ON AN APPROXIMATION PROBLEM FOR STOCHASTIC INTEGRALS WHERE RANDOM TIME NETS DO NOT HELP

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Abstract. Given a geometric Brownian motion $S = (S_t)_{t\in[0,T]}$ and a Borel function $g : (0,\infty) \to \mathbb{R}$ such that $g(S_T) \in L_2$, we approximate $g(S_T) - \mathbb{E}g(S_T)$ by $\sum_{i=1}^n v_{i-1}(S_{\tau_i} - S_{\tau_{i-1}})$ where $0 = \tau_0 \le \cdots \le \tau_n = T$ is an increasing sequence of stopping times and the v_i are $\mathcal{F}_{\tau_i}$-measurable random variables such that $\mathbb{E}v_{i-1}^2(S_{\tau_i} - S_{\tau_{i-1}})^2 < \infty$. In case that g is not almost surely linear, we show that one gets a lower bound for the L_2-approximation rate of $1/\sqrt{n}$ if one optimizes over all nets consisting of $n+1$ stopping times. This lower bound coincides with the upper bound for all reasonable functions g, in case deterministic time-nets are used. Hence random time-nets do not improve the rate of convergence in this case. The same result holds true for the Brownian motion instead of the geometric Brownian motion.

1. Introduction and result

The question, we are dealing with, arises from Stochastic Finance, where one is interested in the L_2-error which occurs while replacing a continuously adjusted portfolio by a discretely adjusted one. Assume a finite time horizon $T > 0$ and a standard Brownian motion $B = (B_t)_{t\in[0,T]}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $B_0 \equiv 0$, continuous paths for all $\omega \in \Omega$, and $\mathcal{F}$ being the completion of $\sigma(B_t : t \in [0,T])$. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the usual augmentation of the natural filtration generated by B and $S = (S_t)_{t\in[0,T]}$ be the standard geometric Brownian motion

$$S_t := e^{B_t - \frac{t}{2}}.$$

Zhang [8], Gobet and Temam [5] and others considered the approximation error

$$\inf \mathbb{E}\left(g(S_T) - \mathbb{E}g(S_T) - \sum_{i=1}^n v_{i-1}(S_{t_i} - S_{t_{i-1}})\right)^2 \tag{1}$$

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as the time knots $(t_i)_{i=0}^n$ are deterministic and the v_i are certain $\mathcal{F}_{t_i}$-measurable random variables. In [7] stopping times were used and the identification of the optimal strategy was considered. Quantitative bounds for the approximation error were not proved. In the present paper we give a lower estimate of expression (1) for stopping times $0 = \tau_0 \le \cdots \le \tau_n = T$. It turns out, that for a large class of g this lower estimate is, up to a factor, the same as the upper bound obtained for deterministic nets, so that one cannot take advantage from random time nets in this case. Our main result is

Theorem 1.1. *Let $(M_t)_{t\in[0,T]}$ be either the Brownian motion $(B_t)_{t\in[0,T]}$ or the geometric Brownian motion $(S_t)_{t\in[0,T]}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel-function with $\mathbb{E}g(M_T)^2 < \infty$. If there are no constants $c_0, c_1 \in \mathbb{R}$ with $g(M_T) = c_0 + c_1 M_T$ a.s., then there is some $c > 0$ such that*

$$\inf \left\| [g(M_T) - \mathbb{E}g(M_T)] - \sum_{i=1}^n v_{i-1}(M_{\tau_i} - M_{\tau_{i-1}}) \right\|_{L_2} \ge \frac{c}{\sqrt{n}}$$

for all $n = 1, 2, ...,$ where the infimum is taken for each n over all sequences of stopping times $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_n = T$ and $\mathcal{F}_{\tau_{i-1}}$-measurable $v_{i-1} : \Omega \to \mathbb{R}$ with $\mathbb{E}v_{i-1}^2(M_{\tau_i} - M_{\tau_{i-1}})^2 < \infty$.

The theorem is proved in Section 2. First we indicate to what extend the lower bound is sharp. Let $\mathbb{E}g(Y_T)^2 < \infty$,

$$\alpha(x) := \begin{cases} 1 & : \quad M = B \\ x & : \quad M = S \end{cases}$$

for $x \in \mathbb{R}$, and consider

$$G(t, x) := \mathbb{E}_{M_t = x} g(M_T)$$

on $[0, T) \times \mathbb{R}$ for $M = B$ and $[0, T) \times (0, \infty)$ for $M = S$, so that $G \in C^\infty$ on its range of definition [1],

$$\frac{\partial G}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 G}{\partial x^2} = 0, \tag{2}$$

and

$$g(M_T) = \mathbb{E}g(M_T) + \int_0^T \frac{\partial G}{\partial x}(u, M_u) dM_u \quad \text{a.s.}$$

by Itô's formula. The following upper estimate was proved for the geometric Brownian motion.

[1]In fact $G \in C^\infty((-\varepsilon, T) \times \mathbb{R})$ and $G \in C^\infty((-\varepsilon, T) \times (0, \infty))$, respectively, for some $\varepsilon > 0$, which follows by the arguments of [3] (Lemma A.2) given for $g \ge 0$ and $M = S$.

Theorem 1.2 ([3]). *Let $g : (0,\infty) \to \mathbb{R}$ be a Borel function such that $\mathbb{E}g(S_T)^2 < \infty$ and $G(t,x) := \mathbb{E}g(xS_{T-t})$ for $(t,x) \in [0,T) \times (0,\infty)$. Assume that there exists some $\theta \in [0,1)$ such that*

$$\sup_{t\in[0,T)} (T-t)^{\theta} \left\| S_t^2 \frac{\partial^2 G}{\partial x^2}(t,S_t) \right\|_{L_2} < \infty. \tag{3}$$

Then there exists some $c > 0$ such that for all $n = 1, 2, ...$ there are deterministic nets $0 = t_0 < t_1 < \cdots < t_n = T$ such that

$$\left\| g(S_T) - \mathbb{E}g(S_T) - \sum_{i=1}^{n} \frac{\partial G}{\partial x}(t_{i-1}, S_{t_{i-1}}) \left(S_{t_i} - S_{t_{i-1}}\right) \right\|_{L_2} \le \frac{c}{\sqrt{n}}.$$

Remark 1.3. (i) Under $\mathbb{E}g(S_T)^2 < \infty$, we know from [3] and [2] that $t \to \left\| S_t^2 \frac{\partial^2 G}{\partial x^2}(t,S_t) \right\|_{L_2}$, $t \in [0,T)$, is continuous and increasing, and that

$$\int_0^T (T-t) \left\| S_t^2 \frac{\partial^2 G}{\partial x^2}(t,S_t) \right\|_{L_2}^2 dt < \infty.$$

This is close to condition (3). A detailed investigation of (3) can be found in [4] (cf. also [1]) and shows in terms of interpolation spaces that for all 'reasonable' g there is a $\theta \in [0,1)$ such that (3) is satisfied. Important examples are $g(y) := (y-K)^+$ and $g(y) := \chi_{[K,\infty)}(y)$ for $K > 0$.

(ii) Concerning Theorem 1.2 an analogous situation for $M = B$ is outlined in [6]. Moreover, it is shown there that $\mathbb{E}g(S_T)^2 < \infty$ without an additional assumption (like for example (3)) does not imply the conclusion of Theorem 1.2.

(iii) The papers [3] and [2] are formulated for non-negative g because of their interpretation as pay-off function. The proofs are valid for $g : (0,\infty) \to \mathbb{R}$, as stated here, without modification.

2. Proof of Theorem 1.1

Sometimes we shall use $\mathbb{E}_\sigma(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_\sigma)$ for σ being a stopping time. Given a sequence of stopping times $0 \le \tau_0 \le \cdots \le \tau_n \le T$ and $M \in \{B, S\}$, we consider the Kunita-Watanabe type projection

$$P^M_{(\tau_i)} : L_2 \to L_2 \quad \text{given by} \quad P^M_{(\tau_i)} Z := \sum_{i=1}^{n} v_{i-1}(\tau, M)(M_{\tau_i} - M_{\tau_{i-1}})$$

with

$$v_{i-1}(\tau, M) := \frac{\mathbb{E}\left(Z(M_{\tau_i} - M_{\tau_{i-1}})|\mathcal{F}_{\tau_{i-1}}\right)}{\mathbb{E}\left((M_{\tau_i} - M_{\tau_{i-1}})^2|\mathcal{F}_{\tau_{i-1}}\right)}\chi_{A_i}$$

and $A_i := \left\{\mathbb{E}\left((M_{\tau_i} - M_{\tau_{i-1}})^2|\mathcal{F}_{\tau_{i-1}}\right) \neq 0\right\}$. This projection solves our approximation problem since

$$\|Z - P^M_{(\tau_i)}Z\|_{L_2} = \inf\left\{ \left\| Z - \sum_{i=1}^n v_{i-1}(M_{\tau_i} - M_{\tau_{i-1}}) \right\|_{L_2} \mid \right.$$
$$\left. \mathbb{E}v^2_{i-1}(M_{\tau_i} - M_{\tau_{i-1}})^2 < \infty, v_{i-1} \text{ is } \mathcal{F}_{\tau_{i-1}}\text{-measurable} \right\}.$$

In the proof of Theorem 1.1 we want to restrict ourselves to sequences of stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = T$ with

$$\sup_{\omega,i} |\tau_i(\omega) - \tau_{i-1}(\omega)| \leq \frac{2T}{n}.$$

For this we need the following observation.

Lemma 2.1. *Assume that $\sigma_0, \sigma_1, ..., \sigma_N : \Omega \to [0,T]$, $N \geq 1$, are stopping times. Then*

$$\eta_l := \max\left\{\min\left(\sigma_{j_0}, ..., \sigma_{j_{N-l}}\right\} \bigm| 0 \leq j_0 < \cdots < j_{N-l} \leq N\right\}$$

defines a sequence of stopping times $0 \leq \eta_0 \leq \eta_1 \leq \cdots \leq \eta_N \leq T$ such that for all $\omega \in \Omega$ one has

$$\{\sigma_0(\omega), \ldots, \sigma_N(\omega)\} = \{\eta_0(\omega), \ldots, \eta_N(\omega)\}.$$

The proof is obvious. Moreover, we need

Lemma 2.2. *Let $0 \leq \tau_0 \leq \cdots \leq \tau_n \leq T$ and $0 \leq \eta_0 \leq \cdots \leq \eta_N \leq T$ be stopping times such that*

$$\{\tau_0(\omega), \ldots, \tau_n(\omega)\} \subseteq \{\eta_0(\omega), \ldots, \eta_N(\omega)\}$$

for all $\omega \in \Omega$. Then, given $Z \in L_2$, one has that

$$\inf \mathbb{E}\left(Z - \sum_{k=1}^N u_{k-1}\left(M_{\eta_k} - M_{\eta_{k-1}}\right)\right)^2$$
$$\leq \inf \mathbb{E}\left(Z - \sum_{i=1}^n v_{i-1}\left(M_{\tau_i} - M_{\tau_{i-1}}\right)\right)^2$$

where the infima are taken over all $\mathcal{F}_{\eta_{k-1}}$- measurable u_{k-1} and $\mathcal{F}_{\tau_{i-1}}$- measurable v_{i-1} such that

$$\mathbb{E}u_{k-1}^2\left(M_{\eta_k}-M_{\eta_{k-1}}\right)^2<\infty \quad \textit{and} \quad \mathbb{E}v_{i-1}^2\left(M_{\tau_i}-M_{\tau_{i-1}}\right)^2<\infty.$$

Proof. Assume we are given v_{i-1}, $i=1,...,n$, as above. If we choose

$$u_{k-1}:=\sum_{i=1}^n v_{i-1}1\!\!1_{\{\tau_{i-1}\le\eta_{k-1}<\tau_i\}}$$

for $k=1,\ldots,N$, then it follows that u_{k-1} is $\mathcal{F}_{\eta_{k-1}}$-measurable. Since (η_k) is a refinement of (τ_i), it holds

$$\sum_{k=1}^N u_{k-1}\left(M_{\eta_k}-M_{\eta_{k-1}}\right)=\sum_{i=1}^n v_{i-1}\left(M_{\tau_i}-M_{\tau_{i-1}}\right).$$

Moreover, one quickly checks that

$$\left\|u_{k-1}(M_{\eta_k}-M_{\eta_{k-1}})\right\|_{L_2}\le\sum_{i=1}^n\left\|v_{i-1}(M_{\tau_i}-M_{\tau_{i-1}})\right\|_{L_2}<\infty.$$

□

Finally, the following lemma is needed for technical reason.

Lemma 2.3. *For a Borel function $g:\mathbb{R}\to\mathbb{R}$ such that $g(M_T)\in L_2$, $k,l\in\{0,1,2,...\}$, $j\in\{1,2\}$, and $b\in[0,T)$ one has that*

$$\mathbb{E}\sup_{0\le s\le t\le b}\left(\alpha(M_t)^k\alpha(M_s)^l\frac{\partial^j G}{\partial x^j}(s,M_s)\right)^2<\infty.$$

Proof. Letting $1<p,q<\infty$ with $1=(1/p)+(1/q)$ we get that

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le t\le b}\left(\alpha(M_t)^k\alpha(M_s)^l\frac{\partial^j G}{\partial x^j}(s,M_s)\right)^2\\ &\le\left(\mathbb{E}\sup_{0\le s\le t\le b}\left|\frac{\alpha(M_t)^{2k}}{\alpha(M_s)^{2j-2l}}\right|^p\right)^{\frac{1}{p}}\left(\mathbb{E}\sup_{0\le s\le b}\left|\alpha(M_s)^j\frac{\partial^j G}{\partial x^j}(s,M_s)\right|^{2q}\right)^{\frac{1}{q}}.\end{aligned}$$

By Hölder's inequality one checks that the first factor is finite for all $1<p<\infty$. Hence we have to find an appropriate $1<q<\infty$ such that the second factor is finite as well. Here we use the principal idea of [3] (Lemma A.3) and combine this with the hyper-contraction property of the Ornstein-Uhlenbeck semi-group. □

Proof of Theorem 1.1. (a) Let us first assume $\delta \in (0,T)$, $n \in \{1,2,...\}$ with $n \geq 12T$, and a sequence of stopping times

$$0 = \sigma_1 \leq \cdots \leq \sigma_n = T - \delta \text{ such that } \sup_{\omega,i} |\sigma_i(\omega) - \sigma_{i-1}(\omega)| \leq \frac{2T}{n}. \tag{4}$$

By the Kunita-Watanabe projection we know that the optimal v_i in

$$\inf \left\{ \left\| \int_0^{T-\delta} \frac{\partial G}{\partial x}(u, M_u) dM_u - \sum_{i=1}^n v_{i-1}(M_{\sigma_i} - M_{\sigma_{i-1}}) \right\|_{L_2} \mid \right.$$
$$\left. \mathbb{E} v_{i-1}^2 (M_{\sigma_i} - M_{\sigma_{i-1}})^2 < \infty, v_{i-1} \text{ is } \mathcal{F}_{\sigma_{i-1}}\text{-measurable} \right\}$$

are given by

$$v_{i-1}(\sigma, M) := \frac{\mathbb{E}\left(\int_0^{T-\delta} \frac{\partial G}{\partial x}(u, M_u) dM_u (M_{\sigma_i} - M_{\sigma_{i-1}}) | \mathcal{F}_{\sigma_{i-1}} \right)}{\mathbb{E}\left((M_{\sigma_i} - M_{\sigma_{i-1}})^2 | \mathcal{F}_{\sigma_{i-1}} \right)} \chi_{A_i}.$$

with $A_i := \left\{ \mathbb{E}\left((M_{\sigma_i} - M_{\sigma_{i-1}})^2 | \mathcal{F}_{\sigma_{i-1}} \right) \neq 0 \right\}$.

(b) Now we decompose $\frac{\partial G}{\partial x}(t, M_t)$. This is done differently in the case of the Brownian motion and the geometric Brownian motion. In order to distinguish between the two cases we denote G in the case of the Brownian motion by G_1 and in the case of the geometric Brownian motion by G_2. From (2) it follows that

$$\frac{\partial^2 G_1}{\partial x \partial t} + \frac{1}{2}\frac{\partial^3 G_1}{\partial x^3} = 0 \quad \text{and} \quad \frac{\partial^2 G_2}{\partial x \partial t} + \frac{x^2}{2}\frac{\partial^3 G_2}{\partial x^3} + x\frac{\partial^2 G_2}{\partial x^2} = 0$$

on $[0,T) \times \mathbb{R}$ and $[0,T) \times (0,\infty)$, respectively. For $0 \leq s < t < T$ Itô's formula yields, a.s.,

$$\begin{aligned}
\frac{\partial G_1}{\partial x}(t, B_t) &= \frac{\partial G_1}{\partial x}(s, B_s) + \int_s^t \frac{\partial^2 G_1}{\partial x^2}(u, B_u) dB_u \\
&= \frac{\partial G_1}{\partial x}(s, B_s) + \frac{\partial^2 G_1}{\partial x^2}(s, B_s)(B_t - B_s) \\
&\quad + \int_s^t \left(\frac{\partial^2 G_1}{\partial x^2}(u, B_u) - \frac{\partial^2 G_1}{\partial x^2}(s, B_s) \right) dB_u \\
&=: (h_0^1 + h_1^1 + h_2^1)(s,t)
\end{aligned}$$

and, a.s.,

$$\begin{aligned}
\frac{\partial G_2}{\partial x}(t, S_t) &= \frac{\partial G_2}{\partial x}(s, S_s) + \int_s^t \frac{\partial^2 G_2}{\partial x^2}(u, S_u) dS_u \\
&\quad - \int_s^t S_u \frac{\partial^2 G_2}{\partial x^2}(u, S_u) du
\end{aligned}$$

$$\begin{aligned} &= \frac{\partial G_2}{\partial x}(s,S_s) + \frac{\partial^2 G_2}{\partial x^2}(s,S_s)\,(S_t - S_s) \\ &\quad + \int_s^t \left(\frac{\partial^2 G_2}{\partial x^2}(u,S_u) - \frac{\partial^2 G_2}{\partial x^2}(s,S_s) \right) dS_u \\ &\quad - \int_s^t S_u \frac{\partial^2 G_2}{\partial x^2}(u,S_u) du \\ &=: (h_0^2 + h_1^2 + h_2^2 + h_3^2)(s,t). \end{aligned}$$

Given stopping times $0 \le \sigma \le \tau \le T$ and a path-wise continuous and adapted process $Z = (Z_u)_{u\in[\sigma,\tau]}$ with $\mathbb{E} \int_\sigma^\tau Z_u^2 \alpha(M_u)^2 du < \infty$, we let

$$\begin{aligned} (5)\quad & P(Z,M;\sigma,\tau) \\ &:= \mathbb{E}_\sigma \left(\int_\sigma^\tau Z_u dM_u - \frac{\mathbb{E}_\sigma \left(\int_\sigma^\tau Z_v dM_v (M_\tau - M_\sigma) \right)}{\mathbb{E}_\sigma (M_\tau - M_\sigma)^2} \chi_A (M_\tau - M_\sigma) \right)^2 \end{aligned}$$

with $A := \{\mathbb{E}_\sigma (M_\tau - M_\sigma)^2 \neq 0\}$. We obtain

Fact 2.4. *For $\delta \in (0,T)$, $\varepsilon \in (0,1/3)$, and stopping times $0 \le \sigma \le \tau \le T - \delta$ with $\tau - \sigma \le \varepsilon$ one has, a.s.,*

(i) $P(h_1(\sigma,\cdot),M;\sigma,\tau) \ge \frac{1}{c} \left(\frac{\partial^2 G}{\partial x^2}(\sigma,M_\sigma) \right)^2 \mathbb{E}_\sigma \left(\langle M \rangle_\tau - \langle M \rangle_\sigma \right)^2,$

(ii) $P(h_2^1(\sigma,\cdot),B;\sigma,\tau) \le$
$$4\varepsilon^2 \mathbb{E}_\sigma \sup_{\sigma \le v \le \tau} \left(\frac{\partial^2 G_1}{\partial x^2}(v,B_v) - \frac{\partial^2 G_1}{\partial x^2}(\sigma,B_\sigma) \right)^2,$$

(iii) $P(h_2^2(\sigma,\cdot),S;\sigma,\tau) \le$
$$\frac{3\varepsilon^2}{1-3\varepsilon} \mathbb{E}_\sigma \sup_{\sigma \le v \le \tau} \left(\frac{\partial^2 G_2}{\partial x^2}(v,S_v) - \frac{\partial^2 G_2}{\partial x^2}(\sigma,S_\sigma) \right)^2 S_v^4,$$

(iv) $P(h_3^2(\sigma,\cdot),S;\sigma,\tau) \le \varepsilon^3 \mathbb{E}_\sigma \sup_{\sigma \le v \le u \le \tau} \left(S_u S_v \frac{\partial^2 G_2}{\partial x^2}(v,S_v) \right)^2,$

where $c > 0$ is an absolute constant and $h_1 := h_1^1$ if $M = B$ and $h_1 := h_1^2$ if $M = S$.

We can assume that the bracket processes and $h_i^k(\sigma,\cdot)$ are continuous for all ω, the latter on $[\sigma,\tau]$. The basic reason for the lower estimate in Theorem 1.1 is the lower estimate from the above item (i). We postpone the proof of the fact and see first how we can use it. From $\sup_i |\sigma_i - \sigma_{i-1}| \le \frac{2T}{n}$, $P(h_0^1(\sigma_{i-1},\cdot),B;\sigma_{i-1},\sigma_i) = 0$ a.s., $(a+b+c)^2 \ge (a^2/2) - 2b^2 - 2c^2$, and Fact 2.4 we get that

$$n \left\| \int_0^{T-\delta} \frac{\partial G_1}{\partial x}(u,B_u) dB_u - \sum_{i=1}^n v_{i-1}(\sigma,B)(B_{\sigma_i} - B_{\sigma_{i-1}}) \right\|_{L_2}^2$$

$$
\begin{aligned}
&= n\sum_{i=1}^n \mathbb{E} P\left(\left(\frac{\partial G_1}{\partial x}(u,B_u)\right)_{u\in[\sigma_{i-1},\sigma_i]}, B;\sigma_{i-1},\sigma_i\right)\\
&\geq n\sum_{i=1}^n \mathbb{E}\Big[\frac{1}{2}P\left(h_1^1(\sigma_{i-1},\cdot),B;\sigma_{i-1},\sigma_i\right)\\
&\qquad -2P\left(h_2^1(\sigma_{i-1},\cdot),B;\sigma_{i-1},\sigma_i\right) - 2P\left(h_0^1(\sigma_{i-1},\cdot),B;\sigma_{i-1},\sigma_i\right)\Big]\\
&\geq \frac{n}{2c}\mathbb{E}\left[\sum_{i=1}^n \left(\frac{\partial^2 G_1}{\partial x^2}(\sigma_{i-1},B_{\sigma_{i-1}})\right)^2 \mathbb{E}_{\sigma_{i-1}}\left(\langle B\rangle_{\sigma_i}-\langle B\rangle_{\sigma_{i-1}}\right)^2\right]\\
&\quad -\frac{32T^2}{n}\mathbb{E}\left[\sum_{i=1}^n \mathbb{E}_{\sigma_{i-1}}\sup_{\sigma_{i-1}\leq v\leq\sigma_i}\left(\frac{\partial^2 G_1}{\partial x^2}(v,B_v)-\right.\right.\\
&\qquad\qquad\qquad\left.\left.-\frac{\partial^2 G_1}{\partial x^2}(\sigma_{i-1},B_{\sigma_{i-1}})\right)^2\right]\\
&\geq \frac{n}{2c}\mathbb{E}\left[\sum_{i=1}^n \left(\frac{\partial^2 G_1}{\partial x^2}(\sigma_{i-1},B_{\sigma_{i-1}})\right)^2 (\sigma_i-\sigma_{i-1})^2\right]\\
&\qquad -32T^2\mathbb{E}\sup_i\sup_{\sigma_{i-1}\leq v\leq\sigma_i}\left(\frac{\partial^2 G_1}{\partial x^2}(v,B_v)-\right.\\
&\qquad\qquad\qquad\left.-\frac{\partial^2 G_1}{\partial x^2}(\sigma_{i-1},B_{\sigma_{i-1}})\right)^2\\
&\geq \frac{1}{2c}\mathbb{E}\left[\sum_{i=1}^n \left|\frac{\partial^2 G_1}{\partial x^2}(\sigma_{i-1},B_{\sigma_{i-1}})\right|(\sigma_i-\sigma_{i-1})\right]^2\\
&\qquad -32T^2\mathbb{E}\sup_i\sup_{\sigma_{i-1}\leq v\leq\sigma_i}\left(\frac{\partial^2 G_1}{\partial x^2}(v,B_v)-\right.\\
&\qquad\qquad\qquad\left.-\frac{\partial^2 G_1}{\partial x^2}(\sigma_{i-1},B_{\sigma_{i-1}})\right)^2.
\end{aligned}
$$

Assume now a sequence of stopping times $\sigma^{(n)} = (\sigma_i^{(n)})_{i=0}^n$ satisfying condition (4). By Lemma 2.3 and Lebesgue's dominated convergence the last term vanishes as $n\to\infty$. Consequently, by Fatou's Lemma,

$$
\begin{aligned}
\liminf_{n\to\infty}\sqrt{n}\left\|\int_0^{T-\delta}\frac{\partial G_1}{\partial x}(u,B_u)dB_u - \sum_{i=1}^n v_{i-1}^{(n)}(B_{\sigma_i^{(n)}}-B_{\sigma_{i-1}^{(n)}})\right\|_{L_2}\\
\geq \sqrt{\frac{1}{2c}}\left\|\left[\int_0^{T-\delta}\left|\frac{\partial^2 G_1}{\partial x^2}(u,B_u)\right|du\right]\right\|_{L_2}.
\end{aligned}
$$

with $v_{i-1}^{(n)} := v_{i-1}(\sigma^{(n)}, B)$. Let us turn to the geometric Brownian motion. Using $(a+b+c+d)^2 \geq (a^2/2) - 4b^2 - 4c^2 - 4d^2$, we get in exactly the same way as in the previous case that

$$\begin{aligned}
& n \left\| \int_0^{T-\delta} \frac{\partial G_2}{\partial x}(u, S_u) dS_u - \sum_{i=1}^n v_{i-1}(\sigma, S)(S_{\sigma_i} - S_{\sigma_{i-1}}) \right\|_{L_2}^2 \\
= \ & n \sum_{i=1}^n \mathbb{E} P\left(\left(\frac{\partial G_2}{\partial x}(u, S_u) \right)_{u \in [\sigma_{i-1}, \sigma_i]}, S; \sigma_{i-1}, \sigma_i \right) \\
\geq \ & n \sum_{i=1}^n \mathbb{E}\Big[\frac{1}{2} P\left(h_1^2(\sigma_{i-1}, \cdot), S; \sigma_{i-1}, \sigma_i\right) - 4P\left(h_2^2(\sigma_{i-1}, \cdot), S; \sigma_{i-1}, \sigma_i\right) \\
& \qquad -4P\left(h_3^2(\sigma_{i-1}, \cdot), S; \sigma_{i-1}, \sigma_i\right) - 4P\left(h_0^2(\sigma_{i-1}, \cdot), S; \sigma_{i-1}, \sigma_i\right) \Big] \\
\geq \ & \frac{n}{2c} \mathbb{E} \left[\sum_{i=1}^n \left(\frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right)^2 \mathbb{E}_{\sigma_{i-1}} \left(\langle S \rangle_{\sigma_i} - \langle S \rangle_{\sigma_{i-1}} \right)^2 \right] \\
& -n^2 \frac{12(2T/n)^2}{1 - 3(2T/n)} \mathbb{E} \sup_i \sup_{\sigma_{i-1} \leq v \leq \sigma_i} \left(\frac{\partial^2 G_2}{\partial x^2}(v, S_v) - \right. \\
& \qquad \left. - \frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right)^2 S_v^4 \\
& \qquad -4n^2 \left(\frac{2T}{n} \right)^3 \mathbb{E} \sup_i \sup_{\sigma_{i-1} \leq v \leq u \leq \sigma_i} \left(S_u S_v \frac{\partial^2 G_2}{\partial x^2}(v, S_v) \right)^2 \\
\geq \ & \frac{1}{2c} \mathbb{E} \left[\sum_{i=1}^n \left| \frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right| \left(\langle S \rangle_{\sigma_i} - \langle S \rangle_{\sigma_{i-1}} \right) \right]^2 \\
& -96 T^2 \mathbb{E} \sup_i \sup_{\sigma_{i-1} \leq v \leq \sigma_i} \left(\frac{\partial^2 G_2}{\partial x^2}(v, S_v) - \frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right)^2 S_v^4 \\
& \qquad - \frac{32 T^3}{n} \mathbb{E} \sup_{0 \leq v \leq u \leq T - \delta} \left(S_u S_v \frac{\partial^2 G_2}{\partial x^2}(v, S_v) \right)^2
\end{aligned}$$

where we have used that $n \geq 12T$. Again, assuming a sequence of stopping times $\sigma^{(n)} = (\sigma_i^{(n)})_{i=0}^n$ satisfying condition (4), we get by Lemma 2.3 and Lebesgue's dominated convergence that the second and the third term are converging to zero as $n \to \infty$, so that

$$\liminf_{n \to \infty} \sqrt{n} \left\| \int_0^{T-\delta} \frac{\partial G_2}{\partial x}(u, S_u) dS_u - \sum_{i=1}^n v_{i-1}^{(n)} (S_{\sigma_i^{(n)}} - S_{\sigma_{i-1}^{(n)}}) \right\|_{L_2}$$

$$\geq \sqrt{\frac{1}{2c}} \left\| \left[\int_0^{T-\delta} \left| \frac{\partial^2 G_2}{\partial x^2}(u, S_u) \right| d\langle S \rangle_u \right] \right\|_{L_2}$$

by Fatou's lemma with $v_{i-1}^{(n)} := v_{i-1}(\sigma^{(n)}, S)$.

(c) Now take sequences of stopping times $\tau^{(n)} = (\tau_i^{(n)})_{i=0}^n$ with

$$0 = \tau_0^{(n)} \leq \cdots \leq \tau_n^{(n)} = T.$$

Stopping additionally at points $\frac{kT}{n}$, $k = 1, ..., n-1$, we get a new sequence $(\eta_k^{(2n-1)})_{k=0}^{2n-1}$ according to Lemma 2.1. Taking $\delta \in (0,T)$ and $\sigma_k^{(2n-1)} := \eta_k^{(2n-1)} \wedge (T-\delta)$ we get sequences of stopping times $\sigma^{(2n-1)} = (\sigma_k^{(2n-1)})_{k=0}^{2n-1}$ with

$$0 = \sigma_0^{(2n-1)} \leq \cdots \leq \sigma_{2n-1}^{(2n-1)} = T - \delta$$

and

$$\sup_{\omega,k} \left| \sigma_k^{(2n-1)}(\omega) - \sigma_{k-1}^{(2n-1)}(\omega) \right| \leq \frac{T}{n} \leq \frac{2T}{2n-1}$$

which is condition (4). By Lemma 2.2 we get

$$\begin{aligned}
& \liminf_n \sqrt{n} \left\| \int_0^T \frac{\partial G}{\partial x}(u, M_u) dM_u - \right. \\
& \qquad \left. - \sum_{i=1}^n v_{i-1}(\tau^{(n)}, M)(M_{\tau_i^{(n)}} - M_{\tau_{i-1}^{(n)}}) \right\|_{L_2} \\
\geq \; & \liminf_n \sqrt{n} \left\| \int_0^T \frac{\partial G}{\partial x}(u, M_u) dM_u - \right. \\
& \qquad \left. - \sum_{i=1}^{2n-1} v_{i-1}(\eta^{(2n-1)}, M)(M_{\eta_i^{(2n-1)}} - M_{\eta_{i-1}^{(2n-1)}}) \right\|_{L_2} \\
\geq \; & \liminf_n \sqrt{n} \left\| \int_0^{T-\delta} \frac{\partial G}{\partial x}(u, M_u) dM_u - \right. \\
& \qquad \left. - \sum_{i=1}^{2n-1} v_{i-1}(\sigma^{(2n-1)}, M)(M_{\sigma_i^{(2n-1)}} - M_{\sigma_{i-1}^{(2n-1)}}) \right\|_{L_2} \\
\geq \; & \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2c}} \left\| \int_0^{T-\delta} \left| \frac{\partial^2 G}{\partial x^2}(u, M_u) \right| d\langle M \rangle_u \right\|_{L_2}.
\end{aligned}$$

(d) Assuming finally

$$\left\| \int_0^T \left| \frac{\partial^2 G}{\partial x^2}(u, M_u) \right| d\langle M \rangle_u \right\|_{L_2} = 0$$

implies

$$\mathbb{E}\int_0^T \left| \left(\alpha^2 \frac{\partial^2 G}{\partial x^2} \right)(u, M_u) \right|^2 du = 0$$

as well, so that the arguments of [3] (Lemma 4.8) give the existence of constants $c_0, c_1 \in \mathbb{R}$ such that $g(M_T) = c_0 + c_1 M_T$ a.s. □

Proof of Fact 2.4. (i) First we remark that

$$\mathbb{E}\int_\sigma^\tau (M_u - M_\sigma)^2 \left[1 + \left(\frac{\partial^2 G}{\partial x^2}(\sigma, M_\sigma) \right)^2 \right] \alpha(M_u)^2 du < \infty,$$

where one can use Lemma 2.3. Moreover, it is easy to see that it is enough to prove assertion (i) with $(\partial^2 G/\partial x^2)(\sigma, M_\sigma)$ replaced by 1. By Itô's formula we get, a.s.,

$$\mathbb{E}_\sigma (M_\tau - M_\sigma)^3 = 3\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\, d\langle M \rangle_u$$

and

$$\mathbb{E}_\sigma (M_\tau - M_\sigma)^4 = 6\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\, d\langle M \rangle_u.$$

Consequently, by Hölder's inequality,

$$\begin{aligned} &\left(\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\, d\langle M \rangle_u \right)^2 \\ &= \frac{1}{9}\left(\mathbb{E}_\sigma (M_\tau - M_\sigma)^3 \right)^2 \\ &\leq \frac{1}{9}\mathbb{E}_\sigma (M_\tau - M_\sigma)^2\, \mathbb{E}_\sigma (M_\tau - M_\sigma)^4 \\ &= \frac{2}{3}\mathbb{E}_\sigma (M_\tau - M_\sigma)^2\, \mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\, d\langle M \rangle_u \text{ a.s.} \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{3}\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\, d\langle M \rangle_u \\ &\leq \mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\, d\langle M \rangle_u - \frac{\left(\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\, d\langle M \rangle_u \right)^2}{\mathbb{E}_\sigma (M_\tau - M_\sigma)^2} \chi_A \text{ a.s.} \end{aligned}$$

On the other hand, the Burkholder-Davis-Gundy and Doob's maximal inequality give

$$\begin{aligned} \frac{1}{c}\mathbb{E}_\sigma \left(\langle M \rangle_\tau - \langle M \rangle_\sigma \right)^2 &\leq \mathbb{E}_\sigma \sup_{\sigma \leq u \leq \tau} (M_u - M_\sigma)^4 \\ &\leq d\, \mathbb{E}_\sigma (M_\tau - M_\sigma)^4 \end{aligned}$$

$$= \quad 6d\, \mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\, d\langle M\rangle_u \;\text{ a.s.}$$

for absolute $c, d > 0$ so that

$$\frac{1}{18cd} \mathbb{E}_\sigma \left(\langle M\rangle_\tau - \langle M\rangle_\sigma\right)^2$$
$$\leq \mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\, d\langle M\rangle_u - \frac{\left(\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\, d\langle M\rangle_u\right)^2}{\mathbb{E}_\sigma (M_\tau - M_\sigma)^2} \chi_A \;\text{ a.s.}$$

and the assertion follows.

(ii) We have, a.s.,

$$\begin{aligned}
& P(h_2^1(\sigma,\cdot), B; \sigma, \tau) \\
& \quad \leq \quad \mathbb{E}_\sigma \int_\sigma^\tau [h_2^1]^2(\sigma,u) du \\
& \quad = \quad \mathbb{E}_\sigma \int_\sigma^\tau \left[\int_\sigma^u \left(\frac{\partial^2 G_1}{\partial x^2}(v, B_v) - \frac{\partial^2 G_1}{\partial x^2}(\sigma, B_\sigma) \right) dB_v \right]^2 du \\
& \quad \leq \quad \varepsilon\, \mathbb{E}_\sigma \sup_{\sigma \leq u \leq \tau} \left[\int_\sigma^u \left(\frac{\partial^2 G_1}{\partial x^2}(v, B_v) - \frac{\partial^2 G_1}{\partial x^2}(\sigma, B_\sigma) \right) dB_v \right]^2 \\
& \quad \leq \quad 4\varepsilon\, \mathbb{E}_\sigma \int_\sigma^\tau \left(\frac{\partial^2 G_1}{\partial x^2}(v, B_v) - \frac{\partial^2 G_1}{\partial x^2}(\sigma, B_\sigma) \right)^2 dv \\
& \quad \leq \quad 4\varepsilon^2\, \mathbb{E}_\sigma \sup_{\sigma \leq v \leq \tau} \left(\frac{\partial^2 G_1}{\partial x^2}(v, B_v) - \frac{\partial^2 G_1}{\partial x^2}(\sigma, B_\sigma) \right)^2,
\end{aligned}$$

where we applied Doob's maximal inequality.

(iii) Let $A(v) := \frac{\partial^2 G_2}{\partial x^2}(v, S_v) - \frac{\partial^2 G_2}{\partial x^2}(\sigma, S_\sigma)$. Then, a.s.,

$$\begin{aligned}
P(h_2^2(\sigma,\cdot), S; \sigma, \tau) \quad & \leq \quad \mathbb{E}_\sigma \int_\sigma^\tau [h_2^2]^2(\sigma,u) S_u^2 du \\
& = \quad \mathbb{E}_\sigma \int_\sigma^\tau \left[\int_\sigma^u A(v) dS_v \right]^2 S_u^2 du.
\end{aligned}$$

For $N \in \{1, 2, ...\}$ we let

$$\tau_N := \inf \left\{ t \in [\sigma, \tau] \,\middle|\, \left| \int_\sigma^u A(v) dS_v \right| > N \right\} \wedge \tau.$$

Now we estimate $\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^u A(v) dS_v \right]^2 S_u^2 du$ from above. We have that

$$\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^u A(v) dS_v \right]^2 S_u^2 du$$

$$\leq \mathbb{E}_\sigma \int_0^\varepsilon \left[\int_\sigma^{(\sigma+u)\wedge\tau_N} A(v)dS_v\right]^2 S^2_{(\sigma+u)\wedge\tau_N} du \text{ a.s.}$$

Itô's formula implies, a.s.,

$$\begin{aligned}
&\mathbb{E}_\sigma \left[\int_\sigma^{(\sigma+u)\wedge\tau_N} A(v)dS_v\right]^2 S^2_{(\sigma+u)\wedge\tau_N} \\
&= \mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} A(t)^2 S_t^4 dt \\
&\qquad + \mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} \left(\int_\sigma^t A(u)dS_u\right)^2 S_t^2 dt \\
&\qquad + 4\mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} \left(\int_\sigma^t A(u)dS_u\right) A(t)S_t^3\, dt \\
&\leq 3\mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} A(t)^2 S_t^4 dt \\
&\qquad + 3\,\mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} \left(\int_\sigma^t A(u)dS_u\right)^2 S_t^2 dt \\
&\leq 3\mathbb{E}_\sigma \int_\sigma^{\tau_N} A(t)^2 S_t^4 dt \\
&\qquad + 3\,\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left(\int_\sigma^t A(u)dS_u\right)^2 S_t^2 dt.
\end{aligned}$$

As a result, a.s.,

$$\begin{aligned}
&\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^u A(v)dS_v\right]^2 S_u^2 du \\
&\leq \mathbb{E}_\sigma \int_0^\varepsilon \left[\int_\sigma^{(\sigma+u)\wedge\tau_N} A(v)dS_v\right]^2 S^2_{(\sigma+u)\wedge\tau_N} du \\
&\leq 3\varepsilon\mathbb{E}_\sigma \int_\sigma^{\tau_N} A(t)^2 S_t^4 dt + 3\varepsilon\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^t A(u)dS_u\right]^2 S_t^2 dt,
\end{aligned}$$

which implies, a.s.,

$$\begin{aligned}
&\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^u A(v)dS_v\right]^2 S_u^2 du \\
&\leq \frac{3\varepsilon}{1-3\varepsilon}\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left(\frac{\partial^2 G_2}{\partial x^2}(u,S_u) - \frac{\partial^2 G_2}{\partial x^2}(\sigma,S_\sigma)\right)^2 S_u^4 du
\end{aligned}$$

$$\leq \frac{3\varepsilon^2}{1-3\varepsilon}\mathbb{E}_\sigma \sup_{\sigma\leq u\leq\tau}\left(\frac{\partial^2 G_2}{\partial x^2}(u,S_u)-\frac{\partial^2 G_2}{\partial x^2}(\sigma,S_\sigma)\right)^2 S_u^4.$$

Letting $N\to\infty$ implies the assertion.

(iv) The last inequality follows from

$$\begin{aligned}
&P(h_3^2(\sigma,\cdot),S;\sigma,\tau))\\
&\quad\leq \mathbb{E}_\sigma\int_\sigma^\tau [h_3^2]^2(\sigma,u)S_u^2du\\
&\quad= \mathbb{E}_\sigma\int_\sigma^\tau\left(\int_\sigma^u S_v\frac{\partial^2 G_2}{\partial x^2}(v,S_v)dv\right)^2 S_u^2du\\
&\quad\leq \varepsilon\mathbb{E}_\sigma\int_\sigma^\tau\int_\sigma^u\left(\frac{\partial^2 G_2}{\partial x^2}(v,S_v)\right)^2 S_v^2dvS_u^2du\\
&\quad\leq \varepsilon^3\mathbb{E}_\sigma\sup_{\sigma\leq v\leq u\leq\tau}\left(S_uS_v\frac{\partial^2 G_2}{\partial x^2}(v,S_v)\right)^2.
\end{aligned}$$

□

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