ON THE CONICAL DENSITY PROPERTIES OF BOREL MEASURES

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Abstract. We compare conical density properties and spherical density properties for general Borel measures on $\mathbb{R}^n$. As consequence, we obtain results for packing and Hausdorff measures $\mathcal{P}_h$ and $\mathcal{H}_h$ provided that the gauge function h satisfies certain conditions.

One consequence of our general results is the following: Let $m, n \in \mathbb{N}$, $0 < s < m \leq n$, $0 < \eta < 1$, and suppose that V is an m-dimensional linear subspace of $\mathbb{R}^n$. Let μ be either the s-dimensional Hausdorff measure or the s-dimensional packing measure restricted to a set A with $\mu(A) < \infty$. Then for μ-almost every $x \in \mathbb{R}^n$, there is $\theta \in V \cap S^{n-1}$ such that

$$\liminf_{r \downarrow 0} r^{-s} \mu (B(x,r) \cap H(x,\theta,\eta)) = 0,$$

where $H(x,\theta,\eta) = \{y \in \mathbb{R}^n : (y-x)\cdot\theta > \eta|y-x|\}$.

1. Introduction

In this note we study the following question: Suppose that μ is a measure on $\mathbb{R}^n$ and $h\colon\ ]0, r_0[\ \to\ ]0, \infty[$ is a function such that for a typical point x the measures $\mu(B(x, r_i))$ of some small balls $B(x, r_i)$ behave roughly like $h(r_i)$. What can be said about measures in cones?

Let us begin with some notation. Let μ be a measure on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$, and $h\colon\ ]0, r_0[\ \to\ ]0, \infty[$. The upper and lower μ-densities of the set A at x with respect to h are defined by

$$\underline{D}_h(\mu, A, x) = \liminf_{r \downarrow 0} \mu(B(x,r) \cap A)/h(r),$$

$$\overline{D}_h(\mu, A, x) = \limsup_{r \downarrow 0} \mu(B(x,r) \cap A)/h(r),$$

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where $B(x,r)$ is the closed ball $B(x,r)=\{y\in\mathbb{R}^n : |y-x|\leq r\}$. Open balls will be denoted by $U(x,r)$ and spheres by $S(x,r)=B(x,r)\setminus U(x,r)$. When $A=\mathbb{R}^n$, we abbreviate $\underline{D}_h(\mu,\mathbb{R}^n,x)=\underline{D}_h(\mu,x)$ and similarly with upper densities. Often $h(r)=r^s$ for some $0\leq s\leq n$. In this case we use the notation $\underline{D}_s(\mu,A,x)$ for $\underline{D}_h(\mu,A,x)$ and so on. For example

$$\underline{D}_s(\mu,x)=\liminf_{r\downarrow 0}\mu(B(x,r))/r^s.$$

Suppose that $m,n\in\mathbb{N}$ are such that $m\leq n$. The collection of m-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $G(n,m)$. A natural metric on $G(n,m)$ is given by

$$d(V,W)=||P_V-P_W|| \qquad V,W\in G(n,m),$$

where P_V is the orthogonal projection onto the subspace V and $||\cdot||$ is the usual operator norm for linear mappings. We also use the notation proj_i for the projections onto the coordinate axis. We will make use of the following equality, see [16, pp. 17-20]:

$$d(V,W)=\sup\{d(x,W): x\in V\cap S^{n-1}\}, \tag{1.1}$$

where $S^{n-1}=S(0,1)$ and on the right hand side $d(y,A)$ stands for the distance between the vector y and the set A. Let $\gamma_{n,m}$ be the unique Radon probability measure on $G(n,m)$ which is invariant with respect to the orthogonal group $O(n)$, see [14, Chapter 3]. Recall that Borel regular and locally finite measures are called Radon measures.

Let $x\in\mathbb{R}^n$, $V\in G(n,m)$, $\theta\in S^{n-1}$, $L_\theta=\{t\theta : t>0\}$, and $0\leq\eta\leq 1$. We define

$$\begin{aligned}
X(x,V,\eta)&=\{y\in\mathbb{R}^n : d(y-x,V)<\eta|y-x|\},\\
X^+(x,\theta,\eta)&=\{y\in\mathbb{R}^n : d(y-x,L_\theta)<\eta|y-x|\},\\
H(x,\theta,\eta)&=\{y\in\mathbb{R}^n : (y-x)\cdot\theta>\eta|y-x|\},\\
H(x,\theta)=H(x,\theta,0)&=\{y\in\mathbb{R}^n : (y-x)\cdot\theta>0\}.
\end{aligned}$$

One easily verifies the following equality

$$H(x,\theta,\eta)=X^+\left(x,\theta,\left(1-\eta^2\right)^{1/2}\right).$$

The H notation is often used for cones that are almost half-spaces, whereas X^+ stands usually for a very narrow cone.

Let $h\colon\ ]0,r_0[\ \to\ ]0,\infty[$ with $\lim_{r\downarrow} h(r) = 0$. Let $\mathcal{H}_h$ be the generalized Hausdorff measure which is constructed using the gauge function h, see [14, 4.9]. If $A \subset \mathbb{R}^n$ has finite $\mathcal{H}_h$ measure, then for $\mathcal{H}_h$-almost every $x \in A$

$$\overline{D}_h(\mathcal{H}_h, A, x) \leq \limsup_{r\downarrow 0} h(2r)/h(r). \tag{1.2}$$

The corresponding result for the spherical density of the generalized packing measure $\mathcal{P}_h$ is

$$\underline{D}_h(\mathcal{P}_h, A, x) \leq \limsup_{r\downarrow 0} h(2r)/h(r), \tag{1.3}$$

which is true for $\mathcal{P}_h$-almost every $x \in A$ provided that $\mathcal{P}_h(A) < \infty$. Inequalities (1.2) and (1.3) can be proved with a similar argument as in the familiar case $h(r) = r^s$, see [14, theorems 6.2, 6.10]. The definition of $\mathcal{P}_h$ can be found for example in [3, definition 2.2].

We will now explain some conical density results that are known for Hausdorff measures. We begin with the following theorem on lower densities. If μ is a measure, the notation $\mu \llcorner A$ stands for the restriction measure, that is $\mu \llcorner A(B) = \mu(A \cap B)$.

Theorem 1.1. *Assume that $0 < s < n$ and let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure. Let $A \subset \mathbb{R}^n$ and $\mathcal{H}^s(A) < \infty$.*

(1) If $\eta > 0$, then for $\mathcal{H}^s$-almost all $x \in A$, there exists $\theta = \theta(x) \in S^{n-1}$ such that $\underline{D}_s(\mathcal{H}^s \llcorner A, H(x,\theta,\eta), x) = 0$.

(2) If $0 < s < 1$ and $\theta \in S^{n-1}$, then $\underline{D}_s(\mathcal{H}^s \llcorner A, H(x,\theta), x) = 0$ for $\mathcal{H}^s$-almost all $x \in A$.

(3) If $n-1 < s < n$, then (1) holds also with the value $\eta = 0$.

Marstrand [9, pp. 293–297] proved Theorem 1.1 in $\mathbb{R}^2$ and his method can be generalized to higher dimensions, see also [4, pp. 56–61]. Besicovitch [1, Theorem 2] had earlier proved (2) in $\mathbb{R}$. He has also shown [2, Theorem 13] that if $A \subset \mathbb{R}^2$ is purely 1-unrectifiable and $\theta \in S^1$, then $\underline{D}_1(\mathcal{H}^1 \llcorner A, H(x,\theta), x) = 0$ for $\mathcal{H}^1$-almost all $x \in A$. Gillis [6] had earlier proved this for cones $H(x,\theta,\eta)$, when $\eta > 0$. Mattila [11] has studied h-densities of singular measures in $\mathbb{R}$ and obtained as a corollary a result somewhat similar to (2) [11, Corollary 9].

Theorem 1.1 (1) does not hold for all $\theta \in S^{n-1}$, see for example 2.5. Recently Lorent [8] showed that one can choose directions θ in (1) to lie on a fixed $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ provided that $0 < s < n-1$ or that A is purely $(n-1)$-unrectifiable. In Section 2 we shall proceed in this direction by showing that for all $m \in \mathbb{N}$, $m \leq n$, the following is true: If A is $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A) < \infty$ and either A is purely m-unrectifiable or $0 < s < m$, then the directions θ in (1) can be chosen to lie on a fixed m-dimensional linear subspace. We will also prove the corresponding result for the s-dimensional packing measure $\mathcal{P}^s$ when $0 < s < m$. Moreover, our results can be applied to many other measures with the property $\underline{D}_h(\mu, x) < \infty$ for μ-almost all $x \in \mathbb{R}^n$, see Theorems 2.1 and 2.2. One could also modify Marstrands methods from [9] to prove claims (1) and (2) of Theorem 1.1 for packing measures $\mathcal{P}^s$.

Concerning upper conical densities for Hausdorff measures the fundamental theorem is as follows:

Theorem 1.2. *Suppose that $m \in \mathbb{N}$, $m < s < n$, $0 < \eta \leq 1$, and $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$.*

(1) If $V \in G(n, n-m)$, then

$$\overline{D}_s(\mathcal{H}^s \llcorner A, X(x, V, \eta), x) > c = c(n, m, \eta) > 0$$

for $\mathcal{H}^s$-almost all $x \in A$.

(2) If above $m = n-1$ and $\theta \in S^{n-1}$, then

$$\overline{D}_s\left(\mathcal{H}^s \llcorner A, X^+(x, \theta, \eta), x\right) > c = c(n, s, \eta) > 0$$

for $\mathcal{H}^s$-almost all $x \in A$.

The preceding theorem is also due to Marstrand [9, Theorem IX] in $\mathbb{R}^2$. Salli [16] generalized Marstrands result to $\mathbb{R}^n$ and proved also a corresponding result for cones generated by open sets $G \subset G(n, m)$ or $S \subset S^{n-1}$. Mattila [12, Theorem 3.3] went even further by proving a very general upper density theorem for Hausdorff measures. Besicovitch [1, Theorem 2] has shown that if $0 < s < 1$ and $A \subset \mathbb{R}$ is $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A) < \infty$, then $\overline{D}_s(\mathcal{H}^s \llcorner A,]x, \infty[, x) = \overline{D}_s(\mathcal{H}^s \llcorner A,]-\infty, x[, x) = 1$ for $\mathcal{H}^s$-almost all $x \in A$.

There are also upper conical density results for purely m-unrectifiable sets. Besicovitch showed in [2] that if $A \subset \mathbb{R}^2$ is purely 1-unrectifiable, then the upper 1-density of the set in opposite angles is strictly positive $\mathcal{H}^1$-almost everywhere in the set A. Federer [5, Theorem 3.3.17] gave a generalization of this result. See also [14, Corollary 15.16].

In Section 3 we shall discuss how Theorem 1.2 can be generalized for packing measures, or more generally, for measures μ such that $\underline{D}_h(\mu, x)$ is finite μ-almost everywhere, with a suitable function h.

The above mentioned conical density theorems have been used in various different ways. Let us briefly explain some of them. Marstrand ([9] and [10]) used Theorem 1.1 to prove the fairly deep fact that for non-integral values of s there are no Radon measures μ so that the positive and finite density, $\lim_{r\downarrow 0} \mu\left(B\left(x,r\right)\right)/r^s$, would exist in a set of positive μ measure. Mattila [12] used his upper density result to find an upper bound for the dimension of strongly porous sets. Theorem 1.1 (1) and Lorent's [8] generalization of it can also be used to give some light on the problem of characterizing removable sets for harmonic functions, see [15] and [8]. Note also that Theorem 1.1 (1) is equivalent to the following statement: Denote $\mu = \mathcal{H}^s \llcorner A$, where $\mathcal{H}^s\left(A\right) < \infty$. Then for μ-almost every x, there is a tangent measure of μ at x which is supported on one side of some $(n-1)$-plane $V \in G\left(n, n-1\right)$. This property is sometimes very useful, see for example [14].

2. Lower densities

The main results of this section are stated in the following two theorems.

Theorem 2.1. *Let $m, n \in \mathbb{N}$ with $m \le n$. Assume that $h\colon \left]0, r_0\right[\to \left]0, \infty\right[$ fulfills the following three conditions:*

(h1) $$\lim_{r\downarrow 0} h\left(r\right) = 0,$$

(h2) $$\lim_{r\downarrow 0} h\left(r\right)/r^m = \infty,$$

(h3) $$h\left(r_1\right) + h\left(r_2\right) \ge h\left(\left(r_1^m + r_2^m\right)^{1/m}\right), \text{ when } r_1^m + r_2^m \le r_0^m.$$

Suppose that $V \in G\left(n, m\right)$, $\eta > 0$ and μ is a Borel measure on $\mathbb{R}^n$ with $\underline{D}_h\left(\mu, x\right) < \infty$ for μ-almost all $x \in \mathbb{R}^n$. Then for μ-almost all $x \in \mathbb{R}^n$, there is $\theta = \theta\left(x\right) \in V \cap S^{n-1}$ such that $\underline{D}_h\left(\mu, H\left(x, \theta, \eta\right), x\right) = 0$.

Theorem 2.2. *Assume that $h\colon \left]0, r_0\right[\to \left]0, \infty\right[$ fulfills the assumptions* (h1)–(h3) *of Theorem 2.1 with $m = 1$. If $\theta \in S^{n-1}$ and μ is a Borel measure on $\mathbb{R}^n$ with $\underline{D}_h\left(\mu, x\right) < \infty$ for μ-almost all $x \in \mathbb{R}^n$, then $\underline{D}_h\left(\mu, H\left(x, \theta\right), x\right) = 0$ for μ-almost all $x \in \mathbb{R}^n$.*

Before the proofs, let us make some remarks concerning Theorems 2.1 and 2.2

If $0 < s < m$, $h\left(r\right) = r^s$, and $\mu = \mathcal{H}^s \llcorner A$, where A is a subset of $\mathbb{R}^n$ with $\mathcal{H}^s\left(A\right) < \infty$, then the assumptions of Theorem 2.1 are clearly satisfied, recall (1.2). If in addition $s < 1$, then the assumptions of Theorem 2.2 are valid. Above one can also replace Hausdorff measures by packing measures. So we have:

Corollary 2.3. *Let μ be either the s-dimensional Hausdorff measure or the s-dimensional packing measure and let $A \subset \mathbb{R}^n$ with $\mu(A) < \infty$.*

(1) If $\eta > 0$, $0 < s < m$, and $V \in G(n,m)$, then for μ-almost all $x \in A$, there is $\theta = \theta(x) \in V \cap S^{n-1}$ such that $\underline{D}_s(\mu \llcorner A, H(x,\theta,\eta), x) = 0$.

(2) If $0 < s < 1$ and $\theta \in S^{n-1}$, then $\underline{D}_s(\mu \llcorner A, H(x,\theta), x) = 0$ for μ-almost all $x \in A$.

There are also many other functions than r^s to which Theorems 2.1 and 2.2 can be applied. For example $h(r) = r^s \log(r^{-1})$ fulfills (h1)–(h3) when $0 < s \le m$. Using density bounds (1.2) and (1.3), we see that the statements of Corollary 2.3 remain true if we let μ to be $\mathcal{H}_h$ or $\mathcal{P}_h$ provided that h fulfills (h1)–(h3) (with $m = 1$ in (2)).

If in Theorem 2.1 or 2.2 also $\underline{D}_h(\mu, x) > 0$ for μ-almost every x, then the densities $\underline{D}_h(\mu, H(x,\theta,\eta), x)$ can be replaced by $\underline{D}_\mu(H(x,\theta,\eta), x)$, where

$$\underline{D}_\mu(H(x,\theta,\eta), x) = \liminf_{r\downarrow 0} \mu(H(x,\theta,\eta) \cap B(x,r)) / \mu(B(x,r))$$

is the conical density with respect to μ.

It remains open, if assertion (3) of Theorem 1.1 is valid for packing measures. However, if we know that the assumptions of Theorem 2.1 hold, $\overline{D}_h(\mu, x) < \infty$ for μ-almost every $x \in \mathbb{R}^n$, and

$$\lim_{\alpha\downarrow 0} \limsup_{r\downarrow 0} \alpha^{1-n} h(\alpha r)/h(r) = 0, \tag{2.1}$$

then one can prove, with the help of Theorem 2.1 and a generalization of [9, Lemma 29], that for μ-almost every $x \in \mathbb{R}^n$, there is $\theta \in S^{n-1}$ so that $\underline{D}_h(\mu, H(x,\theta), x) = 0$. The condition (2.1) is valid for example if $n - 1 < s < n$ and $h(r) = r^s \log(r^{-1})$.

Suppose that h is differentiable and satisfies (h1). If the derivative h' is non-increasing and $\lim_{r\downarrow 0} h'(r) = \infty$, then (h2) and (h3) are valid with $m = 1$.

Condition (h3) may be weakened a little bit. It suffices to assume that there is a constant $c > 0$ such that

$$\sum_i h(r_i) \ge c\, h\left(\left(\sum_i r_i^m\right)^{1/m}\right), \text{ when } \sum_i r_i^m \le r_0^m. \tag{2.2}$$

This is immediate due to the following lemma.

Lemma 2.4. *If $h\colon\,]0,r_0[\, \to\,]0,\infty[$ fulfills inequality (2.2), then there is a function $\tilde{h}$ that satisfies (h3) and inequalities $c\,h \le \tilde{h} \le h$.*

Proof. Define $\tilde{h}\,(r) = \inf\{\sum_i h\,(r_i) : \sum_i r_i^m = r^m\}$. Now the desired properties clearly hold for $\tilde{h}$. □

Assumptions (h1) and (h2) for the function h in Theorems 2.1 and 2.2 are well justified but it is natural to ask whether assumption (h3), or (2.2), is really needed. In example 2.13 we shall construct a Radon measure μ and a function h which show that assumptions (h1) and (h2) alone are not enough to guarantee the assertions of Theorems 2.1 and 2.2. Of course, this does not exclude the possibility that assumption (h3) could be weakened and as P. Mattila pointed out to me, it is an interesting question whether it can be replaced by a doubling condition on h, see Theorem 3.1.

Theorem 2.1 is in a sense sharp for Hausdorff measures. This is shown by the following simple example.

Example 2.5. The assertion of Theorem 2.1 is not valid for measures $\mathcal{H}^s\llcorner A$, $\mathcal{H}^s\,(A) < \infty$, when $s \ge m$: Consider a standard Cantor set $C \subset \mathbb{R}^{n-m}$ such that $0 < \mathcal{H}^{s-m}\,(C) < \infty$. Let $A = U\,(0,1) \cap \mathbb{R}^m \times C \subset \mathbb{R}^n$, $\mu = \mathcal{H}^s \llcorner A$, and $0 \le \eta < 1$. Then $\underline{D}_s\,(\mu, H\,(x,\theta,\eta)\,, x) > 0$ for all $x \in A$, $\theta \in V \cap S^{n-1}$, where $V = \{x \in \mathbb{R}^n : \mathrm{proj}_{m+1}\, x = \ldots = \mathrm{proj}_n\, x = 0\} \in G\,(n,m)$.

We shall next present some basic lemmas. The following Vitali type covering theorem will be used a few times. It follows from [5, Corollary 2.8.15].

Lemma 2.6. *Suppose that μ is a locally finite Borel measure on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$, and $\mathcal{B}$ is a collection of closed balls such that for all $x \in A$ we have*

$$\inf\{r : B\,(x,r) \in \mathcal{B}\} = 0.$$

Then there are disjoint balls $B_i \in \mathcal{B}$ so that $\mu\,(A \setminus \bigcup_i B_i) = 0$.

Lemma 2.7. *Assume that μ is a Borel measure on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ is a Borel set, and $h\colon\,]0,r_0[\, \to\,]0,\infty[$. If for μ-almost every $x \in A$, there is $r > 0$ such that $\mu\,(B\,(x,r)) < \infty$, then for μ-almost all $x \in A$*

(1) $\underline{D}_h\,(\mu, A, x) = \underline{D}_h\,(\mu, x)$,
(2) $\overline{D}_h\,(\mu, A, x) = \overline{D}_h\,(\mu, x)$.

Proof. We prove (1). Claim (2) can be established similarly. Because μ-almost all of A is contained in a countable union of open balls, each of finite μ measure, we can assume that μ is finite.

Clearly $\underline{D}_h(\mu, A, x) \leq \underline{D}_h(\mu, x)$ for all $x \in A$. Lemma 2.6 yields that

$$\lim_{r \downarrow 0} \mu(B(x,r) \cap A) / \mu(B(x,r)) = 1$$

for μ-almost every $x \in A$. Take such a point x and fix radii $r_i \downarrow 0$ such that $\mu(B(x,r_i) \cap A)/h(r_i) \longrightarrow \underline{D}_h(\mu, A, x)$ as $i \to \infty$. Now

$$\frac{\mu(B(x,r_i))}{h(r_i)} = \frac{\mu(B(x,r_i) \cap A)}{h(r_i)} \frac{\mu(B(x,r_i))}{\mu(B(x,r_i) \cap A)} \longrightarrow \underline{D}_h(\mu, A, x)$$

as $i \to \infty$ and thus $\underline{D}_h(\mu, A, x) \geq \underline{D}_h(\mu, x)$. □

Proof of Theorem 2.2. We may assume that $\theta = e_1$. Let $\alpha, \beta \in]0, \infty[$ and define

$$A = \{x \in \mathbb{R}^n : \mu(B(x,r) \cap H(x,\theta)) \geq \alpha\, h(r) \ \text{for } 0 < r < \beta\}. \tag{2.3}$$

We will begin by showing that A is a Borel set. Let us state this as a lemma for later use.

Lemma 2.8. *A is a Borel set.*

Proof of Lemma 2.8. Using assumptions (h1) and (h3), it is easily seen that if $0 < r < r_0$ and $q_i \uparrow r$ as $i \to \infty$, then

$$\liminf_{i \to \infty} h(q_i) \geq h(r).$$

Using this one obtains

$$A = \bigcap_{\substack{0<q<\beta \\ q \in \mathbb{Q}}} \{x \in \mathbb{R}^n : \mu(U(x,q) \cap H(x,\theta)) \geq \alpha\, h(q)\}.$$

Fix $q > 0$. It remains to show that the mapping

$$f : x \mapsto \mu(U(x,q) \cap H(x,\theta))$$

is lower semicontinuous. Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$\mu(\{y \in U(x,q) \cap H(x,\theta) : d(y, \partial(U(x,q) \cap H(x,\theta))) > \delta\}) > f(x) - \varepsilon.$$

If $|y - x| < \delta$, then clearly $f(y) > f(x) - \varepsilon$ and the lemma follows. □

We continue to prove Theorem 2.2. Suppose that $F \subset A$ is closed. It suffices to show that $\mu(F) = 0$. Assume on the contrary that $\mu(F) > 0$. By

Lemma 2.7 (1), we find $x \in F$ such that $\underline{D}_h(\mu, F, x) = \underline{D}_h(\mu, x) = c < \infty$. Let $\varepsilon < \alpha\, 4^{-1}(2c+1)^{-1}$. Fix $0 < r < r_0$ such that $\mu(B(x, 2r)) < \infty$ and

$$(1-\varepsilon)\, c\, h(r) < \mu(B(x,r) \cap F) \leq \mu(B(x,r)) < (1+\varepsilon)\, c\, h(r),$$

whence

$$\mu(B(x,r) \setminus F) < 2\, c\, \varepsilon\, h(r). \tag{2.4}$$

Let $\gamma > 0$. According to assumption (h2), there is $\eta > 0$ such that $r < \gamma\, h(r)$, whenever $0 < r < \eta$. We can now apply Lemma 2.6 to the collection

$$\{B(y,\rho) : y \in F \cap B(x,r),\ \rho < \min\{r, \beta, \eta\}\}$$

to find disjoint balls $B_1, \ldots, B_N$ from this collection such that

$$\mu\left(B(x,r) \cap F \setminus \bigcup_{i=1}^{N} B_i\right) < \varepsilon\, h(r). \tag{2.5}$$

We may assume that $x \in \bigcup_i B_i$. Reducing γ enough, we are led to the estimate

$$\begin{aligned}\sum_{i=1}^{N} r_i < \gamma \sum_{i=1}^{N} h(r_i) \leq \gamma\, \alpha^{-1} \sum_{i=1}^{N} \mu(B_i \cap H(x_i, \theta)) \leq \gamma\, \alpha^{-1} \mu(B(x, 2r))\\ < r/8.\end{aligned} \tag{2.6}$$

We now choose points $x_i \in B_i \cap F$ for $i = 1, \ldots, N$ such that

$$\mathrm{proj}_1(x_i) = \sup\{\mathrm{proj}_1(y) : y \in B_i \cap F\}.$$

Next we select recursively points $y_1, y_2, \ldots, y_l \in \{x_i\}$ and corresponding radii $\delta_1, \delta_2, \ldots, \delta_l$. Let U_i denote the open ball that has double radius but same centre as B_i. We take $y_1 \in \{x_i\}_{i=1}^N$ which maximizes $\mathrm{proj}_1(x_i)$, and define $\delta_1 = \max\{0, r - |x - y_1|\}$. If points $y_1, \ldots, y_k$ and radii $\delta_1, \ldots, \delta_k$ have been selected, then we let $0 < a_k \leq |x - y_k| - \delta_k$ to be the greatest radius such that $S(x, a_k) \cap \bigcup_{i=1}^N U_i = \emptyset$ (If such an a_k does not exist, then we finish our selection). We now choose $y_{k+1} \in \{x_i\} \cap B(x, a_k)$ which maximizes $\mathrm{proj}_1(x_i)$ and define $\delta_{k+1} = a_k - |x - y_{k+1}|$. When this selection terminates, we have defined points $y_1, \ldots, y_l$ and radii $\delta_1, \ldots, \delta_l$ for some $l \leq N$. As a result of the above construction, and with the help of (2.6), we get

$$r \leq \sum_{i=1}^{N} 4r_i + \sum_{i=1}^{l} 2\delta_i \leq r/2 + 2\sum_{i=1}^{l} \delta_i.$$

For technical reasons, we make numbers δ_i small enough to satisfy

$$\sum_{i=1}^{l} \delta_i = r/4. \tag{2.7}$$

It also follows from the construction that for all $i = 1, \dots, l$

$$B(y_i, \delta_i) \cap H(y_i, \theta) \cap F \cap \bigcup_{j=i}^{N} B_j = \emptyset. \tag{2.8}$$

We are now ready to estimate the measure of $B(x,r) \setminus F$. Using the fact that the half balls $B(y_i, \delta_i) \cap H(y_i, \theta) \subset B(x,r)$ are disjoint, (2.8), (2.3), (2.5), (2.7), (h3), and the definition of ε, we deduce

$$\begin{aligned}
&\mu(B(x,r) \setminus F) \\
&\geq \sum_{i=1}^{l} \mu(B(y_i,\delta_i) \cap H(y_i,\theta)) - \mu\left(B(x,r) \cap F \setminus \bigcup_{i=1}^{N} B_i\right) \\
&> \alpha \sum_{i=1}^{l} h(\delta_i) - \varepsilon\, h(r) \geq \alpha\, h(r/4) - \varepsilon\, h(r) \\
&\geq (\alpha/4 - \varepsilon)\; h(r) > 2\, c\, \varepsilon\, h(r).
\end{aligned}$$

This contradicts with (2.4). $\square$

Several lemmas are needed for the proof of Theorem 2.1. Before them, let us define one more conical object. If $V \in G(n,m)$, $x \in \mathbb{R}^n$, $\alpha > 0$, and $r > 0$, then

$$Y_V(x,\alpha,r) = \{y \in \mathbb{R}^n : |P_{V^\perp}(y-x)| \leq \alpha(r - |P_V(y-x)|)\}$$

It follows readily from the above definition that if $y \in Y_V(x,\alpha,r)$, then

$$Y_V(x + P_{V^\perp}(y-x), \alpha, |P_V(y-x)|) \subset Y_V(x,\alpha,r), \tag{2.9}$$

and if $y \in \{x\} + V$ with $|y-x| \leq r$, then

$$Y_V(y, \alpha, r - |y-x|) \subset Y_V(x,\alpha,r). \tag{2.10}$$

Lemma 2.9. *Let $\eta \in]0,1[$, $V \in G(n,m)$, $x \in \mathbb{R}^n$, $y \in \{x\} + S(0,r) \cap V$, and $\theta = (x-y)/|x-y|$, then*

$$B\left(y, \eta^2 r\right) \cap H(y,\theta,\eta) \subset Y_V\left(x, \eta^{-1}\left(1-\eta^2\right)^{1/2}, r\right).$$

Proof. In this proof we denote $\mathrm{proj}_i(x)$ by x_i. We may assume without loss of generality that $r = 1$, $y = 0$, $x = e_1 = (1,0,\dots,0)$, and also that

$V = \{x : x_{m+1} = \ldots = x_n = 0\}$. Let $z \in H(y, \theta, \eta)$, then $z \cdot \theta = z_1 > \eta |z|$, and it follows that $z_1^2 \eta^{-2} (1 - \eta^2) > \sum_{i=2}^{n} z_i^2$. Using this, we deduce

$$|P_{V^\perp}(z - x)| = \left(\sum_{i=m+1}^{n} z_i^2 \right)^{1/2} < \left(z_i^2 \eta^{-2} \left(1 - \eta^2\right) - \sum_{i=2}^{m} z_i^2 \right)^{1/2}. \tag{2.11}$$

We also have

$$r - |P_V(z - x)| = 1 - \left((1 - z_1)^2 + \sum_{i=2}^{m} z_i^2 \right)^{1/2}. \tag{2.12}$$

If $|z| \leq \eta^2$, then a straightforward calculation shows that

$$\left(z_i^2 \frac{1 - \eta^2}{\eta^2} - \sum_{i=2}^{m} z_i^2 \right)^{1/2} \leq \frac{(1 - \eta^2)^{1/2}}{\eta} \left(1 - \left((1 - z_1)^2 + \sum_{i=2}^{m} z_i^2 \right)^{1/2} \right). \tag{2.13}$$

The lemma follows from (2.11)–(2.13). □

Lemma 2.10. *Suppose that $V \in G(n, m)$, $F \subset \mathbb{R}^n$ is a closed set with $x \in F$, and $\mathcal{H}^m(P_V(F \cap Y_V(x, \alpha, r))) = 0$. Then there is a disjoint collection $\{Y_i = Y_V(x_i, \alpha, r_i)\}_i$ such that $\sum_i r_i^m \geq 10^{-m} r^m$, and for all i we have $Y_i \subset Y_V(x, \alpha, r)$, and further*

$$F \cap \{z \in Y_i : |z - x_i| < r_i\} = \emptyset, \tag{2.14}$$

$$F \cap \{z \in Y_i : |z - x_i| = r_i\} \neq \emptyset. \tag{2.15}$$

Proof. We assume that $x \in V$. We will prove that if $y \in V \cap B(x, r/2) \setminus P_V(F \cap Y_V(x, \alpha, r))$, then there is a cone $Y_V(z, \alpha, \delta) \subset Y_V(x, \alpha, r)$ which satisfies conditions (2.14) and (2.15) such that $y \in P_V(Y_V(z, \alpha, \delta))$. The assertion follows then from this by applying the $5r$-covering theorem [14, Theorem 2.1] to the projections $P(Y_V(z, \alpha, \delta))$.

Fix $y \in V \cap B(x, r/2) \setminus P_V(F \cap Y_V(x, \alpha, r))$. Then there exists a radius $0 < \delta < |y - x|$ such that $F \cap \operatorname{int} Y(y, \alpha, \delta) = \emptyset$ but $F \cap \partial Y_V(y, \alpha, \delta) \neq \emptyset$. Take $z \in F \cap \partial Y_V(y, \alpha, \delta)$ such that

$$|P_V(z - y)| = \min\{|P_V(z - y)| : z \in F \cap \partial Y_V(y, \alpha, \delta)\}.$$

Let

$$Y = Y_V(y + P_{V^\perp}(z - y), \alpha, |P_V(z - y)|).$$

Now $y \in P_V(Y)$ and by (2.9), (2.10), we get

$$Y \subset Y_V(y, \alpha, \delta) \subset Y_V(x, \alpha, r).$$

Because $z \in F \cap \{w \in Y : |w - (y + P_{V^\perp}(z-y))| = r_i\}$, and $\operatorname{int} Y \subset \operatorname{int} Y_V(y, \alpha, \delta)$, we conclude that (2.14) and (2.15) are valid for Y. This completes the proof. $\square$

The next lemma follows directly from Lemmas 2.9 and 2.10.

Lemma 2.11. *Let $V \in G(n,m)$, $\eta \in]0,1[$, and suppose that $F \subset \mathbb{R}^n$ is a closed set such that $x \in F$ and*

$$\mathcal{H}^m\left(P_V\left(F \cap Y_V\left(x, \eta^{-1}\left(1-\eta^2\right)^{1/2}, r\right)\right)\right) = 0.$$

Then there are disjoint cones

$$B(x_i, r_i) \cap H(x_i, \theta_i, \eta) \subset Y_V\left(x, \eta^{-1}\left(1-\eta^2\right)^{1/2}, r\right) \setminus F$$

such that $x_i \in F$, $\theta_i \in V \cap S^{n-1}$ for all i and $\sum_i r_i^m \geq 10^{-m}\eta^{2m}r^m$.

Proof of Theorem 2.1. We can assume, as in Lemma 2.7, that μ is finite. We shall first prove that for all numbers $\alpha, \eta \in]0,1[$, and $0 < \beta < r_0$, the set

$$A = \{x : \mu(B(x,r) \cap H(x,\theta,\eta)) \geq \alpha\, h(r) \text{ when } 0 < r < \beta \text{ and } \theta \in V \cap S^{n-1}\}$$

is of μ measure zero and then show how this implies our theorem.

By modifying the proof of Lemma 2.8, it is rather easy to see that A is a Borel set. Hence it is sufficient to show that if $F \subset A$ is closed, then $\mu(F) = 0$. Assume on the contrary that $\mu(F) > 0$. Using Lemma 2.7 (1), we find $x \in F$ such that $\underline{D}_h(\mu, F, x) = \underline{D}_h(\mu, x) = c < \infty$. For technical reasons, we assume that $10^{-m}\eta^{3m} = 2^{-k/m}$ for some $k \in \mathbb{N}$. By iteration of assumption (h3), we obtain a constant $c' > 0$ that depends only on η and m (one can take $c' = 2^{-k}$) such that

$$h\left(10^{-m}\eta^{3m}r\right) \geq c'\, h(r) \tag{2.16}$$

whenever $r < r_0$. Let $0 < \varepsilon < \alpha\, c'/(2c)$. Fix $0 < r < \beta$ such that

$$c(1-\varepsilon)\, h(r) < \mu(B(x,r) \cap F) \leq \mu(B(x,r)) < c(1+\varepsilon)\, h(r)$$

then also

$$\mu(B(x,r) \setminus F) < 2\varepsilon\, c\, h(r). \tag{2.17}$$

Our next step is to prove that

$$\mathcal{H}^m(B(x,r) \cap F) = 0. \tag{2.18}$$

Let $\gamma > 0$. Using (h2), and the $5r$-covering theorem [14, Theorem 2.1], it is possible to select a collection of disjoint balls,

$$\{B_i = B(x_i, r_i)\}_i, \quad x_i \in B(x,r) \cap F, \ r_i^m < 5^{-m} \min\{1, \gamma\, h(r_i)\},$$

such that $F \subset \bigcup_i B(x_i, 5r_i)$. We obtain

$$\sum_i (5r_i)^m < \gamma \sum_i h(r_i) < \alpha^{-1}\gamma \sum_i \mu(B_i) < \alpha^{-1}\gamma\, \mu(B(x, r+1)).$$

Since the right hand side of the above inequality tends to zero as $\gamma \downarrow 0$, this yields (2.18).

Since a projection can not increase Hausdorff measure, we observe that

$$\mathcal{H}^m(P_V(B(x,r) \cap F)) = 0. \tag{2.19}$$

Using the inclusion

$$Y_V\left(x, \eta^{-1}\left(1-\eta^2\right)^{1/2}, \eta r\right) \subset B(x,r)$$

combined with Lemma 2.11, we find disjoint cones $B(x_i, r_i) \cap H(x_i, \theta_i, \eta) \subset B(x,r) \setminus F$ such that $x_i \in F$, $\theta_i \in V$ for all i, and

$$\sum_i r_i^m = 10^{-m} \eta^{3m} r^m.$$

Using the above fact, the definition of A, (h3), (2.16), and our choice of ε, we get

$$\begin{aligned}
\mu(B(x,r) \setminus F) &\geq \sum_{i=1}^{\infty} \mu(B(x_i, r_i) \cap H(x_i, \theta_i, \eta)) \\
&\geq \alpha \sum_{i=1}^{\infty} h(r_i) \geq \alpha\, h\left(10^{-m}\eta^{3m} r\right) \\
&\geq \alpha\, c'\, h(r) > 2\varepsilon\, c\, h(r).
\end{aligned}$$

This contradicts with (2.17) and thus $\mu(A) = 0$.

Fix $\eta > 0$ and $\alpha_i > 0$ such that $\alpha_i \downarrow 0$ as $i \to \infty$. We can now find, for μ-almost every $x \in \mathbb{R}^n$, directions $\theta_i \in V \cap S^{n-1}$, and numbers $r_i > 0$ such that $r_i \downarrow 0$ as $i \to \infty$, and

$$\mu(B(x, r_i) \cap H(x, \theta_i, \eta/2)) < \alpha_i\, h(r_i) \tag{2.20}$$

for all i.

Suppose that (2.20) holds for x. By going to a subsequence, we may assume that $\theta_i \to \theta \in V \cap S^{n-1}$ as $i \to \infty$ and that $H(x,\theta,\eta) \subset H(x,\theta_i,\eta/2)$ for all i. Inequality (2.20) yields

$$\mu\left(B(x,r_i) \cap H(x,\theta,\eta)\right)/h(r_i) < \alpha_i \longrightarrow 0\,,$$

as $i \to \infty$. This completes the proof. □

The last theorem of this section deals with purely m-unrectifiable sets. As mentioned before, it has been proved by Gillis [6] when $n = 2$ and by Lorent [8] when $m = n-1$. We recall that a set $A \subset \mathbb{R}^n$ is purely m-unrectifiable if $\mathcal{H}^m(A \cap E) = 0$ whenever E is a Lipschitz image of $\mathbb{R}^m$.

Theorem 2.12. *Let $m, n \in \mathbb{N}$ such that $m < n$ and suppose that $A \subset \mathbb{R}^n$ is purely m-unrectifiable with $\mathcal{H}^m(A) < \infty$. If $\eta > 0$ and $V \in G(n,m)$, then for $\mathcal{H}^m$-almost all $x \in F$, there is $\theta \in V \cap S^{n-1}$ such that $\underline{D}_m(\mathcal{H}^m \llcorner A, H(x,\theta,\eta), x) = 0$.*

If $m = 1$ and $\theta \in S^{n-1}$, then $\underline{D}_1(\mathcal{H}^1 \llcorner A, H(x,\theta,\eta), x) = 0$ for μ-almost all $x \in \mathbb{R}^n$.

Proof. The proof is based on our proof of Theorem 2.1 together with the Besicovitch-Federer projection theorem, see for example [14, Theorem 18.1], which says that

$$\mathcal{H}^m(P_V(A)) = 0 \qquad (2.21)$$

for $\gamma_{n,m}$-almost all $V \in G(n,m)$.

By the Borel regularity of $\mathcal{H}^m$, we can assume that A is a Borel set. Then the assumptions of Theorem 2.1 are valid for $\mathcal{H}^m \llcorner A$ and $h(r) = r^m$, except for (h2). But assumption (h2) in Theorem 2.1 was only used to obtain (2.19) and if $V \in G(n,m)$ is such that (2.21) holds, then also (2.19) holds and the proof of Theorem 2.1 gives the assertion for V.

Now fix arbitrary $V \in G(n,m)$. According to the above facts, we can find, for μ-almost every $x \in \mathbb{R}^n$, m-planes $V_i \in G(n,m)$, directions $\theta_i \in V_i \cap S^{n-1}$, and radii $r_i > 0$ such that

$$\mathcal{H}^m\left(B(x,r_i) \cap H(x,\theta_i,\eta/2) \cap A\right)/r_i^s \longrightarrow 0,$$

$r_i \downarrow 0$, and $V_i \to V$ in $G(n,m)$ as $i \to \infty$. Using (1.1), we can find a subsequence of $(\theta_i)_i$, which we denote by the same symbols, such that $\theta_i \to \theta \in V \cap S^{n-1}$ and $H(x,\theta,\eta) \subset H(x,\theta_i,\eta/2)$ for all i. The first assertion of the theorem follows.

The second statement can be verified by slightly modifying the argument. We omit the details, see also the discussion below. □

As noted in the introduction, Besicovitch [2, Theorem 13] proved that when $m = 1$ and $n = 2$, then one can take $\eta = 0$ in Theorem 2.12. His method can be modified to prove that this is true for all $n \in \mathbb{N}$. It remains unsolved, if this is true for $m > 1$.

We shall finish this section by the example that was mentioned earlier in this section. We perform a modification of a construction that has turned out to be useful in many connections related to fractal sets and measures, see for example [13, Example 4.4] or [7, Example 3.1].

Example 2.13. There exists a nondecreasing function $h\colon]0,1[\to]0,\infty[$ which fulfills conditions (h1) and (h2) of Theorem 2.1 and a Radon measure μ on $\mathbb{R}^n$ such that for all $\theta \in S^{n-1}$, $0 < \eta < 1$, and for μ-almost every $x \in \mathbb{R}^n$,

$$\underline{D}_h(\mu, x) < \infty, \tag{2.22}$$

$$\underline{D}_h(\mu, H(x,\theta,\eta), x) > c, \tag{2.23}$$

where $c > 0$ is a constant depending on η.

Construction. We assume that $n = 1$. A similar construction works also in higher dimensions.

Define numbers q_k and l_k for $k = 2, 3, 4, \ldots$ by setting $q_k = 1 - 1/k^2$ and $l_k = k^6$. It is a simple matter to check that

$$\prod_{k=2}^{\infty} q_k > 0, \tag{2.24}$$

$$1 - q_k = k l_k^{-1/2}. \tag{2.25}$$

We also define numbers r_k and R_k for $k \in \mathbb{N}$ by setting $R_1 = 1$, $r_k = R_k/l_{k+1}$, and $R_{k+1} = r_k/l_{k+1} = R_k/l_{k+1}^2$. By these definitions

$$l_2 \cdots l_k R_k^{1/2} = 1. \tag{2.26}$$

Let $I_{1,1} = [0,1]$ and $Q_{1,1} = [1/2 - r_1/2, 1/2 + r_1/2]$. Divide $Q_{1,1}$ into l_2 closed subintervals of equal length and denote them by $I_{2,1}, \ldots, I_{2,l_2}$. For every $k = 1, \ldots, l_2$, let $Q_{2,k}$ be the closed interval with same centre as $I_{2,k}$ and with length r_2. Proceed the construction by dividing each $Q_{2,k}$ into l_3 closed subintervals with equal length and so on. For every $k = 2, 3, 4, \ldots$, we get $l_2 \cdots l_k$ closed intervals $I_{k,i}$ and $Q_{k,i}$ with lengths R_k and r_k, respectively.

We now define h and μ. For $k \in \mathbb{N}$ let

$$h(r) = \begin{cases} r_k^{1/2} & \text{if } r_k \le r < R_k, \\ r^{1/2} & \text{if } R_{k+1} \le r \le r_k. \end{cases}$$

The measure μ is defined on the set

$$A = \bigcap_{k=2}^{\infty} \bigcup_{i=1}^{l_2 \cdots l_k} Q_{k,i}$$

by repeated subdivision, that is $\mu(A \cap Q_{k,i}) = (l_2 \cdots l_k)^{-1}$. This measure clearly extends to a Radon probability measure on $\mathbb{R}$.

Clearly $\lim_{r \downarrow 0} h(r) = 0$ and further

$$\begin{aligned} \liminf_{r \downarrow 0} h(r)/r &= \lim_{k \to \infty} r_k^{1/2}/R_k = \lim_{k \to \infty} (l_{k+1} R_k)^{-1/2} \\ &= \lim_{k \to \infty} l_2 \cdots l_k / l_{k+1}^{1/2} = \lim_{k \to \infty} (k!)^6 / (k+1)^3 = \infty. \end{aligned}$$

Thus h satisfies (h1) and (h2). By (2.26) and the definition of μ, we deduce that if $x \in A$, then

$$\liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{h(r)} \le \limsup_{k \to \infty} \frac{\mu(B(x,R_k))}{h(R_k)} \le \frac{3(l_2 \cdots l_k)^{-1}}{R_k^{1/2}} = 3 < \infty$$

and so also (2.22) is valid.

It remains to verify (2.23). For every $l \in \mathbb{N}$ we define a set $B_l \subset A$ as follows: When selecting subintervals $I_{k+1,j}$ inside interval $Q_{k,i}$, for $k \ge l$, we leave $\ulcorner (1 - q_{k+1}) l_{k+1}/2 \urcorner$ intervals $I_{k+1,j}$ outside B_l from the left-hand side and also from the right-hand side. Here $\ulcorner a \urcorner$ denotes the integer part of a. Formally (with a suitable enumeration of the intervals $Q_{k,i}$),

$$B_l = \bigcap_{k=l+1}^{\infty} \bigcup_{i=1}^{i_k} Q_{k,i},$$

where $i_k = l_2 \cdots l_{k-1} (l_k - 2 \ulcorner l_k (1 - q_k)/2 \urcorner) \ge l_2 \cdots l_k q_k$. It follows from (2.24), that $\mu(B_l) \to 1 = \mu(\mathbb{R})$ as $l \to \infty$.

Fix $x \in B_l$ and assume that $\theta = 1$. The case $\theta = -1$ can be handled similarly. Let $R_{k+1} \le r < R_k$ for some $k \ge l$. Now x belongs to one of the intervals $Q_{k,i}$ which we denote by $[\alpha, \beta]$. Let $d = \beta - x$. Recall that then $d < r_k$. If $d \le r < R_k$, then $[(\alpha + \beta)/2, \beta] \setminus B_l \subset [x, x+r]$ and we compute using (2.25), (2.26), and definitions

$$\begin{aligned}
\mu\left([x,x+r]\right)/h\left(r\right) &\geq \mu\left(\left[\left(\alpha+\beta\right)/2,\beta\right]\setminus B_l\right) r_k^{-1/2}\\
&= \mu\left(Q_{k,i}\setminus B_l\right) 2^{-1} r_k^{-1/2}\\
&= \left(l_2\cdots l_k\right)^{-1} \ulcorner l_{k+1}\left(1-q_{k+1}\right)/2\urcorner\, l_{k+1}^{-1} r_k^{-1/2}\\
&\geq \left(l_2\cdots l_k\right)^{-1}\left(2^{-1}\left(k+1\right) l_{k+1}^{1/2}-1\right) l_{k+1}^{-1} r_k^{-1/2}\\
&\geq 2^{-1}k\,\left(l_2\cdots l_k\right)^{-1} R_k^{-1/2}\\
&= k/2.
\end{aligned}$$

If $R_{k+1}\leq r<d$, we observe using (2.26), that

$$\frac{\mu\left([x,x+r]\right)}{h\left(r\right)} > \frac{1}{2}\,\frac{r}{R_{k+1}}\,\frac{\mu\left(I_{k+1,i}\right)}{h\left(r\right)} = \frac{1}{2}\,\frac{r^{1/2}}{R_{k+1}^{1/2}} \geq \frac{1}{2}.$$

We conclude that $\underline{D}_h\left(\mu,H\left(x,1\right),x\right)\geq 1/2>0$ for all $x\in B_l$. □

3. Upper densities

Theorem 1.2, and also the upper density results of Salli [16, Theorems 3.1, 3.7 and 3.8] and Mattila [12, Theorem 3.3], are readily generalized for Radon measures μ that satisfy $0<\overline{D}_s\left(\mu,x\right)<\infty$ for μ-almost every $x\in\mathbb{R}^n$. This arises from the fact that these measures behave in small scales very much like Hausdorff measures, see [14, Theorem 6.9]. Of course, one has to replace the constants $c\left(n,m,\eta\right)$ and $c\left(n,s,\eta\right)$ in Theorem 1.2 by $\overline{D}_s\left(\mu,x\right)c\left(n,m,\eta\right)$ and $\overline{D}_s\left(\mu,x\right)c\left(n,s,\eta\right)$, respectively.

Our goal in this section is to give a generalization of Theorem 1.2 that applies for Borel measures which satisfy μ-almost everywhere the condition $\underline{D}_h\left(\mu,x\right)<\infty$, provided that the function h fulfills some natural conditions. Our proof is a modification of a proof that was given by Salli [16, Theorem 3.1]. Notice that we may well have $\overline{D}_h\left(\mu,x\right)=\infty$ for μ-almost every $x\in\mathbb{R}^n$.

Theorem 3.1. *Let $h\colon\ ]0,r_0[\ \to\ ]0,\infty[$ be a nondecreasing function which satisfies, for some $0<c_1<\infty$, the doubling condition*

(d) $$h\left(2r\right)<c_1h\left(r\right).$$

Assume also that for some $m \in \mathbb{N}$ *and for all* $r < r_0$, $0 < t < 1$,

$$h(tr) \leq t^m h(r), \tag{h4}$$

$$\lim_{t \downarrow 0} t^{-m} h(tr) = 0. \tag{h5}$$

If μ *is a Borel measure on* $\mathbb{R}^n$ *such that* $\underline{D}_h(\mu, x) < \infty$ *for* μ*-almost all* $x \in \mathbb{R}^n$, $V \in G(n, n-m)$, *and* $0 < \eta \leq 1$, *then*

$$\overline{D}_h(\mu, X(x, V, \eta), x) \geq c\overline{D}_h(\mu, x)$$

for μ*-almost all* $x \in \mathbb{R}^n$. *Above the constant* $c > 0$ *depends only on* m, n, η, *and* c_1.

If $m < s < n$, then $h(r) = r^s$ clearly satisfies conditions (d), (h4), and (h5). So the statement of Theorem 3.1 holds for $\mu = \mathcal{P}^s \llcorner A$, when $m < s < n$, $h(r) = r^s$, and $A \subset \mathbb{R}^n$ has finite $\mathcal{P}^s$ measure. Theorem 3.1 can also be applied for measures $\mathcal{H}_h$ and $\mathcal{P}_h$ provided that h fulfills the required assumptions, recall (1.2) and (1.3). One can take for example $h(r) = r^s \log(r^{-1})$, where $m < s < n$, or $h(r) = r^m (\log(r^{-1}))^{-1}$.

Proof of Theorem 3.1. It follows from doubling condition (d) that there is a constant $c_2 < \infty$ such that

$$h\left(2\left(n - m + \eta^2/16\right)^{1/2} r\right) < c_2 h(r), \tag{3.1}$$

when $r > 0$ is small enough. We can assume that this is true for all $0 < r < r_0$. Define c by

$$c = 2^{-1} c_2^{-1} \left(4m^{1/2}\eta^{-1} + 1\right)^{-m}. \tag{3.2}$$

Recall that then $c \geq c'(n, m, c_1)\,\eta^m$.

We may assume that $V = \{x : \mathrm{proj}_1 x = \ldots = \mathrm{proj}_m x = 0\}$. Fix $M \in \,]0, \infty[$ and define

$$B = \{x \in \mathbb{R}^n : \overline{D}_h(\mu, x) > M \text{ and } \underline{D}_h(\mu, x) < \infty\}.$$

It is sufficient to show that $\overline{D}_h(\mu, X(x, V, \eta), x) \geq c\,M$ for μ-almost all $x \in B$. We will show that for any $\alpha > 0$, the set

$$F = \{x \in B : \mu(B(x, r) \cap X(x, V, \eta)) \leq c\,M\,h(r) \quad \text{when } 0 < r < \alpha\}$$

is of μ measure zero. Assume on the contrary that $\mu(F) > 0$. One can use quite standard methods, see Lemma 2.8, to show that F is a Borel set, and hence we can assume it to be closed.

Fix $k \in \mathbb{N}$ such that

$$4\,m^{1/2}/k < \eta \le 4m^{1/2}/\left(k-1\right). \tag{3.3}$$

According to Lemma 2.7 (2), there is $x_0 \in F$ such that $\overline{D}_h\left(\mu, F, x_0\right) > M$. We may assume that $x_0 = 0$. We can now fix $r_1 > 0$ such that

$$\mu\left(\left[-r_1, r_1\right]^n \cap F\right) > M\,h\left(r_1\right).$$

We will next select recursively cubes $R_j \subset \mathbb{R}^{n-m}$ and $Q_j \subset \mathbb{R}^m$. Let $R_0 = \left[-r_1, r_1\right]^{n-m}$ and $Q_0 \subset \left[-r_1, r_1\right]^m$ such that $\ell\left(Q_0\right) = 2r_1k^{-1}$ (ℓ=side length) and $\mu\left(R_0 \times Q_0 \cap F\right) > k^{-m}M\,h\left(r_1\right)$. Suppose that cubes R_j and Q_j have been selected. Denote by S one of the minimal cubes $S \subset R_j$ such that $P_V\left(R_j \times Q_j \cap F\right) \subset S$. If

$$\ell\left(S\right) > \ell\left(R_j\right)/2, \tag{3.4}$$

then we finish our selection. Otherwise we select $R_{j+1} \subset R_j$ such that $S \subset R_{j+1}$ and $\ell\left(R_{j+1}\right) = \ell\left(R_j\right)/2$. We also choose $Q_{j+1} \subset Q_j$ such that $\ell\left(Q_{j+1}\right) = \ell\left(Q_j\right)/2$ and $\mu\left(R_{j+1} \times Q_{j+1} \cap F\right) \ge 2^{-m}\mu\left(R_j \times Q_j \cap F\right)$.

If j is an index such that R_j and Q_j are selected, then

$$\mu\left(R_j \times Q_j \cap F\right) > 2^{-jm}M\,k^{-m}\,h\left(r_1\right) \tag{3.5}$$

and

$$\begin{aligned} d\left(R_j \times Q_j\right) &= 2^{-j}d\left(R_0 \times Q_0\right) = 2^{-j+1}r_1\left(n + m\left(k^{-2} - 1\right)\right)^{1/2} \\ &< \left(n - m + \eta^2/16\right)^{1/2} 2^{-j+1}r_1. \end{aligned} \tag{3.6}$$

The next step is to show that for some index j_0 our selection comes to an end. If this is not the case, then we let $\{x_1\} = \bigcap_{j=1}^{\infty} R_j \times Q_j \cap F$ and denote $d_j = d\left(R_j \times Q_j\right)$. Let $d_{j+1} \le r < d_j$. Using (3.5) and (3.6), we get

$$\begin{aligned} \mu\left(B\left(x_1, r\right)\right)/h\left(r\right) &\ge \mu\left(R_{j+1} \times Q_{j+1} \cap F\right)/h\left(d_j\right) \\ &> Mk^{-m}2^{-jm}h\left(r_1\right)/h\left(\left(n - m + \eta^2/16\right)^{1/2} 2^{-j+1}r_1\right) \end{aligned}$$

It follows from (h5), that the right hand side of the above inequality tends to infinity as $j \to \infty$. This yields $\underline{D}_h\left(\mu, x_1\right) = \infty$, which is impossible since $x \in B$.

Let j_0 be an index such that (3.4) holds. We abbreviate $R = R_{j_0}$, $Q = Q_{j_0}$ and $d = d_{j_0}$. Pick $y, z \in R \times Q \cap F$ such that $\mathrm{proj}_i\left(y - z\right) > \ell\left(R\right)/2$ for some $i \in \{1, \ldots, n-m\}$. We will show that

$$R \times Q \subset X\left(y, V, \eta\right) \cup X\left(z, V, \eta\right). \tag{3.7}$$

Let $x \in R \times Q$. If $\mathrm{proj}_i(x) \geq \mathrm{proj}_i(y+z)/2$, then

$$|x-z| \geq \mathrm{proj}_i(x-z) \geq \mathrm{proj}_i(y-z)/2 > \ell(R)/4 = 2^{-j_0-1} r_1 .$$

Using this and (3.3), we deduce

$$d(x-z,V) = P_{V^\perp}(x-z) \leq d(Q) = 2\, r_1\, k^{-1} m^{1/2} 2^{-j_0} < \eta |x-z| .$$

Hence $x \in X(z,V,\eta)$. If $\mathrm{proj}_i(x) < \mathrm{proj}_i(y+z)/2$, then by symmetry $x \in X(y,V,\eta)$. Now (3.7) holds and we know that with $x=y$ or $x=z$ we have

$$\mu(X(x,V,\eta)) \geq \mu(R \times Q)/2. \tag{3.8}$$

Finally we compute, using (3.2)–(3.8), and (h4),

$$\begin{aligned}
&\mu(X(x,V,\eta) \cap B(x,d))/h(d) \\
&\geq 2^{-1}\mu(R_{j_0} \times Q_{j_0} \cap F)/h(d) \\
&> M\, k^{-m} 2^{-j_0 m-1} h(r_1)/h\left(\left(n-m+\eta^2/16\right)^{1/2} 2^{-j_0+1} r_1\right) \\
&\geq k^{-m} 2^{-1} c_2^{-1} M \\
&\geq c\, M\, .
\end{aligned}$$

This contradicts with the definition of F. □

When $m = n-1$ it is natural to ask whether one can replace the cones $X(x,V,\eta)$ in the above theorem by their one half. The next theorem shows that the answer is positive, at least with a little bit stronger assumptions on h. For example the functions $h(r) = r^s \log(r^{-1})$, where $n-1 < s < n$, satisfy assumption (h6) of the following theorem.

Theorem 3.2. *Assume that $h\colon\,]0,r_0[\, \to\,]0,\infty[$ is a nondecreasing function satisfying doubling condition* (d). *Suppose also that for some $s > n-1$ and for all $r < r_0$, $0 < t < 1$,*

$$h(t\,r) \leq t^s h(r)\, . \tag{h6}$$

If μ is a Borel measure on $\mathbb{R}^n$ such that $\underline{D}_h(\mu,x) < \infty$ for μ-almost all $x \in \mathbb{R}^n$, $0 < \eta \leq 1$, and $\theta \in S^{n-1}$, then

$$\overline{D}_h\left(\mu, X^+(x,\theta,\eta), x\right) \geq c\overline{D}_h(\mu,x)$$

for μ-almost all $x \in \mathbb{R}^n$, where the constant $c > 0$ depends only on n, s, and c_1.

Proof. Define c_2 as in the previous proof and set

$$c = c_2^{-1}\left(1 - 2^{-(s-n+1)/2}\right)\left(4\,(n-1)\,\eta^{-1} + 1\right)^{1-n}.$$

Then $c \geq c'\,(n, c_1)\,(s-n+1)\,\eta^{n-1}$. We may assume that $\theta = e_1$. Fix $M \in \,]0, \infty[$ and define sets B and F, numbers k and r_1, and point $x_0 \in \mathbb{R}^n$ in a corresponding manner as in the previous proof. Assume again, to simplify notation, that $x_0 = 0$.

We begin to choose intervals $I_j \subset \mathbb{R}$ and cubes $Q_j \subset \mathbb{R}^{n-1}$ by setting $I_0 = [-r_1, r_1]$ and selecting $Q_0 \subset [-r_1, r_1]^{n-1}$ such that $\ell\,(Q_0) = 2r_1k^{-1}$ and $\mu\,((I_0 \times Q_0) \cap F) > k^{1-n}M\,h\,(r_1)$. Assume that I_j and Q_j have been selected. Let $a_j = \inf \operatorname{proj}_1\,(I_j \times Q_j \cap F)$ and $b_j = \sup \operatorname{proj}_1\,(I_j \times Q_j \cap F)$. If the conditions

$$b_j - a_j > \frac{\ell\,(I_j)}{2}, \tag{3.9}$$

$$\mu\,([(a_j + b_j)\,/2, b_j] \times Q_j \cap F) > \left(1 - 2^{-\frac{1}{2}(s-n+1)}\right)\mu\,(I_j \times Q_j \cap F) \tag{3.10}$$

hold, then we finish our selection. If (3.9) is not valid, then we choose $I_{j+1} \subset I_j$ such that $[a_j, b_j] \subset I_{j+1}$ and $\ell\,(I_{j+1}) = \ell\,(I_j)\,/2$. We also take $Q_{j+1} \subset Q_j$ such that $\ell\,(Q_{j+1}) = \ell\,(Q_j)\,/2$ and

$$\mu\,(I_{j+1} \times Q_{j+1} \cap F) \geq 2^{1-n}\mu\,(I_j \times Q_j \cap F)\,.$$

If (3.9) holds but (3.10) does not, then we select $I_{j+1} \subset I_j$ such that $\ell\,(I_{j+1}) = \ell\,(I_j)\,/2$, and

$$\mu\,(I_{j+1} \times Q_j \cap F) \geq 2^{-\frac{1}{2}(s-n+1)}\mu\,(I_j \times Q_j \cap F)\,,$$

and further $Q_{j+1} \subset Q_j$ such that $\ell\,(Q_{j+1}) = \ell\,(Q_j)\,/2$ and

$$\mu\,(I_{j+1} \times Q_{j+1} \cap F) \geq 2^{1-n}2^{-\frac{1}{2}(s-n+1)}\mu\,(I_j \times Q_j \cap F)\,.$$

For every I_j and Q_j

$$\mu\,(I_j \times Q_j \cap F) > 2^{j(-n-s+1)/2}M\,k^{1-n}\,h\,(r_1)\,,$$

$$d\,(I_j \times Q_j) = 2^{-j}d\,(I_0 \times Q_0) < \left(1 + \eta^2/16\right)^{1/2}2^{-j+1}r_1.$$

Similar reasoning as in the proof of Theorem 3.1 combined with assumption (h6) yields that our process of selecting intervals I_j and cubes Q_j must terminate. Thus, for some index j_0, sets I_{j_0} and Q_{j_0} satisfy (3.9) and (3.10). Abbreviate $a = a_{j_0}$, $b = b_{j_0}$ and so on. Let $y \in I \times Q \cap F$ be such that $\operatorname{proj}_1 y = a$, then

$$[(a+b)\,/2, b] \times Q \subset X^+\,(y, \theta, \eta) \cap B\,(y, d)\ .$$

We obtain, as in the proof of Theorem 3.1, that

$$\mu\left(X^{+}(y,\theta,\eta)\cap B(y,d)\right)/h(d)\geq c\,M.$$

This leads to a contradiction. □

Let $\alpha,\beta>0$. It is clear from the proofs that assumption (h6) in Theorem 3.2 can be weakened to $h(tr)\leq\alpha t^{s}h(r)$. Also, in Theorems 3.1–3.2 it suffices to assume that (h4) and (h6) hold for $0<t<\beta$. However, under these slightly weaker assumptions, constant c will depend also on α and β.

On the real line the question of upper densities is easier. We state the following theorem without a proof. A somewhat similar calculation as that of [11, Theorem 7] applies.

Theorem 3.3. *If μ is a locally finite Borel measure on $\mathbb{R}$, then for any $h\colon\,]0,\infty[\,\to\,]0,\infty[$,*

$$\overline{D}_h(\mu,[x,\infty[\,,x)=\overline{D}_h(\mu,]-\infty,x]\,,x)\geq\overline{D}_h(\mu,x)/2$$

for μ-almost every $x\in\mathbb{R}$.

The local finiteness of μ in the above result can be replaced for example by assuming that $\mu(\{x\})=0$ for every $x\in\mathbb{R}$, and $\underline{D}_h(\mu,x)<\infty$ for μ-almost all $x\in\mathbb{R}$.

If μ is a measure on $\mathbb{R}^n$, $A\subset\mathbb{R}^n$, and $x\in\mathbb{R}^n$, we define

$$\overline{D}_\mu(A,x)=\limsup_{r\downarrow 0}\mu(B(x,r)\cap A)/\mu(B(x,r)).$$

If $0<\overline{D}_h(\mu,x)<\infty$ for μ-almost every $x\in\mathbb{R}^n$ in Theorems 3.1–3.3, then they can be used to obtain positive lower bounds for the densities $\overline{D}_\mu(X(x,V,\eta),x)$ and $\overline{D}_\mu(X^{+}(x,\theta,\eta),x)$.

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