THE POINCARÉ INEQUALITY IS AN OPEN ENDED CONDITION

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ABSTRACT. Let $p > 1$, and let (X, d, μ) be a complete metric measure space that admits a $(1, p)$-Poincaré inequality, with μ a doubling Borel regular measure. Then there exists $\epsilon > 0$ such that (X, d, μ) admits a $(1, q)$-Poincaré inequality for every $q > p - \epsilon$, quantitatively.

1. INTRODUCTION

Possibly the most famous example of a self-improving inequality is the reverse Hölder inequality, and was discovered by Gehring ([8]). This example of self-improvement can be vaguely summarized by saying that if a function is good on all balls, as expressed by an integrability condition, then the function is actually a little bit better on all balls than we first thought. In this paper we present an analogous, albeit vastly different, result for metric measure spaces. We show that if the geometry of a metric measure space is well behaved at all locations and scales, as expressed by a Poincaré inequality, then it too is better at all locations and scales than we first thought. We now state this result precisely using terminology explained in Section 2; we then provide several applications, and give an example that demonstrates certain sharpness of the result.

Theorem 1.0.1. *Let $p > 1$, and let (X, d, μ) be a complete metric measure space that admits a $(1, p)$-Poincaré inequality, with μ a doubling Borel regular measure. Then there exists $\epsilon > 0$ such that (X, d, μ) admits a $(1, q)$-Poincaré inequality for every $q > p - \epsilon$, quantitatively.*

There are many natural and exotic examples of metric measure spaces that support a Poincaré inequality; an extensive list can be found in [21, 24]. We emphasize, however, that the results of this paper are not only new in the case of abstract metric measure spaces. Rather, they are also new in the case of Riemannian manifolds, graphs, and measures on Euclidean space that are doubling and admit a Poincaré inequality. Poincaré inequalities have been widely studied in all such settings.

For example, weights on Euclidean space that, when integrated against, give rise to doubling measures that support a $(1, p)$-Poincaré inequality, $p \geq 1$, are known as *p-admissible weights*, and are particularly pertinent in the study of the nonlinear potential theory of degenerate elliptic equations; see [16, 6] (the fact that above definition for p-admissible weights coincides with the one given in [16] is proven in [13]). It is known that the A_p weights of Muckenhoupt, are p-admissible for each $p \geq 1$ (see [16, Chapter 15]). However, the converse is not generally true for any $p \geq 1$ (see [16, p. 10], and also the discussion following [23, Theorem 1.3.10]).

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Nonetheless, we see from the following corollary to Theorem 1.0.1, that p-admissible weights display the same open ended property of Muckenhoupt's A_p weights.

Corollary 1.0.2. *Let $p > 1$ and let w be a p-admissible weight in $\mathbf{R}^n$, $n \geq 1$. Then there exists $\epsilon > 0$ such that w is q-admissible for every $q > p - \epsilon$, quantitatively.*

For complete Riemannian manifolds, Saloff-Coste ([34, 35]) established that supporting a doubling measure and a $(1,2)$-Poincaré inequality is equivalent to admitting the parabolic Harnack inequality, quantitatively (Grigor′yan ([10]) also independently established that the former implies the latter). The latter condition was further known to be equivalent to Gaussian-like estimates for the heat kernel, quantitatively (see for example [35]). Thus by Theorem 1.0.1 we see that each of these conditions is also equivalent to supporting a doubling measure and a $(1, 2-\epsilon)$-Poincaré inequality for some $\epsilon > 0$, quantitatively. We remark that relations between $(1,2)$-Poincaré inequalities, heat kernel estimates, and parabolic Harnack inequalities have been established in the setting of Alexandrov spaces by Kuwae, Machigashira, and Shioya ([32]), and in the setting of complete metric measure spaces that support a doubling Radon measure, by Sturm ([40]).

Heinonen and Koskela ([17, 18, 19], see also [15]) developed the notion of a Poincaré inequality and the Loewner condition for general metric measure spaces; the latter being a generalization of a condition, proved by Loewner ([33]) for Euclidean space, that quantitatively describes metric measure spaces that possess a plentitude of rectifiable curves. Heinonen and Koskela demonstrated that quasiconformal homeomorphisms (the definition of which is given through an infinitesimal metric inequality) display certain global rigidity (that is, are quasisymmetric), when mapping between Loewner metric measure spaces with certain upper and lower measure growth restrictions on balls. They further showed that metric measure spaces with certain upper and lower measure growth restrictions on balls, specifically, Ahlfors α-regular metric measure spaces, $\alpha > 1$, are Loewner if and only if they admit a $(1,\alpha)$-Poincaré inequality, quantitatively. By Theorem 1.0.1 we see then that the following holds:

Theorem 1.0.3. *A complete Ahlfors α-regular metric measure space, $\alpha > 1$, is Loewner if and only if it supports a $(1, \alpha - \epsilon)$-Poincaré inequality for some $\epsilon > 0$, quantitatively.*

We now briefly consider a consequence of the above theorem in Gromov hyperbolic geometry. Laakso and the first author ([25]) demonstrated that complete Ahlfors α-regular metric measure spaces, $\alpha > 1$, can not have their Assouad dimension lowered through quasisymmetric mappings if and only if they possess at least one weak-tangent that contains a collection of non-constant rectifiable curves with positive p-modulus, for some or any $p \geq 1$. There is no need here to pass to weak tangents for complete metric measure spaces that are sufficiently rich in symmetry. Indeed, Bonk and Kleiner ([3]) have shown that such metric measure spaces, as described above, that can be identified as the boundary of a Gromov hyperbolic group, are then Loewner. By Theorem 1.0.3 we see that such metric measure spaces further admit a $(1, \alpha - \epsilon)$-Poincaré inequality for some $\epsilon > 0$, quantitatively.

We now apply Theorem 1.0.3 to obtain a new result for quasiconformal and quasisymmetric homeomorphisms between general metric measure spaces. Heinonen and Koskela [19, Theorem 7.11] have shown that pullback measure of a quasisymmetric homeomorphism, from a complete Ahlfors α-regular metric measure space that supports a p-Poincaré inequality, to a complete Ahlfors α-regular metric space, is an A_∞ weight in the sense of Muckenhoupt if $1 \leq p < \alpha$, quantitatively. This extended classical results of Bojarski ([2]) in $\mathbf{R}^2$ and Gehring ([9]) in $\mathbf{R}^n$, $n \geq 3$. For the critical case, that is, when $p = \alpha$, Heinonen, Koskela, Shanmugalingam, and Tyson ([20, Corollary 8.15]) showed that a quasisymmetric homeomorphism, from a complete Ahlfors α-regular Loewner metric measure space to a complete Ahlfors α-regular metric space, is absolutely continuous with respect to α-Hausdorff measure. This left open the question of whether the given quasisymmetric homeomorphism actually induces an A_∞ weight. By applying Theorem 1.0.3 in conjunction with [19, Theorem 7.11] we see that it does. We now state this formally.

Theorem 1.0.4. *Let (X, d, μ) and (Y, l, ν) be complete Ahlfors α-regular metric measure spaces, $\alpha > 1$, with (X, d, μ) Loewner, and let $f : X \longrightarrow Y$ be a quasisymmetric homeomorphism. Then the the pullback $f^*\nu$ of ν by f is A_∞ related to μ, quantitatively. Consequently there exists a measurable function $w : X \longrightarrow [0, \infty)$ such that $df^*\nu = w d\mu$, and such that*

$$\left(\fint_B w^{1+\epsilon} \, d\mu \right)^{1/(1+\epsilon)} \leq C \fint_B w \, d\mu,$$

for every ball B in X, quantitatively.

We now consider further applications of Theorem 1.0.1 to nonlinear potential theory, but this time with an emphasis on general metric measure spaces. There are several papers on nonlinear potential theory where the authors make the standing hypotheses that a given measure on $\mathbf{R}^n$ is q-admissible, or that a given metric measure space is complete and supports a doubling Borel regular measure and a q-Poincaré inequality, for some $1 < q < p$, where p is the "critical dimension" of analysis. This includes papers by Björn, MacManus, and Shanmugalingam ([1]), and Martio and Kinnunen ([27, 28]). It follows by Theorem 1.0.1 that in each of these cases, the standing assumption can be replaced by the *a priori* weaker assumption that the given complete metric measure space supports a doubling Borel regular measure and a p-Poincaré inequality. Also, Kinnunen and Shanmugalingam ([29]) have shown in the setting of metric measure spaces that support a doubling Borel regular measure and a $(1, q)$-Poincaré inequality (in the sense of Heinonen and Koskela in [19]; see Section 1.2), that quasiminimizers of p-Dirichlet integrals satisfy Harnack's inequality, the strong maximum principle, and are locally Hölder continuous, if $1 < q < p$. Here, unlike the cases mentioned above, in order to apply Theorem 1.0.1 and weaken the hypotheses concerning the Poincaré inequality, we need to impose the additional hypothesis that the given metric measure space is complete. We then obtain the following:

Theorem 1.0.5. *Quasiminimizers of p-Dirichlet integrals, on complete metric measure spaces that support a Borel doubling Borel regular measure and a $(1, p)$-Poincaré inequality, $p > 1$, satisfy Harnack's inequality, the strong maximum principle, and are locally Hölder continuous, quantitatively.*

We now consider applications of Theorem 1.0.1 to the theory of Sobolev spaces on general metric measure spaces. Alternate definitions for Sobolev-type spaces on metric measure spaces have been introduced by a variety of authors. Here we consider the Sobolev space $H_{1,p}(X)$, $p \geq 1$, introduced by Cheeger in [4], the Newtonian space $N^{1,p}(X)$ introduced by Shanmugalingam in [39], and the Sobolev space

$M^{1,p}(X)$ introduced by Hajłasz in [11] (we have used the same notation as the respective authors, and do not intend to recall the definitions of these Sobolev-type spaces). It is known that in general this last Sobolev-type space does not always coincide with the former two; see [39, Examples 6.9 and 6.10]. Nonetheless, Shanmugalingam has shown that $H_{1,p}(X)$ is isometrically equivalent in the sense of Banach spaces to $N^{1,p}(X)$, for $p > 1$, whenever the underlying measure is Borel regular; and furthermore, that all of the above three spaces are isomorphic as Banach spaces whenever the given metric measure space X supports a doubling Borel regular measure and a $(1, q)$-Poincaré inequality for some $1 \leq q < p$ (in the sense of Heinonen and Koskela in [19]), quantitatively; see [39, Theorem 4.9 and 4.10]. By Theorem 1.0.1 we see then that the following holds:

Theorem 1.0.6. *Let X be a complete metric measure space that supports a doubling Borel regular measure and a $(1,p)$-Poincaré inequality, $p > 1$. Then $H_{1,p}(X)$, $M^{1,p}(X)$, and $N^{1,p}(X)$ are isomorphic, quantitatively.*

1.1. **Self-improvement for pairs of functions.** One might be tempted to hope that results analogous to Theorem 1.0.1 hold for pairs of functions that are linked by Poincaré type inequalities, regardless of whether the given metric measure space supports a Poincaré inequality. Indeed, Hajłasz and Koskela [14, p. 19] have asked if given $u, g \in L^p(X)$ that satisfy a $(1,p)$-Poincaré inequality, where $p > 1$ and (X, d, μ) is a metric measure space with μ a doubling Borel regular measure, whether the pair u, g also satisfy a $(1, q)$-Poincaré inequality for some $1 \leq q < p$. Here, a pair u, g is said to *satisfy a $(1, q)$-Poincaré inequality*, $q \geq 1$, if there exist $C, \lambda \geq 1$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,\lambda r)} g^q \, d\mu \right)^{1/q}, \tag{1}$$

for every $x \in X$ and $r > 0$. The next proposition demonstrates that the answer to this question is no.

Proposition 1.1.1. *There exists an Ahlfors 1-regular metric measure space such that for every $p > 1$, there exists a pair of functions $u, g \in L^p(X)$ and constants $C, \lambda \geq 1$ such that (1) holds with $q = p$ for every $x \in X$ and $r > 0$, and such that there does not exist $C, \lambda \geq 1$ such that (1) holds with $q < p$ for every $x \in X$ and $r > 0$.*

1.2. **A note on the various definitions of a Poincaré inequality.** There are several different definitions for a Poincaré inequality on a metric measure space, that is, definitions that might not necessarily hold for every metric measure space, but that still make sense for every metric measure space. This partly arises in this general setting because the notion of a gradient of a function is not always easily defined, and because it is not clear which class of functions the inequality should be required to hold for. Nonetheless, most reasonable definitions coincide when the metric measure space is complete and supports a doubling Borel regular measure. In particular, the definitions of Heinonen and Koskela in [19], Semmes in [38, Section 2.3], and several other definitions of the first author, including the definition adopted here (Definition 2.2.1), all coincide in this case; some of this is shown by the first author in [22, 23], the rest is shown by Rajala and the first author in [26].

Theorem 1.0.1 would not generally be true if we removed the hypothesis that the given metric measure space is complete, although, this depends on which definition is used for the Poincaré inequality. In particular, it would not generally be true if one used the definition of Heinonen and Koskela in [19]. For each $p > 1$, an example demonstrating this is given by Koskela in [30], consisting of an open set Ω in Euclidean space endowed with the standard Euclidean metric and Lebesgue measure. The main reason that our proof fails in that setting (as it should) is that Lipschitz functions, and indeed any subspace of the Sobolev space $W^{1,p}(\Omega)$ contained in $W^{1,q}(\Omega)$, is not dense in $W^{1,q}(\Omega)$ for any $1 \leq q < p$. (Here $W^{1,r}(\Omega)$, $r \geq 1$, is the completion of the real-valued smooth functions defined on Ω, under the norm $\|\cdot\|_{1,r}$ given by $\|u\|_{1,r} = \|u\|_r + \|\,|\nabla u|\,\|_r$.) Indeed, our proof works at the level of Lipschitz functions, and establishes the self-improvability of the Poincaré inequality only for Lipschitz functions. In the case when the metric measure space is complete and supports a doubling Borel regular measure, we can then appeal to results of the first author and Rajala ([26]), and the first author ([22, 23]), to recover the improved Poincaré inequality for all functions.

The definition adopted in this paper for the Poincaré inequality (Definition 2.2.1) is preserved under taking the completion of the metric measure space, and still holds if one removes any null set with dense complement. Consequently, the assumption in Theorem 1.0.1 that the given metric measure space is complete, is superfluous. We merely included it so as not to cause the reader confusion when comparing against other papers that use a different definition for the Poincaré inequality. Also, the reader may be concerned that we only consider $(1,p)$-Poincaré inequalities in this paper, and not (q,p)-Poincaré inequalities for $q > 1$, that is, where the L^1 average on the left is replaced by an L^q average (see Definition 2.2.1). Our justification for doing this comes from the fact, as proven by Hajłasz and Koskela [14], that a metric measure space that supports a doubling Borel regular measure and a $(1,p)$-Poincaré inequality, $p \geq 1$, then also supports the *a priori* stronger (q,p)-Poincaré inequality, for some $q > p$, quantitatively.

1.3. **Outline.** In Section 2 we recall terminology and known results. The proof of Theorem 1.0.1 is contained in Section 3. Any further discussion required for Theorems 1.0.3, 1.0.4, 1.0.5 and 1.0.6 can be found in Section 4. The proof of Proposition 1.1.1 is also contained in Section 4.

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2. Terminology and standard lemmas

In this section we recall standard definitions and results needed for the proof of Theorem 1.0.1. With regard to language, when we say that a claim holds *quantitatively*, as in Theorem 1.0.1, we mean that the new parameters of the claim depend only on the previous parameters implicit in the hypotheses. For example, in Theorem 1.0.1 we mean that ϵ and the constants associated with the $(1,q)$-Poincaré inequality depend only on the constant p, the doubling constant of μ, and the constants associated with the assumed $(1,p)$-Poincaré inequality. When we say that two positive reals x, y are *comparable* with constant $C > 0$, we mean that $x/C \leq y \leq Cx$. We use $\chi|_W$ to denote the characteristic function on any set W.

2.1. **Metric measure spaces, doubling measures, and Lip.** In this paper (X, d, μ) denotes a metric measure space and μ is always Borel regular. We will use the notation $|E|$ to denote the μ-measure of any measurable set $E \subset X$. The ball with center $x \in X$ and radius $r > 0$ is denoted by

$$B(x, r) = \{y \in X : d(x, y) < r\},$$

and we use the notation

$$\lambda B(x, r) = \{y \in X : d(x, y) < \lambda r\},$$

whenever $\lambda > 0$. We write

$$u_A = \frac{1}{|A|} \int_A u \, d\mu = \fint_A u \, d\mu,$$

for every $A \subset X$ and measurable function $u : X \longrightarrow [-\infty, \infty]$. The measure μ is said to be *doubling* if there is a constant $C > 0$ such that

$$|B(x, 2r)| \leq C|B(x, r)|,$$

for every $x \in X$ and $r > 0$.

Lemma 2.1.1. [15, pp. 103–104] *Let (X, d, μ) be a metric measure space with μ doubling. Then there exists constants $C, \alpha > 0$ that depend only on the doubling constant of μ, such that*

$$\frac{|B(y, r)|}{|B(x, R)|} \geq C \left(\frac{r}{R}\right)^\alpha,$$

whenever $0 < r < R < \operatorname{diam} X$, $x \in X$, and $y \in B(x, R)$.

A function $u : X \longrightarrow \mathbf{R}$ is said to be *L-Lipschitz*, $L \geq 0$, if

$$|u(x) - u(y)| \leq Ld(x, y),$$

for every $x, y \in X$. We often omit mention of the constant L and just describe such functions as being Lipschitz. Given a Lipschitz function $u : X \longrightarrow \mathbf{R}$ and $x \in X$, we let

$$\operatorname{Lip} u(x) = \sup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}.$$

The following lemma can be easily deduced from Lemma 2.1.1; compare with the proof of [22, Proposition 3.2.3].

Lemma 2.1.2. *Let (X, d, μ) be a metric measure space with μ doubling, and let f and g be real-valued Lipschitz functions defined on X. Then* $\operatorname{Lip} f = \operatorname{Lip} g$ *almost everywhere on the set where $f = g$.*

2.2. **The Poincaré inequality and geodesic metric spaces.** We can now state the definition for the Poincaré inequality on metric measure spaces to be used in this paper; see also Section 1.2.

Definition 2.2.1. A metric measure space (X, d, μ) is said to admit a $(1, p)$-*Poincaré inequality*, $p \geq 1$, with constants $C, \lambda \geq 1$, if the following holds: Every ball contained in X has measure in $(0, \infty)$, and we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,\lambda r)} (\operatorname{Lip} u)^p \, d\mu\right)^{1/p}, \tag{2}$$

for every $x \in X$ and $0 < r < \operatorname{diam} X$, and for every Lipschitz function $u : X \longrightarrow \mathbf{R}$.

If (X, d, μ) is complete and supports a $(1, p)$-Poincaré inequality and a doubling measure, then (X, d, μ) is bi-Lipschitz to a geodesic metric space, quantitatively; see [23, Proposition 6.0.7]. We briefly recall what these words mean and refer to [15] for a more thorough discussion. A metric space is *geodesic* if every pair of distinct points can be connected by a path with length equal to the distance between the two points. A map $f : Y_1 \longrightarrow Y_2$ between metric spaces (Y_1, ρ_1) and (Y_2, ρ_2) is *L-bi-Lipschitz*, $L > 0$, if for every $x, y \in Y_1$ we have

$$\frac{1}{L}\rho_1(x,y) \leq \rho_2(f(x), f(y)) \leq L\rho_1(x,y).$$

Two metric spaces are said to be *L-bi-Lipschitz*, or just bi-Lipschitz, if there exists a surjective L-bi-Lipschitz map between them.

One advantage of working with geodesic metric spaces is that if (X, d, μ) is a geodesic metric space that supports a doubling measure and a $(1, p)$-Poincaré inequality, then (X, d, μ) admits a Poincaré inequality with $\lambda = 1$ in (2.2.1), but possibly a different constant $C > 0$, quantitatively; see [15, Theorem 9.5]. The next lemma extends this result.

Lemma 2.2.2. *Let (X, d, μ) be a geodesic metric space, with μ doubling, that admits a $(1, p)$-Poincaré inequality. Then every ball in X, viewed as a metric subspace, admits a $(1, p)$-Poincaré inequality, quantitatively.*

Proof. This is quite standard and so we only give an outline. Let B be a ball in X. Since (X, d) is geodesic, the following holds whenever $x \in B$ is contained in some ball B_1 of the metric subspace B (that is, B_1 is the restriction to B of a ball in X centered in B) : There exists a chain of balls $\{B_i : i = 2, 3, \dots\}$ in X with $B_i \subset B$ for $i \geq 2$, such that for every $i \in \mathbf{N}$, we have

$$2^{-i-2} \operatorname{diam} B \leq \operatorname{diam} B_i \leq 2^{-i+2} \operatorname{diam} B,$$

and the intersection $B_i \cap B_{i+1}$ contains a ball F in X such that $B_i \cup B_{i+1} \subset 10F$. One can now proceed as in the proof of [15, Theorem 9.5]. $\square$

Another convenient property of geodesic metric spaces is that the measure of points sufficiently near the boundary of any ball is small. This claim is made precise by the following result that appears as Proposition 6.12 in [4], where it is accredited to Colding and Minicozzi II [5].

Proposition 2.2.3. *Let (X, d, μ) be a geodesic metric measure space with μ doubling. Then there exists $C > 0$ that depends only on the doubling constant of μ such that*

$$|B(x,r) \setminus B(x,(1-\delta)r)| \leq \delta^C |B(x,r)|,$$

for every $x \in X$ and $\delta, r > 0$.

2.3. **Maximal type operators.** Given a Lipschitz function $u : X \longrightarrow \mathbf{R}$ and $x \in X$, we set

$$M^{\#}u(x) = \sup_B \frac{1}{r} \fint_B |u - u_B| \, d\mu,$$

for every $x \in X$, where the supremum is taken over all $y \in X$ and $r > 0$ such that $B = B(y, r)$ contains x. This sharp fractional maximal operator should not be confused with the uncentered Hardy-Littlewood maximal operator which we denote by

$$Mu(x) = \sup_B \fint_B |u| \, d\mu,$$

for every $x \in X$, where the supremum is taken over all balls B that contain x. The following lemma is folk-lore; a similar proof to a similar fact can be found in [15, p. 73].

Lemma 2.3.1. *Let (X, d, μ) be a metric measure space with μ doubling, and let $u : X \longrightarrow \mathbf{R}$ be Lipschitz. Then there exists $C > 0$ that depends only on the doubling constant of μ such that*

$$|u(x) - u_{B(y,r)}| \le CrM^{\#}u(x),$$

whenever $r > 0$, $y \in X$, and $x \in B(y, r)$. Consequently, the restriction of u to

$$\{x \in X : M^{\#}u(x) \le \lambda\}$$

is $2C\lambda$-Lipschitz.

3. Proof of Theorem 1.0.1

3.1. **The first reduction.** We begin the proof of Theorem 1.0.1 with several reductions. Let (X, d, μ) be a metric measure space, with μ doubling, that admits a $(1, p)$-Poincaré inequality, $p > 1$. In what follows we let $C > 0$ denote a *varying constant* that depends only on the data associated with the assumed $(1, p)$-Poincaré inequality, the doubling constant of μ, and p. This means that C denotes a positive variable, whose value may vary between each usage, but is then fixed, and depends only on the data outlined above.

Observe that our given definition for the $(1, p)$-Poincaré is invariant under taking the completion of the metric measure space, and under removing null sets with dense complement. Thus our assumption in Theorem 1.0.1, that the given metric space is complete, is not necessarily; see Section 1.2. Nonetheless, by taking the completion, we assume from now on that (X, d) is complete. As explained in Section 2.2, under our given hypotheses, we then have (X, d) is bi-Lipschitz to a geodesic metric space. Now, the property of admitting a $(1, q)$-Poincaré inequality, $q \ge 1$, is quantitatively preserved under bi-Lipschitz maps; here we take the measure on the image of the map to be the push forward measure. Thus, without loss of generality, assume that (X, d) is geodesic.

Fix a ball B in X and a Lipschitz function $u : X \longrightarrow \mathbf{R}$. To prove Theorem 1.0.1 it suffices to show that there exists $\epsilon > 0$ such that

$$\fint_B |u - u_B| \, d\mu \le C(\operatorname{diam} B) \left(\fint_B (\operatorname{Lip} u)^{p-\epsilon} \, d\mu \right)^{\frac{1}{p-\epsilon}}. \tag{3}$$

The claim of Theorem 1.0.1 then follows by Hölder's inequality. Without loss of generality, due to the scale invariance of the Poincaré inequality, assume that B has unit diameter and measure. By Lemma 2.2.2 we have that B admits a $(1, p)$-Poincaré inequality, quantitatively. Therefore, without loss of generality, by another bi-Lipschitz change of metric, and by passing to the completion of B, we can further assume that $B = X$. (The bi-Lipschitz change of metric is needed to ensure that we maintain the property that (X, d) is geodesic.)

The next proposition contains the main part of the proof of Theorem 1.0.1. In order to state the proposition we first let

$$U_\lambda = \{x \in X : M^\# u(x) > \lambda\},$$

for every $\lambda \geq 0$. We always consider U_λ to depend on u, so that if we re-scale u, then U_λ will be given by the formula above with the new u.

Proposition 3.1.1. *For every $\alpha \in \mathbf{N}$, there exists $k \in \mathbf{N}$ that depend only on C and α, such that*

$$\begin{aligned} |U_\lambda| \leq & 2^{kp-\alpha}|U_{2^k\lambda}| + 8^{kp-\alpha}|U_{8^k\lambda}| \\ & + 8^{k(p+1)}|\{x \in X : \operatorname{Lip} u(x) > 8^{-k}\lambda\}| \end{aligned} \tag{4}$$

for every $\lambda > 0$.

We will postpone the proof of the above proposition till later, and instead now use it to deduce Theorem 1.0.1. Let $\alpha = 3$ and then let $k \in \mathbf{N}$ be as given by Proposition 3.1.1. Choose $0 < \epsilon < p - 1$ so that $8^{k\epsilon} < 2$. We now integrate (4) against the measure $d\lambda^{p-\epsilon}$, and over the range $(0, \infty)$, to obtain

$$\begin{aligned} \int_0^\infty |U_\lambda|\, d\lambda^{p-\epsilon} \leq & 2^{k\epsilon-3} \int_0^\infty |U_{2^k\lambda}|\, d(2^k\lambda)^{p-\epsilon} \\ & + 8^{k\epsilon-3} \int_0^\infty |U_{8^k\lambda}|\, d(8^k\lambda)^{p-\epsilon} \\ & + 8^{k(p+1)} \int_0^\infty |\{x \in X : \operatorname{Lip} u(x) \geq 8^{-k}\lambda\}|\, d\lambda^{p-\epsilon}. \end{aligned}$$

Consequently, we have

$$\int_X (M^\# u)^{p-\epsilon}\, d\mu \leq \frac{8^{k\epsilon}}{3} \int_X (M^\# u)^{p-\epsilon}\, d\mu + C \int_X (\operatorname{Lip} u)^{p-\epsilon}\, d\mu,$$

and therefore, by our choice of ϵ, we have

$$\int_X (M^\# u)^{p-\epsilon}\, d\mu \leq C \int_X (\operatorname{Lip} u)^{p-\epsilon}\, d\mu.$$

This then implies (3); to see this observe that left hand side of the above equation trivially dominates the left-hand side of (3).

Thus to complete the proof of Theorem 1.0.1, it remains to prove Proposition 3.1.1 in the case when (X, d, μ) is a complete, geodesic metric measure space, with unit diameter and measure, that admits a $(1, p)$-Poincaré inequality. (The reader may wonder why we stated Proposition 3.1.1 with general $\alpha \in \mathbf{N}$ even though we only use $\alpha = 3$. Our justification for doing this is that it makes the proof of Proposition 3.1.1 easier; in particular, it allows us to keep track of accumulating constants in a more relaxed manner, especially during the proof of Lemma 3.2.1 below.)

3.2. **The second reduction.** We proceed with another reduction.

Lemma 3.2.1. *In order to prove Proposition 3.1.1, we can, without loss of generality, further assume that $\lambda = 1$, and that*

$$\int_X |u - u_X|\, d\mu \geq 1. \tag{5}$$

Proof. We will now assume that Proposition 3.1.1 holds for any metric measure space under the assumptions given in and before Lemma 3.2.1, and then use this to prove Proposition 3.1.1 in its entirety.

Fix $\lambda > 0$ and suppose without loss of generality that $U_\lambda \neq \emptyset$. Then let $\mathcal{F}$ be the collection of balls B in X such that

$$\lambda \leq \frac{1}{\operatorname{diam} B} \fint_B |u - u_B| \, d\mu \leq C\lambda. \tag{6}$$

We can choose the constant $C \geq 1$ here so that $\mathcal{F}$ is a cover of U_λ (this usage of C depends only on the doubling constant of μ). Indeed, for any point $x \in U_\lambda$, there is a ball B such that

$$\lambda < \frac{1}{\operatorname{diam} B} \fint_B |u - u_B| \, d\mu.$$

If

$$\frac{1}{\operatorname{diam} 2B} \fint_{2B} |u - u_{2B}| \, d\mu \leq \lambda,$$

then we let B be in the collection. Otherwise we consider $4B$ and so on, and eventually find a ball containing x, and such that the above two inequalities hold; and therefore such that (6) holds.

We use the fact that u has bounded support to deduce that the members of $\mathcal{F}$ have uniformly bounded diameter. This allows us to apply a standard covering argument ([15, Theorem 1.2]) and so obtain a countable subcollection $\{B_i\}_{i \in I}$ of $\mathcal{F}$, consisting of mutually disjoint balls in X such that $U_\lambda \subset \cup_{i \in I} 5B_i$; here $I = \{1, 2, \ldots\}$ is a possibly finite index set. To prove (4) it then suffices to prove that for every $\alpha \in \mathbf{N}$, there exists $k \in \mathbf{N}$ that depend only on C and α, such that

$$\begin{aligned} |B_i| \leq & 2^{kp-\alpha} |U_{2^k\lambda} \cap B_i| + 8^{kp-\alpha} |U_{8^k\lambda} \cap B_i| \\ & + 8^{k(p+1)} |\{x \in B_i : \operatorname{Lip} u(x) > 8^{-k}\lambda\}| \end{aligned} \tag{7}$$

for every $i \in \mathbf{N}$.

Fix $i \in \mathbf{N}$ and let

$$M^{\#i} u(x) = \sup_B \frac{1}{r} \fint_{B \cap B_i} |u - u_{B \cap B_i}| \, d\mu,$$

for every $x \in B_i$, where the supremum is taken over all $w \in B_i$ and $r > 0$ such that $B = B(w, r)$ contains x. (That is, $M^{\#i}$ is given by the usual definition of $M^{\#}$, except that it is taken on B_i viewed as a metric measure subspace). Then define

$$U_s^i = \{x \in B_i : M^{\#i} u(x) \geq s\},$$

for every $s > 0$. By Lemma 2.2.2 and because B_i is a ball in a geodesic space, we have that B_i, when viewed as a metric measure subspace, admits a $(1, p)$-Poincaré inequality with constants that depend only on C. By the remarks following Definition 2.2.1 we have the closure of B_i is bi-Lipschitz to a geodesic metric space.

Thus we can apply the assumed hypotheses to B_i. (Here we use the fact that the claim of Proposition 3.1.1 is invariant under bi-Lipschitz maps.) It then follows by a standard re-scaling argument, that for every $\alpha \in \mathbf{N}$, there exist $k \in \mathbf{N}$ that depend only on C and α, such that

$$\begin{aligned}|B_i| \leq & 2^{kp-\alpha}|U^i_{2^k\lambda}| + 8^{kp-\alpha}|U^i_{8^k\lambda}| \\ & + 8^{k(p+1)}|\{x \in B_i : \operatorname{Lip} u(x) > 8^{-k}\lambda\}|.\end{aligned}$$

To see that this implies the desired estimate (7), it suffices to prove that $U^i_s \subset U_{Cs}$ for every $s > 0$. However, this inclusion follows from the estimate

$$M^{\#i}u(x) \leq CM^{\#}u(x)$$

for every $x \in B_i$, which is itself an easy consequence of the fact that μ is doubling, and that B_i is a ball in a geodesic metric space. This completes the proof. □

3.3. **The crux of the proof.** We continue the proof of Proposition 3.1.1 under the further assumptions outlined in and before Lemma 3.2.1. Fix $\alpha \in \mathbf{N}$ and suppose, in order to achieve a contradiction, that (4) does not hold with $\lambda = 1$. We continue to let $C > 0$ denote a varying constant (the meaning of which was explained at the beginning of Section 3.1), but now one that depends only on the data associated with the assumed $(1,p)$-Poincaré inequality and the doubling constant of μ, and also α. During the remainder of the proof we will specify a fixed and finite number of lower bounds for k, required for the proof to work. These lower bounds depend only on C. To realize the contradiction at the end of the proof, and thereby prove Lemma 3.2.1, we then take k to be equal to the maximum of this finite collection of lower bounds. Keeping all this in mind, let $k \in \mathbf{N}$.

We begin by making the observation that since, as assumed above, the estimate (4) does not hold with $\lambda = 1$, and since (5) does hold, and X has unit measure, it follows that

$$\begin{gathered}|U_{2^k}| < 2^{-kp+\alpha}, \quad |U_{8^k}| < 8^{-kp+\alpha}, \quad \text{and} \\ |\{x \in X : \operatorname{Lip} u(x) > 8^{-k}\}| < 8^{-k(p+1)}.\end{gathered} \tag{8}$$

The next lemma demonstrates that u has some large scale oscillation outside U_{2^k}.

Lemma 3.3.1. *We have*

$$\int_{X \setminus U_{2^k}} |u - u_{X\setminus U_{2^k}}|\, d\mu \geq C.$$

Proof. We exploit (8). Without loss of generality, by a translation in the range of u, we can assume that $u_{X\setminus U_{2^k}} = 0$. Thus we need to show that

$$\int_{X \setminus U_{2^k}} |u|\, d\mu \geq C.$$

Let $\mathcal{G}$ be the collection of balls B in X such that

$$|B \setminus U_{2^k}| \geq |B|/4 \quad \text{and} \quad |B \cap U_{2^k}| \geq |B|/4. \tag{9}$$

For later use we observe that (8), together with Lemma 2.1.1, implies that each such B satisfies

$$\operatorname{diam} B \leq C^{-kp+\alpha}. \tag{10}$$

Since (X, d) is geodesic, we can apply Proposition 2.2.3 in a straightforward manner to conclude that $\mathcal{G}$ is a cover of U_{2^k}. Indeed, fix $x \in U_{2^k}$, and define $h : (0, 1] \longrightarrow \mathbf{R}$ by

$$h(r) = \frac{|B(x, r) \cap U_{2^k}|}{|B(x, r)|}.$$

Since $M^{\#}$ is uncentered maximal-type operator, we have U_{2^k} is open, and therefore $h(\delta) = 1$ for some $\delta > 0$. We also have $h(1) = |U_{2^k}| \leq 2^{-kp+\alpha}$. Finally, Proposition 2.2.3 implies that h is continuous. Therefore there exists $r > 0$ such that $h(r) = 1/4$. This proves the claim. We now apply a standard covering argument (see [15, Theorem 1.2]) and so obtain a countable subcollection $\{B_i\}_{i \in J}$ of $\mathcal{G}$ consisting of mutually disjoint balls in X such that $U_{2^k} \subset \cup_{i \in J} 5B_i$; here $J = \{1, 2, \dots\}$ is a possibly finite index set.

We now divide U_{2^k} amongst the members of $\{B_i\}_{i \in J}$. Let

$$E_i = 5B_i, \quad E_i^O = B_i \setminus U_{2^k}, \quad \text{and} \quad E_i^I = B_i \cap U_{2^k},$$

for each $i \in J$. Notice that by construction and by (9) we have that

$$|E_i| \leq C \min\{|E_i^O|, |E_i^I|\}, \tag{11}$$

that $\{E_i\}_{i \in J}$ is a cover of U_{2^k}, and that $\{E_i^I\}_{i \in J}$ and $\{E_i^O\}_{i \in J}$ are collections of mutually disjoint measurable sets. Note that I and O stand for inside and outside, respectively.

It follows from these just stated properties and (5) that

$$1 \leq \int_X |u - u_X| \, d\mu \leq 2 \int_X |u| \, d\mu \leq 2 \int_{X \setminus U_{2^k}} |u| \, d\mu + 2 \sum_{i \in J} \int_{E_i} |u| \, d\mu,$$

whereas

$$\begin{aligned} \sum_{i \in J} \int_{E_i} |u| \, d\mu &\leq \sum_{i \in J} |E_i| \, |u_{E_i^O}| + \sum_{i \in J} \int_{E_i} |u - u_{E_i^O}| \, d\mu \\ &\leq C \int_{X \setminus U_{2^k}} |u| \, d\mu + C \sum_{i \in J} \int_{E_i} |u - u_{E_i^O}| \, d\mu, \end{aligned}$$

and therefore

$$1 \leq C \int_{X \setminus U_{2^k}} |u| \, d\mu + C \sum_{i \in J} \int_{E_i} |u - u_{E_i^O}| \, d\mu. \tag{12}$$

Consequently, to complete the proof we need to show that for sufficiently large $k \in \mathbf{N}$, that depends only on C, that the right-hand most term in (12) is less than $1/2$. We use (11), and then the fact that E_i intersects the complement of U_{2^k}, to obtain

$$⨍_{E_i} |u - u_{E_i^O}| \, d\mu \leq C ⨍_{E_i} |u - u_{E_i}| \, d\mu \leq C 2^k \operatorname{diam}(E_i),$$

for every $i \in J$. Thus, the right-hand most term of (12) is bounded by

$$C 2^k \left(\sup_{i \in J} \operatorname{diam} B_i \right) \sum_{i \in J} |E_i|.$$

We now apply (10) and (11) to bound the above sum by

$$C 2^k C^{-kp+\alpha} \sum_{i \in J} |E_i^I| \leq C 2^k C^{-kp+\alpha} |U_{2^k}| \leq C 2^{(1-p)k} C^{-kp+\alpha}.$$

We conclude that for sufficiently large $k \in \mathbf{N}$, that depends only on C, that the right-hand most term in (12) is less than $1/2$. This completes the proof. Note that this part of the proof did not really require the fact that $p > 1$. $\square$

We now remove the small scale oscillation from u while still preserving the large scale oscillation.

Lemma 3.3.2. *There exists a $C8^k$-Lipschitz extension f of the restricted function $u|_{X\setminus U_{8^k}}$ to X such that*

$$M^{\#}f(x) \le CM^{\#}u(x) \tag{13}$$

for every $x \in X \setminus U_{8^k}$.

Proof. By Lemma 2.3.1 we have that $u|_{X\setminus U_{8^k}}$ is $C8^k$-Lipschitz. We could now extend u to X using the McShane extension (see [15, Theorem 6.2]). However, it is not clear that this would then satisfy (13). Instead we use another standard extension technique based on a Whitney-like decomposition of U_{8^k}; similar methods of extension also appear in [37, 31, 12, 36]. The novelty here is not the extension, but rather that there is a Lipschitz extension that satisfies (13).

Observe that because $M^{\#}$ is uncentered, we have U_{8^k} is open. We can then apply a standard covering argument ([15, Theorem 1.2]) to the collection

$$\{B(x, \operatorname{dist}(x, X \setminus U_{8^k})/4) : x \in U_{8^k}\},$$

and so obtain a countable subcollection $\mathcal{F} = \{B_i\}_{i\in I}$, where $I = \{1, 2, \ldots\}$ is a possibly finite index set, such that $U_{8^k} = \cup_{i\in\mathbf{N}} B_i$, and such that $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset$ for $i, j \in I$ with $i \neq j$. It then follows from the fact that μ is doubling that

$$\sum_{i\in I} \chi|_{2B_i} \le C, \tag{14}$$

where we use $\chi|_W$ to denote the characteristic function on any set W.

We now construct a partition of unity subordinate to this collection of balls. For each $i \in I$, let $\hat{\psi}_i : X \longrightarrow \mathbf{R}$ be a $C \operatorname{dist}(B_i, X\setminus U_{8^k})^{-1}$-Lipschitz function with $\hat{\psi} = 1$ on B_i and $\hat{\psi} = 0$ on $X \setminus 2B_i$. Then let

$$\psi_i = \frac{\hat{\psi}_i}{\sum_{j\in I} \hat{\psi}_j}.$$

As usual the sum in the denominator is well-defined at each point in X, as because of (14), all but a finite number of terms in the sum are non-zero. Next define $f : X \longrightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} \sum_{i\in I} u_{B_i}\psi_i(x) & \text{if } x \in U_{8^k}, \\ u(x) & \text{if } x \in X \setminus U_{8^k}. \end{cases}$$

We now show that f is $C8^k$-Lipschitz, that is, we show that

$$|f(x) - f(y)| \le C8^k d(x, y) \tag{15}$$

for every $x, y \in X$. By Lemma 2.3.1 we have (15) holds whenever $x, y \in X \setminus U_{8^k}$. Next consider the case when $x \in U_{8^k}$ and $y \in X \setminus U_{8^k}$. By the triangle inequality, and the case considered two sentences back, we can further suppose that

$$d(x, y) \le 2 \operatorname{dist}(x, X \setminus U_{8^k}).$$

Let B be a ball in $\mathcal{F}$ that contains x. Then $B = B(w, r)$ for some $w \in X$ and $r > 0$ such that r and $d(x, y)$ are comparable with constant C. We can then use Lemma 2.3.1 and the doubling property of μ, to deduce that

$$|u_B - u(y)| \leq CrM^{\#}u(y) \leq C8^k d(x, y).$$

The estimate (15) then follows from the definition of f.

Finally we consider the case when $x, y \in U_{8^k}$. Due to the last two cases considered, we can further suppose that $r = \operatorname{dist}(x, X \setminus U_{8^k})$ is comparable to $d(y, X \setminus U_{8^k})$ with comparability constant C, and that $d(x, y) \leq r$. Let $B = B(x, 2r)$ and observe that if $B_i = B(z, s)$ is a ball in $\mathcal{F}$, for some $i \in I$, $z \in X$ and $s > 0$, such that $\{x, y\} \cap B(z, 2s) \neq \emptyset$, then r and s are comparable with comparability constant C, the function ψ_i is r^{-1}-Lipschitz, and $\operatorname{dist}(B, B(z, s)) \leq r$. Consequently, we have

$$|u_{B_i} - u_B| \leq CrM^{\#}u(w) \leq C8^k r,$$

for some $w \in B \setminus U_{8^k}$. It then follows from the fact that $\{\psi_i\}_{i \in I}$ is a partition of unity on U_{8^k}, that

$$\begin{aligned} |f(x) - f(y)| &= \sum_{i \in I} \psi_i(x) u_{B_i} - \sum_{i \in I} \psi_i(x) u_B + \sum_{i \in I} \psi_i(y) u_B - \sum_{i \in I} \psi_i(y) u_{B_i} \\ &= \sum_{i \in I} (\psi_i(x) - \psi_i(y))(u_{B_i} - u_B) \leq C8^k d(x, y), \end{aligned}$$

as desired. This completes the demonstration that f is $C8^k$-Lipschitz.

It remains to establish (13). Fix $x \in X \setminus U_{8^k}$ and suppose that $M^{\#}f(x) > \delta$ for some $\delta > 0$. Thus there exists a ball $B = B(y, r)$ in X containing x, for some $y \in X$ and $r > 0$, such that

$$\fint_B |f - f_B| \, d\mu > \delta r. \tag{16}$$

We would like to show that

$$\fint_{\lambda B} |u - u_B| \, d\mu > C\delta r, \tag{17}$$

for some $\lambda \geq 1$ that depends only on C. Observe that the above two estimates are invariant under a translation in the range of u and f. Furthermore, the construction of f from u is also invariant under a translation in the range of u and f. By this we mean that if u is replaced by $u + \beta$ for some $\beta \in \mathbf{R}$, then the construction above gives $f + \beta$ in place of f. Thus without loss of generality, by making such a translation, we can assume that $u_B = 0$. Since $u = f$ on $X \setminus U_{8^k}$, we can also assume that $B \cap U_{8^k} \neq \emptyset$, otherwise (17) follows trivially from (16).

It then follows directly from the construction of $\mathcal{F}$, that if $B(z,s) \in \mathcal{F}$ for some $z \in U_{8^k}$ and $s > 0$, then $B(z, 2s) \cap B \neq \emptyset$ implies $s \leq r$. Therefore

$$\begin{aligned}
\int_B |f - f_B| \, d\mu &\leq 2 \int_B |f| \, d\mu \\
&\leq 2 \int_{B \setminus U_{8^k}} |f| \, d\mu + \sum_{i \in I} \int_B |\psi_i u_{B_i}| \\
&\leq 2 \int_{B \setminus U_{8^k}} |u| \, d\mu + C \sum_{\substack{i \in I \\ 2B_i \cap B \neq \emptyset}} \int_{B_i} |u| \, d\mu \\
&\leq C \int_{5B} |u| \, d\mu.
\end{aligned}$$

It follows from this last estimate, the doubling property of μ, and our assumption that $u_B = 0$, that (17) holds. This proves (13) and so completes the proof of the lemma. □

The function f can be viewed as a smoothed version of u, that is, with small scale oscillations removed, and large scale oscillations preserved. The following two lemmas conciliate the previous estimates on the oscillation of u and f. Let

$$F_s = \{x \in X : M^{\#} f(x) > s\}$$

for every $s \geq 0$.

Lemma 3.3.3. *We have*

$$\int_{X \setminus U_{8^k}} (\operatorname{Lip} f)^p \, d\mu \leq C 8^{-k}, \tag{18}$$

and

$$|F_s| \leq C s^{-p}, \tag{19}$$

for every $s > 0$.

Proof. We first prove (18). By Lemma 3.3.2 we have $f = u$ on $X \setminus U_{8^k}$, and so Lemma 2.1.2 implies that $\operatorname{Lip} f = \operatorname{Lip} u$ almost everywhere on $X \setminus U_{8^k}$. Since f is $C8^k$-Lipschitz, we therefore have $\operatorname{Lip} u \leq C8^k$ almost everywhere on $X \setminus U_{8^k}$. It follows that

$$\begin{aligned}
\int_{X \setminus U_{8^k}} (\operatorname{Lip} f)^p \, d\mu &= \int_{X \setminus U_{8^k}} (\operatorname{Lip} u)^p \, d\mu \\
&\leq C 8^{kp} \left| \{x \in X : \operatorname{Lip} u(x) > 8^{-k}\} \right| + 8^{-kp}
\end{aligned}$$

The estimate (18) then follows from (8).

We now prove (19). It follows from (8) and (18), that

$$\int_X (\operatorname{Lip} f)^p \, d\mu \leq 8^{kp} |U_{8^k}| + \int_{X \setminus U^{8^k}} (\operatorname{Lip} f)^p \, d\mu \leq C.$$

Now, by the $(1,p)$-Poincaré inequality we have

$$\left(M^{\#} f\right)^p \leq C M\left((\operatorname{Lip} f)^p\right),$$

where M denotes the uncentered Hardy-Littlewood maximal operator. Therefore by the weak-L^1 bound for the uncentered Hardy-Littlewood maximal operator (see [15, Theorem 2.2] in this setting), we get the desired estimate:

$$\begin{aligned}|F_s| &\leq |\{x \in X : M((\operatorname{Lip} f)^p)(x) > Cs^p\}| \\ &\leq Cs^{-p} \int_X (\operatorname{Lip} f)^p \, d\mu \\ &\leq Cs^{-p},\end{aligned}$$

for every $s > 0$. This proves (19), and completes the proof of the lemma. □

Observe that by Lemma 2.3.1, for every $s > 0$, the restricted function $f|_{X \setminus F_s}$ is Cs-Lipschitz. For each $j \in \mathbf{N}$ we let f_j be the McShane extension of $f|_{X\setminus F_{2^j}}$ to a $C2^j$-Lipschitz function f_j on X; see [15, Theorem 6.2]. (We are not fussy about the sort of Lipschitz extension used here, any decent one will do.) Next let

$$h = \frac{1}{k} \sum_{j=2k}^{3k-1} f_j.$$

Lemma 3.3.4. *We have*

$$\int_X (\operatorname{Lip} h)^p \, d\mu \geq C, \tag{20}$$

and we have

$$\operatorname{Lip} h(x) \leq \chi|_{X\setminus U_{8^k}}(x) \operatorname{Lip} f(x) + \frac{C}{k} \sum_{j=2k}^{3k-1} 2^j \chi|_{U_{8^k} \cup F_{2^j}}(x), \tag{21}$$

for almost every $x \in X$.

Proof. We first prove (20). We require $k \in \mathbf{N}$ is sufficiently large as determined by C, so that (13) implies $F_{4^k} \subset U_{2^k}$. We then have $f_j = u$ almost everywhere on $X \setminus U_{2^k}$ for every $2k \leq j \leq 3k$. It then follows from the definition of h, that $h = u$ on $X \setminus U_{2^k}$. Consequently, we can deduce from Lemma 3.3.1, that

$$\int_X |h - h_X| \geq C.$$

Since h is Lipschitz we can apply the $(1,p)$-Poincaré inequality to conclude that (20) holds.

We now prove (21). Fix $j \in \mathbf{N}$. Observe that $f_j = f$ on $X \setminus F_{2^j}$, and therefore Lemma 2.1.2 implies that $\operatorname{Lip} f_j = \operatorname{Lip} f$ almost everywhere on $X \setminus F_{2^j}$. This and the fact that f_j is $C2^j$-Lipschitz, implies that

$$\operatorname{Lip} f_j(x) \leq \chi|_{X\setminus U_{8^k}}(x) \operatorname{Lip} f(x) + C2^j \chi|_{U_{8^k}\cup F_{2^j}}(x),$$

for almost every $x \in X$. The estimate (21) now follows directly from the definition of h. This completes the proof. □

Observe that $F_s \subset F_t$ whenever $0 \le t \le s$. This property with (19) and (8) implies that

$$\begin{aligned}\int_X \left(\frac{1}{k}\sum_{j=2k}^{3k-1} 2^j \chi|_{U_{8k}\cup F_{2j}}\right)^p d\mu &\le \frac{1}{k^p}\int_X \sum_{j=2k}^{3k-1}\left(\sum_{i=2k}^{j} 2^i\right)^p \chi|_{U_{8k}\cup F_{2j}}\, d\mu \\ &\le \frac{C}{k^p}\sum_{j=2k}^{3k-1} 2^{(j+1)p}2^{-jp} \\ &= Ck^{1-p}.\end{aligned}$$

This with Lemma 3.3.4 and (18) implies that

$$\begin{aligned}C &\le \int_X (\operatorname{Lip} h)^p\, d\mu \\ &\le C\int_{X\setminus U_{8k}} (\operatorname{Lip} f)^p\, d\mu + C\int_X \left(\frac{1}{k}\sum_{j=2k}^{3k-1} 2^j\chi|_{U_{8k}\cup F_{2j}}\right)^p d\mu \\ &\le C8^{-k} + Ck^{1-p}.\end{aligned}$$

Since $p > 1$, we achieve a contradiction when $k \in \mathbf{N}$ is sufficiently large as determined by C. This completes the proof of Proposition 3.1.1 and thereby Theorem 1.0.1.

4. Proof of Proposition 1.1.1 and Theorems 1.0.3 to 1.0.6

Theorem 1.0.4 and 1.0.5 can be easily deduced from Theorem 1.0.1 together with [19, Theorem 7.11], or the main results of [29], respectively. Similarly, Theorem 1.0.6 is easily deduced from [39, Theorem 4.9 and 4.10]. To see that Theorem 1.0.3 follows from Theorem 1.0.1 and [19, Theorem 5.13] (see also [15, Theorem 9.6, and Theorem 9.8]), we need to recall that complete Ahlfors regular spaces (defined below) are proper, and that complete metric measure spaces that support a doubling measure and a Poincaré inequality are quasi-convex, quantitatively. These results are stated in [23, Proposition 6.0.7] and the discussion that follows. The meaning of these words, and the words used in the statement of Theorems 1.0.3, 1.0.4, 1.0.5, and 1.0.6, can be found in the respective references given above.

Before proving Proposition 1.1.1, we recall that a metric measure space (X, d, μ) is *Ahlfors α-regular*, $\alpha > 0$, if μ is Borel regular and there exists $C \ge 1$ such that

$$\frac{1}{C}r^\alpha \le \mu(B(x,r)) \le Cr^\alpha,$$

for every $x \in X$ and $0 < r \le \operatorname{diam} X$.

Proof of Proposition 1.1.1. Let X be the cantor set, which we identify with the collection of all sequences (a_n) where $a_n = 0$ or 1 for every $n \in \mathbf{N}$. Define a metric d on X by $d((a_n),(b_n)) = 2^{-k}$ for any $(a_n),(b_n) \in X$, where if $a_1 \ne b_1$ then we set $k = 0$, and otherwise we let k be the greatest integer such that $a_i = b_i$ for each $1 \le i \le k$. For every $x \in X$ and $r > 0$, let

$$Q(x,r) = \{y \in X : d(x,y) \le r\},$$

and call such sets *cubes*. We further let

$$\lambda Q(x,r) = \{y \in X : d(x,y) \le \lambda r\},$$

for every $\lambda > 0$. Next let μ be the Borel measure on X determined by the condition that $\mu(Q(x, 2^{-k})) = 2^{-k}$ for every $x \in X$ and $k \in \mathbf{N}$; this can be defined using Carathéodory's construction, see [7, Theorem 2.10.1]. Then (X, d, μ) is an Ahlfors 1-regular metric measure space.

For each $n \in \mathbf{N}$, let $Q_n = Q(x_n, 2^{-n})$ where $x_n \in X$ is the sequence consisting of $n-1$ zeroes followed by a one, and then followed by zeroes. Notice that (Q_n) is a sequence of mutually disjoint sets, with union equal to X. It is now easy to construct a Borel function $g : X \longrightarrow \mathbf{R}$ such that

$$\int_{Q_n} g^p \, d\mu = 2^{-n} \quad \text{and} \quad \int_{Q_n} g^{p-1/n} \, d\mu = 4^{-n},$$

for every $n \in \mathbf{N}$. Observe that $g \in L^p(X)$. Moreover, we have

$$\fint_{Q_n} g^p \, d\mu = 1, \tag{22}$$

for every $n \in \mathbf{N}$, and that

$$\lim_{m \to \infty} \fint_{\lambda Q_m} g^{p-1/m} \, d\mu = 0, \tag{23}$$

for every $\lambda > 0$.

Define $u : X \longrightarrow \mathbf{R}$ by the condition $u(x) = 2^{-n}$ whenever $x \in X$ satisfies $x \in Q_n$ for some $n \in \mathbf{N}$. As required we have $u \in L^p(X)$. We claim that there exists $C, \lambda \geq 1$ such that (1) holds, with $q = p$, for every $x \in X$ and $r > 0$. It suffices to show that there exists $C \geq 1$ such that

$$\frac{1}{\operatorname{diam} R} \fint_R |u - u_R| \, d\mu \leq C \left(\fint_R g^p \, d\mu \right)^{1/p}, \tag{24}$$

for every cube R in X. Fix a cube R in X. If $R \subset Q_n$ for some $n \in \mathbf{N}$, then u is constant on R and (24) is trivially true. Otherwise, we have $R = 2Q_n$ for some $n \in \mathbf{N}$, and therefore

$$2^{-4} \leq \frac{1}{\operatorname{diam} R} \fint_R |u - u_R| \, d\mu \leq 1. \tag{25}$$

It follows from this and (22), that (24) holds with $\lambda = 1$ and $C = 2^p$. This proves the above claim. Furthermore, since (25) holds with $R = 2Q_n$ for every $n \in \mathbf{N}$, we deduce from (23) that there does not exist $1 \leq q < p$ and $C, \lambda \geq 1$ such that (1) holds for all $x \in X$ and $r > 0$. This completes the proof. $\square$

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