QUASIHYPERBOLIC BOUNDARY CONDITIONS AND CAPACITY: UNIFORM CONTINUITY OF QUASICONFORMAL MAPPINGS

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ABSTRACT. We prove that quasiconformal maps onto domains which satisfy a suitable growth condition on the quasihyperbolic metric are uniformly continuous when the source domain is equipped with the internal metric. The obtained modulus of continuity and the growth assumption on the quasihyperbolic metric are shown to be essentially sharp. As a tool we prove a new capacity estimate.

1. INTRODUCTION

It is well-known that a quasiconformal mapping between sufficiently nice domains is uniformly continuous. In this paper we consider the following question. Suppose that we are given the growth condition

$$k_\Omega(x_0, x) \le \phi\left(\frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)}\right) \tag{1.1}$$

on the quasihyperbolic metric k_Ω of a domain, where $\phi : (0, \infty) \to (0, \infty)$ is an increasing function and x_0 is a fixed point in Ω. Under which conditions on Ω' and ϕ can we conclude that each quasiconformal mapping $f : \Omega' \to \Omega$ is uniformly continuous when Ω is equipped with the euclidean metric and Ω' with the internal distance $\delta(x, y)$, i.e., the infimum of lengths of curves that join x to y in Ω', or some other natural metric on Ω'.

Let us comment on the history of the problem. Becker and Pommerenke [3] proved that a conformal mapping f from the unit disk onto a domain Ω is uniformly α-Hölder continuous, $0 < \alpha \le 1$, if (and only if) the hyperbolic metric ρ_Ω in Ω satisfies a logarithmic growth condition

$$\rho_\Omega(z_0, z) \le \frac{1}{\alpha} \log \frac{\operatorname{dist}(z_0, \partial\Omega)}{\operatorname{dist}(z, \partial\Omega)} + C_0, \tag{1.2}$$

where $z_0 = f(0)$ and $C_0 < \infty$. Here $\operatorname{dist}(\cdot, \partial\Omega)$ denotes the Euclidean distance to the boundary of Ω. Notice that the hyperbolic metric ρ_Ω above is comparable to k_Ω by the Koebe distortion theorem. Gehring and Martio [5] proved a version of

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this statement in all dimensions for possibly multiply connected domains. Their assumption on on Ω was that

$$k_\Omega(x_0,x) \le \frac{1}{\beta}\log\frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{dist}(x,\partial\Omega)} + C_0 \tag{1.3}$$

for some (each) $x_0 \in \Omega$ and a constant $C_0 = C_0(x_0) < \infty$. They showed that this rather easily imples that, given Ω' and a quasiconformal mapping $f : \Omega' \to \Omega$, f is uniformly Hölder continuous on each (open) ball $B \subset \Omega'$ with an exponent α and constant M which depend only on n, K, and the constants β and C_0 but are independent of B. Given points x, y were then assumed to be joinable by a "nice" curve γ along which this estimate was applied in order to obtain a global distortion estimate. The "niceness" of γ required that the length of it was no more than a constant times $|x-y|$ and that γ stayed sufficiently far away from the boundary, when measured in a certain averaged sense.

Very recently, Koskela, Onninen, and Tyson [13] improved on the above results by showing that, under the logarithmic growth condition (1.3), a mapping f as above is always uniformly Hölder continuous when Ω' is equipped with the internal metric $\delta(x,y)$ and Ω with the euclidean metric.

Before stating our first result, let us indicate what kinds of functions ϕ in the growth condition (1.1) are critical. Consider the unbounded simply connected domain

$$\Omega = \{[x,y] \in \mathbf{R}^2 : \ x \ge -1,\ |y| \le \exp(-x)\}$$

Then (1.1) holds with $x_0 = (0,0)$ and $\phi(t) = Ct + C$ and there exists a conformal mapping from the unit disk onto Ω. Thus this growth order does not guarantee uniform continuity even when Ω' is the unit disk. On the other hand, using the Koebe distortion theorem, one easily checks that the growth order $\phi(t) = t^s + C$ is sufficient for the uniform continuity of conformal mappings from the disk onto Ω, for any $s < 1$. In fact, even the integrability condition

$$\int_0^\infty \frac{dt}{\phi^{-1}(t)} < \infty \tag{1.4}$$

suffices. Our first result comes rather close to verifying that this condition is always sufficient.

In what follows, we denote by $\delta_\Omega(x,y)$ the internal distance between a pair of points $x, y \in \Omega$, i.e., the infimum of the lengths of curves in Ω joining x to y. For technical reasons, it is more convenient to write the growth condition (1.1), following Gotoh [8], in the form

$$\operatorname{dist}(x,\partial\Omega) \le \operatorname{dist}(x_0,\partial\Omega)\varphi(k_\Omega(x_0,x)), \tag{1.5}$$

where we assume that $\varphi : [0,\infty) \to (0,\infty)$ is non-increasing, and that

$$\int_0^\infty \varphi(t)\ dt < \infty. \tag{1.6}$$

To relate (1.1) to (1.5), simply notice that we may take

$$\varphi(t) = \frac{1}{\phi^{-1}(t)}$$

for t large. Given $a \in \mathbf{R}$ we define

$$\Phi(s) = \int_s^\infty \varphi(t)\, dt \text{ and } \psi_a(t) = \left(\Phi^{-1}(t)\right)^{-a}. \tag{1.7}$$

Theorem 1.1. *Let $\Omega, \Omega' \subsetneq \mathbf{R}^n$, $n \geq 2$ be domains and assume that Ω satisfies* (1.5) *and that* (1.6) *holds. Suppose that there is $a > n-1$ such that the function ψ_a is concave. Then each quasiconformal mapping $f : \Omega' \to \Omega$ satisfies*

$$|f(x') - f(y')| \leq C\Phi\Big(C \log \frac{C}{\delta_{\Omega'}(x', y')}\Big) \tag{1.8}$$

for every pair x', y' of distinct points in Ω'.

The above concavity condition does not hold for $\varphi(t) = t^{-s} + C$, when $s = n$, in other words, when $\phi(t) = t^{1/n} + C$. We do however believe that Theorem 1.1 holds even without this condition (also see [13] pp. 429–430). The modulus of continuity given in (1.8) is essentially sharp even when Ω' is the disk, see Section 5 below. For example, when $\phi(t) = t^s + C$, $0 < s < 1$, we may take $\Phi(t) = t^{1-\frac{1}{s}}$, and when $\phi(t) = C \log t + C$, we have $\Phi(t) = \exp(-Ct)$, and we recover Hölder continuity. Moreover, the integrability condition is really needed because otherwise Ω may fail to be bounded.

Our second result allows us to omit the concavity condition, but we need to use another distance function on Ω'.

Theorem 1.2. *Let $\Omega, \Omega' \subsetneq \mathbf{R}^n$, $n \geq 2$ be domains and assume that Ω satisfies* (1.5) *and that* (1.6) *holds. Then each quasiconformal mapping $f : \Omega' \to \Omega$ satisfies*

$$|f(x') - f(y')| \leq C\Phi\Big(C \log \frac{C}{\operatorname{diam}[x', y']_{\Omega'}}\Big) \tag{1.9}$$

for every $x', y' \in \Omega'$, where $[x', y']_{\Omega'}$ is any quasihyperbolic geodesic joining x' to y' in Ω'.

Note that this gives us the result of Theorem 1.1 without the concavity assumption if we assume instead that the Gehring-Hayman inequality is satisfied in Ω'. The Gehring-Hayman inequality states that there is a constant M so that

$$\operatorname{diam}[x', y']_{\Omega'} \leq M\delta(x', y') \tag{1.10}$$

for every $x', y' \in \Omega'$. It is known to hold in many natural situations [4], [10], [2], in particular for simply connected domains and more generally for domains quasiconformally equivalent to uniform domains. Again the given modulus of continuity is essentially sharp, see Section 5.

Our proofs of Theorem 1.1 and Theorem 1.2 rely on certain capacity estimates in domains satisfying the given growth condition on the quasihyperbolic metric. Specifically, we establish the following result.

Theorem 1.3. *Let Ω be a proper subdomain of $\mathbf{R}^n$, $n \geq 2$, with diameter one which satisfies* (1.5) *and that* (1.6) *holds. Suppose that there is $a > n-1$ such that the function ψ_a is concave. Let Q_0 denote a fixed Whitney cube containing the basepoint x_0. Then*

$$\operatorname{cap}(E, Q_0; \Omega) \geq C(\Phi^{-1}(C \operatorname{diam} E))^{1-n} \tag{1.11}$$

for all continua $E \subset \Omega$.

Here $\operatorname{cap}(E, F; \Omega)$ denotes the n-capacity between a pair of disjoint continua E and F in the domain Ω, see Section 2. The given bound on the capacity is sharp, but we do not know if one could omit the concavity assumption.

We prove Theorem 1.3 by a chaining argument involving the Poincaré inequality on Whitney cubes in Ω. Our argument here is an improvement on the techniques introduced in [13] and in [11]. Theorem 1.2 will be obtained from a version of Theorem 1.3, which is proven without the concavity assumption. In this analog, only continua E of a special type are considered.

This paper is organized as follows. Section 2 reviews the notations used in the paper and gives most of the necessary definitions. In Section 3, we introduce the quasihyperbolic metric and prove some of it's important properties. Section 4 contains the proofs of the capacity estimates and the final section, Section 5, is devoted to the discussion of uniform continuity.

2. Notations and definitions

We denote by $\mathbf{R}^n$, $n \geq 1$, the euclidean space of dimension n. The Lebesgue measure in $\mathbf{R}^n$ is denoted by m, although we usually abbreviate $dm(x) = dx$. We write $\#M$ for the number of elements of a set M.

For a cube $Q \subset \mathbf{R}^n$ with center x and side length $s(Q)$ and for a factor $\lambda > 0$, we denote by λQ the dilated cube which is again centered at x but has side length $\lambda s(Q)$. The center of a cube Q is denoted by c_Q. Given an integrable function u on an open set $A \subset \mathbf{R}^n$ we write

$$u_A = \fint_A u(x)dx = \frac{1}{m(A)} \int_A u(x)dx.$$

For an increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, we denote by $\mathcal{H}^\infty_\varphi$ the *Hausdorff φ-content*: $\mathcal{H}^\infty_\varphi(E) = \inf \sum_i \varphi(r_i)$, where the infimum is taken over all coverings of $E \subset \mathbf{R}^n$ with balls $B(x_i, r_i)$, $i = 1, 2, \ldots$. When $\varphi(t) = t^s$ for some $0 < s < \infty$ we write $\mathcal{H}^\infty_s = \mathcal{H}^\infty_\varphi$.

For disjoint compact sets E and F in the domain Ω, we denote by $\operatorname{cap}(E,F;\Omega)$ the *conformal* (or n-) *capacity* of the pair (E,F);

$$\operatorname{cap}(E,F;\Omega)=\inf_u\int_\Omega|\nabla u|^n\,dx,$$

where the infimum is taken over all continuous functions u in the Sobolev space $W^{1,n}_{loc}(\Omega)$ which satisfy $u(x)\le 0$ for $x\in E$ and $u(x)\ge 1$ for $x\in F$.

For $K\ge 1$ and Ω,Ω' as above, we say that a homeomorphism $f:\Omega\to\Omega'$ is K-*quasiconformal* if

$$\frac{1}{K}\operatorname{cap}(E,F;\Omega)\le\operatorname{cap}(E',F';\Omega')\le K\operatorname{cap}(E,F;\Omega)$$

whenever E and F are disjoint compact sets in Ω, where $E'=f(E)$ and $F'=f(F)$. For the basic theory of quasiconformal maps, we refer the reader to the monograph [16] of Väisälä.

Let Ω be a bounded domain in $\mathbf{R}^n$, $n\ge 2$. Set $s(\Omega)=n^{-1/2}\operatorname{diam}\Omega$. We denote by $\mathcal{W}=\mathcal{W}(\Omega)$ a *Whitney decomposition* of the domain Ω into *Whitney cubes* Q, i.e., $\Omega=\cup_{Q\in\mathcal{W}}Q$, the cubes in $\mathcal{W}$ have pairwise disjoint interiors, vertices in the set

$$2^{-N}s(\Omega)\cdot Z^n:=\{(2^{-j}s(\Omega)l_1,\ldots,2^{-j}s(\Omega)l_n):j\in N,l_1,\ldots,l_n\in Z\},$$

and satisfy $\operatorname{diam}Q\le\operatorname{dist}(Q,\partial\Omega)\le 4\operatorname{diam}Q$ for each $Q\in\mathcal{W}$. For the existence of such a decomposition, we refer to Stein's book [15, VI.1]. For any λ, $1<\lambda<5/4$, the expanded collection of cubes $\{\lambda Q:Q\in\mathcal{W}\}$ has bounded overlap, specifically,

$$\sup_{x\in\Omega}\sum_{Q\in\mathcal{W}}\chi_{\lambda Q}(x)\le 12^n<\infty.$$

See, e.g., [15, VI.1.3, Proposition 3]. For $j\in\mathbf{N}$ we set

$$\mathcal{W}_j=\{Q\in\mathcal{W}:\ j\le k_\Omega(c_Q,x_0)<j+1\}. \tag{2.1}$$

We will also suppose that x_0 is the center of the Whitney cube Q_0.

By C we will denote various positive constants that may depend on n, φ, a, M, K, $\operatorname{dist}(x_0,\Omega)$ and $\operatorname{dist}(f^{-1}(x_0),\partial\Omega')$. These constants may vary from expression to expression as usual.

3. Preliminary results on the quasihyperbolic metric

Throughout this section, Ω will denote a proper subdomain in the euclidean space $\mathbf{R}^n$, $n\ge 2$. Recall that the *quasihyperbolic metric* k_Ω in the domain Ω is defined to be

$$k_\Omega(x,y)=\inf_\gamma k_\Omega\text{- length}(\gamma),$$

where the infimum is taken over all rectifiable curves γ in Ω which join x to y and

$$k_\Omega\text{- length}(\gamma) = \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial\Omega)}$$

denotes the *quasihyperbolic length* of γ in Ω. This metric was introduced by Gehring and Palka in [7]. A curve γ joining x to y for which $k_\Omega\text{- length}(\gamma) = k_\Omega(x, y)$ is called a *quasihyperbolic geodesic*. Quasihyperbolic geodesics joining any two points of a proper subdomain of $\mathbf{R}^n$ always exist, see [6, Lemma 1]. If γ is a quasihyperbolic geodesic in Ω and $x', y' \in \gamma$, we denote by $\gamma(x', y')$ the portion of γ which joins x' to y'. Given any two points $x, y \in \Omega$ we denote by $[x, y]_\Omega$ the quasihyperbolic geodesic that joins x and y.

When x and y are sufficiently far apart, $k_\Omega(x, y)$ is roughly equal to the number $N(x, y)$ of Whitney cubes Q that intersect a quasihyperbolic geodesic γ joining x to y. More precisely,

$$N(x, y)/C \leq k_\Omega(x, y) \leq CN(x, y)$$

for all $x, y \in \Omega$ with $|x - y| \geq \operatorname{dist}(x, \partial\Omega)/2$, where $C = C(n)$.

The following estimate is always true for all $x \in \Omega$ (see [7])

$$k_\Omega(x_0, x) \geq \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)}. \tag{3.1}$$

Lemma 3.1. *Let γ be a quasihyperbolic geodesic in Ω starting at the basepoint x_0. Then*

$$\operatorname{card}\{Q \in \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_j : \ Q \cap \gamma \neq \emptyset\} \leq Cj$$

for all $j \in \mathbf{N}$.

Proof. This follows from the fact that the length of a geodesic is comparable with the number of Whitney cubes it intersects and the definition of $\mathcal{W}_j$ (2.1). □

For each cube $Q \in \mathcal{W}$ we fix a quasihyperbolic geodesic γ joining x_0 with c_Q. We denote by $P(Q)$ the collection of all Whitney cubes which intersect γ. The shadow $S(Q)$ of a Whitney cube Q is defined by

$$S(Q) = \bigcup_{\substack{Q_1 \in \mathcal{W} \\ Q \in P(Q_1)}} Q_1.$$

Using only Lemma 3.1 we obtain analogously to [13, Lemma 2.3] the following lemma.

Lemma 3.2. *For every $j \in \mathbf{N}$ and $x \in \Omega$ we have*

$$\sum_{Q \in \mathcal{W}_1 \cup \ldots \cup W_j} \chi_{S(Q)}(x) \leq Cj.$$

Lemma 3.3. *Suppose that Ω satisfies* (1.5) *and that* (1.6) *holds. Then for every $Q \in \mathcal{W}$*

$$\operatorname{diam} S(Q) \le C_1 \operatorname{dist}(x_0, \partial\Omega)\Phi(Ck_\Omega(c_Q, x_0)).$$

Proof. Clearly for every $\tilde{Q} \in S(Q)$ inequality (1.5) and $\operatorname{dist}(c_{\tilde{Q}}, \partial\Omega) \sim \operatorname{diam} \tilde{Q}$ give us

$$\operatorname{diam} \tilde{Q} \le C \operatorname{dist}(x_0, \partial\Omega)\varphi(k_\Omega(x_0, c_{\tilde{Q}})) \le C \operatorname{dist}(x_0, \partial\Omega)\Phi\big(k_\Omega(x_0, c_{\tilde{Q}})/2\big). \tag{3.2}$$

Now it is enough to show that for every $Q_1, Q_2 \in S(Q)$ we have $\operatorname{dist}(c_{Q_1}, c_{Q_2}) \le C\Phi(Ck_\Omega(x_0, c_Q))$. Let γ_i be a quasihyperbolic geodesic joining x_0 and c_{Q_i} for $i = 1, 2$ and choose points $x_i \in Q \cap \gamma_i$. Clearly

$$|c_{Q_1} - c_{Q_2}| \le \operatorname{length}(\gamma_1(x_1, c_{Q_1})) + \operatorname{diam} Q + \operatorname{length}(\gamma_2(x_2, c_{Q_2})).$$

Now the first inequality in (3.2) implies for $i = 1, 2$ that

$$\operatorname{length}(\gamma_i(x_i, c_{Q_i})) \le C \sum_{\substack{\tilde{Q} \in \mathcal{W} \\ \tilde{Q} \cap \gamma_i(x_i, c_{Q_i}) \ne \emptyset}} \operatorname{diam} \tilde{Q}$$

$$\le C \operatorname{dist}(x_0, \partial\Omega) \sum_{j=\max\{0, k_\Omega(x_0, c_Q) - C\}}^{\infty} \varphi(j) \le C \operatorname{dist}(x_0, \partial\Omega)\Phi(Ck_\Omega(x_0, c_Q)).$$

Together with (3.2) for $\tilde{Q} = Q$ this implies the desired result. $\square$

As a corollary to this lemma we obtain the following known fact (see [8]).

Corollary 3.4. *Suppose that $\Omega \subsetneq \mathbf{R}^n$ satisfies* (1.5) *and that* (1.6) *holds. Then Ω is a bounded domain (because $S(Q_0)$ is bounded).*

Lemma 3.5. *Suppose that Ω satisfies* (1.5) *and φ satisfies* (1.6). *Assume that a curve γ in Ω satisfies*

$$\#\{Q \in \mathcal{W} :\ Q \cap \gamma(x_1, x_2) \ne \emptyset\} \le Ck_\Omega(x_1, x_2) + C \tag{3.3}$$

for every $x_1, x_2 \in \gamma$. Then there are $j \in \mathbf{N}$ and a cube $Q \in \mathcal{W}_j$ such that $Q \cap \gamma \ne \emptyset$ and

$$\operatorname{diam}(\gamma) \le C \operatorname{dist}(x_0, \partial\Omega)\Phi(j/2). \tag{3.4}$$

Proof. Denote $j = \min\{i \in \mathbf{N}_0 :\ \text{there is } Q \in \mathcal{W}_i \text{ such that } Q \cap \gamma \ne \emptyset\}$. From the properties of Whitney cubes, (1.5) and definition of j we have

$$\operatorname{diam}(\gamma) \le \sum_{\substack{Q \in \mathcal{W} \\ Q \cap \gamma \ne \emptyset}} \operatorname{diam} Q \le C \sum_{\substack{Q \in \mathcal{W} \\ Q \cap \gamma \ne \emptyset}} \operatorname{dist}(c_Q, \partial\Omega) \tag{3.5}$$

$$\le C \operatorname{dist}(x_0, \partial\Omega) \sum_{\substack{Q \in \mathcal{W} \\ Q \cap \gamma \ne \emptyset}} \varphi(k_\Omega(x_0, c_Q)) \le C \operatorname{dist}(x_0, \partial\Omega) \sum_{i=j}^{\infty} M_i \varphi(i)$$

where $M_i = \#\{Q \in \mathcal{W}_i : Q \cap \gamma \ne \emptyset\}$.

Denote $A_m = \{Q : Q \in \mathcal{W}_i \text{ for some } i \leq m \text{ and } Q \cap \gamma \neq \emptyset\}$ for any $m \geq j$. We claim that

$$\#A_m = M_j + \ldots + M_m \leq Cm. \tag{3.6}$$

This inequality is clearly true if $\#A_m < 3$ and thus we may assume that A_m contains at least three cubes. Find $Q_1^m, Q_2^m \in A_m$ which lay on γ as far as possible from each other and fix $x_1 \in Q_1^m \cap \gamma$ and $x_2 \in Q_2^m \cap \gamma$. That is, for every $Q \in A_m \setminus \{Q_1^m, Q_2^m\}$ we have $Q \cap \gamma \subset \gamma(x_1, x_2)$. From (3.3) we obtain

$$C(M_j + \ldots + M_m) = C\#A_m \leq Ck_\Omega(x_1, x_2) + C$$

$$\leq C + C\big(k_\Omega(x_1, c_{Q_1^m}) + k_\Omega(c_{Q_1^m}, x_0) + k_\Omega(x_0, c_{Q_2^m}) + k_\Omega(c_{Q_2^m}, x_2)\big) \leq C + Cm$$

which gives us (3.6).

Now we multiply (3.6) by $a_m = \varphi(j) - \varphi(j+1)$ and sum these inequalities for m from j to infinity. The left hand side is equal to

$$\sum_{m=j}^{\infty}\Big(a_m \sum_{i=j}^{m} M_i\Big) = \sum_{i=j}^{\infty}\left(M_i \sum_{m=i}^{\infty}\Big(\varphi(m) - \varphi(m+1)\Big)\right) = \sum_{i=j}^{\infty} M_i \varphi(i) \tag{3.7}$$

and is less or equal to the right-hand side

$$C \sum_{m=j}^{\infty} m\Big(\varphi(m) - \varphi(m+1)\Big) \tag{3.8}$$

$$= Cj\varphi(j) + C \sum_{m=j+1}^{\infty} \varphi(m) \leq C \int_{j/2}^{\infty} \varphi(t)\, dt.$$

From (3.5), (3.7) and (3.8) we obtain (3.4). $\square$

In order to later apply the previous lemma in connection to quasiconformal mappings we record the following fact.

Proposition 3.6. *Let $\Omega, \Omega' \subsetneq \mathbf{R}^n$, $n \geq 2$ be domains and assume that $f : \Omega' \to \Omega$ is a quasiconformal mapping. Let γ' be a quasihyperbolic geodesic in Ω' and denote $\gamma = f(\gamma')$. Then*

$$\#\{Q \in \mathcal{W} :\ Q \cap \gamma(x_1, x_2) \neq \emptyset\} \leq Ck_\Omega(x_1, x_2) + C$$

for every $x_1, x_2 \in \gamma$.

This result is essentially contained in [6, Theorem 3], according to which quasiconformal mappings are bi-Lipschitz for large distances in the quasihyperbolic metric. To deduce the above statement, also use the facts that the quasihyperbolic metric can be estimated in terms of the number of the Whitney cubes that intersect the corresponding geodesic and that a quasiconformal mapping maps Whitney cubes to Whitney type objects.

4. Capacitary estimates

Proof of Theorem 1.3. This proof goes analogously to the proofs of theorems 1.4. and 4.2. in [13]. For the completeness of our proofs we give the details. Let $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ be a test function for the n-capacity for the pair (Q_0, E), i.e., $u : \Omega \to [0,1]$ is a continuous function and $u(x) = 1$ for $x \in E$ and $u(x) = 0$ for $x \in Q_0$.

For each $x \in E$, let $Q(x)$ denote the Whitney cube containing x. We define a subchain $P'(Q(x)) \subset P(Q(x))$ as follows: $P'(Q(x)) = \{Q_s, \dots, Q_f\}$ consists of a chain of Whitney cubes, which begins with the terminal cube $Q_s = Q(x)$ and continues back along the chain $P(Q(x))$ until it reaches the first cube for which $Q_f \in W_{j_0}$ where j_0 is a fixed integer such that $j_0 \sim \Phi^{-1}(\operatorname{diam} E)$. From now on let j_0 be fixed.

We claim that without loss of generality we can suppose that $u_{Q(x)} \geq 2/3$ and $u_{Q_f} \leq 1/3$. First, suppose that $u_{Q(x)} < 2/3$ for some $x \in E$. Then we can use some known results from the literature ([17], [9, Theorem 5.9]) to deduce that

$$\int_\Omega |\nabla u(x)|^n \, dx \geq C \left(\log \frac{1}{\operatorname{diam} E} \right)^{1-n}$$

(see [13] for details). Note that in view of (3.1) and (1.5) we obtain $\varphi(t) \geq e^{-t}$ and thus $\Phi(t) \geq e^{-t}$. Since Φ is non-increasing, this implies that this estimate of capacity is no worse than (1.11).

Next suppose that the final cube Q_f in the path $P'(Q(x))$ satisfies $u_{Q_f} > 1/3$. Then we have two cubes Q_0 and Q_f in the domain Ω and a continuous $W^{1,n}_{\mathrm{loc}}$ -function u satisfying $u = 0$ on Q_0 and $u_{Q_f} > 1/3$. We can find a chain of Whitney cubes which joins Q_0 to Q_f i.e. cubes $\{Q_i\}_{i=0}^{l}$ such that Q_0 is the central cube, $\overline{Q_i} \cap \overline{Q_{i+1}} \neq \emptyset$ for all $i \in \{0, \dots, l-1\}$, $Q_l = Q_f$ and

$$l \leq C j_0 \leq C \Phi^{-1}(\operatorname{diam} E) \tag{4.1}$$

where the last inequality follows from the fact that the length of a geodesic is comparable with the number of Whitney cubes it intersects and the choice of j_0. In this situation a straightforward chaining argument involving Poincaré inequality on the cubes from $P'(Q(x))$ (c.f. [12] pp 519–520 or [14] Lemma 8) gives us the estimate

$$1 \leq C \sum_{i=1}^{l} \operatorname{diam} Q \fint_{Q_i} |\nabla u(y)| dy. \tag{4.2}$$

By applying Hölder's inequality twice we obtain

(4.3)
$$1 \leq C \sum_{i=1}^{l} \Big(\int_{Q_i} |\nabla u|^n \Big)^{1/n} \leq C \left(\sum_{i=1}^{l} \int_{Q_i} |\nabla u|^n \right)^{\frac{1}{n}} l^{\frac{n-1}{n}} \leq C \Big(\int_\Omega |\nabla u|^n \Big)^{\frac{1}{n}} l^{\frac{n-1}{n}}.$$

Thus (4.1) gives us (1.11) in this situation.

Therefore we may really assume that $u_{Q(x)} \geq 2/3$ and $u_{Q_f} \leq 1/3$. As before, for a chain $P'(Q(x))$ of cubes we may obtain an estimate

$$1 \leq C \sum_{Q \in P'(Q(x))} \operatorname{diam} Q \fint_{Q_i} |\nabla u(y)| dy. \tag{4.4}$$

We now choose a Frostman measure μ on the continuum E with the growth function $\psi_a(t)$ (see e.g. [1, Theorem 5.1.12]), i.e., a Borel measure supported on E satisfying

$$\mu(E \cap B(x,r)) \leq \psi_a(r) \tag{4.5}$$

for all balls $B(x,r)$ and

$$\mu(E) \geq C\mathcal{H}^{\infty}_{\psi_a}(E) \geq C\psi_a(\operatorname{diam} E),$$

where the last inequality follows from the concavity of ψ_a.

Integrating (4.4) over the set E with respect to the Frostman measure μ and applying Hölder's inequality we obtain

$$\mu(E) \leq C \int_E \sum_{Q \in P'(Q(x))} \Big(\int_Q |\nabla u|^n \Big)^{1/n} d\mu(x).$$

By interchanging the order of summation and integration we obtain in view of the choise of Q_f that

$$\mu(E) \leq C \sum_{j=j_0}^{\infty} \sum_{Q \in W_j} \mu(S(Q) \cap E) \Big(\int_Q |\nabla u|^n \Big)^{1/n}.$$

Hölder's inequality gives us

$$\mu(E) \leq C \Big(\sum_{j=j_0}^{\infty} \sum_{Q \in W_j} \mu(S(Q) \cap E)^{1+1/(n-1)} \Big)^{1-1/n} \Big(\int_\Omega |\nabla u|^n \Big)^{1/n}.$$

We require an estimate for the double sum on the right hand side, which we give in the following lemma:

Lemma 4.1. *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with diameter one which satisfies* (1.5) *and suppose* (1.6) *holds. Assume that the function ψ_a is concave. Suppose that μ is a Borel measure on $\mathbf{R}^n$ which satisfies the growth condition $\mu(B(x,r)) \leq \psi_a(r)$ where $a > 1/\delta$. Then for any set $E \subset \Omega$ we have*

$$\sum_{j=j_0}^{\infty} \sum_{Q \in W_j} \mu(S(Q) \cap E)^{1+\delta} \leq C\mu(E) j_0^{1-a\delta}.$$

We defer the proof of this lemma momentarily. The measure μ satisfies the assumptions of the previous lemma for $\delta = 1/(n-1)$ and thus

$$\mu(E) \leq C\mu(E)^{1-1/n}\left(j_0^{1-a/(n-1)}\right)^{1-1/n}\left(\int_\Omega |\nabla u|^n\right)^{1/n}.$$

Therefore $j_0 \sim \Phi^{-1}(\operatorname{diam} E)$ and the choice of the growth function for the Frostman measure μ gives us

$$\int_\Omega |\nabla u|^n \geq C\mu(E)\big(\Phi^{-1}(\operatorname{diam} E)\big)^{1-n+a} \geq C\big(\Phi^{-1}(\operatorname{diam} E)\big)^{1-n}.$$

$\square$

Proof of Lemma 4.1. From the growth condition on μ, Lemma 3.3, concavity of ψ_a (we can suppose that $C_1 \operatorname{dist}(x_0, \Omega) > 1$) and definition (1.7) of ψ_a we have

$$\begin{aligned}
&\sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j} \mu(S(Q)\cap E)\mu(S(Q)\cap E)^\delta \\
&\qquad\leq \sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j} \mu(S(Q)\cap E)\psi_a(\operatorname{diam} S(Q))^\delta \\
&\qquad\leq \sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j} \mu(S(Q)\cap E)\psi_a\big(C_1 \operatorname{dist}(x_0,\Omega)\Phi(Cj)\big)^\delta \\
&\qquad\leq C\sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j} \mu(S(Q)\cap E)\psi_a\big(\Phi(Cj)\big)^\delta \\
&\qquad\leq C\sum_{j=j_0}^{\infty} j^{-a\delta}\sum_{Q\in\mathcal{W}_j} \mu(S(Q)\cap E)
\end{aligned}$$

Set $a_j = \sum_{Q\in\mathcal{W}_j} \mu(S(Q)\cap E)$ and let $A_j = a_1 + \ldots + a_j$. Thanks to the elementary inequality $\left|j^{-a\delta} - (j+1)^{-a\delta}\right| \leq Cj^{-1-a\delta}$ we can apply summation by parts to obtain

$$\sum_{j=j_0}^{\infty} j^{-a\delta}a_j \leq C\sum_{j=j_0}^{\infty} j^{-1-a\delta}A_j.$$

By Lemma 3.2, $A_j \leq C\mu(E)j$ for each j so

$$\sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j} \mu(S(Q)\cap E)^{1+\delta} \leq C\mu(E)\sum_{j=j_0}^{\infty} j^{-a\delta} \leq C\mu(E)j_0^{1-a\delta}.$$

$\square$

Theorem 4.2. *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with diameter one which satisfies* (1.5) *and suppose* (1.6) *holds. Assume that a curve γ in Ω satisfies*

$$\#\{Q \in \mathcal{W} :\ Q\cap\gamma(x_1,x_2) \neq \emptyset\} \leq Ck_\Omega(x_1,x_2) + C$$

for every $x_1, x_2 \in \gamma$. Then

$$\operatorname{cap}(\gamma, Q_0; \Omega) \geq C \left(\Phi^{-1}\big(C \operatorname{diam} \gamma\big)\right)^{1-n}. \tag{4.6}$$

Proof. Let a function $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ be a test function for the n-capacity for the pair (Q_0, γ), i.e., $u : \Omega \to [0, 1]$ is a continuous function and $u(x) = 1$ for $x \in \gamma$ and $u(x) = 0$ for $x \in Q_0$.

Thanks to Lemma 3.5 we can find a sequence of cubes which joins Q_0 to γ, i.e., cubes $\{Q_i\}_{i=0}^{l}$ such that Q_0 is the central cube, $\overline{Q_i} \cap \overline{Q_{i+1}} \neq \emptyset$ for all $i \in \{0, \dots, l-1\}$, $Q_l \cap \gamma \neq \emptyset$ and

$$l \leq C\Phi^{-1}\left(C \frac{\operatorname{diam} \gamma}{\operatorname{dist}(x_0, \partial\Omega)}\right) \leq C\Phi^{-1}\big(C \operatorname{diam} \gamma\big). \tag{4.7}$$

Without loss of generality we may suppose that $u_{Q_l} \geq 1/2$ (see proof of Theorem 1.3) and clearly $u_{Q_0} = 0$. Analogously to (4.2) and (4.3) we obtain

$$1 \leq C\Big(\int_\Omega |\nabla u|^n\Big)^{\frac{1}{n}} l^{\frac{n-1}{n}}.$$

This inequality and (4.7) imply (4.6). □

5. Uniform continuity

Proof of Theorem 1.1. We follow the ideas from [13, Theorem 1.1] and again we give some details for the completeness of our arguments. Since Ω is bounded (Corollary 3.4) we may scale it to have diameter one. Fix a Whitney cube $F = Q_0$ in Ω with center x_0 and let $F' = f^{-1}(F)$ and $\tilde{F} = f^{-1}(\frac{3}{2}Q_0)$. By elementary properties of quasiconformal mappings, there exists $\delta = \delta(n, K) > 0$ so that the set of points $x \in \mathbf{R}^n$ with $\operatorname{dist}(x, F') \leq \delta \operatorname{diam} F'$ is contained in $\tilde{F}$.

Let $x', y' \in \Omega'$. Since f is automatically Hölder continuous on $\tilde{F}$ we may assume that either x' or y' (say x') is in $\Omega' \setminus \tilde{F}$. Recall that in view of (3.1) and (1.5) the inequality (1.8) cannot give us a better modulus of continuity than Hölder continuity.

Next, note that if $\delta_{\Omega'}(x', y') \geq \frac{1}{4}\delta \operatorname{diam} F'$, then

$$\frac{|f(x') - f(y')|}{\delta_{\Omega'}(x', y')} \leq C(n, K)\frac{\operatorname{diam} \Omega}{\operatorname{diam} F'} \leq C \tag{5.1}$$

which gives us again a modulus of continuity that is no worse than (1.8). Thus it suffices to verify (1.8) only in the case $4\delta_{\Omega'}(x', y') < \delta \operatorname{diam} F' \leq \operatorname{dist}(x', F')$. Choose a continuum $E' \subset \Omega'$ joining x' to y' with $\operatorname{diam} E' \leq 2\delta_{\Omega'}(x', y') < \frac{1}{2}\delta \operatorname{diam} F'$. Then $\operatorname{diam} E' < \frac{1}{2}\delta \operatorname{diam} F' \leq \operatorname{dist}(E', F')$. A fundamental property of the conformal capacity (see Fact 3.1(e) of [11]) states that in this case

$$\operatorname{cap}(E', F'; \Omega') \leq C(n) \left(\log \frac{\operatorname{dist}(E', F')}{\operatorname{diam} E'}\right)^{1-n}.$$

Set $E = f(E')$. By Theorem 1.3,

$$\operatorname{cap}(E, F; \Omega) \geq C\left(\Phi^{-1}\big(C \operatorname{diam} E\big)\right)^{1-n}. \tag{5.2}$$

Therefore

$$\begin{aligned}\left(\Phi^{-1}\big(C \operatorname{diam} E\big)\right)^{1-n} &\leq CK\left(\log \frac{\operatorname{dist}(E', F')}{\operatorname{diam} E'}\right)^{1-n}\\ &\leq CK\left(\log \frac{\frac{1}{2}\delta \operatorname{diam} F'}{\operatorname{diam} E'}\right)^{1-n},\end{aligned}$$

which implies

$$\operatorname{diam} E \leq C\Phi\Big(C \log \frac{C}{\operatorname{diam} E'}\Big).$$

Since $|f(x') - f(y')| \leq \operatorname{diam} E$ and $\operatorname{diam} E' \leq 2\delta_{\Omega'}(x', y')$, the proof of Theorem 1.1 is complete. $\square$

Proof of Theorem 1.2. This proof is analogous to the previous one and therefore we only point out the details which are different. Boundedness of Ω follows from Lemma 3.5. If $\operatorname{diam}[x', y']_{\Omega'} \geq \delta \operatorname{diam} F'/2$, then analogously to (5.1), we obtain (1.9). In this proof we denote $E' = [x', y']_{\Omega'}$ and $E = f(E')$. Thanks to Proposition 3.6 we can use Theorem 4.2 for $\gamma = f([x', y']_{\Omega'})$ to conclude the analog of (5.2). The other details and computations are similar. $\square$

Let us close the paper by commenting on the sharpness of the estimates in Theorem 1.1, Theorem 1.2, and Theorem 1.3.

Example 5.1. *Let $\varphi : [0, \infty) \to (0, \infty)$ be a non-increasing continuous function so that $\int_0^\infty \varphi = 1$ and*

$$c_1\varphi(t) \leq \varphi(t + 1) \text{ for every } t > 0. \tag{5.3}$$

Define $\Phi : [0, \infty) \to [0, 1]$ by formula (1.7) and set $f(x) = \varphi(\Phi^{-1}(x))$ for every $x \in (0, 1]$. Define

$$\Omega = \Big\{x \in \mathbf{R}^n : \ x_1 \in (0, 1], \sum_2^n x_i^2 < f^2(x_1)\Big\} \cup \Big(B(e_1, \varphi(0)) \cap \{x_1 > 1\}\Big),$$

where e_1 is the unit vector $(1, 0, \cdots, 0)$. Set $x_0 = e_1$ and $E_a = \{te_1 : t \in [a/2, a]\}$ for $a > 0$. Then

$$\operatorname{dist}(x, \partial\Omega) \leq C\varphi(Ck_\Omega(x_0, x)) \text{ for every } x \in \Omega \text{ and} \tag{5.4}$$

$$\operatorname{cap}\big(E_a, Q_0; \Omega\big) \leq C(\Phi^{-1}(Ca))^{1-n} \text{ for } a \text{ small enough.}$$

In Example 5.1, condition (1.5) holds, (for $\tilde{\varphi}(t) = C\varphi(Ct)$) and the bound (1.11) for capacity of a continuum is sharp. The growth assumption (5.3) is in practise harmless.

Proof. The function f is clearly increasing and continuous. For $k \in \mathbf{N}$ set $y_k = \Phi(k)$. From (5.3) we obtain

$$y_{k+1} - y_{k+2} = \int_{k+1}^{k+2} \varphi \geq c_1 \varphi(k+1) = c_1 f(y_{k+1}). \tag{5.5}$$

Given $x \in (0, 1/2]$, we can find $k \in \mathbf{N}$ such that $y_{k+1} \leq x \leq y_k$. Then $f(x) \leq f(y_k) = \varphi(k) \leq \varphi(k+1)/c_1 = f(y_{k+1})/c_1$ and therefore (5.5) and (5.3) give us

$$\begin{aligned} f\big(x - c_1^2 f(x)\big) &\geq f\big(y_{k+1} - c_1 f(y_{k+1})\big) \geq f(y_{k+2}) = \\ &= \varphi(k+2) \geq c_1^2 \varphi(k) = c_1^2 f(y_k) \geq c_1^2 f(x). \end{aligned}$$

Because (5.3) easily shows that $f(x) \leq Cx$, this inequality implies that $\operatorname{dist}(xe_1, \partial\Omega) \sim f(x)$ for every $x \in (0, 1/2]$.

It is not difficult to verify from $\operatorname{dist}(xe_1, \partial\Omega) \sim f(x)$, $f(y_k) = \varphi(k)$ and (5.3) that

$$k_\Omega(y_k e_1, y_{k+1} e_1) = \int_{y_{k+1}}^{y_k} \frac{dt}{\operatorname{dist}(te_1, \partial\Omega)} \sim \frac{(y_k - y_{k+1})}{\varphi(k)} \sim C.$$

Therefore $k_\Omega(x_0, y_k e_1) \sim k$. Recall that $x \in [y_{k+1}, y_k]$. Then $\Phi^{-1}(x) \sim k \sim k_\Omega(x_0, xe_1)$ and we have

$$\operatorname{dist}(xe_1, \partial\Omega) \leq f(x) = \varphi(\Phi^{-1}(x)) \leq \varphi(C k_\Omega(x_0, xe_1)).$$

Because f is increasing, we easily conclude that (5.4) holds for all $x \in \Omega$. Fix $k_0 \in \mathbf{N}$ such that $y_{k_0} e_1 \notin Q_0$. For $0 < a < 1/2$ we find $A \in \mathbf{N}$ such that $y_{A+1} \leq a \leq y_A$ and therefore, when a is sufficiently small, $A \geq \Phi^{-1}(Ca)$. If a is small enough we further have $A > 2k_0$. Define a function

$$u([x_1, \ldots, x_n]) = \begin{cases} 0 & \text{if } x_1 \leq y_A, \\ \frac{A-1-k}{A-k_0} + \frac{1}{A-k_0} \frac{x_1 - y_{k+1}}{y_k - y_{k+1}} & \text{if } y_{k+1} \leq x_1 \leq y_k \\ & \text{where } k \in \{k_0, \ldots, A-1\}, \\ 1 & \text{if } y_{k_0} \leq x_1. \end{cases}$$

Clearly $u \in W^{1,n}_{\mathrm{loc}}$, $u \equiv 1$ on Q_0 and $u \equiv 0$ on E_a. Since $f(y_{k+1}) \sim f(y_k) \sim \varphi(k)$ we obtain using (5.5) that

$$|\Omega \cap \{x \in \mathbf{R}^n : y_{k+1} < x_1 < y_k\}| \sim (y_k - y_{k+1}) \varphi(k)^{n-1} \sim (y_k - y_{k+1})^n.$$

Therefore

$$\begin{aligned} \int_\Omega |\nabla u|^n &= \sum_{k=k_0}^{A-1} \int_{\Omega \cap \{x : y_{k+1} < x_1 < y_k\}} \frac{1}{(A-k_0)^n (y_k - y_{k+1})^n} \leq \\ &\leq \frac{C}{(A-k_0)^{n-1}} \leq C A^{1-n} \leq C(\Phi^{-1}(Ca))^{1-n}. \end{aligned}$$

$\square$

Regarding Theorem 1.1 and Theorem 1.2, we let $n = 2$ and consider a conformal mapping of the disk onto Ω, where Ω is given in Example 5.1. Using the fact that the capacity is preserved under conformal mappings and the fact that the lower bound for the capacity is optimal, as discussed above, one easily checks that the given modulus of continuity is optimal. It also seems plausible that domains of this type are, in higher dimensions, quasiconformally equivalent to the unit ball, which would show that the dimension independent modulus of continuity given in Theorem 1.1 and Theorem 1.2 is sharp in all dimensions.

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