{\vec{j}})((k_2-k_1)e_1, x, y, -(k_1+k_2)e_1) \tag{4.19}$$

where $\mathcal{F}_w$ denotes the Fourier transform in the w variable,

$$\mathcal{F}_w c_5^{\vec{j}}(\eta,x,y,\xi) = \int_{\mathbb{R}^2} e^{-i\eta\cdot w} c_5^{\vec{j}}(w,x,y,\xi)\,dw.$$

As the symbol $c_5^{\vec{j}}(w,x,y,\xi)$ is C^∞ smooth and compactly supported in the (x,y,w) variables, we see that $|D_w^\alpha c_5^{\vec{j}}(w,x,y,\xi)| \le c_\alpha(1+|\xi|)^{-2}$ for all $|\alpha|\ge 0$, where c_α is independent of $(w,x,y)\in\mathbb{R}^6$. This implies that after n integrations by parts

$$|I_{\vec{j}}(k_1,k_2,x,y)| \le c_n\frac{1}{1+|k_1+k_2|^2}k_1^{-j_1-j_2-1}k_2^{-\ell_1-\ell_2-1}(1+|k_1-k_2|)^{-n}$$

for all $n\ge 0$. By considering separately the cases $|k_1-k_2|\le |k_1+k_2|/2$ and $|k_1-k_2|\ge|k_1+k_2|/2$ we deduce that

$$|I_{\vec{j}}(k_1,k_2,x,y)| \le c_n'(1+|k_1-k_2|)^{-n}(1+|k_1+k_2|)^{-4-j_1-j_2-l_1-l_2}, \quad n>0. \tag{4.20}$$

This verifies the estimate (4.5).

Before proving (4.6) we consider the asymptotics when $k_1=k_2=k$. We denote $\mathbf{0}=(0,0,0,0)$. For $\vec{j}\ne\mathbf{0}$ we have $I_{\vec{j}}(k,k,x,y)=\mathcal{O}(k^{-5})$. Thus, in order to establish (4.7) it is enough to consider $I_{\mathbf{0}}(k,k,x,y)$. To obtain the leading order asymptotics of $I_{\mathbf{0}}$, we consider the contributions of the principal symbol and the lower order remainder terms separately. Write

$$c_5^{\mathbf{0}}(w,x,y,\xi) = c_5^{\mathbf{0}p}(w,x,y,\xi) + c_r(w,x,y,\xi),$$

where $c_r(w,x,y,\xi)\in S_{1,0}^{-3}(\mathbb{R}^6\times\mathbb{R}^2)$ is smooth and compactly supported in the (w,x,y) variables. Thus $|D_w^\alpha c_r(w,x,y,\xi)|\le c_\alpha(1+|\xi|)^{-3}$ for all multi-indices α and we infer as above that

$$|\mathcal{F}_w c_r(0,x,y,-2ke_1)| = \mathcal{O}(k^{-3}),$$

for all $n > 0$. Thus the contribution of c_r to $I_{\mathbf{0}}$ is estimated by the right hand side of (4.7). Hence it is enough to consider the principal part. To this end, we substitute the principal symbol (4.18) into formula (4.19) and obtain

$$\begin{aligned}(4.21) \qquad & I_{\mathbf{0}}(k,k,x,y) = \\ & k^{-2}\theta(x)\theta(y)\int_{\mathbb{R}^2} \frac{\mu(z_1(0,w,x,y))H^{\mathbf{0}}(0,w,x,y)J(w,x,y)}{1+4k^2|((\kappa_{11}'(0,w,x,y))^{-1})^t e_1|^2}\,dw + \mathcal{O}(k^{-5}).\end{aligned}$$

Since one may compute that $a = 4|((\kappa_{11}'(0,w,x,y))^{-1})^t e_1|^2 \neq 0$ we may apply for for large k the development $(1+k^2a)^{-1} = a^{-1}k^{-2}\sum_{j=0}^{\infty} k^{-2j}(-a)^{-j}$. We obtain the desired formula (4.7) with

$$R(x,y) = \frac{1}{4}\int_{\mathbb{R}^2} \frac{\mu(z_1(0,w,x,y))H^{\mathbf{0}}(0,w,x,y)J(w,x,y)}{|((\kappa_{11}'(0,w,x,y))^{-1})^t e_1|^2}\,dw.$$

Moreover, for $y = x$ we compute $\kappa_{11}' = \begin{pmatrix} \cos\alpha + \alpha\sin\alpha & -\sin\alpha \\ \sin\alpha - \alpha\cos\alpha & \cos\alpha \end{pmatrix}$, where $\alpha = w_2/w_1$. It follows that $J(w,x,x) = \det(\kappa_{11}') = 1$ and $|((\kappa_{11}')^{-1})^t e_1| = 1$. Moreover, we also have $\det(\tau'(0,w,x,x)) = \frac{1}{4}$, and $(\det(\frac{dz_1}{dw}(v,w,x,x)|_{v=0}))^{-1} = 4$. Put together, these observations yield (4.8) for $R(x,x)$.

Finally we prove estimate (4.6). Observe that $\mathbb{E}\,(\widetilde{u}_1(x,y,k_1)\widetilde{u}_1(x,y,k_2))$ is given by a linear combination of terms $\widetilde{I_{\vec{j}}}$ analogous to (4.10) where, in addition to changing constants d_j, we only replace k_2 with $-k_2$. Notice also that in the proof of formula (4.19) one may allow k_2 to be negative, whence the estimate (4.6) follows immediately. $\square$

Lemma 4.2. *In the decomposition (4.4) the random variable $b(x,y,k)$ satisfies a.s. the condition*

$$|b(x,y,k)| \le c'(1+|k|)^{-3}, \quad x,y \in U,\ k > 0$$

where the constant c' depends only on $H_0^{-1,1}(D)$-norm of $q(z,\omega)$.

Proof. By (4.3), $||\Phi_k(\cdot - x)||_{H^{1,\infty}(D)} + ||\Phi_k^{(3)}(\cdot - y)||_{H^{1,\infty}(D)} \le ck^{1/2}$ for $k > 1$, $x,y \in U$. This implies

$$\begin{aligned}|b(x,y,k)| \le\ & ||q||_{H_0^{-1,1}(D)}\Big(||\Phi_k(\cdot - x) - \Phi_k^{(3)}(\cdot - x)||_{H^{1,\infty}(D)}||\Phi_k(\cdot - y)||_{H^{1,\infty}(D)} + \\ & + ||\Phi_k^{(3)}(\cdot - x)||_{H^{1,\infty}(D)}||\Phi_k(\cdot - y) - \Phi_k^{(3)}(\cdot - y)||_{H^{1,\infty}(D)}\Big) \\ \le\ & c||q||_{H_0^{-1,1}(D)}(1+|k|)^{-1-m}.\end{aligned}$$

$\square$

The above results have the following corollary that plays a crucial role in sequel.

Corollary 4.3. *Assume that $k_1, k_2 > 1$ and $x, y \in U$. Then*

$$\mathbb{E}\,\left|Re\,(k_1^2\widetilde{u}_1(x,y,k_1))Re\,(k_2^2\widetilde{u}_1(x,y,k_2))\right| \le c_n(1+|k_1-k_2|)^{-n}, \quad n > 0,$$

where c_n is independent of $x, y \in U$, and one may replace one or both of the real parts by imaginary parts.

Proof. When $k_1, k_2 > 1$ we have that $|k_1 + k_2|^{-n} \leq |k_1 - k_2|^{-n}$. The claim now follows immediately from estimates (4.5) and (4.6) by simply observing that for any $a, b, c, d \in \mathbb{R}$ we may recover all the products ac, ad, bc and bd as linear combinations of real or imaginary parts of the numbers $(a + ib)(c \pm id) = (ac \mp bd) + i(bc \pm ad)$. $\square$

5. Higher order terms

5.1. **The second term.** In this subsection we consider the second term u_2 of the Born series (4.1), given by

$$u_2(x, y, k) = \int_D \int_D \Phi_k(x - z_1)q(z_1)\Phi_k(z_1 - z_2)q(z_2)\Phi_k(z_2 - y)\, dz_1 dz_2. \tag{5.1}$$

Our aim is to provide estimates that will be used later to show that the contribution of u_2 can be ignored in the measurement (1.3). It turns out that the estimation of the size of $u_j(x, y, k)$ is most difficult in the case $j = 2$. However, the following result yields exactly the needed estimate.

Theorem 5.1. *For all $x, y \in U$ it holds almost surely that*

$$\lim_{K \to \infty} \frac{1}{K - 1} \int_1^K |k^2 u_2(x, y, k, \omega)|^2 dk = 0.$$

Proof. One may assume that $x = 0$, so we will abbreviate $u_2(k) = u_2(0, y, k, \omega)$ (the dependence on y and w is suppressed in the notation). A main reduction will be that we replace the Hankel functions, one by one, in (5.1) by the principal terms in the asymptotics (4.2). It will be useful to abbreviate

$$f_k(z) = \Phi_k(z) - d_0(k|z|)^{-1/2} e^{ik|z|} \quad \text{and} \quad g_k(z) = d_0(k|z|)^{-1/2} e^{ik|z|}, \tag{5.2}$$

where the constant d_0 comes from the asymptotics (4.2). We need two auxiliary results. The first one collects together useful knowledge on the behaviour of f_k, g_k, and $\Phi_k(\cdot - y)$ with increasing k (in this section we always assume that $k \geq 1$).

Lemma 5.2. *Let $\varepsilon \in [0, 1]$. Then*

(i) $\|g_k\|_{H^{\varepsilon,p}(D)}, \|\Phi_k(\cdot - y)\|_{H^{\varepsilon,p}(D)} \leq c_p k^{\varepsilon - 1/2}$ *for $p > 1$ and $y \in U$.*

(ii) $\|f_k(\cdot - y)\|_{H^{\varepsilon,p}(D)} \leq c_p k^{2\varepsilon - 3/2}$ *for $p > 1$ and $y \in U$.*

(iii) $\|f_k(z_1 - z_2)\|_{H^{\varepsilon,p}(D \times D)} \leq c_p k^{2\varepsilon - 3/2}$ *for $p \in (1, 4/3)$.*

Proof. Assume $y \in U$ and recall that $\operatorname{dist}(U, D) =: d > 0$. Recall that

$$|H_0^{(1)}(t)| \leq \frac{c}{\sqrt{t}}, \quad |\frac{d}{dt} H_0^{(1)}(t)| \leq \frac{c}{\sqrt{t}}, \quad t \geq d. \tag{5.3}$$

Then by denoting $R = \sup\{|y-z| : \ y \in U, z \in D\}$ we have

$$\int_D |f_y(z)|^p dz \ \leq \int_{d \leq |u| \leq R} \Phi_k(u)^p du \leq c_p k^{-p/2}(R^{1-p/2} - d^{1-p/2}) \ = \ ck^{-p/2}$$

where c is independent of y, and by the same manner one may estimate the gradient $\nabla\Phi_k(\cdot - y)$. We thus have

$$\sup_{y\in U} ||\Phi_k(\cdot - y)||_{L^p(D)} \leq ck^{-1/2}, \quad \sup_{y\in U} ||\nabla\Phi_k(\cdot - y)||_{L^p(D)} \leq ck^{1/2}. \tag{5.4}$$

These estimates interpolate for $0 \leq s \leq 1$ to what was claimed for Φ_k. Next, the statement (i) for g_k is obtained similarly by noting that direct computation of ∇g_k shows that g_k obeys bounds similar to (5.3).

In order to prove (ii), observe that the asymptotics (4.3) yield the estimates

$$\|f_k\|_{L^\infty(D)} \leq ck^{-3/2}, \quad \|\nabla f_k\|_{L^\infty(D)} \leq ck^{1/2}, \tag{5.5}$$

from which the claim again follows by interpolation.

To prove (iii), we again use the asymptotics (4.3) and observe that $|\nabla\Phi_k(z)| \leq ck^{1/2}\max(|z|^{-1}, |z|^{-1/2})$ for all z. Moreover, by direct computation $|\nabla g_k(z)| \leq ck^{1/2}\max(|z|^{-3/2}, |z|^{-1/2})$. These together yield that

$$|\nabla f_k(z)| \leq ck^{1/2}\max(|z|^{-1/2}, |z|^{-3/2}), \quad k \geq 1. \tag{5.6}$$

By direct computation, we obtain $||f_k||_{L^p(B)} \leq c_{p,B}(k^{-2} + k^{-\frac{3p}{2}})^{\frac{1}{p}}$ for $p < 4/3$ in any bounded domain B. By combining this with (5.6) we obtain by interpolation the counterpart of (iii) for the function $z \mapsto f_k(z)$ on any bounded subdomain of $\mathbb{R}^2$. The estimate for the map $(z_1, z_2) \mapsto f_k(z_1 - z_2)$ follows since D is bounded. $\square$

In order to state the second Lemma, recall that the operator H_k was defined through (3.11) in Section 3. We also need to consider the operator K_k which combines the multiplication operator with q to H_k, i.e. $K_k f = H_k(qf)$.

Lemma 5.3. *For any $p > 1$, $s \in (0,1)$ and $k \geq 1$ it holds that*

$$||K_k||_{H^{s,2p}\to H^{s,2p}} \leq ck^{-1+2(s+(1-1/p))}, \tag{5.7}$$

$$||K_k||_{H^{s,2p}\to L^\infty} \leq ck^{1+2s-1/p}, \tag{5.8}$$

where the constant $c = c(\omega)$ is finite almost surely.

Proof. For $0 < s < 1$ and $1 \leq p \leq 2 \leq r \leq \infty$ one has that $H_k : H_0^{-s,p}(D) \to H^{s,r}(D)$ with the norm estimate

$$||H_k||_{H_0^{-s,p}(D)\to H^{s,r}(D)} \leq ck^{-1+2(s+(1/p-1/r))}. \tag{5.9}$$

This estimate follows easily from the proof of Theorem 3.1 in [36]. An application of (5.9) together with Lemma 3.1 and Theorem 2.3 immediately yields the claim. $\square$

Let us now replace the left-most Hankel-factor in the integral (5.1) defining $u_2(k)$ by the approximation g_k and consider

$$u_{2,\ell}(k) := d_0 k^{-1/2} \int_D \int_D e^{ik|z_1|} q(z_1)|z_1|^{-1/2} \Phi_k(z_1 - z_2) q(z_2) \Phi_k(z_2 - y)\, dz_1 dz_2.$$

By the definition of the operator K_k we have

$$|u_{2,\ell}(k) - u_2(k)| = |\langle q, f_k K_k(\Phi_k(\cdot - y))\rangle|,$$

where the brackets refer to distribution duality. According to the previous Lemmata we may estimate the left hand side above by

$$\begin{aligned} & \|q\|_{H^{-\varepsilon,(1+\varepsilon)'}(D)} \|f_k K_k(\Phi_k(\cdot - y))\|_{H^{\varepsilon,1+\varepsilon}(D)} \\ \leq\ & c_0 \|f_k\|_{H^{\varepsilon,2+2\varepsilon}(D)} \parallel K_k \parallel_{H^{\varepsilon,2+2\varepsilon}(D) \to H^{\varepsilon,2+2\varepsilon}(D)} \|\Phi_k(\cdot - y)\|_{H^{\varepsilon,2+2\varepsilon}(D)} \\ \leq\ & c_0 k^{2\varepsilon - 3/2} k^{-1+2(\varepsilon + (1-(1+\varepsilon)^{-1}))} k^{\varepsilon - 1/2}, \end{aligned}$$

where $c_0 = c_0(\omega)$. Here Theorem 2.3 verifies that $\|q\|_{H^{-\varepsilon,(1+\varepsilon)'}(D)} < \infty$ almost surely. Thus we obtain that

$$u_2(k) = u_{2,\ell}(k) + O(k^{\varepsilon' - 3}) \quad \text{for all } \varepsilon' > 0. \tag{5.10}$$

We next apply the same asymptotics to the Φ_k-term on the right and consider

$$u_{2,r}(k) := d_0^2 k^{-1} \int_D \int_D e^{ik(|z_1| + |z_2|)} \Phi_k(z_1 - z_2) q(z_1) q(z_2)(|z_1||z_2 - y|)^{-1/2}\, dz_1 dz_2.$$

In this approximation the induced error term to $u_{2,\ell}(k)$ is given by

$$|u_{2,r}(k) - u_{2,\ell}(k)| = |\langle q, h_k K_k(f_k(\cdot - y)\rangle|,$$

where $h_k(z_1) = d_0 e^{ik|z_1|} |k z_1|^{-1/2}$. Note that $|z_1|^{-1/2}$ is smooth on D. Clearly $\|h_k\|_{H^{\varepsilon,\infty}(D)} \leq c k^{\varepsilon - 1/2}$, and hence we may apply again Lemma 5.2 and obtain analogously to the previous computation

$$\begin{aligned} u_{2,\ell}(k) &= u_{2,r}(k) + O(k^{\varepsilon - 1/2} k^{-1+2(\varepsilon + (1-(1+\varepsilon)^{-1}))} k^{-3/2+2\varepsilon}) \\ &= u_{2,r}(k) + O(k^{\varepsilon' - 3}) \quad \text{for all } \varepsilon' > 0. \end{aligned} \tag{5.11}$$

To complete the reduction, we apply the same asymptotics to the remaining Φ_k-term in the integral defining $u_{2,r}(k)$ and consider

$$v(k) := d_0^3 \int_D \int_D e^{ik(|z_1| + |z_2| + |z_1 - z_2|)} q(z_1) q(z_2)(|z_1 - z_2||z_1||z_2 - y|)^{-1/2}\, dz_1 dz_2.$$

Following our definitions, this integral is understood in the sense that one first does the integration (distributional duality) with respect to the z_1 variable. However, one verifies without difficulty that we also have $u_{2,r}(k) = \langle \Phi_k(\cdot - \cdot), s_k\rangle$, where

$$s_k(z_1, z_2) = d_0^2 k^{-1} q(z_1) q(z_2) e^{ik(|z_1| + |z_2|)} (|z_1||z_2 - y|)^{-1/2}, \quad (z_1, z_2) \in D \times D.$$

One easily verifies that $a_1(z_1) a_2(z_2) \in H^{-2\varepsilon,\infty}(\mathbb{R}^4)$ whenever $a_1, a_2 \in H^{-\varepsilon,\infty}(\mathbb{R}^2)$, and $\epsilon > 0$. Thus, according to Theorem 2.3 and the Sobolev embedding theorem we have that $q_1 \otimes q_2 \in H_0^{-\varepsilon,\infty}(D \times D)$ for all $\varepsilon > 0$. Observe that $\|e^{ik(|z_1|+|z_2|)}\|_{H^{\varepsilon,\infty}(D\times D)} \leq ck^{\varepsilon}$. A simple duality argument using (3.5) shows that

$$\|s_k\|_{H_0^{-\varepsilon,\infty}(D\times D)} \leq ck^{\varepsilon-1}, \quad \text{for all } \varepsilon > 0.$$

Combining this estimate with Lemma 5.2 we obtain

$$\begin{aligned} |k^{-3/2}v(k) - u_{2,r}(k)| &= |\langle f_k(\cdot - \cdot), s_k\rangle| \leq \|f_k(\cdot - \cdot)\|_{H^{\varepsilon,5/4}(D\times D)} \|s_k\|_{H_0^{-\varepsilon,5}(D\times D)} \\ &\leq ck^{3\varepsilon - 5/2}. \end{aligned}$$

In conjunction with (5.10) and (5.11) this finally gives

$$u_2(k) = k^{-3/2}v(k) + O(k^{\varepsilon'-5/2}) \quad \text{for all } \varepsilon' > 0. \tag{5.12}$$

We now enter the main difficulty of the proof, that is, the estimation of $v(k)$. Our basic idea is to circumvent pointwise estimates with respect to k altogether. Namely, in order to prove the Theorem it will be enough to show that

$$\int_1^\infty |v(k)|^2 dk < \infty \quad \text{a.s.} \tag{5.13}$$

To see this, we notice that by choosing $\varepsilon' = 1/4$ in (5.12) one may write

$$\begin{aligned} \frac{1}{K}\int_1^K |k^2 u_2(k)|^2 dk &\leq 2\int_1^K \frac{k}{K}|v(k)|^2 dk + O(K^{-1/2}) \\ &\leq 2\int_1^\infty \min(1, \frac{k}{K})|v(k)|^2 dk + O(K^{-1/2}). \end{aligned}$$

This last integral converges a.s. to zero by the dominated convergence theorem as $k \to \infty$.

Towards (5.13) we will shortly express v as a one-dimensional Fourier transform and get rid of the variable k. Before that a couple of auxiliary considerations are needed. First of all, in order not to have problems with interpreting the distribution dualities that will emerge, we introduce the modification v_δ, $\delta > 0$, of v that is obtained by replacing q by the standard mollification $q_\delta := q * \rho_\delta$, where $\rho_\delta(x) = \delta^{-2}\rho(x/\delta)$ and $\rho \in C_0^\infty(\mathbb{R}^2)$ is a radially symmetric function satisfying $\int \rho(x)dx = 1$. We denote the mollification operator by $M_\delta : f \mapsto f * \rho_\delta$. Observe for later use that the covariance operator of q_δ equals $C_\delta = M_\delta C M_\delta$. Clearly $v_\delta(k) \to v(k)$ as $\delta \to 0$. In order to verify (5.13) it is enough to show that

$$\sup_{\delta\in(0,1)} \int_1^\infty \mathbb{E}\,|v_\delta(k)|^2 dk < \infty, \tag{5.14}$$

since an application of the Fubini theorem and Fatou's lemma then yields that $\mathbb{E}\,(\int_1^\infty |v(k)|^2 dk) < \infty$, which immediately implies (5.13).

We need to take a closer look at the phase function $A(z_1, z_2) := |z_1| + |z_1 - z_2| + |z_2 - y|$. First of all, A is smooth on $D \times D$ apart from the subset where $z_1 = z_2$.

Moreover, the gradient of A is bounded from below and above;

$$0 < c_1 \leq |\nabla A(z_1, z_2)| \leq c_2 < \infty \quad \text{for } (z_1, z_2) \in D \times D, \quad z_1 \neq z_2. \tag{5.15}$$

The upper bound is evident. For the lower bound it is enough to observe that a simple computation yields

$$(z_1, z_2) \cdot \nabla A(z_1, z_2) \geq c_0 > 0 \quad \text{for} \quad z_1 \neq z_2, \quad z_1, z_2 \in D. \tag{5.16}$$

Moreover, it shows that the surfaces

$$\Gamma_t := \{(z_1, z_2) \in D \times D \mid A(z_1, z_2) = t\}, \quad t > 0$$

are locally boundaries of starshaped domains with respect to the origin (observe that Γ_t need not to be connected). One also deduces by using (5.15) and (5.16) that there is a radial strech B_t yielding a bi-Lipschitz chart $B_t : H_t \to \Gamma_t$ over a subdomain $A_t := H_t^{-1}(\Gamma_t)$ of the unit ball. We hence see that the surfaces Γ_t, with varying t provide a fairly regular foliation of the domain $D \times D$.

The considerations in the preceding paragraph justify a generalized co-area formula for integrals on $D \times D$:

$$\int_{D\times D} g(z_1, z_2)\, dz_1 dz_2 = \int_{t_0}^{t_1} \left(\int_{\Gamma_t} g(z_1, z_2) \frac{1}{|\nabla A(z_1, z_2)|}\, d\mathcal{H}^3(z_1, z_2) \right) dt, \tag{5.17}$$

where the inner integral is with respect to the 3-dimensional Hausdorff measure on Γ_t, and g is any integrable Borel-function on $D \times D$. Above the maximal and minimal values $t_0 = t_0(y) > 0$ and $t_1 = t_1(y)$ certainly depend on $y \in U$ (as the whole foliation does).

Lemma 5.4. *Given $\gamma \in (0, 2)$ there is a finite constant c such that for every $t \in [t_1(y), t_2(y)]$ we have*

$$\int_{\Gamma_t} |z_1 - z_2|^{-\gamma} d\mathcal{H}^3(z_1, z_2) \leq c \quad \text{and}$$
$$\int_{\Gamma_t \times \Gamma_t} |z_j - z_k'|^{-\gamma} d\mathcal{H}^3(z_1, z_2) d\mathcal{H}^3(z_1', z_2') \leq c \quad \text{for } k, j = 1, 2.$$

Proof. For any $(z_1, z_2) \in \Gamma_t$ let us denote $(w_1, w_2) = B_t^{-1}(z_1, z_2)$, whence $(w_1, w_2) = (\lambda(z_1, z_2)z_1, \lambda(z_1, z_2)z_2)$, where the scalar factor $\lambda = \lambda(z_1, z_2) \in \mathbb{R}^+$ satisfies uniformly $\lambda \geq a > 0$. Since the map B_t is bi-Lipschitz, the 3-dimensional Hausdorff measure changes only by a bounded factor under B_t. Hence, a change of variables in the first integral in the assertion yields that

$$\int_{\Gamma_t} |z_1 - z_2|^{-\gamma} d\mathcal{H}^3(z_1, z_2) \leq \int_{B_t^{-1}(\Gamma_t)} a^{-\gamma} |w_1 - w_2|^{-\gamma} c\, d\mathcal{H}^3(w_1, w_2).$$

The last written integral can be estimated by the integral over the whole sphere S^3, which is easily seen to be finite.

In order to treat the second integral we first note that by elementary geometry one has $|x - y| \geq |x||x^0 - y^0|/2$, where $x^0 = x/|x|$ (and similarly for y) stands for the corresponding unit vector. For the values of interest it holds that $|z_j|, |z'_k| \geq b > 0$, where b is the distance of D to the origin. Hence the second integral is bounded from above by

$$2^\gamma b^{-\gamma} \int_{\Gamma_t(0,x)\times\Gamma_t(0,x)} |(z_j)^0 - (z'_k)^0|^{-\gamma} d\mathcal{H}^3(z_1,z_2) d\mathcal{H}^3(z'_1,z'_2). \tag{5.18}$$

Using again the change of variables as before with respect to both z and z' we obtain that (5.18) is dominated by the expression

$$2^\gamma b^{-\gamma} c^2 \int_{H\times H} |(w_j)^0 - (w'_k)^0|^{-\gamma} d\mathcal{H}^3(w_1,w_2) d\mathcal{H}^3(w'_1,w'_2),$$

where $H = \{(z_1,z_2) \in S^3 : |z_1|, |z_2| > b\}$. The last written integral is readily seen to be finite by an application of the Fubini theorem. $\square$

We return to our main theme and use (5.17) to write

$$v_\delta(k) = \widehat{S}_\delta(k), \quad k > 0, \tag{5.19}$$

where for $t \in \mathbb{R}$ one has

$$S_\delta(t) := \int_{\Gamma_t} q_\delta(z_1) q_\delta(z_2)(|z_1 - z_2||z_1||z_2 - y|)^{-1/2} |\nabla A(z_1,z_2)|^{-1}\, d\mathcal{H}^3(z_1,z_2).$$

We claim that (5.14) follows as soon as we verify that there is a finite constant $M < \infty$ such that

$$\mathbb{E}\,|S_\delta(t)|^2 \leq M \quad \text{for all } \delta \in (0,1) \quad \text{and } t \in [t_1,t_2]. \tag{5.20}$$

Namely, we then have

$$\mathbb{E}\,\|S_\delta\|^2_{L^2(\mathbb{R})} \leq M(t_2 - t_1) < \infty,$$

and (5.14) will be a consequence of Parseval's formula.

It remains to estimate $\mathbb{E}\,|S_\delta(t)|^2$. In fact, by using the well-known Wick formulae for the expectation of n-fold products of centered Gaussian variables we obtain

$$\begin{gathered}\mathbb{E}\,(q_\delta(z_1)q_\delta(z_2)q_\delta(z'_1)q_\delta(z'_2)) = \\ C_\delta(z_1,z_2)C_\delta(z'_1,z'_2) + C_\delta(z_1,z'_1)C_\delta(z_2,z'_2) + C_\delta(z_1,z'_2)C_\delta(z_2,z'_1)\end{gathered}$$

and thus

$$\begin{aligned}\mathbb{E}\,|S_\delta(t)|^2 = \int_{\Gamma_t\times\Gamma_t} &(|z_1 - z_2||z_1||z_2 - y|)^{-1/2} |\nabla A(z_1,z_2)|^{-1} \cdot \\ &\cdot (|z'_1 - z'_2||z'_1||z'_2 - y|)^{-1/2} |\nabla A(z'_1,z'_2)|^{-1} \\ \cdot(C_\delta(z_1,z_2)C_\delta(z'_1,z'_2) &+ C_\delta(z_1,z'_1)C_\delta(z_2,z'_2) + C_\delta(z_1,z'_2)C_\delta(z_2,z'_1)) \\ &\cdot d\mathcal{H}^3(z_1,z_2)\, d\mathcal{H}^3(z'_1,z'_2).\end{aligned}$$

From Proposition 2.2 it is immediate that for any given $a > 0$ there is a finite constant c_a such that $|C_\delta(z_1, z_2)| \le c_a |z_1 - z_2|^{-a}$ for any $\delta \in (0,1)$ and $(z_1, z_2) \in D \times D$. By (5.15) we obtain for any $t \in [t_1, t_2]$ the estimate

$$\begin{aligned} &\sup_{\delta\in(0,1)} \mathbb{E}\,|S_\delta(t)|^2 \\ &\le c \int_{\Gamma_t\times\Gamma_t} R(z_1, z_2, z_1', z_2') T(z_1, z_2, z_1', z_2')\, d\mathcal{H}^3(z_1, z_2) d\mathcal{H}^3(z_1', z_2'), \end{aligned} \tag{5.21}$$

where

$$R(z_1, z_2, z_1', z_2') = |z_1 - z_2|^{-1/2} |z_1' - z_2'|^{-1/2}$$

and

$$T(z_1, z_2, z_1', z_2') = \sum_{\{r_1,r_2,r_3,r_4\}=\{z_1,z_2,z_1',z_2'\}} (|r_1 - r_2||r_3 - r_4|)^{-a}.$$

In this last formula we sum over all permutations of the four-element set. It is now clear by symmetry, Lemma 5.2, and an application of Hölder's inequality on (5.21) that the integral (5.21) is finite for all $t \in [t_1, t_2]$ as soon as a is chosen small enough. This completes the proof of of the first part of the Theorem 5.1. □

5.2. **The convergence of the Born series.** In this subsection we verify that the Born-series converges to the solution (if k is large enough) and that the higher order terms decay in an appropriate way.

Theorem 5.5. **(i)** *There is a (random) index $k_0 = k_0(\omega)$ such that $k_0 < \infty$ almost surely and, if $k \ge k_0$ then the Born series* (4.1) *converges for any $x, y \in U$ to the solution $u(x, y, k)$.*

(ii) *For any $\epsilon > 0$ and $k \ge k_0$ there exist $c = c(\epsilon, \omega)$, finite almost surely, such that*

$$\sum_{n=3}^{\infty} \sup_{x,y\in U} |u_n(x, y, k)| \le c k^{-5/2+\epsilon}.$$

Proof. We start from the expression $u_n(x, y, k) = (K_k^n \Phi_k(\cdot - y))(x)$. By Lemma 5.2 (i) and Lemma 5.3 we may estimate that

$$\begin{aligned} ||u_n(\cdot, y, k)||_{L^\infty(U)} &\le ||K_k||_{H^{s,2p}\to L^\infty} ||K_k||^{n-1}_{H^{s,2p}\to H^{s,2p}} ||\Phi_k(\cdot - y)||_{H^{s,2p}} \\ &\le c^n k^{1+2s-1/p} k^{(n-1)(-1+2(s+1-1/p))} k^{-1/2+s}. \end{aligned} \tag{5.22}$$

Here the constant $c = c(\omega)$ is independent of y and thus the desired estimate follows.

Let us denote $s - 1 + \frac{1}{p} = \epsilon_1$ and $2(s + 1 - \frac{1}{p}) = \epsilon_2$, whence we can take $\epsilon_1 > 0$ and $\epsilon_2 > 0$ arbitrarily small. With these choices (5.22) yields that

$$||u_n(\cdot, \cdot, k)||_{L^\infty(U\times U)} \le c^n k^{1/2+\epsilon_1-n(1-\epsilon_2)}$$

and consequently

$$\sum_{n=3}^{\infty} ||u_n(\cdot,\cdot,k)||_{L^\infty(U\times U)} \le c^3 k^{-5/2+(\epsilon_1+3\epsilon_2)} \frac{1}{1-ck^{\epsilon_2-1}}.$$

This proves (ii) as soon as we choose k_0 large enough so that $ck_0^{\varepsilon_2-1} < 1/2$.

To obtain (i), observe that an iteration of the Lippmann-Schwinger equation yields the n:th remainder term in the form $(K_k)^{n+1}u$, which converges to zero by the operator norm estimate for K_k used in (5.22). □

6. Existence of the measurement: convergence of the ergodic averages

Now we are ready to analyze the measurement $m(x,y,\omega)$.

Theorem 6.1. *For $x,y\in U$ the limit* (1.3) *exist almost surely and equals*

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K k^4 |u_s(x,y,k,\omega)|^2 dk = R(x,y) \tag{6.1}$$

where $R(x,y)$ is the smooth function on $U\times U$ given in Proposition 4.1.

Before giving the proof we first describe the philosophy behind Theorem 6.1. Let us write

$$u_s(x,y,k) = \widetilde{u}_1(x,y,k) + u_r(x,y,k), \tag{6.2}$$

where $u_r = (b + u_2 + u_3 + u_4 + \ldots)$ stands for the remainder term (recall that $u_1 = \widetilde{u}_1 + b$). The results of the previous section will yield that the contribution of u_r is negligible in the measurement, whence it remains to understand the mean behaviour of $|\widetilde{u}_1|^2$. The analytic estimates of Section 4 show that the expectation $\mathbb{E}\, k^4|\widetilde{u}_1(x,y,k)|^2$ tends to a limit as $k\to\infty$. In addition, the same estimates verify that the terms $k_1^2\widetilde{u}_1(x,y,k_1)$ and $k_2^2\widetilde{u}_1(x,y,k_2)$ become asymptotically independent as k_2 grows towards infinity (see the figure below). This makes it plausible that one could recover $\lim_{k\to\infty}\mathbb{E}\,|k^4\widetilde{u}_1(x,y,k)|^2$ as a suitable ergodic average, in view of the strong law of large numbers, and this turns out to be true.

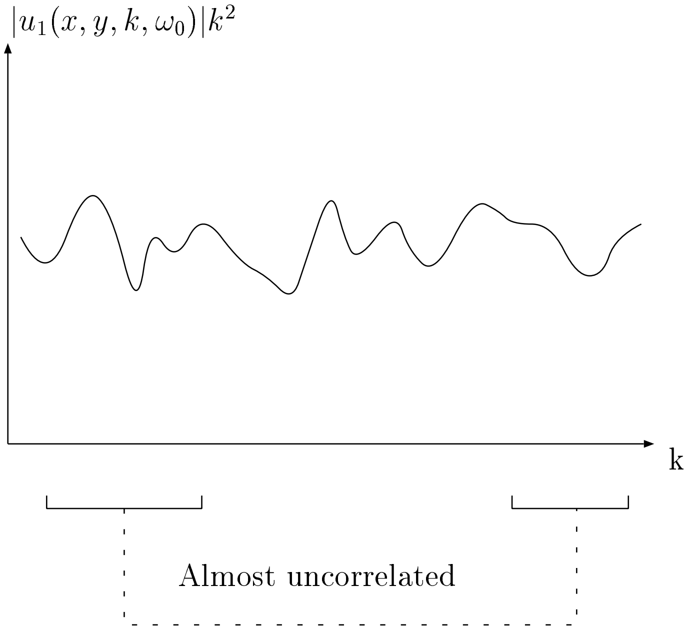
We record an elementary lemma.

Lemma 6.2. *Let X and Y be zero-mean Gaussian random variables. Then*

$$\mathbb{E}\,((X^2-\mathbb{E}\,X^2)(Y^2-\mathbb{E}\,Y^2)) = 2(\mathbb{E}\,XY)^2.$$

Proof. By scaling one may obviously assume that $\mathbb{E}\,X^2 = \mathbb{E}\,Y^2 = 1$. Denote $\mathbb{E}\,XY = \cos\alpha \in [-1,1]$. Then (X,Y) and $(X, \cos(\alpha)X + \sin(\alpha)Y')$ have the same distribution, where Y' is an independent copy of X. The result follows now by a straightforward computation. □

Let us recall an ergodic theorem suitable for our purposes. The following is obtained e.g. as an immediate corollary of [13].



Theorem 6.3. *Let X_t, $t \geq 0$ be a real valued stochastic process with continuous paths. Assume that for some positive constants $c, \varepsilon > 0$ the condition*

$$|\mathbb{E}\, X_t X_{t+r}| \leq c(1+r)^{-\varepsilon}$$

holds for all $t, r \geq 0$. Then almost surely

$$\lim_{K\to\infty} \frac{1}{K} \int_1^K X_t dt = 0.$$

The ergodicity of the term $\widetilde{u}_1$ is verified in the following proposition.

Proposition 6.4. *For any $x, y \in U$ we have almost surely*

$$\lim_{K\to\infty} \frac{1}{K-1} \int_1^K k^4 |\widetilde{u}_1(x,y,k)|^2 dk = R(x,y). \tag{6.3}$$

Proof. According to Lemma 4.1 we have $\lim_{k\to\infty} \mathbb{E}\,(k^4|\widetilde{u}_1(x,y,k)|^2) = R(x,y)$. Hence it is clear that the claim follows as soon as we show that

$$\lim_{K\to\infty} \frac{1}{K-1} \int_1^K Y(x,y,k) dk = 0, \tag{6.4}$$

where $Y(x,y,k) = k^4(|\widetilde{u}_1(x,y,k)|^2 - \mathbb{E}\,|\widetilde{u}_1(x,y,k)|^2)$. Since

$$\begin{aligned} Y(x,y,k) = k^4 \big(&(\mathrm{Re}\,\widetilde{u}_1(x,y,k))^2 - \mathbb{E}\,(\mathrm{Re}\,\widetilde{u}_1(x,y,k))^2) + \\ &+ (\mathrm{Im}\,\widetilde{u}_1(x,y,k))^2 - \mathbb{E}\,(\mathrm{Im}\,\widetilde{u}_1(x,y,k))^2) \big), \end{aligned}$$

we may combine Corollary 4.3 together with Lemma 6.2 to obtain

$$E|Y(x,y,k_1)Y(x,y,k_2)| \leq \frac{c}{1+|k_1-k_2|^2},$$

for any $k_1, k_2 \geq 1$. Statement (6.4) now follows immediately from Theorem 6.3. $\square$

We are ready for

Proof of Theorem 6.1. By denoting $u_r(x,y,k) = b(x,y,k) + u_2(x,y,k) + u_R(x,y,k)$ we may decompose

$$u_s(x,y,k) = \widetilde{u}_1(x,y,k) + b(x,y,k) + u_2(x,y,k) + u_R(x,y,k).$$

According to Lemma 4.2 and Theorem 5.5 we have a.s. $\lim_{k\to\infty} k^2(b(x,y,k) + u_R(x,y,k)) = 0$. Together with Theorem 5.1 this yields that almost surely

$$\lim_{K\to\infty} \frac{1}{K-1} \int_1^K k^4 |u_r(x,y,k)|^2 \, dk = 0. \tag{6.5}$$

The desired statements now follow directly by combining (6.5) and Proposition 6.4, as the obtained cross term may be estimated with the aid of the Cauchy-Schwartz inequality in the space $[1,K]$ equipped with the weight $(K-1)^{-1}dk$. $\square$

7. Conclusion: proof of Theorem 1.3

The results obtained so far (Theorem 6.1 from the previous section) prove directly parts (i) and (ii) of our main result, Theorem 1.3: the measurement (1.3) is almost surely well defined for any $x, y \in U$.

It remains to prove part (iii) of the Theorem, which deals with the recovery of μ from the measurements. Observe that in our case $m_0(x,x) = R(x,x)$ for any $x \in U$, and, by the formula (4.8) in Section 4, we have that

$$R(x,x) = \frac{1}{2^8\pi^2} \int_D \frac{1}{|x-z|^2} \mu(z) \, dz. \tag{7.1}$$

Especially, the function $x \mapsto R(x,x)$ is continuous. Hence, by performing measurements in a dense set of points $x \in U$, Theorem 6.1 shows that almost surely we can recover $R(x,x)$ for *all* $x \in U$.

Thus, the relation (7.1) shows that we are left with a simple deconvolution problem: the values of the convolution

$$H(x) := (h * \mu)(x), \quad h(z) := \frac{1}{2^8\pi^2|z|^2}$$

are known in a open set U that has a positive distance to the support of $\mu \in C_0^\infty(\mathbb{R}^2)$, and we are to show that this knowledge is enough to recover μ. For that end, observe first that $\Delta_z(|z|^{-2p}) = 4p^2|z|^{-2p-2}$. Thus our data determines also the convolutions

$$c_p \Delta_x^p H(x) \quad = \quad \int_D \frac{1}{|x-z|^{2p}} \mu(z) \, dz$$

for $p > 1$ and $x \in U$. Let us denote

$$S(x,r) \quad = \quad \int_{|z-x|=r} \mu(z) \, d|z|,$$

which corresponds to the Radon transform along circles. Fix any $x \in U$. It follows that we are able to recover the integrals

$$\int_{\mathbb{R}_+} \frac{S(x,r)}{r^2} Q(\frac{1}{r^2})\, dr,$$

where $Q(t) = \sum_{j=0}^{p} a_j t^j$, $p \geq 0$. The support of the continuous function $r \mapsto S(x,r)$ lies in a finite interval $[a,b]$ with $a, b > 0$, and obviously the functions of the form $Q(1/r^2)$ are dense in $C([a,b])$. Thus the function $S(x,r)$ is uniquely determined for all $r > 0$.

The observation that we just made can be stated in another form: the data yields the knowledge of integrals of μ over all circles that are centered in the open set U. This is a classical problem of integral geometry, of the Radon type, which can be solved in a simple manner, cf. eg. [4] and the extensive list of references therein. Namely, let $g(z) = \exp(-|z|^2/2)$ for $z \in \mathbb{R}^2$, and observe that knowing the integrals over the above mentioned circles we may compute the convolution $g * \mu(z)$ for $z \in U$. However, $g * \mu$ is clearly real analytic and the set U is open, whence we know $g * \mu$ everywhere. As the Fourier transform of g is smooth and non-zero all over $\mathbb{R}^2$, it follows that we can recover μ uniquely. This completes the proof of our main result. $\square$

Remark 1. Above we have assumed that the potential is Gaussian and centered, but it is possible to dispense with the assumption that $\mathbb{E}\, q = 0$ in Theorem 1.3. Namely, assume that $\mathbb{E}\, q = p \in C_0^\infty(D)$ and denote $q_0 = q - p$. Then

$$\mathbb{E}\,(q(z_1)q(z_2)) = \mathbb{E}\,(q_0(z_1)q_0(z_2)) + p(z_1)p(z_2). \tag{7.2}$$

We briefly analyze how the above proof should be modified for this case. We have again that $q \in H_0^{-\varepsilon,p}(D)$ a.s. Thus the results for the direct scattering problem given in Section 3 are valid without any change, and we see in particular that the higher order Born terms $u_3 + u_4 + \ldots$ do not contribute to measurement (1.3).

When the term $p(z_1)p(z_2)$ in formula (7.2) is added to the covariance operator in formula (4.9), we see that this causes only a $S_{1,0}^{-\infty}$ perturbation for the symbol of the covariance operator C_q. Hence the proof of Proposition 4.1 remains unchanged. With small modifications the considerations in Subsection 5.1 remain valid, too. Finally, as the stationary phase method yields $\mathbb{E}\,\widetilde{u}_1(x,y,k) = o(k^{-\infty})$, we obtain Theorem 1.3 by finishing the proof as in Sections 6 and 7.

Remark 2. One may also consider as the measurement the average

$$\lim_{K\to\infty} \frac{1}{K-1} \int_1^K \int_U k^4 |u_s(x,x,k,\omega)|^2 \phi(x) dx dk$$

with $\phi \in C_0^\infty(U)$. The main result can also be stated in terms of this kind of 'distributional measurements'. In this setup the proof of Theorem 1.3 remains essentially unchanged. One should also note that the function $R(x,x)$ is uniquely determined from integrals $\int_U R(x,x)\phi(x)dx$ against a countable dense set of smooth test functions ϕ.

Remark 3. It is interesting to compare non-stability of the stochastical inverse problem with the the deterministic one. In Theorem 1.3 the operator T is linear and thus the reconstruction of μ requires solving of a linear ill-posed inverse problem. More precisely, by the observations in the present section, T corresponds to a Radon transform over circles, which gives a pretty clear picture of the ill-posedness. This is markedly different from the corresponding deterministic problems. Let us assume that in the deterministic case one possesses additional information as data: assume that besides the amplitude $|u_s(x,y,k)|$ one also knows the phase $\arg u_s(x,y,k)$ and this data is given for pairs $(x,y) \in U \times U$. Then one can in principle find from the extended data $u_s(x,y,k)$, $x,y \in U$, $k \in \mathbb{R}_+$ the solution $u_s(x,y,k)$ for $x,y \in \mathbb{R}^2 \setminus D$, $k \in \mathbb{R}_+$ by using analytic continuation. This yields the far field data for all θ, ω, and k, which is finally known to determine the Fourier transform of $\hat{q}(\xi)$ (cf. e.g. [39, pp. x-xi], or [48] for an alternative solution based on exponentially growing plane waves).

Remark 4. We mention that in the backscattering case $y = x$ it is possible to avoid the use of the pseudodifferential calculus in Section 4, although the proof remains fairly technical. By this manner it is possible to relax somewhat the assumption of smoothness of μ.

Appendix A: Markov random fields

We follow here the monograph of Rozanov [44] and recall in more detail the relation of local operators to the Markov random fields.

As in Section 1.1 we consider a bounded domain $D \subset \mathbb{R}$ and a Gaussian centered random variable q having values in $\mathcal{D}'(D)$. We simply call q a field on D. The Markov property of q was defined previously in Definition 1.4. One can also characterize the Markov property in terms of conditional independence. Recall that of three sub-σ-algebras Σ_i, $i = 0,1,2$ of Σ the σ-algebras Σ_1 and Σ_2 are called *conditionally independent with respect to* Σ_0 if the conditional probabilities satisfy

$$P(A_1 \cap A_2|\Sigma_0) = P(A_1|\Sigma_0)P(A_2|\Sigma_0)$$

for any $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$. Let S_1, S_2, S_ε be as in Section 1.1. Thus, $S_1 \subset D$ is an arbitrary open subset, $S_2 = D \setminus \overline{S_1}$, and $S_\varepsilon = \{x \in D | d(x, \partial S_1) < \varepsilon\}$. We have

Theorem 7.1. *A generalized random field q is Markov, if and only if the σ-algebras $\mathcal{B}(S_1)$ and $\mathcal{B}(S_2)$ are conditionally independent with respect to $\mathcal{B}(S_\varepsilon)$, for any S_1, S_2 and S_ε as above.*

The above claim follows readily from the definition and basic properties of the conditional probability and generalized random variable. We refer to [44, pp.54, 56, 97] for the proof and the definition of a Gaussian Markov fields.

In order to be able to express the Markov property in terms of the covariance operator we need to introduce the notion of a biorthogonal field p. A centered Gaussian field p on D is biorthogonal to the field q if

$$\mathbb{E}\left(\langle q,\psi_1\rangle\langle p,\psi_2\rangle\right)=(\psi_1,\psi_2),\quad \psi_i\in C_0^\infty(D)\ ,i=1,2, \tag{7.3}$$

and in addition there is the equality $H_q(D)=H_p(D)$. Here, given a field η on D and open subset $S\subset D$ we denote

$$H_\eta(S)=\overline{\mathrm{span}}\{\langle \eta,\phi\rangle|\phi\in C_0^\infty(S)\},$$

where the closure in taken in $L^2(\Omega)$, i.e. in the space of square integrable random variables. By the definition, the biorthogonal function is unique, and the covariance operator C_p can be thought as a (partial) inverse operator of the covariance operator C_q of our original field. We have the following:

Theorem 7.2. *If the field q is Markov, then the covariance operator C_p of the biorthogonal field is* <u>*local*</u> *in the sense that $\langle C_p\psi_1,\psi_2\rangle=0$ if $\psi_1,\psi_2\in C_0^\infty(D)$ have disjoint supports.*

See [44, pp.112-113] for this fact.

In order to obtain a converse statement we must assume slightly more than just biorthogonality from the relation between q and p. That is, p must be *dual* to q. Let us define $H_\eta^+(S)=\bigcap_{\varepsilon>0}H_\eta(S_\varepsilon)$, where the intersection is taken over all ε-neighborhoods S_ε of the subset $S\subset D$. The biorthogonal field p is *the dual field* of q if the equality

$$H_p(S)=H_q^+(D\setminus S)^\perp \tag{7.4}$$

holds for all open subsets $S\subset D$. There are useful sufficient conditions ([44, Lemma 1-2, pp. 108-109]) for the duality to hold. We then have (c.f. [44, Theorem on p. 112]) the converse result

Theorem 7.3. *Assume that the field p is dual to the field q (i.e. (7.4) holds). Then q is Markov if the covariance operator C_p is local.*

Put together, if the duality holds, then the Markov property is equivalent to the locality of the dual field.

By using above results, one easily verifies that there are always Markov fields q on D such that the inverse of the covariance operator $C_q^{-1}=C_p$ has the principal part $-a(x)\Delta$, assuming that $a\in C^\infty(D)$ satisfies $\inf_D a(x)>0$. Actually, by considering the operator $-\nabla\cdot a(x)\nabla$, we obtain this statement easily from the following result in [44].

Theorem 7.4. *Assume that the covariance operator C_p of the biorthogonal field p is a partial differential operator of order $2\ell\geq 1$ with smooth coefficients, and it* *satisfies the coercivity inequality* ($c > 0$)

$$\langle Q_p\phi, \phi\rangle \geq c \sum_{|\alpha|\leq \ell} \|D^\alpha \phi\|^2_{L^2(D)}, \quad \phi \in C_0^\infty(D).$$

Then the field q is Markov.

See [44, Theorem 3 p.129] for this statement.

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