POROSITIES IN MANDELBROT PERCOLATION

ARTEMI BERLINKOV[1] AND ESA JÄRVENPÄÄ[2]

Abstract. We study porosities in Mandelbrot percolation. We prove that almost surely the lower porosity of the set and the natural measure are equal to zero, the upper porosity of the set is equal to one half, and the upper porosity of the natural measure is equal to one. We also show that almost surely the lower and upper mean porosities of the set and the lower mean porosity of the measure are positive and less than one for all parameter values.

1. Introduction

The porosity of a set describes the size of holes in the set. It dates back to 1920's when Denjoy introduced a notion which he called index (see [De]). In today's terminology his index is called the upper porosity (see Definition 3.1). The name porosity was introduced by Dolženko in [Do]. If the upper porosity of a set is α, then one can find holes of relative size α at arbitrarily small distances. The upper porosity has been used for describing properties of exceptional sets, for example, for measuring sizes of sets where certain functions are non-differentiable. For more details about the upper porosity, the reader is referred to an article of Zajíček ([Z]).

As the upper porosity says that one can find holes of certain size at arbitrarily small distances, the lower porosity (see Definition 3.1) guarantees the existence of holes of certain size at all small enough distances. Mattila [M] used it to find upper bounds for the Hausdorff dimension of a set, and Salli [S] extended the result to packing and box counting dimensions.

It turns out that the upper porosity cannot be used to estimate the dimension of a set. However, there are sets which are not lower porous but which nevertheless contain so many holes that their dimension is smaller than the dimension of the ambient space. In order to study boundary behaviour of conformal and quasiconformal mappings, Koskela and Rohde [KR] introduced the notion of mean porosity of a set which guarantees that certain percentage of scales, that is, distances which are integer powers of some fixed number, contain holes of fixed relative size. They showed that if a set in m-dimensional Euclidean

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space is mean porous, then its Hausdorff and packing dimensions are smaller than m. For a modification of their definition see Definition 3.4.

Whilst the notion of the porosity of a set is already quite old, the porosity of a measure is relatively new. The lower porosity was introduced by Eckmann and E. and M. Järvenpää in [EJJ], and the upper one by Mera and Morán in [MM] (see Definition 3.2). In [EJJ] it was shown that the packing dimension of any doubling measure can be estimated above by a function depending on the lower porosity of the measure. In [JJ] Järvenpääs extended this result to the Hausdorff dimension of all measures. Just like in the case of sets the upper porosity is too weak for this purpose. In [MM] it was shown that the upper porosity may attain only the values 0, 1/2, and 1. In [BS] Beliaev and Smirnov found a unifying approach for dimension estimates mentioned above by introducing the mean porosity of a measure (see Definition 3.4). They proved that the packing dimension of a measure can be estimated above by a function depending on the mean porosity of the measure. Their upper bound is the best possible one as far as the asymptotic behaviours near m and $m-1$ are concerned.

Porosity is also an interesting concept on its own right and not just as a tool for other purposes. For example sets with same dimension may have different porosities. In [JJM] Järvenpääs and Mauldin, and in [U] Urbański characterized deterministic iterated function systems whose attractors have positive porosity. Porosities of random recursive constructions were studied in [JJM]. Especially interesting random constructions are those for which the copies of the seed set are glued together in such a way that there are no holes left. Thus the corresponding deterministic system would be non-porous. So the important question is whether the randomness in the construction makes the set or measure porous. The most natural model of these kind of random constructions is the Mandelbrot percolation (see section 2). In [JJM] it was shown that almost surely both the zero and one half porous points are dense in the limit set. However, the question about the porosity of typical points and the porosity of the natural measure remained open.

In this paper we prove the conjecture made in [JJM] by proving a stronger result according to which almost surely the lower porosity of the limit set and the measure are equal to zero, the upper porosity of the set is equal to one half, and the upper porosity of the measure is equal to one (see Theorem 4.8). We also show that almost surely the lower and upper mean porosities of the set and the lower mean porosity of the measure are positive and less than one for all parameter values (see Propositions 4.1, 4.3 and 4.6).

The article is organized as follows. In section 2 we explain the basic facts about Mandelbrot percolation. In section 3 we define several notions of porosities and compare them to those already existing in the literature. Finally, in section 4 we prove our results about the porosities and mean porosities in Mandelbrot percolation.

2. Mandelbrot percolation

In this section we give the basic facts about Mandelbrot percolation in m-dimensional Euclidean space $\mathbb{R}^m$. Let $k \geq 2$ be an integer, $I = \{0, 1, \ldots, k^m\}$, and $I^* = \cup_{i=0}^{\infty} I^i$. Here $I^0 = \emptyset$. The length of an element $\sigma \in I^i$ is $|\sigma| = i$, and for $\sigma \in I^i$ and $\sigma' \in I^j$ we denote by $\sigma * \sigma'$ the element of I^{i+j} whose first i coordinates are those of σ and the last j coordinates are those of σ'. For $k \in \mathbb{N}$ and $\sigma \in I^* \cup I^{\mathbb{N}}$, denote by $\sigma|_k$ a sequence in I^* formed by first k elements of σ. We shall write $\sigma \prec \tau$ if the sequence τ starts with σ. Denote by Ω the set of functions $\omega : I^* \to \{c, n\}$. Each $\omega \in \Omega$ can be thought of as a code that tells us which cubes we choose (c) and which we neglect (n). More precisely, let $\omega \in \Omega$. We take a cube of diameter 1, denote it by $J_\emptyset$, and divide it into k^m closed cubes with diameter k^{-1}, enumerate them with letters from alphabet I, and repeat this procedure inside each subcube. For all $\sigma \in I^i$ we use the notation J_σ for the unique closed subcube of $J_\emptyset$ with diameter k^{-i} coded by σ. The image of $\eta \in I^{\mathbb{N}}$ under the natural projection from $I^{\mathbb{N}}$ to $[0,1]^m$ is denoted by $x(\eta)$, that is,

$$x(\eta) = \bigcap_{i=0}^{\infty} J_{\eta|_i}$$

where $\eta|_0 = \emptyset$. Here $\mathbb{N} = \{1, 2, \ldots\}$. If $\omega(\sigma) = n$ for $\sigma \in I^i$, then $J_\sigma(\omega) = \emptyset$, and if $\omega(\sigma) = c$, then $J_\sigma(\omega) = J_\sigma$. Define

$$K_\omega = \bigcap_{i=0}^{\infty} \bigcup_{\sigma \in I^i} J_\sigma(\omega).$$

Fix $0 \leq p \leq 1$. We make the above construction random by demanding that if J_σ is chosen then $J_{\sigma * j}$, $j = 1, \ldots, k^m$, are chosen independently with probability p. Let P be the natural probability measure on Ω, that is, for all $\sigma \in I^*$ and $j = 1, \ldots, k^m$

$$\begin{aligned}
&P(\omega(\emptyset) = c) = 1, \\
&P(\omega(\sigma * j) = c \mid \omega(\sigma) = c) = p, \\
&P(\omega(\sigma * j) = n \mid \omega(\sigma) = n) = 1.
\end{aligned}$$

(We will use abbreviations like $P(\omega(\emptyset) = c) = P(\{\omega \in \Omega \mid \omega(\emptyset) = c\})$.)

It is a well-known result in the theory of branching processes that if the expectation of the number of chosen cubes of diameter k^{-1} does not exceed one, then the limit set K_ω is P-almost surely an empty set or a point (see [AN, Theorem 1, p.7, and Lemma 1, p.4]). In our case this expectation equals $k^m p$. Thus we assume that $k^{-m} < p < 1$. According to [MW, Theorem 1.1], the Hausdorff dimension of K_ω is P-almost surely equal to $d = m + \log p / \log k$ provided $K_\omega \neq \emptyset$. Moreover, for

P-almost all $\omega \in \Omega$ there exists a natural Radon measure ν_ω on K_ω (see [MW, Theorem 3.2]). There is also a natural Radon probability measure Q on $I^{\mathbb{N}} \times \Omega$ such that for every Borel set $B \subset I^{\mathbb{N}} \times \Omega$ we have

$$Q(B) = \int \mu_\omega(B_\omega) dP(\omega)$$

where $B_\omega = \{\eta \in I^{\mathbb{N}} \mid (\eta, \omega) \in B\}$, and ν_ω is the image of μ_ω under the natural projection (see [GMW, (1.13)]).

For a finite sequence $\sigma \in I^*$ consider the martingale $\{N_{j,\sigma} k^{-jd}\}_{j \in \mathbb{N}}$, where $N_{j,\sigma} = \operatorname{card}\{\tau \in I^* \mid \sigma \prec \tau, |\tau| = |\sigma| + j, \text{ and } \omega(\tau) = c\}$ and denote its almost sure limit by $X_\sigma(\omega)$. For $l \in \mathbb{N}$, define random variable X_l on $I^{\mathbb{N}} \times \Omega$ by $X_l(\eta, \omega) = X_{\eta|_l}(\omega)$. For $l = 0$, we set $X_l(\eta, \omega) = X_\emptyset(\omega)$. It is easy to see that for P-almost all $\omega \in \Omega$

$$X_\sigma(\omega) = \sum_{|\tau| = j} k^{-jd} X_{\sigma * \tau}(\omega) \mathbf{1}_{\{\omega(\sigma * \tau) = c\}}.$$

Further, for $\sigma, \tau \in I^*$, X_σ and X_τ are identically distributed, and if $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, they are also independent. Thus X_l, $l \in \mathbb{N} \cup \{0\}$, have the same distribution.

Expectations with respect to measures P and Q are connected in the following way (see [GMW, (1.16)]): if $j \in \mathbb{N}$ and $Y : I^{\mathbb{N}} \times \Omega \to \mathbb{R}$ is a random variable such that $Y(\eta, w) = Y(\eta', w)$ provided $\eta|_j = \eta'|_j$, then

$$E_Q[Y] = E_P\Big[\sum_{|\sigma| = j,\, \omega(\sigma) = c} k^{-jd} X_\sigma Y(\sigma, \cdot) \Big].$$

Hence we have $Q(X_l = 0) = 0$ and $E_Q[X_l] = E_P[X_0^2] < \infty$ for all $l \in \mathbb{N} \cup \{0\}$ (see [MW, Theorem 2.1]).

Lemma 2.1. *Let H be a translate of a coordinate hyperplane. Then*

$$P(K_\omega \cap H \neq \emptyset) = 0.$$

Proof. We proceed with induction on the dimension m. Let $m = 1$ and $x(\eta) \in [0, 1]$. The probability that K_ω intersects the $k^{-l}/2$-neighbourhood of $x(\eta)$ is smaller than $2p^l$ since either $J_{\eta|_l}$ or its neighbour must have been chosen. As l tends to infinity, this probability tends to zero for any $p < 1$.

Observe that projecting K_ω onto a coordinate hyperplane, one obtains a random set which is a Mandelbrot percolation in $\mathbb{R}^{m-1}$ where one neglects a cube with probability $(1-p)^k$ and thus choose a cube with probability $p' = 1 - (1-p)^k < 1$. Let π be the projection onto a coordinate hyperplane perpendicular to H. Then

$$P(\{\omega \in \Omega \mid K_\omega \cap H \neq \emptyset\}) = P'(\{\omega' \in \Omega' \mid K_{\omega'} \cap \pi(H) \neq \emptyset\}) = 0$$

where the objects with prime refer to the Mandelbrot percolation in $\mathbb{R}^{m-1}$ with choosing probability p'. This concludes the induction step. $\square$

3. Porosities

In this section we define several notions of porosities and compare them with other definitions appearing in the literature.

Definition 3.1. Let $A \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$. The local porosity of A at x at distance r is

$$\begin{aligned}\operatorname{por}(A,x,r) = \sup\{\alpha \geq 0 \mid \text{ there is } z \in \mathbb{R}^n \text{ such that} \\ B(z,\alpha r) \subset B(x,r) \setminus A\}.\end{aligned}$$

The upper and lower porosities of A at x are defined as

$$\overline{\operatorname{por}}(A,x) = \limsup_{r\to 0} \operatorname{por}(A,x,r) \text{ and } \underline{\operatorname{por}}(A,x) = \liminf_{r\to 0} \operatorname{por}(A,x,r),$$

respectively.

Definition 3.2. The lower and upper porosities of a Radon measure μ on $\mathbb{R}^m$ at a point $x \in \mathbb{R}^m$ are defined by

$$\begin{aligned}\underline{\operatorname{por}}(\mu,x) &= \lim_{\varepsilon\to 0} \liminf_{r\to 0} \operatorname{por}(\mu,x,r,\varepsilon) \text{ and} \\ \overline{\operatorname{por}}(\mu,x) &= \lim_{\varepsilon\to 0} \limsup_{r\to 0} \operatorname{por}(\mu,x,r,\varepsilon),\end{aligned}$$

respectively, where for all $r, \varepsilon > 0$

$$\begin{aligned}\operatorname{por}(\mu,x,r,\varepsilon) = \sup\{\alpha \geq 0 \mid \text{ there is } z \in \mathbb{R}^m \text{ such that} \\ B(z,\alpha r) \subset B(x,r) \text{ and } \mu(B(z,\alpha r)) \leq \varepsilon\mu(B(x,r))\}.\end{aligned}$$

Remark 3.3. Instead of demanding that the whole empty ball is inside the reference ball $B(x,r)$ one may assume that only the centre of the empty ball is inside the reference ball $B(x,r)$, that is, define

$$\begin{aligned}\widetilde{\operatorname{por}}(A,x,r) = \sup\{\alpha \geq 0 \mid \text{ there is } z \in B(x,r) \text{ such that} \\ B(z,\alpha r) \cap A = \emptyset\}.\end{aligned}$$

This will only rescale the upper and lower porosities by the transformation $\alpha \mapsto \alpha/(1-\alpha)$.

Unlike the dimension, the porosity is sensitive to the metric used. For example, if one wants to use cubes instead of balls in the definition, there is no formula to convert ball-porosity to cube-porosity or vice versa. It is true that if the lower porosity is positive by one of these definitions, then it is positive for the other one also. But it is easy to construct a set such that the cube-porosity attains its maximum value (at some point) but the ball-porosity will not. Take for example the union of the x- and y-axes in the plane.

In general metric spaces it is sometimes useful to demand that the empty ball is inside the reference ball also algebraically, that is, $d(x,z)+\alpha r \leq r$. For further discussion about this matter see [MMPZ].

The upper and lower porosities tell the sizes of largest and smallest holes, respectively. Sometimes it is useful to know how often there exists a hole of certain size as the distance tends to zero. This leads to the notion of mean porosity. Mean porosity was used in [KR] and [BS] to find upper bounds for dimensions of sets and measures. The concepts of mean porosity used in [KR] and [BS] are slightly different from each other. Since there seems to be no commonly accepted definition for the mean porosity we try to clarify this situation by suggesting a definition which does not contain any hidden or dependent parameters like those in [KR] and [BS]. Below we give definitions which contain the phrase "there exists an α-hole at scale j near x". There are several ways the make this phrase rigorous and all of them will give a possibly different notion of mean porosity. We give first a definition of mean porosity without defining exactly the sentence "there exists an α-hole at scale j near x", and then we discuss different alternatives. After that we fix the interpretation used in the rest of this paper (see Definition 3.8). However, we now restrict the meaning of "at scale j near x" by demanding that one considers distances from x which are comparable to k^{-j} where $k > 1$ is some fixed number.

Definition 3.4. Let $k > 1$ and $0 \leq \alpha, \rho \leq 1$. A set $A \subset \mathbb{R}^m$ is said to be lower or upper (k, α, ρ)-mean porous at x if

$$\liminf_{i\to\infty} \frac{N_i(k,\alpha,x)}{i} \geq \rho \text{ or } \limsup_{i\to\infty} \frac{N_i(k,\alpha,x)}{i} \geq \rho,$$

respectively, where

$$N_i(k,\alpha,x) = \operatorname{card}\{j \in \mathbb{N} \mid j \leq i \text{ and there exists an } \alpha\text{-hole at scale } j \text{ near } x\}.$$

A Radon measure μ on $\mathbb{R}^m$ is said to be lower or upper (k, α, ρ)-mean porous at x if

$$\lim_{\varepsilon\to 0}\liminf_{i\to\infty} \frac{\tilde{N}_i(k,\alpha,x,\varepsilon)}{i} \geq \rho \text{ or } \lim_{\varepsilon\to 0}\limsup_{i\to\infty} \frac{\tilde{N}_i(k,\alpha,x,\varepsilon)}{i} \geq \rho,$$

respectively, where

$$\tilde{N}_i(k,\alpha,x,\varepsilon) = \operatorname{card}\{j \in \mathbb{N} \mid j \leq i \text{ and there exists an } (\alpha,\varepsilon)\text{-hole at scale } j \text{ near } x\}.$$

The lower (k, α)-mean porosity of A at x is

$$\underline{\kappa}(A,x,k,\alpha) = \sup\{\rho \in [0,1] \mid A \text{ is lower } (k,\alpha,\rho)\text{-mean porous at } x\}$$

and the upper (k, α)-mean porosity of A at x is

$$\overline{\kappa}(A,x,k,\alpha) = \sup\{\rho \in [0,1] \mid A \text{ is upper } (k,\alpha,\rho)\text{-mean porous at } x\}.$$

The lower and upper (k, α)-mean porosities of μ are defined analogously.

Remark 3.5. Definitions 3.1-3.4 can be used in any metric space but since we will study Mandelbrot percolation only in $\mathbb{R}^m$ we stated them only in the Euclidean setting.

Remark 3.6. The definition of the mean porosity in [BS] is slightly different from ours. According to [BS] a set A is mean (α, ρ)-porous at x if there exist N and n_0 such that for all $n > n_0$ there is an α-hole for at least ρN scales among scales $n+1, \dots, n+N$. Note that this notation of mean porosity is stronger than the one in Definition 3.4. Indeed, we will prove that Mandelbrot percolation is not mean porous in the sense of [BS] although it is mean porous in the sense of our definition (see Remark 4.4).

Now we have to define what is the existence of an α-hole at scale j. It is natural to demand that there exists $z \in \mathbb{R}^n$ such that $B(z, \alpha k^{-j} r_0) \subset B(x, k^{-j} r_0) \setminus A$ for some (or for all) $k^{-1} < r_0 \leq 1$. The following example shows that the choice of r_0 matters. It also shows that with this definition there is no simple relation between mean porosities defined using steps of size k, and, for example, steps of size k^3.

Example 3.7. Consider annuli centred at $x \in \mathbb{R}^m$ with outer radius k^{-i} and inner radius $k^{-(i+1)}$. Fill in two annuli out of every three successive annuli. If one does this in a regular fashion then $\kappa(A, x, k, (1-k^{-1})/2) = \frac{1}{3}$ and $\kappa(A, x, k^3, (1-k^{-1})/2) = 1$. (Note that the lower and upper porosities are equal.) Here one has to choose r_0 such that the empty annulus is the first one among the three k-annuli inside a k^3-annulus. But one can also fill in the annuli in such a way that $\kappa(A, x, k, (1-k^{-1})/2) = \frac{1}{3}$ and $\kappa(A, x, k^3, (1-k^{-1})/2) = 0$. In fact, first fill in the first and the second annulus inside N_1 successive k^3-annuli, then the first and the third one inside N_2 successive k^3-annuli, then the second and third one inside N_3 successive k^3-annuli, and so on. Now let N_i tend to infinity fast enough (N_i is much larger than the sum $N_1 + \cdots + N_{i-1}$).

The problem illustrated in Example 3.7 can be circumvented by demanding that there exists an α-hole at scale j near x if there exists a point z in the annulus centred at x with inner radius $k^{-(j+1)}$ and outer radius k^{-j} such that $B(z, \alpha|z-x|)$ does not intersect the set. In this definition the size of the hole is determined using the distance between the point and the centre of the hole. A disadvantage of this definition is that if k is large, then for example a $\frac{1}{2}$-hole at scale i may be quite small compared to k^{-i}. Thus it does not correspond to the intuition of a large hole at scale i.

From the point of Mandelbrot percolation the cubes are more natural objects than balls. Particularly natural are the construction cubes. Also the role of the starting distance r_0 in the above definition is a bit annoying. So we use the following definition where the role of the starting distance r_0 has disappeared.

Definition 3.8. Let $A \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$. Let $k \geq 2$ be an integer and $0 \leq \alpha \leq 1$. There is an α-hole at scale j near x if there is a point

$z \in Q_j^k(x) \setminus Q_{j+1}^k(x)$ such that

$$B_\varrho(z, \alpha k^{-j}/2) \subset Q_j^k(x) \setminus A.$$

Here $Q_j^k(x)$ is the (half open) k-adic cube of side-length k^{-j} containing x, ϱ is the maximum metric, that is, $\varrho(x,y) = \max_i\{|x_i - y_i|\}$, and $B_\varrho(y,r)$ is the open ball in the metric ϱ centred at y and with radius r. Recall that the balls in the maximum metric are cubes whose faces are parallel to the coordinate planes.

Let μ be a Radon measure on $\mathbb{R}^m$ and $\varepsilon > 0$. There is an (α, ε)-hole at scale j near x if there is a point $z \in Q_j^k(x) \setminus Q_{j+1}^k(x)$ such that

$$B_\varrho(z, \alpha k^{-j}/2) \subset Q_j^k(x) \text{ and } \mu(B_\varrho(z, \alpha k^{-j}/2)) \le \varepsilon \mu(Q_j^k(x)).$$

Remark 3.9. Note that unlike in Definition 3.1 we have divided the radius of the empty ball by 2 so α may attain values between 0 and 1. The reason for this is that the point x may be arbitrarily close to the boundary of $Q_j^k(x)$ and if the whole cube $Q_j^k(x)$ is empty it is natural to say that there is a hole of relative size 1.

4. Results

In this section we study Mandelbrot percolation in $\mathbb{R}^m$ and k will be a fixed integer as in section 2. The almost sure Hausdorff dimension of the limit set K_ω is denoted by d. The porosities (see Definition 3.8) are defined using k-adic cubes subordinate to the seed set $J_\emptyset$. We start with a proposition which states that the infima and the suprema of the mean porosities are almost surely constant.

Proposition 4.1. *For every* $l \in \mathbb{N}$ *there exist* $0 \le \underline{c}(k^l, \alpha) \le \underline{C}(k^l, \alpha) \le 1$ *such that for* P*-almost all* $\omega \in \Omega$ *with* $K_\omega \neq \emptyset$

$$\nu_\omega\text{-}\operatorname*{ess\,inf}_{x \in K_\omega} \underline{\kappa}(K_\omega, x, k^l, \alpha) = \underline{c}(k^l, \alpha) \text{ and}$$
$$\nu_\omega\text{-}\operatorname*{ess\,sup}_{x \in K_\omega} \underline{\kappa}(K_\omega, x, k^l, \alpha) = \underline{C}(k^l, \alpha).$$

Same statement holds for the upper mean porosity, $\overline{\kappa}(K_\omega, x, k^l, \alpha)$*. Corresponding constants are denoted by* $\overline{c}(k^l, \alpha)$ *and* $\overline{C}(k^l, \alpha)$*.*

Proof. For a code $\tau \in I^l$, let $y_\tau(\omega) = \nu_\omega\text{-}\operatorname*{ess\,inf}_{x \in K_\omega \cap J_\tau} \underline{\kappa}(K_\omega \cap J_\tau, x, k^l, \alpha)$. By Lemma 2.1, $\nu_\omega\text{-}\operatorname*{ess\,inf}_{x \in K_\omega} \underline{\kappa}(K_\omega, x, k^l, \alpha) \ge c$ if and only if $y_\tau(w) \ge c$ for all $|\tau| = l$, and the events $\{y_\tau \ge c\}$ are independent. By a zero-one law [BM, Theorem 2],

$$\underline{c}(k^l, \alpha) = \sup\{c \in \mathbb{R} \mid P\big(K_\omega \neq \emptyset \text{ and } \nu_\omega\text{-}\operatorname*{ess\,inf}_{x \in K_\omega} \underline{\kappa}(K_\omega, x, k^l, \alpha) \ge c\big) > 0\}.$$

The other cases are treated analogously. □

Next we prove a little lemma.

Lemma 4.2. *For every $l \in \mathbb{N}$ we have:*

(i) *If $\underline{c}(k^l, \alpha) > 0$ then $\underline{c}(k, \alpha) > 0$.*
(ii) *If $\overline{C}(k^l, \alpha) < 1$ then $\overline{C}(k, \alpha) < 1$.*

Proof. Let $x \in K_\omega$. Suppose that there exists $z \in Q_j^{k^l}(x) \setminus Q_{j+1}^{k^l}(x)$ such that $B_\varrho(z, \alpha k^{-jl}/2) \subset Q_j^{k^l}(x) \setminus K_\omega$. Let $jl \leq i < (j+1)l$ be the unique integer satisfying $z \in Q_i^k(x) \setminus Q_{i+1}^k(x)$. Since $k \geq 2$, there exists $z' \in B_\varrho(z, \alpha k^{-jl}/2) \cap (Q_i^k(x) \setminus Q_{i+1}^k(x))$ with $B_\varrho(z', \alpha k^{-i}/2) \subset Q_i^k(x) \setminus K_\omega$. This proves case (i). Claim (ii) follows similarly. □

Now we are ready to prove that the limit set of the Mandelbrot percolation is mean porous.

Proposition 4.3. *Let $0 < \alpha < 1$. Then for all $l \in \mathbb{N}$*

$$0 < \underline{c}(k^l, \alpha) \leq \overline{C}(k^l, \alpha) < 1, \text{ and } \lim_{l \to \infty} [\overline{C}(k^l, \alpha) - \underline{c}(k^l, \alpha)] = 0.$$

Further, we have $\underline{\kappa}(K_\omega, x, k^l, 0) = 1$ and $\overline{\kappa}(K_\omega, x, k^l, 1) = 0$ for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$.

Proof. By Lemma 4.2 it is enough to prove the claim for all large l. Suppose $l \in \mathbb{N}$ so that $k^{-l} < \min\{1 - \alpha, \alpha/4\}$. Let $Y_i^{k^l,\alpha}(\eta, \omega) = 1$, if $J_{\eta|_{il}}$ is chosen and there exists an α-hole at scale il near $x(\eta)$ which is completely inside $J_{\eta|_{il}} \setminus J_{\eta|_{(i+1)l}}$, and 0 otherwise. Then $Y_i^{k^l,\alpha}$, $i \in \mathbb{N}$, are independent identically distributed random variables with expectation $\tilde{c}(k^l, \alpha) := \int Y_i^{k^l,\alpha}(\eta, \omega) dQ(\eta, \omega) > 0$, guaranteed by the choice of l. According to the strong law of large numbers,

$$\lim_{j \to \infty} \frac{1}{j} \sum_{i=1}^{j} Y_i^{k^l,\alpha}(\eta, \omega) = \tilde{c}(k^l, \alpha)$$

for Q-almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$. This proves the lower bound since $\underline{c}(k^l, \alpha) \geq \tilde{c}(k^l, \alpha)$.

Note that if there is an α-hole at scale il near $x(\eta)$, then at least half of this hole is in $J_{\eta|_{il}} \setminus J_{\eta|_{(i+1)l}}$. Thus there is no α-hole at scale il if $Y_i^{k^l,\alpha/2}(\eta, \omega) = 0$. Since $\int Y_i^{k^l,\alpha/2}(\eta, \omega) dQ(\eta, \omega) < 1$ by the choice of l, we have (by the strong law of large numbers) that

$$\overline{C}(k^l, \alpha) \leq 1 - (1 - \int Y_i^{k^l,\alpha/2}(\eta, \omega) dQ(\eta, \omega)) < 1.$$

Consider $\delta > 0$, $\alpha' < \alpha$, and $l \in \mathbb{N}$ such that $k^{-l} < \delta$. There is an α-hole at scale il near $x(\eta)$ if the entire α-hole is in $J_{\eta|_{il}} \setminus J_{\eta|_{(i+1)l}}$ (i.e. $Y_i^{k^l,\alpha}(\eta, \omega) = 1$), or if $Y_i^{k^l,\alpha}(\eta, \omega) = 0$, $Y_i^{k^l,\alpha'}(\eta, \omega) = 1$, and a part of the α-hole is in $J_{\eta|_{(i+1)l}}$ adjacent to the α'-hole in $J_{\eta|_{il}} \setminus J_{\eta|_{(i+1)l}}$. By the choice of l, the latter is possible only if $\alpha' \geq \alpha - \delta$ and

(i) the limit set intersects the $(\alpha-\alpha')k^{-il}$-neighbourhood of one of the faces of the α'-hole in $J_{\eta|_{il}} \setminus J_{\eta|_{(i+1)l}}$ (since one cannot increase the α'-hole up to an α-hole inside $J_{\eta|_{il}} \setminus J_{\eta|_{(i+1)l}}$), or

(ii) the distance to $J_{\eta|_{(i+1)l}}$ from a face of $J_{\eta|_{il}}$ is more than $k^{-il}\alpha'$ but less than $k^{-il}\alpha$ (in this case any α-hole containing the α'-hole would intersect either $J_{\eta|_{(i+1)l}}$ or the complement of $J_{\eta|_{il}}$).

Both cases yield a series of events whose indicator functions are independent identically distributed random variables, and whose expectations tend to zero as l tends to infinity (in the first case by Lemma 2.1). Hence $\lim_{l\to\infty}[\overline{C}(k^l,\alpha)-\underline{c}(k^l,\alpha)]=0$.

The claim $\underline{\kappa}(K_\omega,x,k^l,0)=1$ is obvious. Finally, if the upper mean porosity $\overline{\kappa}(K_\omega,x,k^l,1)>0$, then there exists a scale j such that there is a 1-hole near x at scale j. But this signifies that x is in the boundary of the hole and $J_{\eta|_j}$, and so there is a 1-hole near x at all scales larger than j. Thus $\underline{\kappa}(K_\omega,x,k^l,\alpha)=1$ for all $\alpha\le 1$. Since $\underline{\kappa}(K_\omega,x,k^l,\alpha)<1$ for ν_ω-almost all x, the last claim follows. □

Remark 4.4. Mandelbrot percolation is not mean porous in the sense of [BS]. Indeed, assume that it is. Then one can find α, ρ, N, n_0, and $A\subset I^{\mathbb{N}}\times\Omega$ such that $Q(A)>0$ and N and n_0 are like in Remark 3.6 for all $(\eta,\omega)\in A$. Taking $l>N$, one finds $\alpha'>0$ such that $\underline{\kappa}(K_\omega,x,k^l,\alpha')=1$ for all $(\eta,\omega)\in A$. Since $Q(A)>0$ this is a contradiction with Proposition 4.3.

To compare the mean porosities of the limit set and the construction measure, we need the following lemma concerning the random variables X_j (see section 2).

Lemma 4.5. *For all $s>0$, the sequence $\mathbf{1}_{\{X_j\le s\}}$ satisfies the (weak) law of large numbers.*

Proof. Since the joint distribution of (X_i,X_j) depends on $j-i$ only, it is enough to prove that $\lim_{j\to\infty}\operatorname{Cov}(\mathbf{1}_{\{X_0\le s\}},\mathbf{1}_{\{X_j\le s\}})=0$ for the Bernstein's theorem to ensure the law of large numbers. Let $Y_j=X_0-k^{-jd}X_j$, $q=Q(X_0\le s)$, and $\varepsilon>0$. It is easy to notice that Y_j and X_j are Q-independent. From [AN] we know that the distribution of X_j is continuous on $(0,\infty)$, therefore there exists $\delta>0$ such that $Q(X_0\le s+\delta)\le(1+\varepsilon)q$. For all j such that $E_Q[X_j]k^{-jd}/\delta<\varepsilon$ (note that $E_Q[X_j]=E_Q[X_0]$ for all $j\in\mathbb{N}$), we have

$$\begin{aligned}
\operatorname{Cov}(\mathbf{1}_{\{X_0\le s\}},\mathbf{1}_{\{X_j\le s\}}) &= Q(X_0\le s \text{ and } X_j\le s\})-q^2\\
&\le q\big(Q(Y_j\le s)-q\big)\\
&= q\big(Q(Y_j\le s \text{ and } k^{-jd}X_j\le\delta\})\\
&\quad +Q(Y_j\le s \text{ and } k^{-jd}X_j>\delta)-q\big)\\
&\le q((1+\varepsilon)q-q+E_Q[X_j]k^{-jd}/\delta)\\
&= \varepsilon q(q+1).
\end{aligned}$$

The result follows. $\square$

Proposition 4.6. *Let* $0 < \alpha < 1$. *For all* $\delta > 0$ *there exists* $l_0 \in \mathbb{N}$ *such that for all* $l \geq l_0$ *corresponds* $\xi(l) \geq 0$ *such that*

$$\underline{\kappa}(\nu_\omega, x(\eta), k^l, \alpha + \delta) \leq \underline{\kappa}(K_\omega, x(\eta), k^l, \alpha)$$

for all $(\eta, \omega) \in (I^{\mathbb{N}} \times \Omega) \setminus E$ *with* $Q(E) \leq \xi(l)$. *Here* $\xi(l)$ *tends to zero as* l *tends to infinity.*

Proof. Let $\delta, \zeta > 0$ and take l_0 such that $k^{-l_0} < \delta/2$ and $\overline{C}(k^l, \alpha) - \underline{c}(k^l, \alpha) < \zeta/8$ for all $l \geq l_0$ (see Proposition 4.3). Suppose that $q = Q(\Lambda) > 0$ for some $\zeta > 0$ and $l \geq l_0$ where

$$\Lambda = \{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid \underline{\kappa}(\nu_\omega, x(\eta), k^l, \alpha + \delta) - \underline{\kappa}(K_\omega, x(\eta), k^l, \alpha) \geq \zeta\}.$$

Let

$$\begin{aligned}\hat{N}_i(k^l, \alpha, x, \varepsilon) = \operatorname{card}\{&j \in \mathbb{N} \mid j \leq i \text{ and there exists an } (\alpha+\delta, \varepsilon)\text{-hole} \\ &\text{at scale } jl \text{ near } x \text{ but there is no } \alpha\text{-hole at scale } jl \text{ near } x\}.\end{aligned}$$

Then for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ and $\Lambda' \subset \Lambda$ such that $Q(\Lambda') > q/2$ and $\hat{N}_i(k^l, \alpha, x(\eta), \varepsilon)/i > \zeta/2$ for all $i \geq n_0$ and for all $(\eta, \omega) \in \Lambda'$.

Considering the relative positions of an $(\alpha + \delta, \varepsilon)$ hole H at scale jl near $x(\eta)$ and $J_{\eta|_{(j+1)l}}$, when there is no α-hole at the same scale near the same point, we arrive at three possibilities:

(i) $J_{\eta|_{(j+1)l}} \subset H$, thus $\nu_\omega(J_{\eta|_{(j+1)l}}) \leq \varepsilon \nu_\omega(J_{\eta|_{jl}})$,
(ii) $J_{\eta|_{(j+1)l}} \cap H = \emptyset$, thus there exists a code τ_j of length $(j+1)l$ such that $J_{\tau_j} \cap J_{\eta|_{(j+1)l}} = \emptyset$, $J_{\tau_j} \subset H$ and $K_\omega \cap J_{\tau_j} \neq \emptyset$, and
(iii) $J_{\eta|_{(j+1)l}}$ and H intersect only partially, then one can still find a code τ_j like in the case (ii) because otherwise there would exist an α-hole.

According to [MW, Theorems 3.2 and 3.3], $\nu_\omega(J_\sigma) = \operatorname{diam}(J_\sigma)^d X_\sigma$ where the diameter of a set A is denoted by $\operatorname{diam}(A)$. Denote by A_j the event that $\nu_\omega(J_{\eta|_{(j+1)l}}) \leq \varepsilon \nu_\omega(J_{\eta|_{jl}})$, i.e.

$$A_j = \Big\{ X_{\eta|_{(j+1)l}} \leq \frac{\varepsilon}{1-\varepsilon} \sum_{\substack{\eta|_{jl} \prec \tau,\ |\tau| = (j+1)l \\ \tau \neq \eta|_{(j+1)l},\ \omega(\tau) = c}} X_\tau \Big\}.$$

Let for all $s > 0$

$$A^s_{j,1} = \Big\{ X_{\eta|_{(j+1)l}} \leq \frac{s\varepsilon}{1-\varepsilon} \Big\} \text{ and } A^s_{j,2} = \Big\{ \sum_{\substack{\eta|_{jl} \prec \tau,\ |\tau| = (j+1)l \\ \tau \neq \eta|_{(j+1)l},\ \omega(\tau) = c}} X_\tau > s \Big\}.$$

Then $\mathbf{1}_{A_j} \leq \mathbf{1}_{A^s_{j,1}} + \mathbf{1}_{A^s_{j,2}}$. Since the indicator functions of events $A^s_{j,2}$, $j \in \mathbb{N}$, are independent identically distributed random variables and

$\lim_{s\to\infty} Q(A^s_{j,2}) = 0$, the strong law of large numbers implies that

$$\lim_{j\to\infty} \frac{1}{j}\sum_{i=1}^{j} \mathbf{1}_{A^s_{i,2}} < \frac{\zeta}{8}$$

Q-almost surely for all s large enough. By Lemma 4.5 we can choose ε small enough so that for all large enough n

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{A^s_{i,1}} < \frac{\zeta}{8}$$

with probability at least $1 - q/8$.

In cases (ii) and (iii) let

$$A_j = \{\exists\, \tau \succ \eta|_{jl},\ \tau \neq \eta|_{(j+1)l},\ |\tau| = (j+1)l \text{ and } X_{\eta|_{jl}} \geq \varepsilon^{-1} k^{ld} X_\tau > 0\}.$$

Defining

$A^s_{j,1} = \{X_{\eta|_{jl}} > s\}$ and

$A^s_{j,2} = \{\exists\, \tau \succ \eta|_{jl},\ \tau \neq \eta|_{(j+1)l},\ |\tau| = (j+1)l \text{ and } s \geq \varepsilon^{-1} k^{ld} X_\tau > 0\}$,

we have $\mathbf{1}_{A_j} \leq \mathbf{1}_{A^s_{j,1}} + \mathbf{1}_{A^s_{j,2}}$. Recall that for any $\tau \in I^*$, $P(X_\tau > 0 \mid K_\omega \cap J_\tau \neq \emptyset) = 1$ by [MW, Theorem 3.4]. Again by Lemma 4.5, we can choose s so large that for all large enough n

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{A^s_{i,1}} < \frac{\zeta}{8}$$

with probability at least $1 - q/8$. Then by the strong law of large numbers we can choose ε small enough so that

$$\lim_{j\to\infty} \frac{1}{j}\sum_{i=1}^{j} \mathbf{1}_{A^s_{i,2}} < \frac{\zeta}{8}$$

Q-almost surely. This yields a contradiction. □

Remark 4.7. By Propositions 4.3 and 4.6 $0 < \underline{\kappa}(\nu_\omega, x, k^l, \alpha) < 1$ almost surely for all $l \in \mathbb{N}$ and $0 < \alpha < 1$ (see also Lemma 4.2). Whether $\overline{\kappa}(\nu_\omega, x, k^l, \alpha) < 1$, $c(k^l, \alpha) = C(k^l, \alpha)$, and $\kappa(K_\omega, x, k^l, \alpha) = \kappa(\nu_\omega, x, k^l, \alpha)$ almost surely remains an open problem.

Although our methods are not strong enough to give the complete understanding of the mean porosities in the Mandelbrot percolation, they are powerful enough from the point of lower and upper porosities. Indeed, the following theorem solves the Conjecture 3.2 stated in [JJM].

Theorem 4.8. *For P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$ we have*

$$\underline{\mathrm{por}}(K_\omega, x) = \underline{\mathrm{por}}(\nu_\omega, x) = 0, \quad \overline{\mathrm{por}}(K_\omega, x) = \frac{1}{2}, \ \textit{and}\ \overline{\mathrm{por}}(\nu_\omega, x) = 1.$$

Proof. If the mean porosities were defined using balls instead of construction cubes, the first claim would follow directly from Propositions 4.3 and 4.6. However, it is possible that there is an α-hole at scale il outside the construction cube $Q^k_{il}(x)$. Consider $l > 8$ large enough so that $k^{-l} < \alpha/2$, and the event that $x(\eta)$ is within the ϱ-distance $k^{-l(i+1)}/4$ from the centre of the cube $J_{\eta|_{il}}$ and there does not exist an α-hole in $J_{\eta|_{il}} \setminus J_{\eta|_{(i+1)l}}$. Proceeding like in the proof of Proposition 4.3, we get that for any $\alpha, \varepsilon > 0$ there are infinitely many scales i such that

$$\mathrm{por}(K_\omega, x, k^{-il}) < \alpha \text{ and } \mathrm{por}(\nu_\omega, x, k^{-il}, \varepsilon) < \alpha.$$

Thus we have

$$\underline{\mathrm{por}}(K_\omega, x) = 0 = \underline{\mathrm{por}}(\nu_\omega, x)$$

for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$. The inequality $c(k^l, \alpha) > 0$ for any $\alpha < 1$ implies

$$\overline{\mathrm{por}}(K_\omega, x) = \frac{1}{2}$$

for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$. It is easy to see that the measure ν_ω is non-doubling Q-almost surely. Thus the last claim follows from [MM, Proposition 4]. □

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland[1,2]

E-mail address: `artemiberlinkov@pochtamt.ru`[1]

E-mail address: `esaj@maths.jyu.fi`[2]