ON NATURAL INVARIANT MEASURES ON GENERALISED ITERATED FUNCTION SYSTEMS

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Abstract. We consider the limit set of iterated function systems. Under the assumption of a natural potential, the so called cylinder function, we prove the existence of the invariant probability measure satisfying the equilibrium state. We motivate this approach by showing that for typical self–affine sets there exists an ergodic invariant measure having the same Hausdorff dimension as the set itself.

1. Introduction

It is well known that applying methods of thermodynamical formalism, we can find ergodic invariant measures on self–similar and self–conformal sets satisfying the equilibrium state and having the same Hausdorff dimension as the set itself. See, for example, Bowen [3], Hutchinson [10] and Mauldin and Urbański [14]. In this work we try to generalise this concept. Our main objective is to study iterated function systems (IFS) even though we develop our theory in a more general setting.

We introduce the definition of a cylinder function, which is a crucial tool in developing the corresponding concept of thermodynamical formalism for our setting. The use of the cylinder function provides us a sufficiently general framework to study iterated function systems. We could also use the notation of subadditive thermodynamical formalism like in Falconer [5], [7] and Barreira [2], but we feel that in studying iterated function systems we should use more IFS–style notation. We can think that the idea of the cylinder function is to generalise the mass distribution, which is well explained in Falconer [6]. Falconer proved in [5] that for each approximative equilibrium state there exists an approximative equilibrium measure, that is, there is a k–invariant measure for which the approximative topological pressure equals to the sum of the corresponding entropy and energy. More precisely, using the notation of this work, for each $t \geq 0$ there exists a Borel probability measure μ_k such that

$$\tfrac{1}{k} P^k(t) = \tfrac{1}{k} h^k_{\mu_k} + \tfrac{1}{k} E^k_{\mu_k}(t). \tag{1.1}$$

Letting now $k \to \infty$, the approximative equilibrium state converges to the desired equilibrium state, but unfortunately we will lose the invariance. However, Barreira [2] showed that the desired equilibrium state can be attained as a supremum, that is,

$$P(t) = \sup\big(h_\mu + E_\mu(t)\big), \tag{1.2}$$

where the supremum is taken over all invariant Borel regular probability measures. Using the concept of generalised subadditivity, we show that it is possible to attain the supremum in (1.2). We also prove that this equilibrium measure is ergodic.

Date: 20th August 2003.

1991 *Mathematics Subject Classification.* Primary 28A80; Secondary 37C45.

The author is supported by the Academy of Finland, projects 23795 and 53292.

We start developing our theory in the symbol space and after proving the existence of the equilibrium measure, we begin to consider the geometric projections of the symbol space and the equilibrium measure. The use of the cylinder function provides us with a significant generality in producing equilibrium measures for different kind of settings. A natural question now is: What can we say about the Hausdorff dimension of the projected symbol space, the so called limit set? To answer this question we have to assume something on our geometric projection. We use the concept of an iterated function system for getting better control of cylinder sets, the sets defining the geometric projection. To be able to approximate the size of the limit set, we also need some kind of separation condition for cylinder sets to avoid too much overlapping among these sets. Several separation conditions are introduced and relationships between them are studied in detail. We also study a couple of concrete examples, namely the similitude IFS, the conformal IFS and the affine IFS, and we look how our theory turns out in these particular cases. As an easy consequence we notice that the Hausdorff dimension of equilibrium measures of the similitude IFS and the conformal IFS equals to the Hausdorff dimension of the corresponding limit sets, the self–similar set and the self–conformal set. After proving the ergodicity and studying dimensions of the equilibrium measure in our more general setting, we obtain the same information for "almost all" affine IFS's by applying Falconer's result for the Hausdorff dimension of self–affine sets. This gives a partially positive answer to the open question proposed by Kenyon and Peres [12].

Before going into more detailed preliminaries, let us fix some notation. As usual, let I be a finite set with at least two elements. Put $I^* = \bigcup_{n=1}^\infty I^n$ and $I^\infty = I^\mathbb{N} = \{(i_1, i_2, \ldots) : i_j \in I \text{ for } j \in \mathbb{N}\}$. Thus, if $\mathtt{i} \in I^*$, there is $k \in \mathbb{N}$ such that $\mathtt{i} = (i_1, \ldots, i_k)$, where $i_j \in I$ for all $j = 1, \ldots, k$. We call this k the *length* of $\mathtt{i}$ and we denote $|\mathtt{i}| = k$. If $\mathtt{j} \in I^* \cup I^\infty$, then with the notation $\mathtt{i},\mathtt{j}$ we mean the element obtained by juxtaposing the terms of $\mathtt{i}$ and $\mathtt{j}$. If $\mathtt{i} \in I^\infty$, we denote $|\mathtt{i}| = \infty$, and for $\mathtt{i} \in I^* \cup I^\infty$ we put $\mathtt{i}|_k = (i_1, \ldots, i_k)$ whenever $1 \le k < |\mathtt{i}|$. We define $[\mathtt{i}; A] = \{\mathtt{i},\mathtt{j} : \mathtt{j} \in A\}$ as $\mathtt{i} \in I^*$ and $A \subset I^\infty$ and we call the set $[\mathtt{i}] = [\mathtt{i}, I^\infty]$ the *cylinder set of level* $|\mathtt{i}|$. We say that two elements $\mathtt{i},\mathtt{j} \in I^*$ are *incomparable* if $[\mathtt{i}] \cap [\mathtt{j}] = \emptyset$. Furthermore, we call a set $A \subset I^*$ incomparable if all its elements are mutually incomparable. For example, the sets I and $\{(i_1, i_2), (i_1, i_1, i_2)\}$, where $i_1 \ne i_2$, are incomparable subsets of I^*.

Define

$$|\mathtt{i} - \mathtt{j}| = \begin{cases} 2^{-\min\{k-1 \,:\, \mathtt{i}|_k \ne \mathtt{j}|_k\}}, & \mathtt{i} \ne \mathtt{j} \\ 0, & \mathtt{i} = \mathtt{j} \end{cases} \tag{1.3}$$

whenever $\mathtt{i},\mathtt{j} \in I^\infty$. Then the couple $(I^\infty, |\cdot|)$ is a compact metric space. Let us call $(I^\infty, |\cdot|)$ a *symbol space* and an element $\mathtt{i} \in I^\infty$ a *symbol*. If there is no danger of misunderstanding, let us call also an element $\mathtt{i} \in I^*$ a symbol. Define the *left shift* $\sigma : I^\infty \to I^\infty$ by setting

$$\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots). \tag{1.4}$$

Clearly σ is continuous and surjective. If $\mathtt{i} \in I^n$ for some $n \in \mathbb{N}$, then with the notation $\sigma(\mathtt{i})$ we mean the symbol $(i_2, \ldots, i_n) \in I^{n-1}$. Sometimes, without mentioning it explicitly, we work also with "empty symbols", that is, symbols with zero length.

For each cylinder we define a cylinder function $\psi_{\mathtt{i}}^t : I^\infty \to (0,\infty)$ depending also on a given parameter $t \ge 0$. The exact definition is introduced at the beginning of the second chapter. To follow this introduction, the reader is encouraged to keep in mind the idea of the mass distribution. With the help of the cylinder function we define a *topological pressure* $P : [0,\infty) \to \mathbb{R}$ by setting

$$P(t) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}), \tag{1.5}$$

where $\mathtt{h} \in I^\infty$ is some fixed point. Denoting with $\mathcal{M}_\sigma(I^\infty)$ the collection of all Borel regular probability measures on I^∞ which are *invariant*, that is, $\mu([\mathtt{i}]) = \sum_{i\in I} \mu([i,\mathtt{i}])$ for every $\mathtt{i} \in I^*$, we define an *energy* $E_\mu : [0,\infty) \to \mathbb{R}$ by setting

$$E_\mu(t) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \psi_{\mathtt{i}}^t(\mathtt{h}) \tag{1.6}$$

and an *entropy* h_μ by setting

$$h_\mu = -\lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \mu([\mathtt{i}]). \tag{1.7}$$

For the motivation of these definitions, see, for example, Mauldin and Urbański [14] and Falconer [8]. For every $\mu \in \mathcal{M}_\sigma(I^\infty)$ we have $P(t) \ge h_\mu + E_\mu(t)$, and if there exists a measure $\mu \in \mathcal{M}_\sigma(I^\infty)$ for which

$$P(t) = h_\mu + E_\mu(t), \tag{1.8}$$

we call this measure a t–*equilibrium measure*. Using the generalised subadditivity, we will prove the existence of the t–equilibrium measure. We obtain the ergodicity of that measure essentially because $\mu \mapsto h_\mu + E_\mu(t)$ is an affine mapping from a convex set whose extreme points are ergodic and then recalling Choquet's theorem. Applying now Kingman's subadditive ergodic theorem and the theorem of Shannon–McMillan, we notice that

$$P(t) = \lim_{n\to\infty} \tfrac{1}{n} \log \frac{\psi_{\mathtt{i}|_n}^t(\mathtt{h})}{\mu([\mathtt{i}|_n])} \tag{1.9}$$

for μ–almost all $\mathtt{i} \in I^\infty$ as μ is the t–equilibrium measure. Following the ideas of Falconer [5], we introduce an equilibrium dimension $\dim_\psi$ for which $\dim_\psi(I^\infty) = t$ exactly when $P(t) = 0$. Using the ergodicity, we will also prove that $\dim_\psi(A) = t$ if $P(t) = 0$ and $\mu(A) = 1$, where μ is the t–equilibrium measure. In other words, the equilibrium measure μ is ergodic, invariant and has full equilibrium dimension.

To project this setting into $\mathbb{R}^d$ we need some kind of geometric projection. With the geometric projection here we mean mappings obtained by the following construction. Let $X \subset \mathbb{R}^d$ be a compact set with nonempty interior. Choose then a collection $\{X_{\mathtt{i}} : \mathtt{i} \in I^*\}$ of nonempty closed subsets of X satisfying

(1) $X_{\mathtt{i},i} \subset X_{\mathtt{i}}$ for every $\mathtt{i} \in I^*$ and $i \in I$,
(2) $d(X_{\mathtt{i}}) \to 0$, as $|\mathtt{i}| \to \infty$.

Here d means the diameter of a given set. We define a *projection mapping* to be the function $\pi : I^\infty \to X$, for which

$$\{\pi(\mathtt{i})\} = \bigcap_{n=1}^{\infty} X_{\mathtt{i}|_n} \tag{1.10}$$

as $\mathtt{i} \in I^\infty$. The compact set $E = \pi(I^\infty)$ is called a *limit set*, and if there is no danger of misunderstanding, we call also the sets $\pi([\mathtt{i}])$, where $\mathtt{i} \in I^*$, cylinder sets. In general, it is really hard to study the geometric properties of the limit set, for example, to determine the Hausdorff dimension. We might come up against the following problems: There is too much overlapping among the cylinder sets and it is too difficult to approximate the size of these sets. Therefore we introduce geometrically stabile IFS's. With the *iterated function system (IFS)* we mean the collection $\{\varphi_i : i \in I\}$ of contractive injections from Ω to Ω, for which $\varphi_i(X) \subset X$ as $i \in I$. Here $\Omega \supset X$ is an open subset of $\mathbb{R}^d$. We set $X_{\mathtt{i}} = \varphi_{\mathtt{i}}(X)$, where $\varphi_{\mathtt{i}} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_{|\mathtt{i}|}}$ as $\mathtt{i} \in I^*$, and making now a suitable choice for the mappings φ_i, we can have the limit set E to be a self–similar set or a self–affine set, for example. Likewise, changing the choice of the cylinder function, we can have the equilibrium measure μ to have different kind of properties, and thus, making a suitable choice, the measure $m = \mu \circ \pi^{-1}$ might be useful in studying the geometric properties of the limit set. If there is no danger of misunderstanding, we call also the projected equilibrium measure m an equilibrium measure. We say that IFS is *geometrically stabile* if it satisfies a bounded overlapping condition and the mappings of IFS satisfy the following bi–Lipschitz condition: for each $\mathtt{i} \in I^*$ there exist constants $0 < \underline{s}_{\mathtt{i}} < \overline{s}_{\mathtt{i}} < 1$ such that

$$\underline{s}_{\mathtt{i}}|x - y| \le |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le \overline{s}_{\mathtt{i}}|x - y| \tag{1.11}$$

for every $x, y \in \Omega$. The exact definition of these constants is introduced in Chapter 3. To follow this introduction the reader can think for simplicity that for each $i \in I$ there exist such constants and $\underline{s}_{\mathtt{i}} = \underline{s}_{i_1} \cdots \underline{s}_{i_{|\mathtt{i}|}}$ and $\overline{s}_{\mathtt{i}} = \overline{s}_{i_1} \cdots \overline{s}_{i_{|\mathtt{i}|}}$ as $\mathtt{i} \in I^*$. The upper and lower bounds of the bi–Lipschitz condition are crucial for getting upper and lower bounds for the size of the cylinder sets. The *bounded overlapping* is satisfied if the cardinality of the set $\{\mathtt{i} \in I^* : \varphi_{\mathtt{i}}(X) \cap B(x, r) \ne \emptyset \text{ and } \underline{s}_{\mathtt{i}} < r \le \underline{s}_{\mathtt{i}|_{|\mathtt{i}|-1}}\}$ is bounded as $x \in X$ and $0 < r < r_0 = r_0(x)$.

The class of geometrically stabile IFS's includes many interesting cases of IFS's, for example, a conformal IFS satisfying the OSC and the so called boundary condition and an affine IFS satisfying the SSC. The open set condition (OSC) and the strong separation condition (SSC) are commonly used examples of separation conditions we need to use for having not too much overlapping among the cylinder sets. We prove that for the Hausdorff dimension of the limit set of geometrically stabile IFS's, there exist natural upper and lower bounds obtained from the bi–Lipschitz constants. It is now very tempting to guess that for geometrically stabile IFS's, making a good choice for the cylinder function, it could be possible to have the same equilibrium dimension and Hausdorff dimension for the limit set, and thus it would be possible to obtain the Hausdorff dimension from the behaviour of the topological pressure. It has been already proved that this is true for similitude and conformal IFS's and also for "almost all" affine IFS's. Recalling now that the equilibrium measure has full equilibrium dimension, we conclude that in many cases, like in “almost all” affine IFS's, making a good choice for the cylinder function, we can have an ergodic invariant measure on the limit set having full Hausdorff dimension.

Acknowledgement. The author is deeply indebted to Professor Pertti Mattila for his valuable comments and suggestions for the manuscript.

2. Cylinder function and equilibrium measure

In this chapter we introduce the definition of the cylinder function. Using the cylinder function we are able to define tools of thermodynamical formalism. In this setting we prove the existence of a so called equilibrium measure.

Take $t \geq 0$ and $\mathtt{i} \in I^*$. We call a function $\psi_{\mathtt{i}}^t : I^\infty \to (0,\infty)$ a *cylinder function* if it satisfies the following three conditions:

(1) There exists $K_t \geq 1$ depending only on t such that
$$\psi_{\mathtt{i}}^t(\mathtt{h}) \leq K_t \psi_{\mathtt{i}}^t(\mathtt{j}) \tag{2.1}$$
for any $\mathtt{h}, \mathtt{j} \in I^\infty$.

(2) For every $\mathtt{h} \in I^\infty$ and integer $1 \leq j < |\mathtt{i}|$ we have
$$\psi_{\mathtt{i}}^t(\mathtt{h}) \leq \psi_{\mathtt{i}|_j}^t\big(\sigma^j(\mathtt{i}), \mathtt{h}\big)\psi_{\sigma^j(\mathtt{i})}^t(\mathtt{h}). \tag{2.2}$$

(3) For any given $\delta > 0$ there exist constants $0 < \underline{s}_\delta < 1$ and $0 < \overline{s}_\delta < 1$ depending only on δ such that
$$\psi_{\mathtt{i}}^t(\mathtt{h})\underline{s}_\delta^{|\mathtt{i}|} \leq \psi_{\mathtt{i}}^{t+\delta}(\mathtt{h}) \leq \psi_{\mathtt{i}}^t(\mathtt{h})\overline{s}_\delta^{|\mathtt{i}|} \tag{2.3}$$
for every $\mathtt{h} \in I^\infty$. We assume also that $\underline{s}_\delta, \overline{s}_\delta \nearrow 1$ as $\delta \searrow 0$ and that $\psi_{\mathtt{i}}^0 \equiv 1$.

Note that when we speak about one cylinder function, we always assume there is a collection of them defined for $\mathtt{i} \in I^*$ and $t > 0$. Let us comment on these conditions. The first one is called the *bounded variation principle (BVP)* and it says that the value of $\psi_{\mathtt{i}}^t(\mathtt{h})$ cannot vary too much; roughly speaking, $\psi_{\mathtt{i}}^t$ is essentially constant. The second condition is called the *submultiplicative chain rule for the cylinder function* or just *subchain rule* for short. If the subchain rule is satisfied with equality, we call it a *chain rule*. The third condition is there just to guarantee the nice behaviour of the cylinder function with respect to the parameter t. It also implies that
$$\underline{s}_t^{|\mathtt{i}|} \leq \psi_{\mathtt{i}}^t(\mathtt{h}) \leq \overline{s}_t^{|\mathtt{i}|} \tag{2.4}$$
with any choice of $\mathtt{h} \in I^\infty$.

For each $k \in \mathbb{N}$, $\mathtt{i} \in I^{k*} := (I^k)^*$ and $t \geq 0$ define a function $\psi_{\mathtt{i}}^{t,k} : I^\infty \to (0,\infty)$ by setting
$$\psi_{\mathtt{i}}^{t,k}(\mathtt{h}) = \prod_{j=0}^{|\mathtt{i}|/k-1} \psi_{\sigma^{jk}(\mathtt{i})|_k}^t\big(\sigma^{(j+1)k}(\mathtt{i}), \mathtt{h}\big) \tag{2.5}$$
as $\mathtt{h} \in I^\infty$. Clearly, now $\psi_{\mathtt{i}}^t(\mathtt{h}) \leq \psi_{\mathtt{i}}^{t,k}(\mathtt{h})$ for every $k \in \mathbb{N}$ and $\mathtt{i} \in I^{k*}$ using the subchain rule. Note that if the chain rule is satisfied, then $\psi_{\mathtt{i}}^t(\mathtt{h}) = \psi_{\mathtt{i}}^{t,k}(\mathtt{h})$ for every $k \in \mathbb{N}$ and that we always have $\psi_{\mathtt{i}}^t(\mathtt{h}) = \psi_{\mathtt{i}}^{t,|\mathtt{i}|}(\mathtt{h})$.

It is very tempting to see these functions as cylinder functions satisfying the chain rule on I^{k*}. Indeed, straight from the definitions we get the chain rule and condition (3) satisfied. However, to get the BVP for $\psi_{\mathtt{i}}^{t,k}$ we need better information on the local behaviour of the function $\psi_{\mathtt{i}}^t$. More precisely, we need better control over the variation of $\psi_{\mathtt{i}}^t$ in small scales. We call a cylinder function from which we get the BVP for $\psi_{\mathtt{i}}^{t,k}$ with any choice of $k \in \mathbb{N}$ *smooth cylinder function.* We say that a mapping $f : I^\infty \to \mathbb{R}$ is a *Dini function* if

$$\int_0^1 \frac{\omega_f(\delta)}{\delta} d\delta < \infty, \tag{2.6}$$

where

$$\omega_f(\delta) = \sup_{|\mathtt{i}-\mathtt{j}|\le\delta} |f(\mathtt{i}) - f(\mathtt{j})| \tag{2.7}$$

is the *modulus of continuity.* Observe that Hölder continuous functions are always Dini.

Proposition 2.1. *Suppose the cylinder function is Dini. Then it is smooth and functions $\psi_{\mathtt{i}}^{t,k}$ are cylinder functions satisfying the chain rule on I^{k*}.*

Proof. It suffices to verify the BVP. For each $k \in \mathbb{N}$ we denote $\omega_k(\delta) = \max_{\mathtt{i}\in I^k} \omega_{\psi_{\mathtt{i}}^t}(\delta)$. Using now the assumption and the definitions we have for each $\mathtt{i} \in I^{k*}$

$$\begin{aligned}
\log \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) - \log \psi_{\mathtt{i}}^{t,k}(\mathtt{j}) &= \sum_{j=0}^{|\mathtt{i}|/k-1} \log\left(\frac{\psi^t_{\sigma^{jk}(\mathtt{i})|_k}\big(\sigma^{(j+1)k}(\mathtt{i}),\mathtt{h}\big)}{\psi^t_{\sigma^{jk}(\mathtt{i})|_k}\big(\sigma^{(j+1)k}(\mathtt{i}),\mathtt{j}\big)} \right) \\
&= \sum_{j=0}^{|\mathtt{i}|/k-1} \log\left(1 + \frac{\psi^t_{\sigma^{jk}(\mathtt{i})|_k}\big(\sigma^{(j+1)k}(\mathtt{i}),\mathtt{h}\big) - \psi^t_{\sigma^{jk}(\mathtt{i})|_k}\big(\sigma^{(j+1)k}(\mathtt{i}),\mathtt{j}\big)}{\psi^t_{\sigma^{jk}(\mathtt{i})|_k}\big(\sigma^{(j+1)k}(\mathtt{i}),\mathtt{j}\big)} \right) \\
&\le \underline{s}_t^{-k} \sum_{j=0}^{|\mathtt{i}|/k-1} \left| \psi^t_{\sigma^{jk}(\mathtt{i})|_k}\big(\sigma^{(j+1)k}(\mathtt{i}),\mathtt{h}\big) - \psi^t_{\sigma^{jk}(\mathtt{i})|_k}\big(\sigma^{(j+1)k}(\mathtt{i}),\mathtt{j}\big) \right| \\
&\le \underline{s}_t^{-k} \sum_{j=0}^{|\mathtt{i}|/k-1} \omega_k\big(2^{-(|\mathtt{i}|-(j+1)k)}\big) \qquad (2.8) \\
&\le \underline{s}_t^{-k} \int_0^\infty \omega_k\big(2^{-(\eta-1)k}\big) d\eta \\
&= \frac{1}{\underline{s}_t^k k \log 2} \int_0^1 \frac{\omega_k(\delta)}{\delta} d\delta,
\end{aligned}$$

whenever $\mathtt{h}, \mathtt{j} \in I^\infty$ by substituting $\eta = -\frac{1}{k}(\log_2 \delta) + 1$ and $d\eta = -(\delta k \log 2)^{-1} d\delta$. This gives

$$\frac{\psi_{\mathtt{i}}^{t,k}(\mathtt{h})}{\psi_{\mathtt{i}}^{t,k}(\mathtt{j})} \le K_{t,k}, \tag{2.9}$$

where the logarithm of $K_{t,k}$ equals to the finite upper bound found in (2.8). $\square$

Of course, a cylinder function satisfying the chain rule is always smooth. In this case the previous proposition gives us a sufficient condition for the BVP to hold. Next, we introduce an important property of functions of the following type. We say that a function $a : \mathbb{N} \times \mathbb{N} \cup \{0\} \to \mathbb{R}$ satisfies the *generalised subadditive condition* if

$$a(n_1 + n_2, 0) \le a(n_1, n_2) + a(n_2, 0) \tag{2.10}$$

and $|a(n_1, n_2)| \le n_1 C$ for some constant C. Furthermore, we say that this function is *subadditive* if in addition $a(n_1, n_2) = a(n_1, 0)$ for all $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N} \cup \{0\}$.

Lemma 2.2. *Suppose that a function $a : \mathbb{N} \times \mathbb{N} \cup \{0\} \to \mathbb{R}$ satisfies the generalised subadditive condition. Then*

$$\tfrac{1}{n} a(n, 0) \le \tfrac{1}{kn} \sum_{j=0}^{n-1} a(k, j) + \tfrac{3k}{n} C \tag{2.11}$$

for some constant C whenever $0 < k < n$. Moreover, if this function is subadditive, then the limit $\lim_{n\to\infty} \frac{1}{n} a(n, 0)$ exists and equals to $\inf_n \frac{1}{n} a(n, 0)$.

Proof. We follow the ideas found in Lemma 4.5.2 of Katok and Hasselblatt [11]. Fix $n \in \mathbb{N}$ and choose $0 < k < n$. Now for each integer $0 \le q < k$ we define $\alpha(q) = \lfloor \frac{n-q-1}{k} \rfloor$ to be the integer part of $\frac{n-q-1}{k}$. Straight from this definition we shall see that α is non–increasing,

$$n - k - 1 < \alpha(q)k + q \le n - 1 \tag{2.12}$$

and

$$\tfrac{n}{k} - 2 < \alpha(q) \le \tfrac{n-1}{k} \tag{2.13}$$

whenever $0 \le q < k$. Temporarily fix q and take $0 \le l < \alpha(q)$ and $0 \le i < k$. Now

$$q - 1 < lk + q + i < \alpha(q)k + q \tag{2.14}$$

and therefore,

$$\{0, \ldots, n-1\} = \{lk + q + i : 0 \le l < \alpha(q),\ 0 \le i < k\} \cup S_q, \tag{2.15}$$

where S_q is the union of the sets $S_q^1 = \{0, \ldots, q-1\}$ and $S_q^2 = \{\alpha(q)k + q, \ldots, n-1\}$. Using (2.12), we notice that $1 \le \#S_q^2 \le k$. It follows from (2.13) that $\alpha(q)$ can attain at maximum two values, namely $\lfloor \frac{n-1}{k} \rfloor$ and $\lfloor \frac{n-1}{k} \rfloor - 1$. Let q_0 be the largest integer for which $\alpha(q_0) = \lfloor \frac{n-1}{k} \rfloor$. Then clearly,

$$\{lk + q : 0 \le l \le \alpha(q),\ 0 \le q < k\} = \{0, \ldots, \alpha(q_0)k + q_0\}. \tag{2.16}$$

By the choice of q_0 it holds also that $\alpha(q_0) = (n - q_0 - 1)/k$ and thus $\alpha(q_0)k + q_0 = n - 1$.

It is clear that $\#S_q^1 = q$. It is also clear that $S_q^2 = \{n - k + q, \ldots, n-1\}$ if $q_0 = k - 1$. But if not, we notice that $\alpha(q_0 + 1) = \alpha(q_0) - 1 = (n - q_0 - k - 1)/k$, and thus $\alpha(q_0 + 1)k + q_0 + 1 = n - k$. Therefore, defining a bijection η between sets $\{0, \ldots, k-1\}$ and $\{1, \ldots, k\}$ by setting

$$\eta(q) = \begin{cases} q_0 - q + 1, & 0 \le q \le q_0 \\ q_0 - q + k + 1, & q_0 < q < k, \end{cases} \tag{2.17}$$

we have $\#S_q^2 = \eta(q)$ for all $0 \le q < k$.

Since n is of the form $\eta(q)+\alpha(q)k+q$ for any $0 \le q < k$, we get, using the assumption several times that

$$\begin{aligned} a(n,0) &= a\big(\eta(q),\alpha(q)k+q\big) + \sum_{l=1}^{\alpha(q)} a\big(k,(\alpha(q)-l)k+q\big) + a(q,0) \\ &\le \sum_{l=0}^{\alpha(q)-1} a(k,lk+q) + 2kC \qquad (2.18) \\ &\le \sum_{l=0}^{\alpha(q)} a(k,lk+q) + 3kC. \end{aligned}$$

In fact, we have

$$\begin{aligned} \tfrac{1}{n}a(n,0) &\le \tfrac{1}{kn}\sum_{q=0}^{k-1}\Bigg(\sum_{l=0}^{\alpha(q)} a(k,kl+q) + 3kC\Bigg) \\ &= \tfrac{1}{kn}\sum_{j=0}^{n-1} a(k,j) + \tfrac{3k}{n}C \qquad (2.19) \end{aligned}$$

using (2.16).

If our function is subadditive, we have

$$\limsup_{n\to\infty} \tfrac{1}{n}a(n,0) \le \tfrac{1}{k}a(k,0) \qquad (2.20)$$

with any choice of k using (2.19). This also finishes the proof. □

Now we define the basic concepts for thermodynamical formalism with the help of the cylinder function. Fix some $\mathtt{h} \in I^\infty$. We call the following limit

$$P(t) = \lim_{n\to\infty} \tfrac{1}{n}\log \sum_{\mathtt{i}\in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}) \qquad (2.21)$$

the *topological pressure for the cylinder function* or just *topological pressure* for short. For each $k \in \mathbb{N}$ we also denote

$$\begin{aligned} \overline{P}^k(t) &= \limsup_{n\to\infty} \tfrac{1}{n}\log \sum_{\mathtt{i}\in I^{kn}} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) \quad \text{and} \\ \underline{P}^k(t) &= \liminf_{n\to\infty} \tfrac{1}{n}\log \sum_{\mathtt{i}\in I^{kn}} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}). \qquad (2.22) \end{aligned}$$

If they agree, we denote the common value with $P^k(t)$. Recall that the collection of all Borel regular probability measures on I^∞ is denoted by $\mathcal{M}(I^\infty)$. Denote

$$\mathcal{M}_\sigma(I^\infty) = \{\mu \in \mathcal{M}(I^\infty) : \mu \text{ is invariant}\}, \qquad (2.23)$$

where the invariance of μ means that $\mu([\mathtt{i}]) = \mu\big(\sigma^{-1}([\mathtt{i}])\big)$ for every $\mathtt{i} \in I^*$. Now $\mathcal{M}_\sigma(I^\infty)$ is a nonempty closed subset of the compact set $\mathcal{M}(I^\infty)$ in the weak topology.

For given $\mu \in \mathcal{M}_\sigma(I^\infty)$ we define an *energy for the cylinder function* $E_\mu(t)$, or just *energy* for short, by setting

$$E_\mu(t) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \psi_{\mathtt{i}}^t(\mathtt{h}) \tag{2.24}$$

and an *entropy* h_μ by setting

$$h_\mu = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} H\big(\mu([\mathtt{i}])\big), \tag{2.25}$$

where $H(x) = -x\log x$, as $x > 0$, and $H(0) = 0$. Note that H is concave. For each $k \in \mathbb{N}$ we also denote

$$\begin{aligned} \overline{E}_\mu^k(t) &= \limsup_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^{kn}} \mu([\mathtt{i}])\psi_{\mathtt{i}}^{t,k}(\mathtt{h}) \quad \text{and} \\ \underline{E}_\mu^k(t) &= \liminf_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^{kn}} \mu([\mathtt{i}])\psi_{\mathtt{i}}^{t,k}(\mathtt{h}). \end{aligned} \tag{2.26}$$

If they agree, we denote the common value with $E_\mu^k(t)$. Finally, we similarly denote

$$h_\mu^k = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^{kn}} H\big(\mu([\mathtt{i}])\big). \tag{2.27}$$

Let us next justify the existence of these limits using the power of subadditive sequences. We will actually prove a little more than just subadditivity as we can see from the following lemma.

Lemma 2.3. *For any given $\mu \in \mathcal{M}(I^\infty)$ the following functions*

1. $(n_1, n_2) \mapsto \sum_{\mathtt{i}\in I^{n_1}} H\big(\mu\circ\sigma^{-n_2}([\mathtt{i}])\big)$ *and*
2. $(n_1, n_2) \mapsto \sum_{\mathtt{i}\in I^{n_1}} \mu\circ\sigma^{-n_2}([\mathtt{i}]) \log \psi_{\mathtt{i}}^t(\mathtt{h}) + \log K_t$

defined on $\mathbb{N}\times\mathbb{N}\cup\{0\}$ satisfy the generalised subadditive condition. Furthermore, if $\mu \in \mathcal{M}_\sigma(I^\infty)$, the functions are subadditive.

Proof. For every $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}\cup\{0\}$ we have

$$\begin{aligned} \sum_{\mathtt{i}\in I^{n_1+n_2}} H\big(\mu([\mathtt{i}])\big) &= -\sum_{\mathtt{i}\in I^{n_1}} \sum_{\mathtt{j}\in I^{n_2}} \mu([\mathtt{j},\mathtt{i}]) \log \mu([\mathtt{j},\mathtt{i}]) \\ &= -\sum_{\mathtt{i}\in I^{n_1}} \sum_{\mathtt{j}\in I^{n_2}} \mu([\mathtt{j},\mathtt{i}]) \log \frac{\mu([\mathtt{j},\mathtt{i}])}{\mu([\mathtt{j}])} - \sum_{\mathtt{i}\in I^{n_1}} \sum_{\mathtt{j}\in I^{n_2}} \mu([\mathtt{j},\mathtt{i}]) \log \mu([\mathtt{j}]) \\ &= \sum_{\mathtt{i}\in I^{n_1}} \sum_{\mathtt{j}\in I^{n_2}} \mu([\mathtt{j}]) H\left(\frac{\mu([\mathtt{j},\mathtt{i}])}{\mu([\mathtt{j}])}\right) + \sum_{\mathtt{j}\in I^{n_2}} H\big(\mu([\mathtt{j}])\big) \qquad (2.28) \\ &\le \sum_{\mathtt{i}\in I^{n_1}} H\left(\sum_{\mathtt{j}\in I^{n_2}} \mu([\mathtt{j},\mathtt{i}])\right) + \sum_{\mathtt{j}\in I^{n_2}} H\big(\mu([\mathtt{j}])\big) \end{aligned}$$

using the concavity of the function H. Note that while calculating, we can sum over only cylinders with positive measure. Using the concavity again, we get

$$\begin{aligned}\frac{1}{(\#I)^{n_1}}\sum_{\mathtt{i}\in I^{n_1}} H\Bigl(\sum_{\mathtt{j}\in I^{n_2}}\mu([\mathtt{j},\mathtt{i}])\Bigr) &\le H\Bigl(\frac{1}{(\#I)^{n_1}}\sum_{\mathtt{i}\in I^{n_1+n_2}}\mu([\mathtt{i}])\Bigr)\\ &= \frac{1}{(\#I)^{n_1}}\log(\#I)^{n_1},\end{aligned}\tag{2.29}$$

which finishes the proof of (1).

For every $n_1\in\mathbb{N}$ and $n_2\in\mathbb{N}\cup\{0\}$ we have

$$\begin{aligned}\sum_{\mathtt{i}\in I^{n_1+n_2}}\mu([\mathtt{i}])\log\psi^t_{\mathtt{i}}(\mathtt{h}) &\le \sum_{\mathtt{i}\in I^{n_1+n_2}}\mu([\mathtt{i}])\log\psi^t_{\sigma^{n_2}(\mathtt{i})}(\mathtt{h})\\ &\quad+\sum_{\mathtt{i}\in I^{n_1+n_2}}\mu([\mathtt{i}])\log\psi^t_{\mathtt{i}|_{n_2}}(\sigma^{n_2}(\mathtt{i}),\mathtt{h})\\ &\le \sum_{\mathtt{i}\in I^{n_1}}\mu\circ\sigma^{-n_2}([\mathtt{i}])\log\psi^t_{\mathtt{i}}(\mathtt{h})\\ &\quad+\sum_{\mathtt{i}\in I^{n_2}}\mu([\mathtt{i}])\log\psi^t_{\mathtt{i}}(\mathtt{h})+\log K_t\end{aligned}\tag{2.30}$$

using the BVP and the subchain rule. From the condition (3) of the definition of the cylinder function it follows that

$$n_1\log\underline{s}_t\le\sum_{\mathtt{i}\in I^{n_1+n_2}}\mu([\mathtt{i}])\log\psi^t_{\sigma^{n_2}(\mathtt{i})}(\mathtt{h})\le n_1\log\overline{s}_t,\tag{2.31}$$

which finishes the proof of (2).

The last statement follows directly from the definition of the invariant measure. □

Now we can easily conclude the existence of the previously defined limits. Compare the following proposition also with Chapter 3 of Falconer [7].

Proposition 2.4. *For any given $\mu\in\mathcal{M}_\sigma(I^\infty)$ it holds that*

(1) *$P(t)$ exists and equals to $\inf_n\frac{1}{n}\bigl(\log\sum_{\mathtt{i}\in I^n}\psi^t_{\mathtt{i}}(\mathtt{h})+C_t\bigr)$ with any $C_t\ge\log K_t$,*
(2) *$E_\mu(t)$ exists and equals to $\inf_n\frac{1}{n}\bigl(\sum_{\mathtt{i}\in I^n}\mu([\mathtt{i}])\log\psi^t_{\mathtt{i}}(\mathtt{h})+C_t\bigr)$ with any $C_t\ge\log K_t$,*
(3) *h_μ exists and equals to $\inf_n\frac{1}{n}\sum_{\mathtt{i}\in I^n}H\bigl(\mu([\mathtt{i}])\bigr)$,*
(4) *topological pressure is continuous and strictly decreasing and there exists a unique $t\ge 0$ such that $P(t)=0$.*

Furthermore, if the cylinder function is smooth, all the previous conditions hold for $P^k(t)$, $E^k_\mu(t)$ and h^k_μ with any given $k\in\mathbb{N}$. It holds also (even without the smoothness assumption) that

(5) *$P(t)=\lim_{k\to\infty}\frac{1}{k}\overline{P}^k(t)=\lim_{k\to\infty}\frac{1}{k}\underline{P}^k(t)=\inf_k\frac{1}{k}\overline{P}^k(t)=\inf_k\frac{1}{k}\underline{P}^k(t)$,*
(6) *$E_\mu(t)=\lim_{k\to\infty}\frac{1}{k}\overline{E}^k_\mu(t)=\lim_{k\to\infty}\frac{1}{k}\underline{E}^k_\mu(t)=\inf_k\frac{1}{k}\overline{E}^k_\mu(t)=\inf_k\frac{1}{k}\underline{E}^k_\mu(t)$,*
(7) *$h_\mu=\frac{1}{k}h^k_\mu$ for every $k\in\mathbb{N}$.*

Finally, none of these limits depends on the choice of $\mathtt{h}\in I^\infty$.

Proof. Take $\mathtt{h} \in I^\infty$ and $\mu \in \mathcal{M}_\sigma(I^\infty)$. From the subchain rule we get

$$\begin{aligned}\sum_{\mathtt{i}\in I^{n_1+n_2}} \psi^t_{\mathtt{i}}(\mathtt{h}) &\le \sum_{\mathtt{i}\in I^{n_1+n_2}} \psi^t_{\mathtt{i}|_{n_1}}\big(\sigma^{n_1}(\mathtt{i}),\mathtt{h}\big)\psi^t_{\sigma^{n_1}(\mathtt{i})}(\mathtt{h}) \\ &\le K_t \sum_{\mathtt{i}\in I^{n_1}} \psi^t_{\mathtt{i}}(\mathtt{h}) \sum_{\mathtt{i}\in I^{n_2}} \psi^t_{\mathtt{i}}(\mathtt{h}) \end{aligned} \tag{2.32}$$

using the BVP for any choice of $n_1, n_2 \in \mathbb{N}$. Thus, using Lemma 2.2, we get (1). Statements (2) and (3) follow immediately from the invariance of μ and Lemmas 2.3 and 2.2.

Using the assumption (3) in the definition of the cylinder function, we have for fixed $n \in \mathbb{N}$

$$\begin{aligned}\log \underline{s}_\delta + \tfrac{1}{n}\log \sum_{\mathtt{i}\in I^n} \psi^t_{\mathtt{i}}(\mathtt{h}) &\le \tfrac{1}{n}\log \sum_{\mathtt{i}\in I^n} \psi^{t+\delta}_{\mathtt{i}}(\mathtt{h}) \\ &\le \log \overline{s}_\delta + \tfrac{1}{n}\log \sum_{\mathtt{i}\in I^n} \psi^t_{\mathtt{i}}(\mathtt{h}) \end{aligned} \tag{2.33}$$

with any choice of $\delta > 0$. Letting $n \to \infty$, we get $0 < \log \frac{1}{\overline{s}_\delta} \le P(t) - P(t+\delta) \le \log \frac{1}{\underline{s}_\delta}$. This gives the continuity of the topological pressure since $\underline{s}_\delta, \overline{s}_\delta \nearrow 1$ as $\delta \searrow 0$. It says also that the topological pressure is strictly decreasing and $P(t) \to -\infty$, as $t \to \infty$. Since $P(0) = \log \# I$, we have proved (4).

Assuming the cylinder function to be smooth, we notice that $\psi^{t,k}_{\mathtt{i}}$ are cylinder functions on I^{k*} with any choice of $k \in \mathbb{N}$, and, therefore, the previous proofs apply. Using the BVP, we get

$$\begin{aligned}\tfrac{1}{kn}\log \sum_{\mathtt{i}\in I^{kn}} \psi^{t,k}_{\mathtt{i}}(\mathtt{h}) &\le \tfrac{1}{kn}\log K_t^n \sum_{\mathtt{i}\in I^{kn}} \prod_{j=0}^{n-1} \psi^t_{\sigma^{jk}(\mathtt{i})|_k}(\mathtt{h}) \\ &= \tfrac{1}{k}\log K_t + \tfrac{1}{kn}\log\Big(\sum_{\mathtt{i}\in I^k} \psi^t_{\mathtt{i}}(\mathtt{h})\Big)^n \end{aligned} \tag{2.34}$$

for any choice of $k, n \in \mathbb{N}$. Therefore, due to the subchain rule,

$$\begin{aligned}P(t) &\le \tfrac{1}{kn}\log \sum_{\mathtt{i}\in I^{kn}} \psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{kn}\log K_t \\ &\le \tfrac{1}{kn}\log \sum_{\mathtt{i}\in I^{kn}} \psi^{t,k}_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{kn}\log K_t \\ &\le \tfrac{1}{k}\log \sum_{\mathtt{i}\in I^{k}} \psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{k}\log K_t + \tfrac{1}{kn}\log K_t \end{aligned} \tag{2.35}$$

using (1). Now letting $n \to \infty$ and then $k \to \infty$, we get (5). Similarly, using the invariance of μ and the BVP, we have

$$\begin{aligned}\tfrac{1}{kn}\sum_{\mathtt{i}\in I^{kn}}\mu([\mathtt{i}])\log\psi_{\mathtt{i}}^{t,k}(\mathtt{h}) &\le \tfrac{1}{kn}\sum_{\mathtt{i}\in I^{kn}}\mu([\mathtt{i}])\log K_t^n\prod_{j=0}^{n-1}\psi^t_{\sigma^{jk}(\mathtt{i})|_k}(\mathtt{h})\\ &= \tfrac{1}{k}\log K_t + \tfrac{1}{kn}\sum_{j=0}^{n-1}\sum_{\mathtt{i}\in I^{kn}}\mu([\mathtt{i}])\log\psi^t_{\sigma^{jk}(\mathtt{i})|_k}(\mathtt{h})\\ &= \tfrac{1}{k}\log K_t + \tfrac{1}{k}\sum_{\mathtt{i}\in I^k}\mu([\mathtt{i}])\log\psi^t_{\mathtt{i}}(\mathtt{h})\end{aligned} \tag{2.36}$$

for any choice of $k, n \in \mathbb{N}$. Therefore

$$\begin{aligned}E_\mu(t) &\le \tfrac{1}{kn}\sum_{\mathtt{i}\in I^{kn}}\mu([\mathtt{i}])\log\psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{kn}\log K_t\\ &\le \tfrac{1}{kn}\sum_{\mathtt{i}\in I^{kn}}\mu([\mathtt{i}])\log\psi^{t,k}_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{kn}\log K_t\\ &\le \tfrac{1}{k}\sum_{\mathtt{i}\in I^{k}}\mu([\mathtt{i}])\log\psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{k}\log K_t + \tfrac{1}{kn}\log K_t\end{aligned} \tag{2.37}$$

using (2). Now letting $n \to \infty$ and then $k \to \infty$, we get (6). Using the BVP, we get rid of the dependence on the choice of $\mathtt{h} \in I^\infty$ on these limits. Noting that (7) is trivial, we have finished the proof. $\square$

Note that if a cylinder function satisfy the chain rule, we have $P(t) = \frac{1}{k}P^k(t)$ and $E_\mu(t) = \frac{1}{k}E^k_\mu(t)$ for every choice of $k \in \mathbb{N}$ and $\mu \in \mathcal{M}_\sigma(I^\infty)$. With these tools of thermodynamical formalism we are now ready to look for a special invariant measure on I^∞, the so called equilibrium measure. If we denote $\alpha(\mathtt{i}) = \psi^t_{\mathtt{i}}(\mathtt{h})/\sum_{\mathtt{j}\in I^{|\mathtt{i}|}}\psi^t_{\mathtt{j}}(\mathtt{h})$, as $\mathtt{i} \in I^*$, we get, using Jensen's inequality for any $n \in \mathbb{N}$ and $\mu \in \mathcal{M}(I^\infty)$,

$$\begin{aligned}0 = 1\log 1 = \tfrac{1}{n}H\Bigg(\sum_{\mathtt{i}\in I^n}\alpha(\mathtt{i})\frac{\mu([\mathtt{i}])}{\alpha(\mathtt{i})}\Bigg) &\ge \tfrac{1}{n}\sum_{\mathtt{i}\in I^n}\alpha(\mathtt{i})H\Bigg(\frac{\mu([\mathtt{i}])}{\alpha(\mathtt{i})}\Bigg)\\ = \tfrac{1}{n}\sum_{\mathtt{i}\in I^n}\mu([\mathtt{i}])\Bigg(-\log\mu([\mathtt{i}]) + \log\psi^t_{\mathtt{i}}(\mathtt{h}) - \log\sum_{\mathtt{j}\in I^n}\psi^t_{\mathtt{j}}(\mathtt{h})\Bigg)&\end{aligned} \tag{2.38}$$

with equality if and only if $\mu([\mathtt{i}]) = C\alpha(\mathtt{i})$ for some constant $C > 0$. Thus, in the view of Proposition 2.4,

$$P(t) \ge h_\mu + E_\mu(t) \tag{2.39}$$

whenever $\mu \in \mathcal{M}_\sigma(I^\infty)$. We call a measure $\mu \in \mathcal{M}_\sigma(I^\infty)$ as *t–equilibrium measure* if it satisfies an equilibrium state

$$P(t) = h_\mu + E_\mu(t). \tag{2.40}$$

In other words, the equilibrium measure (or state) is a solution for a variational equation $P(t) = \sup_{\mu\in\mathcal{M}_\sigma(I^\infty)}\big(h_\mu + E_\mu(t)\big)$.

Define now for each $k \in \mathbb{N}$ a Perron–Frobenius operator $\mathcal{F}_{t,k}$ by setting

$$\big(\mathcal{F}_{t,k}(f)\big)(\mathtt{h}) = \sum_{\mathtt{i} \in I^k} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) f(\mathtt{i},\mathtt{h}) \tag{2.41}$$

for every continuous function $f : I^\infty \to \mathbb{R}$. Using this operator, we are able to find our equilibrium measure. Assuming $\big(\mathcal{F}_{t,k}^{n-1}(f)\big)(\mathtt{h}) = \sum_{\mathtt{i} \in I^{k(n-1)}} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) f(\mathtt{i},\mathtt{h})$, we get inductively, using the chain rule,

$$\begin{aligned}
\big(\mathcal{F}_{t,k}^{n}(f)\big)(\mathtt{h}) &= \big(\mathcal{F}_{t,k}\big(\mathcal{F}_{t,k}^{n-1}(f)\big)\big)(\mathtt{h}) \\
&= \sum_{\mathtt{i} \in I^k} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) \big(\mathcal{F}_{t,k}^{n-1}(f)\big)(\mathtt{i},\mathtt{h}) \\
&= \sum_{\mathtt{i} \in I^k} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) \sum_{\mathtt{j} \in I^{k(n-1)}} \psi_{\mathtt{j}}^{t,k}(\mathtt{i},\mathtt{h}) f(\mathtt{j},\mathtt{i},\mathtt{h}) \\
&= \sum_{\mathtt{i} \in I^{kn}} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) f(\mathtt{i},\mathtt{h}).
\end{aligned} \tag{2.42}$$

Let us then denote with $\mathcal{F}_{t,k}^*$ the dual operator of $\mathcal{F}_{t,k}$. Due to the Riesz representation theorem it works on $\mathcal{M}(I^\infty)$. Relying now on the definitions of these operators, we may find a special measure using a suitable fixed point theorem. See also Theorem 1.7 of Bowen [3] and Theorem 3.5 of Mauldin and Urbański [14].

Theorem 2.5. *For each $t \geq 0$ and $k \in \mathbb{N}$ there exists a measure $\nu_k \in \mathcal{M}(I^\infty)$ such that*

$$\nu_k([\mathtt{i}; A]) = \Pi_k^{-|\mathtt{i}|/k} \int_A \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) d\nu_k(\mathtt{h}), \tag{2.43}$$

where $\Pi_k > 0$, $\mathtt{i} \in I^{k*}$ and $A \subset I^\infty$ is a Borel set. Moreover, $\lim_{k\to\infty} \Pi_k^{1/k} = e^{P(t)}$ and if the cylinder function is smooth, $\Pi_k = e^{P^k(t)}$ for every $k \in \mathbb{N}$.

Proof. For fixed $t \geq 0$ and $k \in \mathbb{N}$ define $\Lambda : \mathcal{M}(I^\infty) \to \mathcal{M}(I^\infty)$ by setting

$$\Lambda(\mu) = \frac{1}{\big(\mathcal{F}_{t,k}^*(\mu)\big)(I^\infty)} \mathcal{F}_{t,k}^*(\mu). \tag{2.44}$$

Take now an arbitrary converging sequence, say, (μ_n) for which $\mu_n \to \mu$ in the weak topology with some $\mu \in \mathcal{M}(I^\infty)$. Then for each continuous f we have

$$\begin{aligned}
\big(\mathcal{F}_{t,k}^*(\mu_n)\big)(f) &= \mu_n\big(\mathcal{F}_{t,k}(f)\big) \\
&\to \mu\big(\mathcal{F}_{t,k}(f)\big) = \big(\mathcal{F}_{t,k}^*(\mu)\big)(f)
\end{aligned} \tag{2.45}$$

as $n \to \infty$. Thus Λ is continuous. Now the Schauder–Tychonoff fixed–point theorem applies and we find $\nu_k \in \mathcal{M}(I^\infty)$ such that $\Lambda(\nu_k) = \nu_k$. Denoting $\Pi_k = \big(\mathcal{F}_{t,k}^*(\nu_k)\big)(I^\infty)$, we have $\mathcal{F}^*_{t,k}(\nu_k) = \Pi_k \nu_k$. Take now some Borel set $A \subset I^\infty$ and $\mathtt{i} \in I^{k*}$. Then

$$\begin{aligned}
\Pi_k^{|\mathtt{i}|/k} \nu_k([\mathtt{i}; A]) &= \big((\mathcal{F}^*_{t,k})^{|\mathtt{i}|/k}(\nu_k) \big)([\mathtt{i}; A]) = \nu_k\big(\mathcal{F}_{t,k}^{|\mathtt{i}|/k}(\chi_{[\mathtt{i};A]}) \big) \\
&= \int_{I^\infty} \sum_{\mathtt{j} \in I^{|\mathtt{i}|}} \psi_{\mathtt{j}}^{t,k}(\mathtt{h}) \chi_{[\mathtt{i};A]}(\mathtt{j}, \mathtt{h}) d\nu_k(\mathtt{h}) \\
&= \int_{I^\infty} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) \chi_A(\mathtt{h}) d\nu_k(\mathtt{h}) \qquad (2.46) \\
&= \int_A \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) d\nu_k(\mathtt{h}),
\end{aligned}$$

which proves the first claim. It also follows applying the BVP that for each $n \in \mathbb{N}$

$$\begin{aligned}
\Pi_k^n = \Pi_k^n \sum_{\mathtt{i} \in I^{kn}} \nu_k([\mathtt{i}]) &= \int_{I^\infty} \sum_{\mathtt{i} \in I^{kn}} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) d\nu_k(\mathtt{h}) \\
&\le K_t^n \sum_{\mathtt{i} \in I^{kn}} \psi_{\mathtt{i}}^{t,k}(\mathtt{h}) \qquad (2.47)
\end{aligned}$$

and, similarly, the other way around. Taking now logarithms, dividing by kn and taking the limit, we have for each $k \in \mathbb{N}$

$$\tfrac{1}{k} \underline{P}^k(t) - \tfrac{1}{k} \log K_t \le \tfrac{1}{k} \log \Pi_k \le \tfrac{1}{k} \overline{P}^k(t) + \tfrac{1}{k} \log K_t. \qquad (2.48)$$

If the cylinder function is smooth, then for each k there exists a constant $K_{t,k} \ge 1$ for which $\psi_{\mathtt{i}}^{t,k}(\mathtt{h}) \le K_{t,k} \psi_{\mathtt{i}}^{t,k}(\mathtt{j})$ whenever $\mathtt{h}, \mathtt{j} \in I^\infty$ and $\mathtt{i} \in I^{k*}$. Using this in (2.47), we have finished the proof. □

Note that if a cylinder function satisfies the chain rule, then $\nu_k = \nu$ for every $k \in \mathbb{N}$, where

$$\nu([\mathtt{i}; A]) = e^{-|\mathtt{i}| P(t)} \int_A \psi_{\mathtt{i}}^t(\mathtt{h}) d\nu(\mathtt{h}) \qquad (2.49)$$

as $\mathtt{i} \in I^*$ and $A \subset I^\infty$ is a Borel set. The measure ν is called a *t–conformal measure.*

Theorem 2.6. *There exists an equilibrium measure.*

Proof. According to Theorem 2.5, we have for each $n \in \mathbb{N}$ a measure $\nu_n \in \mathcal{M}(I^\infty)$ for which

$$\nu_n([\mathtt{i}]) = \Pi_n^{-1} \int_{I^\infty} \psi_{\mathtt{i}}^t(\mathtt{h}) d\nu_n(\mathtt{h}), \qquad (2.50)$$

where $\mathtt{i} \in I^n$ and $\lim_{n\to\infty} \frac{1}{n} \log \Pi_n = P(t)$. Hence, using the BVP, we get

$$\begin{aligned}
&\tfrac{1}{n} \sum_{\mathtt{i} \in I^n} \nu_n([\mathtt{i}]) \big(-\log \nu_n([\mathtt{i}]) + \log \psi_{\mathtt{i}}^t(\mathtt{h}) \big) \\
&\qquad = \tfrac{1}{n} \sum_{\mathtt{i} \in I^n} \nu_n([\mathtt{i}]) \Big(-\log \Pi_n^{-1} \int_{I^\infty} \psi_{\mathtt{i}}^t(\mathtt{h}) d\nu_n(\mathtt{h}) + \log \psi_{\mathtt{i}}^t(\mathtt{h}) \Big) \\
&\qquad \ge \tfrac{1}{n} \sum_{\mathtt{i} \in I^n} \nu_n([\mathtt{i}]) (\log \Pi_n - \log K_t) \qquad (2.51) \\
&\qquad = \tfrac{1}{n} \log \Pi_n - \tfrac{1}{n} \log K_t
\end{aligned}$$

for every $n \in \mathbb{N}$. Define now for each $n \in \mathbb{N}$ a probability measure

$$\mu_n = \tfrac{1}{n} \sum_{j=0}^{n-1} \nu_n \circ \sigma^{-j} \tag{2.52}$$

and take μ to be some accumulation point of the set $\{\mu_n\}_{n\in\mathbb{N}}$ in the weak topology. Now for any $\mathtt{i} \in I^*$ we have

$$\begin{aligned} \left|\mu_n([\mathtt{i}]) - \mu_n\big(\sigma^{-1}([\mathtt{i}])\big)\right| &= \tfrac{1}{n}\left|\nu_n([\mathtt{i}]) - \nu_n \circ \sigma^{-n}([\mathtt{i}])\right| \\ &\le \tfrac{1}{n} \to 0, \end{aligned} \tag{2.53}$$

as $n \to \infty$. Thus $\mu \in \mathcal{M}_\sigma(I^\infty)$. According to Lemma 2.2 and Proposition 2.3(1), we have, using concavity of H,

$$\begin{aligned} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} H\big(\nu_n([\mathtt{i}])\big) &\le \tfrac{1}{kn} \sum_{j=0}^{n-1} \sum_{\mathtt{i}\in I^k} H\big(\nu_n \circ \sigma^{-j}([\mathtt{i}])\big) + \tfrac{3k}{n} C_1 \\ &\le \tfrac{1}{k} \sum_{\mathtt{i}\in I^k} H\big(\mu_n([\mathtt{i}])\big) + \tfrac{3k}{n} C_1 \end{aligned} \tag{2.54}$$

for some constant C_1 whenever $0 < k < n$. Using then Lemma 2.2 and Proposition 2.3(2), we get

$$\begin{aligned} &\tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \nu_n([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{n} \log K_t \\ &\qquad \le \tfrac{1}{kn} \sum_{j=0}^{n-1} \left(\sum_{\mathtt{i}\in I^k} \nu_n \circ \sigma^{-j}([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) + \log K_t \right) + \tfrac{3k}{n} C_2 \\ &\qquad = \tfrac{1}{k} \sum_{\mathtt{i}\in I^k} \mu_n([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{k} \log K_t + \tfrac{3k}{n} C_2 \end{aligned} \tag{2.55}$$

for some constant C_2 whenever $0 < k < n$. Now putting (2.51), (2.54) and (2.55) together, we have

$$\begin{aligned} \tfrac{1}{n} \log \Pi_n &\le \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} H\big(\nu_n([\mathtt{i}])\big) + \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \nu_n([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{n} \log K_t \\ &\le \tfrac{1}{k} \sum_{\mathtt{i}\in I^k} H\big(\mu_n([\mathtt{i}])\big) + \tfrac{1}{k} \sum_{\mathtt{i}\in I^k} \mu_n([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) \\ &\quad + \tfrac{3k}{n} C_1 + \tfrac{3k}{n} C_2 + \tfrac{1}{k} \log K_t \end{aligned} \tag{2.56}$$

whenever $0 < k < n$. Letting now $n \to \infty$, we get

$$P(t) \le \tfrac{1}{k} \sum_{\mathtt{i}\in I^k} H\big(\mu([\mathtt{i}])\big) + \tfrac{1}{k} \sum_{\mathtt{i}\in I^k} \mu([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) + \tfrac{1}{k} \log K_t \tag{2.57}$$

since cylinder sets have empty boundary. The proof is finished by letting $k \to \infty$. $\square$

Remark 2.7. In order to prove the existence of the equilibrium measure, the use of the Perron–Frobenius operator is not necessarily needed. Indeed, for fixed $\mathtt{h} \in I^\infty$ we could define for each $n \in \mathbb{N}$ a probability measure

$$\nu_n = \frac{\sum_{\mathtt{i} \in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}) \delta_{\mathtt{i},\mathtt{h}}}{\sum_{\mathtt{i} \in I^n} \psi_{\mathtt{i}}^t(\mathtt{h})}, \tag{2.58}$$

where $\delta_{\mathtt{h}}$ is a probability measure with support $\{\mathtt{h}\}$. Now with this measure we have equality in (2.38), which is going to be our replacement for (2.51) in the proof of Theorem 2.6.

Notice that in the simplest case, where the cylinder function is constant and satisfies the chain rule, the conformal measure equals to the equilibrium measure. This can be easily derived from the following theorem. Compare it also with Theorem 3.8 of Mauldin and Urbański [14].

Theorem 2.8. *Suppose the cylinder function satisfies the chain rule. Then*

$$K_t^{-1} \nu(A) \le \mu(A) \le K_t \nu(A) \tag{2.59}$$

for every Borel set $A \subset I^\infty$, where ν is a t–conformal measure and μ is the t–equilibrium measure found in Theorem 2.6.

Proof. Using the BVP, we derive from (2.49)

$$\begin{aligned} 1 = \sum_{\mathtt{i} \in I^n} \nu([\mathtt{i}]) &= e^{-nP(t)} \sum_{\mathtt{i} \in I^n} \int_{I^\infty} \psi_{\mathtt{i}}^t(\mathtt{h}) d\nu(\mathtt{h}) \\ &\le K_t e^{-nP(t)} \sum_{\mathtt{i} \in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}) \end{aligned} \tag{2.60}$$

for all $n \in \mathbb{N}$ and, similarly, the other way around. Thus we have

$$K_t^{-1} e^{nP(t)} \le \sum_{\mathtt{i} \in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}) \le K_t e^{nP(t)} \tag{2.61}$$

for all $n \in \mathbb{N}$. Note that in view of the chain rule we have

$$\begin{aligned} \mu([\mathtt{i}]) &= \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} \nu \circ \sigma^{-j}([\mathtt{i}]) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} \sum_{\mathtt{j} \in I^j} \nu([\mathtt{j},\mathtt{i}]) \\ &= \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} \sum_{\mathtt{j} \in I^j} e^{-|\mathtt{j},\mathtt{i}|P(t)} \int_{I^\infty} \psi_{\mathtt{j},\mathtt{i}}^t(\mathtt{h}) d\nu(\mathtt{h}) \\ &= \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} e^{-(j+|\mathtt{i}|)P(t)} \int_{I^\infty} \psi_{\mathtt{i}}^t(\mathtt{h}) \sum_{\mathtt{j} \in I^j} \psi_{\mathtt{j}}^t(\mathtt{i},\mathtt{h}) d\nu(\mathtt{h}) \end{aligned} \tag{2.62}$$

whenever $\mathtt{i} \in I^*$ since cylinder sets have empty boundary. Now, using (2.61), we get

$$K_t^{-1} \nu([\mathtt{i}]) \le \mu([\mathtt{i}]) \le K_t \nu([\mathtt{i}]) \tag{2.63}$$

for every $\mathtt{i} \in I^*$. Pick a closed set $C \subset I^\infty$ and define $C_n = \{\mathtt{i} \in I^n : [\mathtt{i}] \cap C \ne \emptyset\}$ whenever $n \in \mathbb{N}$. Now sets $\bigcup_{\mathtt{i} \in C_n} [\mathtt{i}] \supset C$ are decreasing as $n = 1, 2, \ldots$, and, therefore,

$\bigcap_{n=1}^{\infty} \bigcup_{\mathtt{i} \in C_n} [\mathtt{i}] = C$. Thus,

$$\begin{aligned} K_t^{-1}\nu(C) = K_t^{-1} \lim_{n\to\infty} \sum_{\mathtt{i}\in C_n} \nu([\mathtt{i}]) &\le \lim_{n\to\infty} \sum_{\mathtt{i}\in C_n} \mu([\mathtt{i}]) \\ &= \mu(C) \le K_t\nu(C). \end{aligned} \tag{2.64}$$

Let $A \subset I^\infty$ be a Borel set. Then, by the Borel regularity of these measures, we may find closed sets $C_1, C_2 \subset A$ such that $\nu(C_1 \setminus A) < \varepsilon$ and $\mu(C_2 \setminus A) < \varepsilon$ for any given $\varepsilon > 0$. Therefore, $\nu(A) \le \nu(C_1) + \varepsilon \le K_t\mu(A) + \varepsilon$ and $\mu(A) \le \mu(C_2) + \varepsilon \le K_t\mu(A) + \varepsilon$. Letting now $\varepsilon \searrow 0$, we have finished the proof. □

3. Equilibrium dimension and iterated function system

In the previous chapter, with the help of the simple structured symbol space using the cylinder function, we found measures with desired properties. In the following we will project this situation into $\mathbb{R}^d$. The natural question now is: What can we say about the Hausdorff dimension of the projected symbol space, the so called limit set? To answer this question, we have to make several extra assumptions, namely, we define the concept of the iterated function system and we introduce a couple of separation conditions. To illustrate our theory, we give concrete examples at the end of this chapter.

For fixed $t \ge 0$ we denote with μ_t a corresponding equilibrium measure. We define for each $n \in \mathbb{N}$

$$\mathcal{G}_n^t(A) = \inf\left\{ \sum_{j=1}^{\infty} \int_{I^\infty} \psi_{\mathtt{i}_j}^t(\mathtt{h}) d\mu_t(\mathtt{h}) : A \subset \bigcup_{j=1}^{\infty} [\mathtt{i}_j],\ |\mathtt{i}_j| \ge n \right\} \tag{3.1}$$

whenever $A \subset I^\infty$. Assumptions in Carathéodory's construction are now satisfied and we have a Borel regular measure $\mathcal{G}^t$ on I^∞ with

$$\mathcal{G}^t(A) = \lim_{n\to\infty} \mathcal{G}_n^t(A). \tag{3.2}$$

Lemma 3.1. *If $\mathcal{G}^{t_0}(A) < \infty$, then $\mathcal{G}^t(A) = 0$ for all $t > t_0$.*

Proof. Let $n \in \mathbb{N}$ and choose a collection of cylinder sets $\{[\mathtt{i}_j]\}_j$ such that $|\mathtt{i}_j| \ge n$ and $\sum_j \int_{I^\infty} \psi_{\mathtt{i}_j}^{t_0}(\mathtt{h}) d\mu_{t_0}(\mathtt{h}) \le \mathcal{G}_n^{t_0}(A) + 1$. Then

$$\begin{aligned} \mathcal{G}_n^t(A) &\le \sum_j \int_{I^\infty} \psi_{\mathtt{i}_j}^t(\mathtt{h}) d\mu_t(\mathtt{h}) \le K_t K_{t_0} \sum_j \int_{I^\infty} \psi_{\mathtt{i}_j}^{t_0}(\mathtt{h}) d\mu_{t_0}(\mathtt{h}) \overline{s}_{t-t_0}^{|\mathtt{i}_j|} \\ &\le K_t K_{t_0} \overline{s}_{t-t_0}^n \big(\mathcal{G}_n^{t_0}(A) + 1\big). \end{aligned} \tag{3.3}$$

By letting $n \to \infty$ we have finished the proof. □

Using this lemma, we may now define

$$\begin{aligned} \dim_\psi(A) &= \inf\{t \ge 0 : \mathcal{G}^t(A) = 0\} \\ &= \sup\{t \ge 0 : \mathcal{G}^t(A) = \infty\} \end{aligned} \tag{3.4}$$

and we call this "critical value" the *equilibrium dimension* of the set $A \subset I^\infty$. Notice that the equilibrium dimension does not depend on the measure μ_t. In fact, defining the measure $\mathcal{G}^t$ by using a fixed $\mathtt{h} \in I^\infty$ instead of the integral average in (3.1), leads us for the same definition of the equilibrium dimension. The most important property of the equilibrium dimension is the following theorem.

Theorem 3.2. *$P(t) = 0$ if and only if $\dim_\psi(I^\infty) = t$.*

Proof. Let us first show that $P(t) < 0$ implies $\dim_\psi(I^\infty) \le t$. Using the BVP, we derive from Theorem 2.5

$$\begin{aligned} 1 = \sum_{\mathtt{i} \in I^n} \nu_n([\mathtt{i}]) &= \Pi_n^{-1} \sum_{\mathtt{i} \in I^n} \int_{I^\infty} \psi_{\mathtt{i}}^t(\mathtt{h}) d\nu_n(\mathtt{h}) \\ &\ge K_t^{-1} \Pi_n^{-1} \sum_{\mathtt{i} \in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}), \end{aligned} \tag{3.5}$$

where $\lim_{n\to\infty} \Pi_n^{1/n} = e^{P(t)}$. Now

$$\limsup_{n\to\infty} \left(\sum_{\mathtt{i} \in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}) \right)^{1/n} \le \lim_{n\to\infty} (K_t \Pi_n)^{1/n} = e^{P(t)} < 1 \tag{3.6}$$

and choosing n_0 big enough, we have

$$\left(\sum_{\mathtt{i} \in I^n} \psi_{\mathtt{i}}^t(\mathtt{h}) \right)^{1/n} < \frac{1 + e^{P(t)}}{2} < 1 \tag{3.7}$$

whenever $n \ge n_0$. Hence, for any given $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{\mathtt{i} \in I^n} \int_{I^\infty} \psi_{\mathtt{i}}^t(\mathtt{h}) d\mu(\mathtt{h}) < \varepsilon \tag{3.8}$$

whenever $n \ge n_1$. This proves the claim.

For the convenience of the reader, to prove the other direction we repeat here the argument of Falconer from [5]. Let us assume that $t > \dim_\psi(I^\infty)$ and $\mathtt{h} \in I^\infty$. Then, clearly, $\mathcal{G}^t(I^\infty) = 0$ and we may choose a finite cover for I^∞ of the form $\{[\mathtt{i}] : \mathtt{i} \in A \subset \bigcup_{j=1}^{n_0} I^j\}$, where $n_0 \in \mathbb{N}$ is large enough and A is some incomparable set such that

$$\sum_{\mathtt{i} \in A} \psi_{\mathtt{i}}^t(\mathtt{h}) < K_t^{-1}. \tag{3.9}$$

Here we can choose a finite cover, since any infinite collection of disjoint cylinders will not cover the whole I^∞. Define now for each integer $n \ge n_0$ a set

$$\begin{aligned} A_n = \{\mathtt{i}_1, \ldots, \mathtt{i}_q \in I^* : \mathtt{i}_j \in A \text{ as } j = 1, \ldots, q \text{ with some } q, \\ |\mathtt{i}_1, \ldots, \mathtt{i}_q| \ge n \text{ and } |\mathtt{i}_1, \ldots, \mathtt{i}_{q-1}| \le n\}. \end{aligned} \tag{3.10}$$

Now, using the subchain rule, we get with any choice of $\mathtt{j} \in I^*$

$$\sum_{\mathtt{i} \in A} \psi_{\mathtt{j},\mathtt{i}}^t(\mathtt{h}) \le K_t \psi_{\mathtt{j}}^t(\mathtt{h}) \sum_{\mathtt{i} \in A} \psi_{\mathtt{i}}^t(\mathtt{h}) \le \psi_{\mathtt{j}}^t(\mathtt{h}) \tag{3.11}$$

whenever $\mathtt{h} \in I^\infty$. Thus, inductively, we get for every $n \ge n_0$

$$\sum_{\mathtt{i} \in A_n} \psi_{\mathtt{i}}^t(\mathtt{h}) \le K_t^{-1}. \tag{3.12}$$

Assuming $\mathtt{i} \in I^{n+n_0}$, we have $\mathtt{i} = \mathtt{j},\mathtt{k}$ for some $\mathtt{j} \in A_n$ and $\mathtt{k} \in I^*$ with $|\mathtt{k}| \le n_0$. Moreover, for each such $\mathtt{j}$ there are at most $(\#I)^{n_0}$ such $\mathtt{k}$. Since $\psi^t_{\mathtt{i}}(\mathtt{h}) \le \psi^t_{\mathtt{j}}(\mathtt{k},\mathtt{h})\overline{s}_t^{|\mathtt{k}|} \le \psi^t_{\mathtt{j}}(\mathtt{k},\mathtt{h})$, we have

$$\sum_{\mathtt{i} \in I^{n+n_0}} \psi^t_{\mathtt{i}}(\mathtt{h}) \le (\#I)^{n_0} K_t \sum_{\mathtt{j} \in A_n} \psi^t_{\mathtt{j}}(\mathtt{h}) \le (\#I)^{n_0} \tag{3.13}$$

for all $n \in \mathbb{N}$. From this we derive that $P(t) \le 0$. This also finishes the proof. $\square$

So far we have worked only in the symbol space. It has provided us with a simple structured environment for finding measures with desired properties. It is, however, more interesting to study geometric projections of these measures and the symbol space. In the following we define what we mean by this geometric projection. Let $X \subset \mathbb{R}^d$ be a compact set with nonempty interior. Choose then a collection $\{X_{\mathtt{i}} : \mathtt{i} \in I^*\}$ of nonempty closed subsets of X satisfying

(1) $X_{\mathtt{i},i} \subset X_{\mathtt{i}}$ for every $\mathtt{i} \in I^*$ and $i \in I$,
(2) $d(X_{\mathtt{i}}) \to 0$, as $|\mathtt{i}| \to \infty$.

Here d means the diameter of a given set. Define now a *projection mapping* $\pi : I^\infty \to X$ such that

$$\{\pi(\mathtt{i})\} = \bigcap_{n=1}^{\infty} X_{\mathtt{i}|_n} \tag{3.14}$$

as $\mathtt{i} \in I^\infty$. It is clear that π is continuous. We call the compact set $E = \pi(I^\infty)$ as the *limit set* of this collection, and if there is no danger of misunderstanding, we also call the projected cylinder set a cylinder set.

We could now define a cylinder function for this collection of sets. But without any additional information the equilibrium dimension has most likely nothing to do with the Hausdorff dimension of the limit set. Therefore, in order to determine the Hausdorff dimension, it is natural to require that the cylinder function somehow represents the size of the subset $X_{\mathtt{i}}$ and also that there is not too much overlapping among these sets. The use of iterated function systems with well–chosen mappings and separation condition will provide us with the sufficient information we need.

Take now $\Omega \supset X$ to be an open subset of $\mathbb{R}^d$. Let $\{\varphi_{\mathtt{i}} : i \in I^*\}$ be a collection of contractive injections from Ω to Ω such that the collection $\{\varphi_{\mathtt{i}}(X) : \mathtt{i} \in I^*\}$ satisfies both properties (1) and (2) above. By *contractivity* we mean that for every $\mathtt{i} \in I^*$ there exists a constant $0 < s_{\mathtt{i}} < 1$ such that $|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le s_{\mathtt{i}}|x-y|$ whenever $x, y \in \Omega$. This kind of collection is called a *general iterated function system.* Furthermore, we call the collection $\{\varphi_i : i \in I\}$ of the same kind of mappings an *iterated function system (IFS).* Defining $\varphi_{\mathtt{i}} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_{|\mathtt{i}|}}$, as $\mathtt{i} \in I^*$, we clearly get the assumptions of general IFS satisfied. In fact, we have $d\big(\varphi_{\mathtt{i}}(X)\big) \le (\max_{i \in I} s_i)^{|\mathtt{i}|} d(X)$.

To avoid too much overlapping, we need a decent separation condition for the subsets $\varphi_{\mathtt{i}}(X)$. We say that a *strong separation condition (SSC)* is satisfied if $\varphi_{\mathtt{i}}(X) \cap \varphi_{\mathtt{j}}(X) = \emptyset$ whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable. For IFS it suffices to require $\varphi_i(X) \cap \varphi_j(X) = \emptyset$ for $i \ne j$. Of course, assuming the SSC would be enough in many cases, but it is a rather restrictive assumption, and usually we do not need that much. We say that an *open set condition (OSC)* is satisfied if $\varphi_{\mathtt{i}}\big(\mathrm{int}(X)\big) \cap \varphi_{\mathtt{j}}\big(\mathrm{int}(X)\big) = \emptyset$ whenever $\mathtt{i}$ and $\mathtt{j}$

are incomparable. Again, for IFS it suffices to require $\varphi_i\big(\mathrm{int}(X)\big) \cap \varphi_j\big(\mathrm{int}(X)\big) = \emptyset$ for $i \neq j$. With the notation $\mathrm{int}(X)$ we mean the interior of X. Furthermore, we say that a general IFS has *weak bounded overlapping* if the cardinality of incomparable subsets of $\{\mathtt{i} \in I^* : x \in \varphi_{\mathtt{i}}(X)\}$ is bounded as $x \in X$. Trivially, a general IFS satisfying the SSC has weak bounded overlapping. Assume now that for each $\mathtt{i} \in I^*$ there exists a constant $0 < \underline{s}_{\mathtt{i}} < 1$ such that $\underline{s}_{\mathtt{i}} \to 0$ as $|\mathtt{i}| \to \infty$. Then we say that a general IFS has *bounded overlapping* if the cardinality of the set $Z(x,r) = \{\mathtt{i} \in Z(r) : \varphi_{\mathtt{i}}(X) \cap B(x,r) \neq \emptyset\}$ is bounded as $x \in X$ and $0 < r < r_0 = r_0(x)$. Here $Z(r)$ is an incomparable subset of $\{\mathtt{i} \in I^* : \underline{s}_{\mathtt{i}} < r \leq \underline{s}_{\mathtt{i}|_{|\mathtt{i}|-1}}\}$ such that $\{[\mathtt{i}] : \mathtt{i} \in Z(r)\}$ is a cover for I^∞. We will choose the constants $\underline{s}_{\mathtt{i}}$ rigorously in a while. Next we study how these separation conditions are related.

Lemma 3.3. *Suppose a general IFS has bounded overlapping. Then it has also weak bounded overlapping.*

Proof. If the weak bounded overlapping is not satisfied, then the cardinality of incomparable subsets of $R(x) = \{\mathtt{i} \in I^* : x \in \varphi_{\mathtt{i}}(X)\}$ is not bounded as $x \in X$. Therefore, $\sup_{x \in X} \#\big(R(x) \cap Z(r)\big) \to \infty$, as $r \searrow 0$. On the other hand, $R(x) \cap Z(r) \subset Z(x,r)$ for all $x \in X$ and $r > 0$, which gives a contradiction. $\square$

It seems that by assuming only the mappings of a general IFS to be Lipschitz it is very difficult to get information about the Hausdorff dimension of the limit set. While the Lipschitz condition provides us with an upper bound for the diameter of the cylinder set, it does not give any kind of lower bound for the size of the cylinder set. Having the lower bound seems to be crucial for getting this kind of information. Assuming the mappings $\varphi_{\mathtt{i}}$ to be bi–Lipschitz, we denote the "maximal derivative" with

$$L_{\mathtt{i}}(x) = \limsup_{y \to x} \frac{|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)|}{|x-y|} \tag{3.15}$$

and the "minimal derivative" with

$$l_{\mathtt{i}}(x) = \liminf_{y \to x} \frac{|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)|}{|x-y|}. \tag{3.16}$$

We say that a general IFS is *bi–Lipschitz* if the mappings $\varphi_{\mathtt{i}}$ are bi–Lipschitz and there exist cylinder functions $\underline{\psi}^t_{\mathtt{i}}$ and $\overline{\psi}^t_{\mathtt{i}}$ satisfying the chain rule such that $\underline{\psi}^t_{\mathtt{i}}(\mathtt{h}) \leq l_{\mathtt{i}}\big(\pi(\mathtt{h})\big)^t$ and $\overline{\psi}^t_{\mathtt{i}}(\mathtt{h}) \geq L_{\mathtt{i}}\big(\pi(\mathtt{h})\big)^t$ for all $\mathtt{h} \in I^\infty$, and in both functions the parameter t is an exponent, that is, $\underline{\psi}^t_{\mathtt{i}}(\mathtt{h}) = \big(\underline{\psi}^1_{\mathtt{i}}(\mathtt{h})\big)^t$ and $\overline{\psi}^t_{\mathtt{i}}(\mathtt{h}) = \big(\overline{\psi}^1_{\mathtt{i}}(\mathtt{h})\big)^t$. We also assume that the bi–Lipschitz constants for the mappings $\varphi_{\mathtt{i}}$ are $\underline{s}_{\mathtt{i}} = \inf_{\mathtt{h} \in I^\infty} \underline{\psi}^1_{\mathtt{i}}(\mathtt{h})$ and $\overline{s}_{\mathtt{i}} = \sup_{\mathtt{h} \in I^\infty} \overline{\psi}^1_{\mathtt{i}}(\mathtt{h})$. From now on, these are the constants $\underline{s}_{\mathtt{i}}$ we will use in the definition of the bounded overlapping.

Lemma 3.4. *A bi–Lipschitz IFS satisfying the SSC has bounded overlapping.*

Proof. We use the idea found in the proof of Proposition 9.7 of Falconer [6]. Denote $q = \min_{i \neq j} d\big(\varphi_i(X), \varphi_j(X)\big)$, where d means the distance between two given sets, and take $x \in E$ and $r > 0$. We can take x from E since otherwise there is nothing to prove. Choose $\mathtt{i} \in I^\infty$ such that $x = \pi(\mathtt{i})$. Since now $\varphi_{\mathtt{i}|_n}(X) \cap B(x,r) \neq \emptyset$ for every

$n \in \mathbb{N}$, we can choose n such that $\mathtt{i}|_n \in Z(x,r)$. Take also an arbitrary $\mathtt{j} \in Z(r)$ such that $\mathtt{j} \neq \mathtt{i}|_n$ and let $0 \le j < n$ be the largest integer for which $\mathtt{j}|_j = \mathtt{i}|_j$. If it were $d\big(\varphi_{\mathtt{i}|_n}(X), \varphi_{\mathtt{j}}(X)\big) < \underline{s}_{\mathtt{i}|_j} q$, there would be $y \in \varphi_{\mathtt{i}|_n}(X)$ and $z \in \varphi_{\mathtt{j}}(X)$ such that

$$|y - z| < \underline{s}_{\mathtt{i}|_j} q. \tag{3.17}$$

The bi–Lipschitz condition implies $|(\varphi_{\mathtt{i}|_j})^{-1}(y) - (\varphi_{\mathtt{i}|_j})^{-1}(z)| < q$, which contradicts the strong separation assumption due to the choice of j. Hence

$$d\big(\varphi_{\mathtt{i}|_n}(X), \varphi_{\mathtt{j}}(X)\big) \ge \underline{s}_{\mathtt{i}|_j} q \ge \underline{s}_{\mathtt{i}|_{n-1}} q \ge rq \tag{3.18}$$

and thus $\mathtt{i}|_n$ is the only symbol in $Z(r)$ with $\varphi_{\mathtt{i}|_n}(X) \cap B(x, rq) \neq \emptyset$. This also means that there exists exactly one $\mathtt{h} \in Z(r/q)$ for which $\varphi_{\mathtt{h}}(X) \cap B(x,r) \neq \emptyset$. Take now an arbitrary $\mathtt{j} \in Z(x,r)$ and assuming $q < 1$ we notice that $\mathtt{j} = \mathtt{h},\mathtt{k}$ for some $\mathtt{k} \in I^*$. Choose the smallest integer k such that $\underline{s}_{\mathtt{k}} < q/\underline{K}_1$ for all $\mathtt{k} \in I^*$ for which $|\mathtt{k}| \ge k$. Here $\underline{K}_t$ is the constant from the BVP of the cylinder function $\underline{\psi}^t_{\mathtt{i}}$. Hence if it were $\mathtt{j} = \mathtt{h},\mathtt{k}$ for some $\mathtt{k} \in I^*$ for which $|\mathtt{k}| > k$, it would hold that

$$\underline{s}_{\mathtt{j}|_{|\mathtt{j}|-1}} \le \underline{K}_1 \underline{s}_{\mathtt{h}} \underline{s}_{\mathtt{k}|_{|\mathtt{k}|-1}} < r \tag{3.19}$$

and therefore $\mathtt{j}$ could not be in $Z(x,r)$. Thus there can be at maximum $(\#I)^k$ of such $\mathtt{k}$ and hence $\#Z(x,r) \le (\#I)^k$. □

It seems to be important that the shape of the open set of the OSC would not be too "wild", and, therefore, the shape of the cylinder sets, or rather the sets $\varphi_{\mathtt{i}}(X)$, is under control. Motivated by this, we say that the *boundary condition* is satisfied if there exists $\varrho_0 > 0$ such that

$$\inf_{x \in \partial X} \inf_{0<r<\varrho_0} \frac{\mathcal{H}^d\big(B(x,r) \cap \operatorname{int}(X)\big)}{\mathcal{H}^d\big(B(x,r)\big)} > 0, \tag{3.20}$$

where ∂X denotes the boundary of the set X. This condition says that the boundary of X cannot be too "thick"; for example, recalling the Lebesgue density theorem, we have $\mathcal{H}^d(\partial X) = 0$. The boundary condition is clearly satisfied if the set X is convex.

Proposition 3.5. *A bi–Lipschitz general IFS satisfying the OSC and the boundary condition has weak bounded overlapping if $\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}}$ is bounded as $\mathtt{i} \in I^*$.*

Proof. Fix $x \in X$ and denote with R some incomparable subset of $\{\mathtt{i} \in I^* : x \in \varphi_{\mathtt{i}}(X)\}$. Put $r_0 = \min\{\varrho_0, d(X, \partial\Omega)\}$, where ϱ_0 is as in the boundary condition. Now there exists $\delta > 0$ such that for every $y \in X$ we have

$$\mathcal{H}^d\big(B(y,r) \cap \operatorname{int}(X)\big) \ge \mathcal{H}^d\big(B(y,\delta r)\big) \tag{3.21}$$

whenever $0 < r < r_0$. Note that the collection $\{\varphi_{\mathtt{i}}\big(\operatorname{int}(X)\big) : \mathtt{i} \in R\}$ is disjoint due to the OSC. For each $\mathtt{i} \in R$ take $y_{\mathtt{i}} \in X$ such that $\varphi_{\mathtt{i}}(y_{\mathtt{i}}) = x$ and choose an increasing sequence of finite sets $R_1 \subset R_2 \subset \cdots$ such that $\bigcup_{j=1}^\infty R_j = R$. Now fix j and choose $r > 0$ small enough such that $r_{\mathtt{i}} := r/\underline{s}_{\mathtt{i}} < r_0$ for all $\mathtt{i} \in R_j$. Using now the boundary condition, bi–Lipschitzness and the OSC, we see that

$$\begin{aligned}
\#R_j r^d &= \sum_{\mathtt{i}\in R_j} \underline{s}_{\mathtt{i}}^d r_{\mathtt{i}}^d \\
&= \big(\alpha(d)\delta^d\big)^{-1} \sum_{\mathtt{i}\in R_j} \underline{s}_{\mathtt{i}}^d \mathcal{H}^d\big(B(y_{\mathtt{i}},\delta r_{\mathtt{i}})\big) \\
&\le \big(\alpha(d)\delta^d\big)^{-1} \sum_{\mathtt{i}\in R_j} \underline{s}_{\mathtt{i}}^d \mathcal{H}^d\big(B(y_{\mathtt{i}}, r_{\mathtt{i}}) \cap \mathrm{int}(X)\big) \\
&\le \big(\alpha(d)\delta^d\big)^{-1} \sum_{\mathtt{i}\in R_j} \mathcal{H}^d\Big(\varphi_{\mathtt{i}}\big(B(y_{\mathtt{i}}, r_{\mathtt{i}}) \cap \mathrm{int}(X)\big)\Big) \qquad (3.22) \\
&\le \big(\alpha(d)\delta^d\big)^{-1} \mathcal{H}^d\Bigg(\bigcup_{\mathtt{i}\in R_j} B(x, \overline{s}_{\mathtt{i}} r_{\mathtt{i}})\Bigg) \\
&\le \delta^{-d} C^d r^d,
\end{aligned}$$

where $\alpha(d)$ is the Hausdorff measure of the unit ball and $\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}} \le C$ as $\mathtt{i} \in I^*$. Hence $\#R = \lim_{j\to\infty} \#R_j \le \delta^{-d} C^d$, where the upper bound does not depend on the choice of $x \in X$. □

Now we define an important class of iterated function systems. We say that a general IFS is *(weakly) geometrically stabile* if it is bi–Lipschitz and it has (weak) bounded overlapping. Geometrically stabile systems are clearly weakly geometrically stabile by Lemma 3.3. If we have a good control over the size of the cylinder sets, the converse is also true.

Proposition 3.6. *Suppose a general IFS is weakly geometrically stabile such that $\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}}$ is bounded as $\mathtt{i} \in I^*$. Then it is also geometrically stabile.*

Proof. Notice first that the weak bounded overlapping assumption implies the existence of the constant C for which $\sum_{\mathtt{i}\in A} \chi_{\varphi_{\mathtt{i}}(X)}(x) < C$ whenever $x \in X$ and the set $A \subset I^*$ is incomparable. Recall that

$$Z(x,r) = \{\mathtt{i} \in Z(r) : \varphi_{\mathtt{i}}(X) \cap B(x,r) \ne \emptyset\} \tag{3.23}$$

is incomparable and notice that $\varphi_{\mathtt{i}}(X) \subset B(x, rd(X)\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}} + r)$ as $\mathtt{i} \in Z(x,r)$. Choosing C big enough such that also $d(X)\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}} + 1 \le C$ whenever $\mathtt{i} \in I^*$, we get

$$\begin{aligned}
\#Z(x,r) r^d &\le \Big(\min_{i\in I} \underline{s}_i^d\Big)^{-1} \sum_{\mathtt{i}\in Z(x,r)} \underline{s}_{\mathtt{i}}^d \\
&\le \Big(\mathcal{H}^d(X) \min_{i\in I} \underline{s}_i^d\Big)^{-1} \sum_{\mathtt{i}\in Z(x,r)} \mathcal{H}^d\big(\varphi_{\mathtt{i}}(X)\big) \qquad (3.24) \\
&\le \Big(\mathcal{H}^d(X) \min_{i\in I} \underline{s}_i^d\Big)^{-1} \int_{B(x,Cr)} \sum_{\mathtt{i}\in Z(x,r)} \chi_{\varphi_{\mathtt{i}}(X)}(x)\, d\mathcal{H}^d(x).
\end{aligned}$$

Since $r^d = \big(\alpha(d)C^d\big)^{-1}\mathcal{H}^d\big(B(x,Cr)\big)$, we conclude

$$\#Z(x,r) \le \frac{\alpha(d)C^{d+1}}{\mathcal{H}^d(X)\min_{i\in I}\underline{s}_i^d}, \tag{3.25}$$

where $\alpha(d)$ is the Hausdorff measure of the unit ball. □

Before studying the Hausdorff dimension of the limit set, we show in the following theorem that with respect to any invariant measure we can have the same structure in the limit set as in the symbol space. Under the weak bounded overlapping assumption, somehow the weakest separation condition, we can project any invariant measure from I^∞ to the limit set E such that the overlapping has measure zero.

Theorem 3.7. *Suppose a general IFS has weak bounded overlapping. Then for $m = \mu\circ\pi^{-1}$, where $\mu\in\mathcal{M}_\sigma(I^\infty)$, we have*

$$m\big(\varphi_{\mathtt{i}}(X)\cap\varphi_{\mathtt{j}}(X)\big) = 0 \tag{3.26}$$

whenever $\mathtt{i}$ and $\mathtt{j}$ are incomparable.

Proof. We use the idea found in the proof of Lemma 3.10 of Mauldin and Urbański [14]. For fixed incomparable $\mathtt{h}$ and $\mathtt{k}$ we denote $A = \varphi_{\mathtt{h}}(X)\cap\varphi_{\mathtt{k}}(X)$ and $A_n = \bigcup_{\mathtt{i}\in I^n}\varphi_{\mathtt{i}}(A)$ as $n\in\mathbb{N}$. Let us first show that $\bigcap_{q=1}^\infty\bigcup_{n=q}^\infty A_n = \emptyset$. Assume contrarily that there exists $x\in\bigcap_{q=1}^\infty\bigcup_{n=q}^\infty A_n$. Then $x\in\bigcup_{n=q}^\infty A_n$ for every q and hence $x\in A_{n_q}$, where $\{n_q\}_{q\in\mathbb{N}}$ is an increasing sequence of indexes. Now for each q there exists a symbol $\mathtt{j}_q\in I^{n_q}$ such that $x\in\varphi_{\mathtt{j}_q,\mathtt{h}}(X)$ and $x\in\varphi_{\mathtt{j}_q,\mathtt{k}}(X)$. Denoting with R_k^* the maximal incomparable subset of $R_k = \{\mathtt{i}\in\bigcup_{q=1}^k(I^{n_q+|\mathtt{h}|}\cup I^{n_q+|\mathtt{k}|}) : x\in\varphi_{\mathtt{i}}(X)\}$, we have $\#R_1\ge 2$ and also $\#R_1^*\ge 2$. Clearly, $\#R_2\ge 4$, and even if it were $\mathtt{j}_2|_{n_1+|\mathtt{h}|} = \mathtt{j}_1,\mathtt{h}$ (or $\mathtt{j}_2|_{n_1+|\mathtt{k}|} = \mathtt{j}_1,\mathtt{k}$), it is still $\#R_2^*\ge 3$ since the two new symbols $\mathtt{j}_2,\mathtt{h}$ and $\mathtt{j}_2,\mathtt{k}$ with the symbol $\mathtt{j}_1,\mathtt{k}$ (or $\mathtt{j}_1,\mathtt{h}$) are incomparable. Observe that for each k the symbol $\mathtt{j}_k$ can be comparable at maximum with one element of R_{k-1}^*. Thus continuing in this manner, we get $\#R_k^*\ge k+1$ as $k\in\mathbb{N}$. The claim is proved since this contradicts the bounded overlapping assumption.

The boundedness assumption also implies $\sum_{\mathtt{i}\in I^n}\chi_{\varphi_{\mathtt{i}}(A)}(x)\le C$ for every $x\in X$ and $n\in\mathbb{N}$ with some constant $C\ge 0$. Thus, using the invariance of μ, we have

$$\begin{aligned} m(A_n) = m\Bigg(\bigcup_{\mathtt{i}\in I^n}\varphi_{\mathtt{i}}(A)\Bigg) &\ge C^{-1}\sum_{\mathtt{i}\in I^n} m\big(\varphi_{\mathtt{i}}(A)\big) \\ &\ge C^{-1}\sum_{\mathtt{i}\in I^n}\mu\big([\mathtt{i};\pi^{-1}(A)]\big) = C^{-1}\mu\circ\sigma^{-n}\big(\pi^{-1}(A)\big) = C^{-1}m(A) \end{aligned} \tag{3.27}$$

whenever $n\in\mathbb{N}$. So, if $m(A)>0$, we get a contradiction immediately since

$$m\Bigg(\bigcap_{q=1}^\infty\bigcup_{n=q}^\infty A_n\Bigg) = \lim_{q\to\infty} m\Bigg(\bigcup_{n=q}^\infty A_n\Bigg) \ge \lim_{q\to\infty} m(A_q) \ge C^{-1}m(A). \tag{3.28}$$

The proof is complete. □

If we assume that the cylinder function satisfy

$$\underline{\psi}_{\mathtt{i}}^t(\mathtt{h}) \le \psi_{\mathtt{i}}^t(\mathtt{h}) \le \overline{\psi}_{\mathtt{i}}^t(\mathtt{h}), \tag{3.29}$$

where $\mathtt{i} \in I^*$, then we clearly have $\dim_{\underline{\psi}}(I^\infty) \le \dim_\psi(I^\infty) \le \dim_{\overline{\psi}}(I^\infty)$, where $\dim_{\underline{\psi}}(I^\infty)$ and $\dim_{\overline{\psi}}(I^\infty)$ are the equilibrium dimensions derived from cylinder functions $\underline{\psi}^t_{\mathtt{i}}$ and $\overline{\psi}^t_{\mathtt{i}}$, respectively. The following theorem guarantees that the similar behaviour occurs also for the Hausdorff dimension with geometrically stabile systems. It is now very tempting to guess that in some cases making a reasonable choice for the cylinder function, it is possible to get $\dim_H(E) = \dim_\psi(I^\infty)$. If there is no danger of misunderstanding, we call also the projected equilibrium measure an equilibrium measure.

Theorem 3.8. *Suppose a general IFS is geometrically stabile. Then it has*

$$\dim_{\underline{\psi}}(I^\infty) \le \dim_H(E) \le \dim_{\overline{\psi}}(I^\infty), \tag{3.30}$$

and, in fact, $\mathcal{H}^t(A) > 0$ as $t \le \dim_{\underline{\psi}}(I^\infty)$ whenever A is a Borel set such that $\underline{m}(A) = 1$ and $\underline{m}$ is the equilibrium measure constructed using the cylinder function $\underline{\psi}^t_{\mathtt{i}}$.

Proof. Let us first prove the right–hand side of (3.30). For each $t \ge 0$ we have

$$\begin{aligned}\mathcal{H}^t(E) &\le \lim_{n\to\infty} \inf\Big\{ \sum_j d\big(\varphi_{\mathtt{i}_j}(E)\big)^t : E \subset \bigcup_j \varphi_{\mathtt{i}_j}(E),\ |\mathtt{i}_j| \ge n \Big\} \\ &\le \lim_{n\to\infty} \inf\Big\{ \sum_j d(E)^t \overline{s}^t_{\mathtt{i}_j} : E \subset \bigcup_j \varphi_{\mathtt{i}_j}(E),\ |\mathtt{i}_j| \ge n \Big\} \\ &\le \overline{K}_t d(E)^t \overline{\mathcal{G}}^t(I^\infty),\end{aligned} \tag{3.31}$$

where $\overline{\mathcal{G}}^t$ is the measure constructed in a similar way as in (3.1) and (3.2) but using the cylinder function $\overline{\psi}^t_{\mathtt{i}}$. Here $\overline{K}_t$ is the constant of the BVP. Thus $\dim_H(E) \le \dim_{\overline{\psi}}(I^\infty)$. Notice that here we did not need any kind of separation condition.

For the left–hand side recall first that the set $Z(r)$ is incomparable and the cardinality of the set $Z(x,r)$ is bounded as $x \in X$ and $0 < r < r_0 = r_0(x)$. Now for fixed $x \in X$ and $0 < r < r_0(x)$ we have, using theorems 2.5 and 2.8,

$$\begin{aligned}\underline{m}\big(B(x,r)\big) &\le \sum_{\mathtt{i}\in Z(x,r)} \underline{m}\big(\varphi_{\mathtt{i}}(X)\big) \\ &\le \underline{K}_t \sum_{\mathtt{i}\in Z(x,r)} \int_{I^\infty} \underline{\psi}^t_{\mathtt{i}}(\mathtt{h}) d\underline{\nu}(\mathtt{h}) \le \underline{K}_t^2 \# Z(x,r) r^t,\end{aligned} \tag{3.32}$$

where $\underline{m}$ and $\underline{\nu}$ are the corresponding equilibrium measure and conformal measure constructed using the cylinder function $\underline{\psi}^t_{\mathtt{i}}$ and $t = \dim_{\underline{\psi}}(I^\infty)$. Taking $A \subset E$ such that $\underline{m}(A) = 1$ and defining $A_k = \{x \in A : \frac{1}{k} < r_0(x)\}$, we have $A = \bigcup_{k=1}^\infty A_k$. Now for each $x \in A_k$ we have

$$\frac{\underline{m}\big(B(x,r)\big)}{r^t} \le \underline{K}_t^2 \# Z(x,r) \tag{3.33}$$

as $0 < r < \frac{1}{k}$, and thus $\mathcal{H}^t(A_k) \ge C\underline{m}(A_k)$ for some positive constant C. Since $\mathcal{H}^t(A) = \lim_{k\to\infty} \mathcal{H}^t(A_k) \ge C > 0$, we have finished the proof. □

Next we introduce a couple of examples of IFS's which have aroused great interest for some time. After each definition we also discuss a little how our theory turns out to be in that particular case.

Definition 3.9. Let the mappings of IFS be *similitudes*, that is, for each $i \in I$ there exists $0 < s_i < 1$ such that $|\varphi_i(x) - \varphi_i(y)| = s_i|x-y|$ whenever $x, y \in \Omega$. We call this kind of setting a *similitude IFS* and the corresponding limit set a *self–similar set*.

If for each $\mathtt{i} \in I^*$ we choose $\psi^t_{\mathtt{i}} \equiv s^t_{\mathtt{i}}$, where $s_{\mathtt{i}} = s_{i_1} \cdots s_{i_{|\mathtt{i}|}}$, then $\psi^t_{\mathtt{i}}$ is a constant cylinder function satisfying the chain rule. Assuming weak bounded overlapping, the similitude IFS is geometrically stabile due to Proposition 3.6, and, thus, with this choice of the cylinder function we get, applying Theorem 3.8, that $\dim_H(E) = \dim_\psi(I^\infty)$ (we clearly have $\underline{s}_{\mathtt{i}} = \overline{s}_{\mathtt{i}} = s_{\mathtt{i}}$). Notice also that Theorem 3.5 provides us with concrete assumptions, namely the OSC and the boundary condition, to obtain the weak bounded overlapping. The definition of this setting goes back to the well known article of Hutchinson [10]. However, the open set condition was first introduced by Moran in [15]. Schief studied in [19], extending ideas of Bandt and Graf [1], the relationship between the OSC and the choice of the mappings of IFS. It also follows from the result of Schief that the weak bounded overlapping implies the OSC since according to Proposition 3.6 and Theorem 3.8 we have $\mathcal{H}^t(E) > 0$, where $t = \dim_H(E)$. For example, using Theorems 2.5, 2.8 and 3.7, we see that the t–equilibrium measure, where $t = \dim_H(E)$, gives us the idea of "mass distribution"; we start with mass 1 and on each level of the construction we divide the mass from cylinder sets of the previous level using the rule obtained by the probability vector $(s^t_i)_{i \in I}$. We could also define our cylinder function using an arbitrary probability vector $(p_i)_{i \in I}$. In this case it is interesting to study how big is the subset of the limit set where the so called local Hausdorff dimension of the equilibrium measure equals to a given number at each point.

Definition 3.10. Let mappings of IFS be *conformal*, that is, $|\varphi'_i|^d = |J_{\varphi_i}|$ for every $i \in I$, where J stands for the usual Jacobian and the norm on the left–hand side is just a standard "sup–norm" for linear mappings. We call this kind of setting a *conformal IFS* and the corresponding limit set a *self–conformal set*.

In the definition of the conformal mapping, the derivative ought to exist for $\mathcal{H}^d$–almost all points. In our case this is true due to the Rademacher's theorem. Observe that the conformal mapping is complex analytic in the plane and, by Liouville's theorem, a Möbius transformation in higher dimensions (see Theorem 4.1 of Reshetnyak [18]). So, in fact, conformal mappings are C^∞ and infinitesimally similitudes. Notice also that it is essential to use the bounded set Ω here since conformal mappings contractive in the whole $\mathbb{R}^d$ are similitudes, as d exceeds 2. If for each $\mathtt{i} \in I^*$ we choose $\psi^t_{\mathtt{i}}(\mathtt{h}) = \left|\varphi'_{\mathtt{i}}\big(\pi(\mathtt{h})\big)\right|^t$, then $\psi^t_{\mathtt{i}}$ is a cylinder function satisfying the chain rule. The BVP for $\psi^t_{\mathtt{i}}$ is guaranteed by the smoothness of mappings $\varphi_{\mathtt{i}}$, Proposition 2.1 and the chain rule. With this choice of the cylinder function we may also call the BVP a *bounded distortion property (BDP)* since it gives information about the distortion of mappings $\varphi_{\mathtt{i}}$. Assuming weak bounded overlapping, the system is geometrically stabile and we get $\dim_H(E) = \dim_\psi(I^\infty)$ like before (we can choose $\underline{\psi}^t_{\mathtt{i}} = \overline{\psi}^t_{\mathtt{i}} = \psi^t_{\mathtt{i}}$). Notice again that, using Theorem 3.5, the OSC and the bounded overlapping provides us with a sufficient condition for the weak bounded overlapping to hold. In the conformal case the equilibrium measure is equivalent to the conformal measure. Peres, Rams, Simon and Solomyak [16] generalised the result of Schief for the conformal setting. Thus, the weak bounded overlapping implies the OSC also in this setting. Mauldin and Urbański [14] have introduced the theory of conformal IFS's for infinite collections of mappings.

Definition 3.11. Let the mappings of IFS be *affine*, that is, $\varphi_i(x) = A_i x + a_i$ for every $i \in I$, where A_i is a contractive non–singular linear mapping and $a_i \in \mathbb{R}^d$. We call this kind of setting an *affine IFS* and the corresponding limit set a *self–affine set*.

Clearly, the products $A_{\mathtt{i}} = A_{i_1} \cdots A_{i_{|\mathtt{i}|}}$ are also contractive and non–singular. Singular values of a non–singular matrix are the lengths of the principle semiaxes of the image of the unit ball. On the other hand, the singular values $1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d > 0$ of a contractive, non–singular matrix A are the non–negative square roots of the eigenvalues of A^*A, where A^* is the transpose of A. Define the singular value function α^t by setting $\alpha^t(A) = \alpha_1\alpha_2\cdots\alpha_{l-1}\alpha_l^{t-l+1}$, where l is the smallest integer greater than or equal to t. For all $t > d$ we put $\alpha^t(A) = (\alpha_1\cdots\alpha_d)^{t/d}$. It is clear that $\alpha^t(A)$ is continuous and strictly decreasing in t. If for each $\mathtt{i} \in I^*$ we choose $\psi_{\mathtt{i}}^t \equiv \alpha^t(A_{\mathtt{i}})$, then $\psi_{\mathtt{i}}^t$ is a constant cylinder function. The subchain rule for $\psi_{\mathtt{i}}^t$ is satisfied by Lemma 2.1 of Falconer [5]. Since in this case we do not have the chain rule, it is still very difficult to say anything "concrete" about the equilibrium measure or the Hausdorff dimension of the limit set. Assuming the SSC, we have bounded overlapping satisfied by Lemma 3.4 and thus we can at least approximate the Hausdorff dimension of the limit set by using Theorem 3.8. We study self–affine sets and equilibrium measures of affine IFS's in more detail in the next chapter. The following example shows us that in the affine setting we cannot allow overlapping even at one single point if we want to have the weak bounded overlapping.

Example 3.12. Put $I = \{1,2\}$, $X = \overline{B(0,1)} \cap \{(x_1,x_2) \in \mathbb{R}^2 : |x_2| \leq x_1\}$ and define two affine mappings (in matrix notation) as follows:

$$\begin{aligned}\varphi_1(x_1,x_2) &= \begin{pmatrix}\cos(\pi/8) & -\sin(\pi/8)\\ \sin(\pi/8) & \cos(\pi/8)\end{pmatrix}\begin{pmatrix}0.9 & 0\\ 0 & 0.3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\\ \varphi_2(x_1,x_2) &= \begin{pmatrix}\cos(\pi/8) & \sin(\pi/8)\\ -\sin(\pi/8) & \cos(\pi/8)\end{pmatrix}\begin{pmatrix}0.9 & 0\\ 0 & 0.3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.\end{aligned} \tag{3.34}$$

The set X is a sector with angle $\pi/2$, and functions φ_1 and φ_2 map this sector into two flattened sectors inside X such that $\varphi_1(X) \cap \varphi_2(X) = \{0\}$. The OSC is therefore satisfied. Since the origin is the only fixed point of both mappings, the limit set is nothing but $\{0\}$. This setting does not satisfy the weak bounded overlapping, because the amount of cylinder sets of the level n including the origin is always 2^n.

Notice that the similitude IFS is always both conformal and affine. Also if we consider the cylinder functions introduced before, we notice that the cylinder function of the similitude IFS is just a special case of both cylinder functions of conformal IFS and affine IFS. We could also study more general limit sets in this manner. Falconer [7] has obtained some dimension results into this direction by using the singular value function for the derivatives of more general mappings. Using the concept of general IFS, it is possible to use bi–Lipschitz mappings for defining geometric constructions for which it is possible easily to determine the Hausdorff dimension of the limit set.

Example 3.13. Consider a bi–Lipschitz general IFS satisfying the OSC and the boundary condition. Suppose that for each $\mathtt{i} \in I^*$ there exist balls $\underline{B}_{\mathtt{i}}$ and $\overline{B}_{\mathtt{i}}$ and a constant

$C > 0$ such that

$$\underline{B}_{\mathtt{i}} \subset \varphi_{\mathtt{i}}(X) \subset \overline{B}_{\mathtt{i}}, \tag{3.35}$$

$l_{\mathtt{i}}(x) \geq Cd(\underline{B}_{\mathtt{i}})$ and $L_{\mathtt{i}}(x) \leq Cd(\overline{B}_{\mathtt{i}})$ as $x \in X$. Now, if the ratio between the radii of $\overline{B}_{\mathtt{i}}$ and $\underline{B}_{\mathtt{i}}$ remains bounded, then

$$\dim_H(E) = t, \tag{3.36}$$

where $t \geq 0$ is the unique number satisfying

$$\lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in I^n} r_{\mathtt{i}}^t = 0 \tag{3.37}$$

and $r_{\mathtt{i}}$ is the radius of either $\overline{B}_{\mathtt{i}}$ or $\underline{B}_{\mathtt{i}}$. This result is easily obtained by first noting that the ratio $\overline{s}_{\mathtt{i}}/\underline{s}_{\mathtt{i}}$ is bounded and then using Propositions 3.5 and 3.6, Theorem 3.8 and recalling the definition of the topological pressure.

The concept of the general IFS is also crucial in the following example, which says that the relative positions of cylinder sets are irrelevant concerning the Hausdorff dimension of the limit set of conformal systems provided that a sufficient separation condition is satisfied.

Example 3.14. Consider a conformal IFS satisfying the OSC and the boundary condition. Choosing $\psi_{\mathtt{i}}^t(\mathtt{h}) = \big|\varphi_{\mathtt{i}}'\big(\pi(\mathtt{h})\big)\big|^t$, we have $\dim_H(E) = \dim_\psi(I^\infty)$. In this setting the placement of cylinder sets is fixed and their relative positions follow from the rule obtained by the mappings φ_i. We could now rearrange the placements and ask what happens to the Hausdorff dimension of the limit set. We define a general IFS by composing our original conformal mappings with isometries such that the OSC remains satisfied. Since this does not affect our cylinder function and composed mappings are still conformal, we will get for the limit set $\tilde{E}$ of this general IFS that $\dim_H(\tilde{E}) = \dim_\psi(I^\infty)$ using Propositions 3.5, 3.6 and Theorem 3.8.

4. Dimension of the equilibrium measure

We say that the Hausdorff dimension of a given Borel probability measure m is $\dim_H(m) = \inf\{\dim_H(A) : A \text{ is a Borel set such that } m(A) = 1\}$. To check if $\dim_H(m) = \dim_H(E)$ is one way to examine how well a given measure m is spread out on a given set E. If we consider similitude and conformal IFS's and we choose cylinder functions to be the ones introduced in the previous chapter, we notice using Proposition 3.6 and Theorem 3.8 that $\dim_H(m) = \dim_H(E) =: t$ provided that the weak bounded overlapping is satisfied. Here m and E are the corresponding t–equilibrium measure and the limit set. It is an interesting question whether we can obtain the same result for the affine setting. In the following we will prove that at least in "almost all" affine cases this is possible. To do that we first have to prove that the equilibrium measure μ is *ergodic*, that is, $\mu(A) = 0$ or $\mu(A) = 1$ for every Borel set A for which $A = \sigma^{-1}(A)$. In the proof we use some ideas found in Zinsmeister [23], Bowen [3] and Phelps [17].

Theorem 4.1. *There exists an ergodic equilibrium measure.*

Proof. Let us first study mappings $\mathcal{P}, \mathcal{Q}_n, \mathcal{Q} : \mathcal{M}_\sigma(I^\infty) \to \mathbb{R}$, for which $\mathcal{P}(\mu) = h_\mu$, $\mathcal{Q}_n(\mu) = \frac{1}{n}\sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \psi_{\mathtt{i}}^t(\mathtt{h})$ and $\mathcal{Q}(\mu) = \lim_{n\to\infty} \mathcal{Q}_n(\mu) = E_\mu(t)$. It is clear that each $\mathcal{Q}_n$ is affine and continuous (basically because cylinder sets have empty boundary) and $\mathcal{Q}$ is affine. We will prove that $\mathcal{P}$ is affine and upper semicontinuous.

Fix $0 \le x_1, x_2 \le 1$ and $\lambda \in [0,1]$ and denote $x = \lambda x_1 + (1-\lambda)x_2$. Now using the concavity of the function $H(x) = -x\log x$, $H(0) = 0$, we have

$$\begin{aligned}
0 &\le -x\log x + \lambda x_1 \log x_1 + (1-\lambda)x_2 \log x_2 \\
&= -\lambda x_1(\log x - \log x_1) - (1-\lambda)x_2(\log x - \log x_2) \\
&= -\lambda x_1\bigl(\log x - \log(\lambda x_1)\bigr) - (1-\lambda)x_2\bigl(\log x - \log((1-\lambda)x_2)\bigr) \\
&\quad - \lambda x_1 \log\lambda - (1-\lambda)x_2\log(1-\lambda) \qquad (4.1)\\
&\le -x_1\lambda\log\lambda - x_2(1-\lambda)\log(1-\lambda) \\
&\le x_1\tfrac{1}{e} + x_2\tfrac{1}{e}
\end{aligned}$$

since $\log x - \log(\lambda x_1)$ and $\log x - \log((1-\lambda)x_2)$ are positive. Hence we get

$$\begin{aligned}
0 &\le \sum_{\mathtt{i}\in I^n} H\bigl(\mu([\mathtt{i}])\bigr) - \lambda\sum_{\mathtt{i}\in I^n} H\bigl(\mu_1([\mathtt{i}])\bigr) - (1-\lambda)\sum_{\mathtt{i}\in I^n} H\bigl(\mu_2([\mathtt{i}])\bigr) \\
&\le \tfrac{1}{e}\sum_{\mathtt{i}\in I^n}\mu_1([\mathtt{i}]) + \tfrac{1}{e}\sum_{\mathtt{i}\in I^n}\mu_2([\mathtt{i}]) = \tfrac{2}{e}, \qquad (4.2)
\end{aligned}$$

where $\mu_1, \mu_2 \in \mathcal{M}_\sigma(I^\infty)$ and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. By the convexity of $\mathcal{M}_\sigma(I^\infty)$ we have $\mu \in \mathcal{M}_\sigma(I^\infty)$ and thus it follows from (4.2) that $h_\mu = \lambda h_{\mu_1} + (1-\lambda)h_{\mu_2}$, and hence, $\mathcal{P}$ is affine. Take next $\varepsilon > 0$ and $\mu \in \mathcal{M}_\sigma(I^\infty)$ and choose n_0 big enough such that

$$\tfrac{1}{n}\sum_{\mathtt{i}\in I^n} H\bigl(\mu([\mathtt{i}])\bigr) \le h_\mu + \tfrac{\varepsilon}{2} \qquad (4.3)$$

whenever $n \ge n_0$. Now we choose arbitrary $\eta \in \mathcal{M}_\sigma(I^\infty)$ for which

$$\tfrac{1}{n}\sum_{\mathtt{i}\in I^n} H\bigl(\eta([\mathtt{i}])\bigr) \le \tfrac{1}{n}\sum_{\mathtt{i}\in I^n} H\bigl(\mu([\mathtt{i}])\bigr) + \tfrac{\varepsilon}{2} \qquad (4.4)$$

for some $n \ge n_0$. This choice can be made just by taking η to be close enough to μ in the weak topology and recalling that cylinder sets have empty boundary. Therefore, using Proposition 2.4(3), we have

$$h_\eta \le \tfrac{1}{n}\sum_{\mathtt{i}\in I^n} H\bigl(\eta([\mathtt{i}])\bigr) \le \tfrac{1}{n}\sum_{\mathtt{i}\in I^n} H\bigl(\mu([\mathtt{i}])\bigr) + \tfrac{\varepsilon}{2} \le h_\mu + \varepsilon \qquad (4.5)$$

for some $n \ge n_0$. We have established the upper semicontinuity of the mapping $\mathcal{P}$.

Denote the set of all ergodic measures of $\mathcal{M}_\sigma(I^\infty)$ with $\mathcal{E}_\sigma(I^\infty)$. Let us now assume contrarily that $\mathcal{P} + \mathcal{Q}$ cannot attain its supremum with an ergodic measure, that is, $(\mathcal{P} + \mathcal{Q})(\eta) < (\mathcal{P} + \mathcal{Q})(\mu)$ for all $\eta \in \mathcal{E}_\sigma(I^\infty)$, where μ is an equilibrium measure. Recalling Theorem 6.10 of Walters [22], we know that the set $\mathcal{M}_\sigma(I^\infty)$ is compact and convex and the set of its extreme points is exactly the set $\mathcal{E}_\sigma(I^\infty)$. An extreme point of a convex set is a point which cannot be expressed as an average of two distinct points. Using Choquet's theorem (see Chapter 3 of [17]), we can get an ergodic decomposition for every invariant measure, namely, for each $\mu \in \mathcal{M}_\sigma(I^\infty)$ there exists a unique Borel regular probability measure τ_μ on $\mathcal{E}_\sigma(I^\infty)$ such that

$$\mathcal{R}(\mu) = \int_{\mathcal{E}_\sigma(I^\infty)} \mathcal{R}(\eta) d\tau_\mu(\eta) \tag{4.6}$$

for every continuous affine $\mathcal{R} : \mathcal{M}_\sigma(I^\infty) \to \mathbb{R}$.

Denoting now $A_k = \{\eta \in \mathcal{E}_\sigma(I^\infty) : (\mathcal{P}+\mathcal{Q})(\mu) - (\mathcal{P}+\mathcal{Q})(\eta) \geq \frac{1}{k}\}$, where μ is an equilibrium measure, we have $\bigcup_{k=1}^\infty A_k = \mathcal{E}_\sigma(I^\infty)$ and thus $\tau_\mu(A_k) > 0$ for some k. Clearly,

$$\begin{aligned}(\mathcal{P}+\mathcal{Q})(\mu) - \int_{\mathcal{E}_\sigma(I^\infty)} (\mathcal{P}+\mathcal{Q})(\eta) d\tau_\mu(\eta) &= \int_{\mathcal{E}_\sigma(I^\infty)} (\mathcal{P}+\mathcal{Q})(\mu) - (\mathcal{P}+\mathcal{Q})(\eta) d\tau_\mu(\eta) \\ &\geq \int_{A_k} \tfrac{1}{k} d\tau_\mu(\eta) = \tfrac{1}{k}\tau_\mu(A_k) \end{aligned} \tag{4.7}$$

for every k and thus

$$(\mathcal{P}+\mathcal{Q})(\mu) > \int_{\mathcal{E}_\sigma(I^\infty)} (\mathcal{P}+\mathcal{Q})(\eta) d\tau_\mu(\eta). \tag{4.8}$$

We will show that this is impossible, and, hence, the contradiction we obtain finishes the proof.

Our goal now is to prove that we can write (4.6) also by using upper semicontinuous affine functions, particularly with $\mathcal{P}+\mathcal{Q}_n$. Fix $n \in \mathbb{N}$ and define $\overline{\mathcal{R}} : \mathcal{M}_\sigma(I^\infty) \to \mathbb{R}$ by setting $\overline{\mathcal{R}}(\mu) = \inf\{\mathcal{R}(\mu) : \mathcal{R} \geq \mathcal{P}+\mathcal{Q}_n$ is continuous and affine$\}$. Let us first prove that for each continuous affine $\mathcal{R}_1, \mathcal{R}_2 > \mathcal{P}+\mathcal{Q}_n$ there exists a continuous affine $\mathcal{R}$ for which $\mathcal{P}+\mathcal{Q}_n < \mathcal{R} \leq \mathcal{R}_1, \mathcal{R}_2$. Since $\mathcal{P}+\mathcal{Q}_n$ is affine and upper semicontinuous, we notice that the set $D = \{(\mu, r) : \mu \in \mathcal{M}_\sigma(I^\infty),\ r \leq (\mathcal{P}+\mathcal{Q}_n)(\mu)\}$ is closed and convex. Since both mappings $\mathcal{R}_i$ are continuous and affine as $i = 1, 2$, we get that both sets $D_i = \{(\mu, r) : \mu \in \mathcal{M}_\sigma(I^\infty),\ r = \mathcal{R}_i(\mu)\}$ are compact and convex. Observe that the convex hull of the union $D_1 \cup D_2$ is compact and disjoint from the set D. Now applying the separation theorem for convex sets (Corollary 1.2 of [4]), we notice there exists a non–zero continuous real–valued linear functional l on $\mathcal{M}_\sigma(I^\infty) \times \mathbb{R}$ and a real number α such that the affine hyperplane

$$A = \{(\mu, r) : \mu \in \mathcal{M}_\sigma(I^\infty),\ l(\mu, r) = \alpha\} \tag{4.9}$$

strictly separates the sets D and the convex hull of $D_1 \cup D_2$. Because of the linearity of l for each $\mu \in \mathcal{M}_\sigma(I^\infty)$, there exists exactly one r for which $(\mu, r) \in A$. Thus there exists a function $\mathcal{R} : \mathcal{M}_\sigma(I^\infty) \to \mathbb{R}$ such that $l\big(\mu, \mathcal{R}(\mu)\big) = \alpha$ as $\mu \in \mathcal{M}_\sigma(I^\infty)$. The function $\mathcal{R}$ is affine and continuous because the functional l is linear and continuous. Since now $l(\mu, r) > \alpha$ for every $(\mu, r) \in D$ and $l(\mu, r) < \alpha$ for every (μ, r) in the convex hull of $D_1 \cup D_2$ (or the other way around), we have $\mathcal{R}(\mu) > (\mathcal{P}+\mathcal{Q}_n)(\mu)$ and $\mathcal{R}(\mu) < \mathcal{R}_1(\mu), \mathcal{R}_2(\mu)$ for each $\mu \in \mathcal{M}_\sigma(I^\infty)$, which is exactly what we wanted. A similar reasoning implies that $\overline{\mathcal{R}} = \mathcal{P}+\mathcal{Q}_n$. Assume contrarily that there exists ν such that $(\mathcal{P}+\mathcal{Q}_n)(\nu) < \overline{\mathcal{R}}(\nu)$. Now the set D is disjoint from the compact convex set $\big\{\big(\nu, \overline{\mathcal{R}}(\nu)\big)\big\}$ and the separation theorem gives us an immediate contradiction. We will next show that

$$\int_{\mathcal{E}_\sigma(I^\infty)} (\mathcal{P}+\mathcal{Q}_n)(\eta)d\tau_\mu(\eta) = \inf\biggl\{ \int_{\mathcal{E}_\sigma(I^\infty)} \mathcal{R}(\eta)d\tau_\mu(\eta) : \mathcal{R} \ge \mathcal{P}+\mathcal{Q}_n \text{ is continuous and affine} \biggr\}. \tag{4.10}$$

Let us denote with γ the right–hand side of (4.10) and choose a sequence $\{\mathcal{R}_i\}_{i\in\mathbb{N}}$ of continuous affine mappings greater than or equal to $\mathcal{P}+\mathcal{Q}_n$ such that

$$\lim_{i\to\infty} \int_{\mathcal{E}_\sigma(I^\infty)} \mathcal{R}_i(\eta)d\tau_\mu(\eta) = \gamma. \tag{4.11}$$

We can assume that this sequence is monotonically decreasing, and hence there exists a Borel measurable function $\mathcal{R} = \lim_{i\to\infty}\mathcal{R}_i$ with $\mathcal{R} \ge \mathcal{P}+\mathcal{Q}_n$ and

$$\int_{\mathcal{E}_\sigma(I^\infty)} \mathcal{R}(\eta)d\tau_\mu(\eta) = \gamma \tag{4.12}$$

using the monotone convergence theorem. If it held that $\tau_\mu\bigl(\{\eta \in \mathcal{E}_\sigma(I^\infty) : \mathcal{R}(\eta) > (\mathcal{P}+\mathcal{Q}_n)(\eta)\}\bigr) > 0$, then there would be real numbers r and q such that also the set $\{\eta \in \mathcal{E}_\sigma(I^\infty) : (\mathcal{P}+\mathcal{Q}_n)(\eta) < r < q < \mathcal{R}(\eta)\}$ has positive measure. By the Borel regularity, this set contains a compact subset C of positive measure. Now for each $\eta \in C$ there is a continuous affine mapping $\tilde{\mathcal{R}} \ge \mathcal{P}+\mathcal{Q}_n$ such that $\tilde{\mathcal{R}}(\eta) < r$. Relying now on compactness and continuity, we can choose a finite number of them, say, $\tilde{\mathcal{R}}_1,\ldots,\tilde{\mathcal{R}}_k$ such that for each $\eta \in C$ there is $1 \le j \le k$ with $\tilde{\mathcal{R}}_j(\eta) < r$. For each $i \in \mathbb{N}$ we choose a continuous affine mapping $\hat{\mathcal{R}}_i$ such that $\mathcal{P}+\mathcal{Q}_n < \hat{\mathcal{R}}_i \le \mathcal{R}_i, \tilde{\mathcal{R}}_1,\ldots,\tilde{\mathcal{R}}_k$. Hence $\hat{\mathcal{R}}_i < r < r + \mathcal{R} - q < \mathcal{R}_i - (q-r)$ on C and $\hat{\mathcal{R}}_i \le \mathcal{R}_i$ elsewhere. Therefore,

$$\gamma \le \int_{\mathcal{E}_\sigma(I^\infty)} \hat{\mathcal{R}}_i(\eta)d\tau_\mu(\eta) \le \int_{\mathcal{E}_\sigma(I^\infty)} \mathcal{R}_i(\eta)d\tau_\mu(\eta) - (q-r)\tau_\mu(C), \tag{4.13}$$

which finishes the proof of (4.10) as we let $i \to \infty$. Using now (4.10) and (4.6), we get that

$$\begin{aligned}\int_{\mathcal{E}_\sigma(I^\infty)} (\mathcal{P}+\mathcal{Q}_n)(\eta)d\tau_\mu(\eta) &= \inf\biggl\{ \int_{\mathcal{E}_\sigma(I^\infty)} \mathcal{R}(\eta)d\tau_\mu(\eta) : \mathcal{R} \ge \mathcal{P}+\mathcal{Q}_n \text{ is continuous and affine} \biggr\} \qquad (4.14)\\ &= \inf\bigl\{ \mathcal{R}(\mu) : \mathcal{R} \ge \mathcal{P}+\mathcal{Q}_n \text{ is continuous and affine} \bigr\} = (\mathcal{P}+\mathcal{Q}_n)(\mu).\end{aligned}$$

Letting $n \to \infty$ and using the dominated convergence theorem, we have shown that (4.8) cannot happen and thus finished the proof. ☐

Notice that in the previous theorem we actually proved that the equilibrium measure can be chosen to be ergodic. The ergodicity of the equilibrium measure is crucial in the following proposition, which, for example, in the similitude and conformal cases gives information about the so called local Hausdorff dimension of the equilibrium measure.

Proposition 4.2. *Suppose $t \geq 0$ and μ is an ergodic t–equilibrium measure. Then*

$$\lim_{n\to\infty} \frac{\log \mu([\mathtt{i}|_n])}{\log \psi^t_{\mathtt{i}|_n}(\mathtt{h})} = 1 - \frac{P(t)}{E_\mu(t)} \tag{4.15}$$

for μ–almost all $\mathtt{i} \in I^\infty$.

Proof. Let us first note that due to the invariance of the equilibrium measure and theorem of Shannon–McMillan (for example, see Chapter 3 of Zinsmeister [23]) we have

$$h_\mu = -\lim_{n\to\infty} \tfrac{1}{n} \log \mu([\mathtt{i}|_n]) \tag{4.16}$$

for μ–almost all $\mathtt{i} \in I^\infty$. We can get a similar kind of expression for the energy as well. Indeed, using Kingman's subadditive ergodic theorem (for example, see Steele [21]) and the BVP, we have

$$\begin{aligned} E_\mu(t) &= \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \psi^t_{\mathtt{i}}(\mathtt{h}) \\ &= \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \int_{[\mathtt{i}]} \log \psi^t_{\mathtt{i}}(\mathtt{h}) d\mu(\mathtt{h}) \\ &= \lim_{n\to\infty} \tfrac{1}{n} \int_{I^\infty} \log \psi^t_{\mathtt{i}|_n}\big(\sigma^n(\mathtt{i})\big) d\mu(\mathtt{i}) \\ &= \lim_{n\to\infty} \tfrac{1}{n} \log \psi^t_{\mathtt{j}|_n}\big(\sigma^n(\mathtt{j})\big) \\ &= \lim_{n\to\infty} \tfrac{1}{n} \log \psi^t_{\mathtt{j}|_n}(\mathtt{h}) \end{aligned} \tag{4.17}$$

for μ–almost all $\mathtt{j} \in I^\infty$. Now the claim follows easily from the fact

$$P(t) = E_\mu(t) + h_\mu. \tag{4.18}$$

□

Now, with the help of this proposition, we can prove the next theorem, our main tool in studying the Hausdorff dimension of the equilibrium measure on affine systems. We define the *equilibrium dimension of a measure* $\mu \in \mathcal{M}(I^\infty)$ by setting $\dim_\psi(\mu) = \inf\{\dim_\psi(A) : A$ is a Borel set such that $\mu(A) = 1\}$.

Theorem 4.3. *Suppose $P(t) = 0$ and μ is an ergodic t–equilibrium measure. Then*

$$dim_\psi(\mu) = t. \tag{4.19}$$

Proof. Let us denote

$$R = \Big\{ \mathtt{i} \in I^\infty : \lim_{n\to\infty} \frac{\log \mu([\mathtt{i}|_n])}{\log \psi^t_{\mathtt{i}|_n}(\mathtt{h})} = 1 \Big\} \tag{4.20}$$

and take an arbitrary Borel set $A \subset I^\infty$ for which $\mu(A) = 1$. Using Proposition 4.2, we also have $\mu(R \cap A) = 1$. Fix $\mathtt{i} \in R \cap A$ and $q < t$. Now it follows from the definition of the cylinder function, Proposition 4.2 and (4.16) that

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log \mu([\mathtt{i}|_n])}{\log \psi^q_{\mathtt{i}|_n}(\mathtt{h})} &\geq \lim_{n\to\infty} \frac{\log \mu([\mathtt{i}|_n])}{\log \psi^t_{\mathtt{i}|_n}(\mathtt{h}) + \log \overline{s}^{-n}_{t-q}} \\ &= \frac{1}{1 + \frac{1}{h_\mu}\log \overline{s}_{t-q}} > 1. \end{aligned} \tag{4.21}$$

Thus there exists $n_0 = n_0(\mathtt{i})$ such that

$$\frac{\log \mu([\mathtt{i}|_n])}{\log \psi^q_{\mathtt{i}|_n}(\mathtt{h})} \geq 1 \tag{4.22}$$

whenever $n \geq n_0$. Denoting $A_k = \{\mathtt{i} \in R \cap A : n_0(\mathtt{i}) < k\}$, we have $R \cap A = \bigcup_{k=1}^\infty A_k$. Hence, using (4.22), we get for each $\mathtt{i} \in A_k$

$$\frac{\mu([\mathtt{i}|_n])}{\psi^q_{\mathtt{i}|_n}(\mathtt{h})} \leq 1 \tag{4.23}$$

whenever $n \geq k$. Take $\{[\mathtt{i}_j]\}_j$ to be any cover for A_k such that $|\mathtt{i}_j| > k$ and $[\mathtt{i}_j] \cap A_k \neq \emptyset$ for every j. We can choose each $\mathtt{i}_j$ to be of the form $\mathtt{i}|_n$ for some $\mathtt{i} \in A_k$ and $n \in \mathbb{N}$. Hence by (4.23)

$$\mu(A_k) \leq \sum_j \mu([\mathtt{i}_j]) \leq \sum_j \psi^q_{\mathtt{i}_j}(\mathtt{h}) \leq K_q \sum_j \int_{I^\infty} \psi^q_{\mathtt{i}_j}(\mathtt{h}) d\mu(\mathtt{h}), \tag{4.24}$$

from which we get $\mathcal{G}^q(A_k) \geq K_q^{-1}\mu(A_k)$. Now, clearly, $\mathcal{G}^q(R \cap A) = \lim_{k\to\infty} \mathcal{G}^q(A_k) \geq K_q^{-1} \lim_{k\to\infty} \mu(A_k) = K_q^{-1}\mu(R \cap A)$, which gives $\mathcal{G}^q(A) > 0$ and $\dim_\psi(A) \geq q$. Since $q < t$ was arbitrary as was the choice of the Borel set A of full measure, we conclude $\dim_\psi(\mu) \geq t$. The proof is finished by recalling Theorem 3.2. □

In the similitude and conformal cases we obtained the desired dimension result easily straight from Theorem 3.8. For the affine IFS we can not apply Theorem 3.8 because in that case it gives only upper and lower bounds for the Hausdorff dimension of the equilibrium measure. But using now Theorem 4.3, we can extend a little the result of Falconer [5] to be more suitable for our purposes.

Theorem 4.4. *Suppose mappings of an affine IFS are of the form $\varphi_i(x) = A_i x + a_i$, where $|A_i| < \frac{1}{2}$, as $i \in I$ and the cylinder function is chosen to be the singular value function, $\psi^t_{\mathtt{i}} \equiv \alpha^t(A_{\mathtt{i}})$. We also assume that $P(t) = 0$, μ is an ergodic t–equilibrium measure and $m = \mu \circ \pi^{-1}$. Then for $\mathcal{H}^{d\#I}$–almost all $a = (a_1, \ldots, a_{\#I}) \in R^{d\#I}$ we have*

$$dim_\psi(I^\infty) = dim_H(A) \tag{4.25}$$

whenever $A \subset E = E(a)$ is a Borel set such that $m(A) = 1$.

Let us make a few comments on this theorem. The constant $\frac{1}{2}$ here is improved from the one used by Falconer. In fact, rather surprisingly, it is proven to be sharp in a sense if $|A_i| \geq \frac{1}{2} + \varepsilon$ for some $i \in I$ and for any $\varepsilon > 0$, then the theorem may fail. See Proposition 3.1 of Solomyak [20] for details. Falconer did not state his theorem for sets with full m–measure, but the same proofs will apply basically because due to Theorems 4.3 and 3.2, we have $\dim_\psi(A) = \dim_\psi(I^\infty)$ whenever $A \subset I^\infty$ has full μ–measure. Notice that in the theorem no separation condition of any kind is assumed. However, there are situations where the equilibrium dimension and the Hausdorff dimension do not coincide if we just assume $|A_i| < \frac{1}{2}$ for every $i \in I$. For example, there is too much overlapping among the sets $\varphi_{\mathtt{i}}(X)$, or these sets are aligned in a way that it is not possible to obtain economical covers using ellipsoids, and, thus, the use of the singular value function does not fit. In the theorem all of these "bad" situations are excluded by the statement "for $\mathcal{H}^{d\#I}$–almost all a". It is an interesting question to find a characterization for these "bad" situations. Hueter and Lalley have provided in [9] with checkable sufficient conditions for the theorem to hold for all a.

The following theorem, which we obtain easily as a corollary from Theorem 4.4, gives a partially positive answer to the open question proposed by Kenyon and Peres in [12]. They asked whether there exists a T–invariant ergodic probability measure on a given compact set, where the mapping T is continuous and expanding, such that it has full dimension. In our case the mapping T is constructed by using inverses of the mappings of IFS.

Theorem 4.5. *Suppose mappings of an affine IFS are of the form $\varphi_i(x) = A_i x + a_i$, where $|A_i| < \frac{1}{2}$, as $i \in I$ and the cylinder function is chosen to be the singular value function, $\psi^t_{\mathtt{i}} \equiv \alpha^t(A_{\mathtt{i}})$. We also assume that $P(t) = 0$, μ is an ergodic t–equilibrium measure and $m = \mu \circ \pi^{-1}$. Then for $\mathcal{H}^{d\#I}$–almost all $a = (a_1, \ldots, a_{\#I}) \in R^{d\#I}$ we have*

$$dim_H(m) = dim_H(E), \tag{4.26}$$

where $E = E(a)$.

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