Mappings of finite distortion: The Rickman-Picard theorem for mappings of finite lower order

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Abstract

We show that an entire mapping f of finite distortion with finite lower order can omit at most finitely many points when the distortion function of f is suitably controlled. The proof uses the recently established modulus inequalities for mappings of finite distortion [12] and comparison inequalities for the averages of the counting function. A similar technique also gives growth estimates for mappings having asymptotic values.

1 Introduction

We call a mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ *a mapping of finite distortion* if it satisfies

$$|Df(x)|^n \leq K(x,f)J(x,f) \qquad \text{a.e.,} \tag{1.1}$$

where $1 \leq K(x,f) < \infty$, and if also $J(\cdot,f) \in L^1_{\text{loc}}(\Omega)$. When $1 \leq K(x,f) \leq K < \infty$ a.e., f is called a K-quasiregular mapping.

Starting from the works of Reshetnyak in the late 1960s, the study of n-dimensional quasiregular mappings has shown that they form, from the geometric function theoretic point of view, a natural generalization of analytic functions to higher dimensions. See the monographs [16] and [21] for the basic theory of quasiregular mappings. One of the most important works in this area is the value distribution theory developed mainly by Rickman. This theory includes the following generalization of Picard's theorem of analytic functions: For every $n \geq 2$ and $K \geq 1$ there exists a finite number $q(n,K)$ so that a non-constant K-quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ can omit at most $q(n,K)$ points. Moreover, in dimension three one has for any $q < \infty$ a $K(q)$-quasiregular mapping $\mathbb{R}^3 \to \mathbb{R}^3$ omitting q points. See [17], [18], [19], [20], [22] and [21] for more information on the value distribution theory of quasiregular mappings.

Recently a systematic study of mappings of finite distortion has been begun. This study has several motivations. One of them is to relax the assumptions of quasiregularity in Reshetnyak's basic results: continuity, discreteness and openness, and the Lusin condition. It has turned out that these results hold true for mappings of finite distortion whose distortion function $K(\cdot,f)$ satisfies some suitable exponential integrability condition. See [6], [8], [9], [10] and [7] for the theory of mappings of finite distortion. We next introduce an assumption that has turned out to be sharp for all of these results (weaker assumptions on the distortion function are sufficient if the mapping is assumed to belong to $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, cf. [3] and [14]).

Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing, differentiable function. We call such a function an Orlicz-function. Now assert the following conditions:

(Φ-1) $\displaystyle\int_1^\infty \frac{\Phi'(t)}{t}\, dt = \infty,$

(Φ-2) $t\,\Phi'(t)$ increases to infinity when $t \to \infty$.

Then the assumption is that for a mapping of finite distortion f there should exist an Orlicz-function Φ, satisfying conditions (Φ-1) and (Φ-2), such that

$$\exp(\Phi(K(\cdot, f))) \in L^1_{\mathrm{loc}}(\Omega). \tag{1.2}$$

A large portion of the theory of quasiregular mappings has been established by using path family methods. In particular, most of the value distribution theory mentioned above relies on these geometric methods. On the other hand, although the theory of mappings of finite distortion has developed rapidly, only analytical methods have been available until very recently. As one of the main goals in this theory has been the extension of the theory of quasiregular mappings into this more general framework of mappings of finite distortion, the lack of geometric methods has been essential. In [12] counterparts of the modulus inequalities of quasiregular mappings are proved. Thus it is now possible to use path family methods also in the study of mappings of finite distortion.

In this paper we apply the path family methods in order to prove a counterpart of the Rickman-Picard theorem for mappings of finite distortion with both growth and distortion function suitably controlled. The main ideas in the proof are from Rickman's proofs ([17], [19], [21]), and they include comparison inequalities for the counting function. An alternative way would be to follow the more recent potential-theoretic proofs of Eremenko and Lewis [2] and Lewis [13] and to prove Harnack-type inequalities related to mappings of finite distortion.

Let us formulate our main result. For mappings satisfying Assumption (1.2), we shall use the following assumption on the distortion function:

$$R^{-n} \int_{B(0,R)} \exp\big(\Phi(K(x, f))\big)\, dx \le A \qquad \forall\, 1 \le R < \infty. \tag{1.3}$$

We shall also assume that the mappings in consideration do not grow too quickly, namely that they are of finite lower order. Set

$$M_f(r) = \max_{x \in \overline{B}(0,r)} |f(x)|.$$

Then the lower order λ_f of a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\lambda_f = \liminf_{r\to\infty} (n-1)\frac{\log\log M_f(r)}{\log r}.$$

The mapping f is said to have finite lower order if $\lambda_f < \infty$, and positive lower order if $\lambda_f > 0$. The main result of this paper is the following theorem. Here A is the constant in Inequality (1.3). Recall that mappings of finite distortion satisfying Assumption (1.2) are continuous and either constant or both open and discrete.

Theorem 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non-constant mapping of finite distortion satisfying Inequality* (1.3). *Suppose that there exists, for each $T > 1$, a constant $C(T) > 1$ and a sequence (R_L) so that $R_L \to \infty$ and*

$$(\log M_f(TR_L))^{n-1} \leq C(T)(\log M_f(R_L))^{n-1} \quad \textit{for all } L \in \mathbb{N}. \tag{1.4}$$

Then f omits at most $q = q(n, \Phi, A, C(T))$ points.

A simple calculation shows that Assumption (1.4) is particularly true when f has finite lower order λ_f, and in this case $C(T)$ depends only on T and λ_f. Thus we have the following corollary.

Corollary 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non-constant mapping of finite distortion satisfying Inequality* (1.3). *Suppose that f has finite lower order λ_f. Then f omits at most $q = q(n, \Phi, A, \lambda_f)$ points.*

A full counterpart of the Rickman-Picard theorem would be one in which no growth conditions are required. It would be very interesting to know if such a theorem is true under any natural integrability assumptions on the distortion function that are weaker than L^∞. In the quasiregular case one is able to use methods like rescaling in order to show that for proving the Rickman-Picard theorem for all quasiregular mappings it suffices to prove it for mappings with finite lower order. In the case of unbounded distortion no such reduction method is available. Hence it seems that new ideas are needed for proving similar results if no growth conditions are assumed. In [12] it is proved (without growth assumptions) that under an assumption slightly weaker than (1.3) at most a set of zero conformal modulus can be omitted.

Notice that the plane case is easier. There are factorization results (cf. [7], Chapter 11) showing that mappings of finite distortion $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying Assumption (1.3), and quasiregular mappings in particular, can omit at most one finite point. Recall also Rickman's example mentioned above, which tells that the number of omitted values may be large in dimension three.

Using methods similar to those used in the proof of the above theorem, we are also able to prove a counterpart of a theorem of Rickman and Vuorinen [23] on quasiregular mappings with at least one asymptotic value. We call a point $b \in \mathbb{R}^n$ an asymptotic value of a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ if there exists a path $\gamma : [0, 1[\to \mathbb{R}^n$ so that $\lim_{t\to 1^-} \gamma(t) = \infty$ and $\lim_{t\to 1^-} f(\gamma(t)) \to b$.

Theorem 1.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non-constant mapping of finite distortion satisfying Inequality* (1.3). *If f has an asymptotic value $b \in \mathbb{R}^n$, then $\lambda_f \geq M(n, \Phi, A) > 0$.*

One could ask if the value of the lower order should increase when the number of the asymptotic values increases, as happens in the case of analytic functions. This turns out to be false for all $n \geq 3$, see [5] and [1].

2 Modulus estimates and the counting function

Let Γ be a path family in a domain Ω. We call a Borel function $\rho : \Omega \to [0, \infty]$ *admissible* for the path family Γ, if

$$\int_\gamma \rho \, ds \geq 1 \qquad \text{for all locally rectifiable } \gamma \in \Gamma.$$

Now let $\omega : \Omega \to [0, \infty]$ be a measurable function. The weighted p-modulus $M_{p,\omega}(\Gamma)$ of Γ is defined by

$$M_{p,\omega}(\Gamma) = \inf \Big\{ \int_{\mathbb{R}^n} \rho^p(x) \, \omega(x) \, dx : \rho : \Omega \to [0, \infty) \text{ is admissible for } \Gamma \Big\}.$$

Note that when $\omega(x) = 1$ for all $x \in \Omega$, we then recover the usual p-modulus M_p. When $p = n$, we write M_ω instead of $M_{n,\omega}$.

When $A \subset \mathbb{R}^n$ is a Borel set and $f : A \to \mathbb{R}^n$ a mapping, we use the notation $N(y, f, A) = \operatorname{card}\{x \in A : f(x) = y\}$. The notation Γ_f is used to denote all locally rectifiable paths in A having a closed subpath on which f is not absolutely continuous. We now have the following counterpart of the K_O-inequality of quasiregular mappings.

Theorem 2.1. *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying Assumption* (1.2). *Let $A \subset \Omega$ be a Borel set with*

$$\sup_{y \in \mathbb{R}^n} N(y, f, A) < \infty.$$

Moreover, let Γ be a family of paths in A. If ρ is an admissible function for $f(\Gamma \setminus \Gamma_f)$, then

$$M_{K^{-1}(\cdot, f)}(\Gamma \setminus \Gamma_f) \leq \int_{\mathbb{R}^n} \rho^n(y) N(y, f, A) \, dy.$$

Moreover, $M_p(\Gamma_f) = 0$ for all $1 < p < n$.

Proof. By Inequality (1.1), Assumption (1.2) and Hölder's inequality, $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for all $1 < p < n$. Then Fuglede's theorem (cf. [21], II 2.3) says that Γ_f is of zero p-modulus for all $1 < p < n$. Now the theorem follows exactly as in the proof of Theorem II 2.4 in [21]. $\square$

In [12] the following counterpart of Väisälä's inequality of quasiregular mappings [25] is proved. Here $i(x, f)$ is the local index of f at a point x. See [21] Chapter I, Section 4 for information on the local index.

Theorem 2.2 ([12], Theorem 4.1). *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying Assumption* (1.2)*. Let Γ be a path family in Ω, Γ' be a path family in $\mathbb{R}^n$, and m be a positive integer such that the following is true. For every path $\beta : I \to \mathbb{R}^n$ in Γ' there are paths $\alpha_1, \dots \alpha_m$ in Γ such that $f \circ \alpha_j \subset \beta$ for all j and such that for every $x \in \Omega$ and $t \in I$ the equality $\alpha_j(t) = x$ holds for at most $i(x, f)$ indices j. Then*

$$M(\Gamma') \leq \frac{M_{K^{n-1}(\cdot,f)}(\Gamma)}{m}.$$

For spherical rings we have the following upper bound for the $K^{n-1}(\cdot, f)$-modulus.

Theorem 2.3 ([12], Theorem 5.3). *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying Assumption* (1.2)*. Suppose that*

$$I_a^R := R^{-n} \int_{B(a,R)} \exp(\Phi(K)) < \infty.$$

Then there exist $C_1, C_2 > 0$ depending on n, Φ and I_a^R such that

$$\begin{aligned} (2.1)\ M_{K^{n-1}(\cdot,f)}(\Gamma) \quad &\leq \quad C_1 \left(\int_{2r}^{R/2} \frac{ds}{s\Phi^{-1}\left(\log(C_2 R^n s^{-n})\right)} \right)^{1-n} \\ &=: \quad \varphi(I_a^R, R/r) \end{aligned}$$

for all $0 < 6r \leq R$, where Γ is the family of all paths connecting $\overline{B}(a, r)$ and $\mathbb{R}^n \setminus B(a, R)$. Moreover, $\varphi(C, \theta) \to 0$ as $\theta \to \infty$.

In what follows, the notation $\varphi(I_a^R, R/r)$ is frequently used. This constant also depends on n and Φ, but in our considerations they will be fixed.

We shall also need a lower bound for the $K^{-1}(\cdot, f)$-modulus. Estimates similar to the one in Theorem 2.5 below have been established in [4]. We consider the p-modulus on spheres. Let Γ be a path family in $S^{n-1}(a, r)$, and let $1 < p < \infty$. We denote

$$\begin{aligned} M_p^S(\Gamma) \quad = \quad \inf \Big\{ & \int_{S^{n-1}(a,r)} \rho^p(x)\, dS(x) : \\ & \rho : S^{n-1}(a, r) \to [0, \infty] \text{ is admissible for } \Gamma \Big\}. \end{aligned}$$

Here dS means integration against the surface measure. The following lemma can be proved by slightly modifying the proof of Lemma 2.3 in [15], see also [24], Theorem 10.2.

Lemma 2.4. *Let $G, H \subset S^{n-1}(a, r)$ be disjoint non-empty sets, and let $n-1 < p \leq n$. Then there exists a constant $C_p > 0$, depending only on p, so that*

$$M_p^S(\Gamma) \geq \frac{C_p}{r^{p+1-n}},$$

where Γ is the family of all paths joining G and H in $S^{n-1}(a,r)$.

Theorem 2.5. *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying Assumption (1.2). Let E and F be two sets in $\mathbb{R}^n$. Assume that there exists a point $a \in \Omega$ so that $S^{n-1}(a,r)$ intersects both E and F for all $0 < t < r < 8t$. If $B(a,8t) \subset\subset \Omega$ and if Γ is the family of all paths joining E and F in $B(a,8t) \setminus \overline{B}(a,t)$, then*

$$M_{K^{-1}(\cdot,f)}(\Gamma \backslash \Gamma_f) \geq C_n \Big(\Phi^{-1}\Big(\log\Big((8t)^{-n}\int_{B(a,8t)} \exp(\Phi(K(x,f)))\,dx\Big)\Big)\Big)^{-1}. \tag{2.2}$$

Proof. In this proof $C_n > 0$ will denote a varying constant depending only on n. Let ρ be an admissible function for $M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f)$ so that

$$\int_\Omega \rho^n(x) K(x,f)^{-1}\,dx < \infty.$$

By writing

$$\int_{B(a,8t)} \rho^p(x)\,dx = \int_{B(a,8t)} \rho^p(x) K^{-p/n}(x,f) K^{p/n}(x,f)\,dx$$

and using Hölder's inequality and Assumption (1.2), we see that $\rho \in L^p(B(0,8t))$ for all $n-1 < p < n$. Fix such p. Consequently,

$$\int_{S^{n-1}(a,r)} \rho^p(x) dS(x) < \infty$$

for almost all $r \in (t,8t)$ (with respect to the linear measure). Now, by Theorem 2.1, $M_p(\Gamma_f) = 0$, and thus $M_p^S(\Gamma_f^r) = 0$ for almost all $r \in (t,8t)$, where $\Gamma_f^r = \{\gamma \in \Gamma_f : |\gamma| \subset S^{n-1}(a,r)\}$. Hence we can apply Lemma 2.4 in order to have that for almost all $r \in (t,8t)$

$$\int_{S^{n-1}(a,r)} \rho^p\,dS(x) \geq M_p^S(\Gamma^r \setminus \Gamma_f^r) = M_p^S(\Gamma^r) \geq \frac{C_p}{r^{p+1-n}},$$

where $\Gamma^r = \{\gamma \in \Gamma : |\gamma| \subset S^{n-1}(a,r)\}$. Hence the inequality

$$1 \leq C_n r \Big(r^{1-n}\int_{S^{n-1}(a,r)} \rho^p(x)\,dS(x)\Big)^{1/p} \tag{2.3}$$

holds for almost all r.

On the other hand, we can choose a constant $C_n > 1$ so that

$$\Big(r^{1-n}\int_{S^{n-1}(a,r)} \rho^p(x)\,dS(x)\Big)^{1/p} \leq C_n \Big(t^{-n}\int_{B(a,8t)} \rho^p(x)\,dx\Big)^{1/p} \tag{2.4}$$

for some $r \in (t,8t)$ for which also (2.3) holds. We use Hölder's inequality in order to have

$$\begin{aligned}
&\left(t^{-n}\int_{B(a,8t)}\rho^p(x)\,dx\right)^{1/p}\\
= &\left(t^{-n}\int_{B(a,8t)}\rho^p(x)K(x,f)^{-p/n}K(x,f)^{p/n}\,dx\right)^{1/p}\\
\leq &\left(t^{-n}\int_{B(a,8t)}\rho^n(x)K(x,f)^{-1}\,dx\right)^{1/n}\\
\cdot &\left(t^{-n}\int_{B(a,8t)}K(x,f)^{p/(n-p)}\,dx\right)^{(n-p)/pn}.
\end{aligned}$$

By our assumptions on the function Φ, the function Λ defined by

$$\Lambda(t)=\exp(\Phi(t^{(n-p)/p}))$$

is convex for $t>t_0$, where t_0 depends on Φ, see [11], Lemma 2.3. As we may without loss of generality assume that $K(x,f)\geq t_0$ for all $x\in B(a,8t)$, we are allowed to use Jensen's inequality in order to have

$$\begin{aligned}
&\left(t^{-n}\int_{B(a,8t)}K(x,f)^{p/(n-p)}\,dx\right)^{(n-p)/pn}=\\
&C_n\left(\Lambda^{-1}\left(\Lambda\left(|B(a,8t)|^{-1}\int_{B(a,8t)}K(x,f)^{p/(n-p)}\,dx\right)\right)\right)^{(n-p)/pn}\leq\\
&C_n\left(\Lambda^{-1}\left(t^{-n}\int_{B(a,8t)}\Lambda(K(x,f)^{p/(n-p)})\,dx\right)\right)^{(n-p)/pn}=\\
&C_n\left(\Phi^{-1}\left(\log\left(t^{-n}\int_{B(a,8t)}\exp(\Phi(K(x,f)))\,dx\right)\right)\right)^{1/n}.
\end{aligned}$$

By combining this estimate with (2.3) and (2.4), we have

$$\begin{aligned}
1\ \leq\ &C_n r\Big((t^{-n}\int_{B(a,8t)}\rho^n(x)K(x,f)^{-1}\,dx\Big)^{1/n}\\
\cdot\ &\Big(\Phi^{-1}\Big(\log\Big(t^{-n}\int_{B(a,8t)}\exp(\Phi(K(x,f)))\,dx\Big)\Big)\Big)^{1/n}
\end{aligned}$$

for some $r\in(t,8t)$. Since $r/t<8$, we can remove the t^{-n} in front of the first integral in order to have

$$\begin{aligned}
&\int_{B(a,8t)}\rho^n(x)K(x,f)^{-1}\,dx\geq\\
&C_n\Big(\Phi^{-1}\Big(\log\Big((8t)^{-n}\int_{B(a,8t)}\exp(\Phi(K(x,f)))\,dx\Big)\Big)\Big)^{-1},
\end{aligned}$$

as desired. □

For a mapping of finite distortion $f : \Omega \to \mathbb{R}^n$ and a Borel set $E \subset \Omega$, define the counting function $n(E, y)$ by

$$n(E, y) = \sum_{x \in f^{-1}(y) \cap E} i(x, f),$$

where $i(x, f)$ is again the local index. For an $(n-1)$-dimensional sphere $S^{n-1}(y, t) \subset \mathbb{R}^n$, define the average $\nu(E, y, t)$ of the counting function $n(E, \cdot)$ over the sphere $S^{n-1}(y, t)$ by

$$\nu(E, y, t) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} n(E, y + tx)\, dx,$$

where S^{n-1} is the unit sphere and ω_{n-1} its surface measure. We shall use the notation $\nu(a, r, y, t)$ if $E = B(a, r)$.

Lemma 2.6. *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying Assumption* (1.2). *Suppose that $B(a, \theta r) \subset\subset \Omega$, $\theta \geq 6$. Moreover, let $y \in \mathbb{R}^n$ and $s, t > 0$. Then*

$$\nu(a, \theta r, y, s) \geq \nu(a, r, y, t) - \omega_{n-1}^{-1} \varphi(I_a^{\theta r}, \theta) |\log(t/s)|^{n-1}.$$

Proof. The proof of Lemma IV 1.1 in [21] can be carried out also in our situation. We only need to replace Väisälä's inequality and the estimate of the modulus of ring domains by Theorem 2.2 and estimate (2.1), respectively. □

The proof of our next lemma is similar to the proof of Lemma IV 2.3 in [21].

Lemma 2.7. *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying Assumption* (1.2). *Let E and F be two disjoint continua in $\overline{B}(a, R)$ so that $fE \subset B(z, s)$ and $fF \subset \mathbb{R}^n \setminus B(z, t)$, $s < t$. Let $\theta \geq 6$ and suppose $B(a, \theta R) \subset\subset \Omega$. Then*

$$\nu(a, \theta R, z, t) \geq \omega_{n-1}^{-1} \Big(\log \frac{t}{s} \Big)^{n-1} \Big(M_{K^{-1}(\cdot, f)}(\Gamma \setminus \Gamma_f) - \varphi(I_a^{\theta R}, \theta) \Big),$$

where Γ is the family of all paths joining E and F in $B(a, R)$.

Proof. We may assume that $z = 0$. Now we can define an admissible function ρ for $f(\Gamma \setminus \Gamma_f)$ by setting

$$\rho(y) = \begin{cases} \frac{1}{(\log \frac{t}{s})|y|} & s < |y| < t, \\ 0 & \textit{elsewhere}. \end{cases}$$

Then by Theorem 2.1 and by the definition of ν,

$$\begin{aligned} M_{K^{-1}(\cdot, f)}(\Gamma \setminus \Gamma_f) &\leq \\ \int_{\mathbb{R}^n} \rho(y)^n n(B(a, R), y)\, dy &= \frac{\omega_{n-1}}{\left(\log \frac{t}{s}\right)^n} \int_s^t \frac{\nu(a, R, 0, r)}{r}\, dr. \end{aligned}$$

By Lemma 2.6,

$$\nu(a,R,0,r) \le \nu(a,\theta R,0,t) + \omega_{n-1}^{-1}\varphi(I_a^{\theta R},\theta)\Big(\log\frac{t}{s}\Big)^{n-1}$$

for all $r \in (s,t)$, and thus

$$M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) \le \omega_{n-1}\nu(a,\theta R,0,t)\Big(\log\frac{t}{s}\Big)^{1-n} + \varphi(I_a^{\theta R},\theta).$$

□

3 Proofs of Theorems 1.3 and 1.1

Proof of Theorem 1.3. Let $\gamma : [0,1[\to \mathbb{R}^n$ be a path so that

$$\lim_{t\to 1^-}\gamma(t) = \infty \quad \text{and} \quad \lim_{t\to 1^-} f(\gamma(t)) \to b.$$

We may assume that $b = 0$. Denote $\gamma([0,1[)$ by E. Let $R > 1$ be large enough so that $f(E\cap(\mathbb{R}^n\setminus B(0,R))) \subset B(0,1)$. Now there exists a component F_R of $f^{-1}(\mathbb{R}^n\setminus\overline{B}(0,M_f(R)))$ so that $\overline{F_R}\cap S^{n-1}(0,R) \neq \emptyset$. This can be seen, for instance, by path lifting, see [21], Chapter II, Section 3 (recall that f is continuous, open and discrete). By Lemma 2.7 and Inequality (1.3) we have, for all $\theta \ge 6$,

$$\begin{aligned}\nu(0,\theta R,0,1) &\ge \omega_{n-1}^{-1}(\log M_f(R))^{n-1}\Big(M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) - \varphi(I_0^{\theta R},\theta)\Big)\\ &\ge \omega_{n-1}^{-1}(\log M_f(R))^{n-1}\Big(M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) - \varphi(A,\theta)\Big),\end{aligned}$$

where Γ is the family of all paths joining E and F_R in $B(0,\theta R)\setminus\overline{B}(0,R)$. By applying Theorem 2.5 repeteadly in annuli $B(0,8R)\setminus\overline{B}(0,R)$, $B(0,64R)\setminus\overline{B}(0,8R)$ and so on, we may choose $\theta = \theta_f$ depending only on n, Φ and the constant A in Inequality (1.3), so that (recall that $\varphi(A,\theta)\to 0$ as $\theta\to\infty$)

$$M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) - \varphi(A,\theta) \ge 2\varphi(A,6).$$

Hence

$$\omega_{n-1}\nu(0,\theta R,0,1) \ge 2\varphi(A,6)(\log M_f(R))^{n-1} \tag{3.1}$$

for our choice of θ. On the other hand, by Lemma 2.6,

$$\begin{aligned}0 &= \nu(0,6\theta R,0,M_f(6\Theta R))\\ &\ge \nu(0,\theta R,0,1) - \omega_{n-1}^{-1}\varphi(I_0^{6\theta R},6)(\log M_f(6\Theta R))^{n-1},\end{aligned}$$

and so

$$(\log M_f(6\theta R))^{n-1} \ge \frac{\omega_{n-1}\nu(0,\theta R,0,1)}{\varphi(I_0^{6\theta R},6)} \ge \frac{\omega_{n-1}\nu(0,\theta R,0,1)}{\varphi(A,6)}. \tag{3.2}$$

Combining the estimates (3.1) and (3.2), we obtain

$$(\log M_f(6\theta R))^{n-1} \ge 2(\log M_f(R))^{n-1}. \tag{3.3}$$

By iterating (3.3), we obtain the estimate

$$\lambda_f \geq \frac{\log 2}{\log 6\theta}.$$

The proof is complete. $\square$

Proof of Theorem 1.1. Let f be as in Theorem 1.1, and suppose that f omits q points $\{a_1, \ldots, a_q\}$. We may assume that $\min_{a_i \neq a_j} |a_i - a_j| = 2$. Choose R large enough, so that $M_f(R) \leq 2M_{f_i}(R) \leq 3M_f(R)$ for all $i = 1, \ldots, q$, where f_i is defined by $f_i(x) := f(x) - a_i$. As in the proof of Theorem 1.3, we see that there exists a $\theta > 1$ depending only on n, Φ and the constant A in Inequality (1.3), so that

$$\nu(0, \theta R, a_i, 1) \geq (\log M_f(R))^{n-1} \quad \text{for all} \quad i = 1, \ldots, q, \tag{3.4}$$

when R is large enough. Notice that the omitted values of f are also asymptotic values (here we use the fact that f can omit at most a set of zero conformal modulus). Set

$$m_i(r) = \min_{x \in \overline{B}(0,r)} |f_i(x)|, \quad i = 1, \ldots, q.$$

By Lemma 2.6, we have

$$\begin{aligned} 0 &= \nu(0, 6\theta R, a_i, m_i(6\theta R)) \\ &\geq \nu(0, \theta R, a_i, 1) - \omega_{n-1}^{-1} \varphi(I_0^{6\theta R}, 6) \Big(\log \frac{1}{m_i(6\theta R)} \Big)^{n-1}, \end{aligned}$$

i.e.

$$\varphi(I_0^{6\theta R}, 6) \Big(\log \frac{1}{m_i(6\theta R)} \Big)^{n-1} \geq \omega_{n-1} \nu(0, \theta R, a_i, 1) \tag{3.5}$$

for each $i \in 1, \ldots, q$. Now, since f omits a_i, there exists for each i an unbounded component F_i of $f^{-1}(B(a_i, m_i(6\theta R)))$ so that $\overline{F_i}$ meets $\overline{B}(0, 6\theta R)$. Consider the sphere $S^{n-1}(0, 24\theta R)$. Then

$$\frac{1}{\theta R} \min_{i \neq j} d(\overline{F_j} \cap S^{n-1}(0, 24\theta R), \overline{F_i} \cap S^{n-1}(0, 24\theta R)) \leq C(q, n), \tag{3.6}$$

where $C(q,n)$ depends only on q and n, and $C(q,n) \to 0$ as $q \to \infty$. Let F_j and F_k be two components that minimize the distance in (3.6), and consider the mapping f_j. We claim that

$$M_{K^{-1}(\cdot, f)}(\Gamma \setminus \Gamma_f) \geq C(n, \Phi, A, C(q,n)) =: C_1,$$

where Γ is the family of all paths joining F_j and F_k in $B(0, 48\theta R)$ and

$$C_1 \to \infty \quad \text{as} \quad q \to \infty.$$

By (3.6) there exists, for a fixed $q \geq 2$, an annulus $B(z, T) \setminus \overline{B}(z, t) \subset B(0, 48\theta R)$ so that $T \geq 16\theta R$, $t \to 0$ as $C(q,n) \to 0$ and so that both

$\overline{F_j}$ and $\overline{F_k}$ intersect $S^{n-1}(z,r)$ for all $r \in (t,T)$. Notice that $I_z^T \leq 3^n A$. We apply Theorem 2.5 to annuli

$$B(z,T)\backslash\overline{B}(z,T/8),\ B(z,T/8)\backslash\overline{B}(z,T/64),\ \ldots, B(z,8^{1-p}T)\backslash\overline{B}(z,8^{-p}T),$$

where p is the largest integer so that $8^{-p}T \geq t$. Now we have

$$M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f) \geq \sum_{m=1}^{p} M_{K^{-1}(\cdot,f)}(\Gamma_m \setminus \Gamma_f) \geq \sum_{m=1}^{p} \eta(I_z^{8^{-m+1}T}) =: S_p,$$

where Γ_m is the family of all paths joining $\overline{F_j}$ and $\overline{F_k}$ in $B(z,8^{1-m}T)\setminus \overline{B}(z,8^{-m}T)$, and where the η:s are as in (2.2). To prove the claim we need to show that $S_p \to \infty$ as $p \to \infty$. We have

$$\eta(I_z^{8^{-m+1}T}) \geq C_n(\Phi^{-1}(\log(C_n' A 8^{mn})))^{-1} \geq C_n(\Phi^{-1}(\log(8^{mD})))^{-1}$$

for some $D > n$ depending only on n and A, and so

$$\eta(I_z^{8^{-m+1}T}) \geq C_n(\Phi^{-1}(m\log(8^D)))^{-1}. \tag{3.7}$$

By the change of variables $t = \Phi^{-1}(s)$ in part $(\Phi - 1)$ of Assumption (1.2), we see that

$$\infty = \int_1^\infty \frac{\Phi'(t)}{t}\,dt = \int_b^\infty \frac{ds}{\Phi^{-1}(s)},$$

which further implies that

$$\sum_{m=1}^{\infty} \frac{1}{\Phi^{-1}(m\log 8^D)} = \infty.$$

Hence the estimate (3.7) shows that $S_p \to \infty$ when $p \to \infty$. This proves the claim.

Application of Lemma 2.7 now gives (note that $B(a_k,1)\cap B(a_j,1) = \emptyset$)

$$\begin{aligned}\nu(0,288\theta R,a_j,1) &\geq \omega_{n-1}^{-1}\Big(\log\frac{1}{m_j(6\theta R)}\Big)^{n-1}\Big(C_1 - \varphi(I_0^{288\theta R},6)\Big)\\ &\geq \omega_{n-1}^{-1}\Big(\log\frac{1}{m_j(6\theta R)}\Big)^{n-1}\Big(C_1 - \varphi(A,6)\Big) \qquad (3.8)\\ &\geq \frac{1}{2\omega_{n-1}}\Big(\log\frac{1}{m_j(6\theta R)}\Big)^{n-1}C_1,\end{aligned}$$

when q is large enough. On the other hand, we use Lemma 2.6 again in order to have

$$\begin{aligned}0 &= \nu(0,17280\theta R,a_j,M_{f_j}(17280\theta R))\\ &\geq \nu(0,288\theta R,a_j,1) - \omega_{n-1}^{-1}\varphi(A,6)\Big(\log M_{f_i}(17280\theta R)\Big)^{n-1},\end{aligned}$$

i.e.

$$\nu(0,288\theta R,a_j,1) \leq \omega_{n-1}^{-1}\varphi(A,6)\Big(\log M_{f_i}(17280\theta R)\Big)^{n-1}. \tag{3.9}$$

Combining (3.4), (3.5), (3.8) and (3.9), we have

$$(\log M_f(1728\theta R))^{n-1} \geq C(n, \Phi, A, q)(\log M_f(R))^{n-1},$$

where $C(n, \Phi, A, q) \to \infty$ as $q \to \infty$. This contradicts Inequality (1.4) when q is large enough compared to the other data. The proof is complete. $\square$

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