AVERAGE HOMOGENEITY AND DIMENSIONS OF MEASURES

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Abstract. We introduce the concept of average homogeneity of a measure by comparing the measure to the uniform distribution in a relatively simple way. This leads to a very general notion which may be regarded as an inverse of porosity. In this paper the emphasis is given to relations between homogeneity and dimensions of measures. First we consider the effect of homogeneity on dimensions by proving an upper bound to the Hausdorff dimension as a function of homogeneity and its order. The opposite question of how dimensions effect homogeneity is solved by giving an upper bound to homogeneity in terms of upper packing dimension. We also illustrate by examples that all our results are the best possible ones.

1. Introduction

Both homogeneity and porosity are quantities that describe irregular structures of sets and measures. The homogeneity, introduced in Definition 2.1, can be regarded as an inverse concept of porosity. Porosity, in turn, gives the relative size of holes in a set at all small scales. The extensive study of dimensional properties of porous sets and measures was pioneered by Mattila [Mat1]. As a consequence of a conical density theorem, he proved that if porosity of a set $A \subset \mathbb{R}^n$ is close to its maximum value, then Hausdorff dimension of A cannot be much bigger than $n-1$. In addition to generalizing the corresponding result to packing dimension, Salli [S] established the correct asymptotics for the dimension bound in terms of large porosity. In [KR] Koskela and Rohde considered a larger class of mean porous sets having holes of certain size at a certain percentage of small scales, and found the correct asymptotics for the dimension estimate when the size of holes is small.

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For measures the notation of porosity, introduced in [EJJ], describes for all small scales the relative size of balls having small relative measure. In [EJJ] the analogy of Mattila's result was proven for measures that satisfy the doubling condition. This in turn was generalized to arbitrary measures in [JJ]. In the comprehensive representation [BS], Beliaev and Smirnov introduced a more general concept of mean porosity for measures, and proved analogous dimension estimates of those of [S] and [KR] for measures. In particular, they verified the correct asymptotics for the dimension estimates.

As pointed out in [BS], porosity is a strong condition, and the assumptions of the asymptotic dimension estimates mentioned above rule out many naturally arising measures that fail to suit to the category of porous measures, but which are nevertheless unevenly distributed, and have strictly smaller dimension than the ambient space. In [BS] Beliaev and Smirnov introduced two generalizations of the notion: weak porosity and weak mean porosity. Weak porosity is a weaker concept than porosity, and weak mean porosity, in turn, is weaker than mean porosity. For more information on different notations of porosity and their intermediate relations, see [BJ], [Mat2], [MM], and [MMPZ].

In this paper we address the problem of obtaining dimension results which are sharp (not only asymtotically) without ruling out any measures. This leads to the notion of average homogeneity which is based on a comparison between an arbitrary measure and the uniform distribution in a relatively simple way. Homogeneity is a weaker concept than weak porosity and weak mean porosity in [BS] (see Remarks 2.2 (1) and 2.4), but it has still interesting relations to dimensions. Intuitively, given positive integers k and j, and a measure μ on $\mathbb{R}^n$, the k-average homogeneity of μ of order j is determined by taking into account the average of the μ-measure of the k-adic cubes having jth smallest μ-measure (see Definition 2.1). Homogeneity is an inverse concept of porosity in the sense that if the porosity of a measure μ is close to its maximum value, then a suitable restriction of μ has small homogeneity (see Remark 2.2 (5)).

In comparison with porosity, homogeneity has the following advantages: Unlike porosity, homogeneity enables us to establish an upper bound for Hausdorff dimension which is valid for all measures, and which is sharp, not only asymptotically when homogeneity tends to either its maximum or minimum value, but also in the general setting. This upper bound is given in terms of homogeneity and its order (see Theorem 3.1). On the other hand, we give a complete answer to the question of how homogeneous structure of a measure depends on its dimension (see Theorem 4.2). It appears that small upper packing dimension always guarantees that the measure is non-homogeneous in the sense that it has small homogeneity. In both these cases the choice of the correct dimension is crucial as indicated by examples in sections 3 and 4. Our main tools include dimension estimates for self-similar measures determined by weights that satisfy certain growth condition, and approximations of arbitrary measures by self-similar measures of this type.

The paper is organized as follows: In section 2, we introduce average homogeneity, prove some of its basic properties, and illustrate it by examples. Relations between homogeneity and different concepts of porosities are also discussed. In section 3, the emphasis is given to dimension estimates in terms of homogeneity, whereas the opposite question of setting an upper bound for homogeneity by means of dimensions is discussed in section 4. Both sections 3 and 4 contain examples which show that all our results are sharp.

2. Preliminary discussion and notation

Given a positive integer k, we say that a half-open cube $K \subset \mathbb{R}^n$ is k-adic if there are integers $l_1, \dots, l_n$ and i such that

$$K = \{x \in \mathbb{R}^n \mid l_j k^{-i} \leq x_j < (l_j + 1)k^{-i} \text{ for } j = 1, \dots, n\}.$$

For all integers i, let $\mathcal{K}_i$ be the family of k-adic cubes in $\mathbb{R}^n$ having side-length k^{-i}. For any $x \in \mathbb{R}^n$, we denote by $K_i(x)$ the unique cube in $\mathcal{K}_i$ containing x.

Let $\underline{\dim}_{\text{loc}}\, \mu(x)$ and $\overline{\dim}_{\text{loc}}\, \mu(x)$ be the lower and upper local dimensions of a finite Radon measure μ at a point $x \in \mathbb{R}^n$, that is,

$$\underline{\dim}_{\text{loc}}\, \mu(x) = \liminf_{i \to \infty} \frac{\log \mu(K_i(x))}{\log k^{-i}},$$

and

$$\overline{\dim}_{\text{loc}}\, \mu(x) = \limsup_{i \to \infty} \frac{\log \mu(K_i(x))}{\log k^{-i}}.$$

Instead of k-adic cubes, one could use balls with centres at x and radii $r > 0$, and let r tend to 0, in the above definitions. Then for μ-almost all points these quantities coincide. (In the case of 2-adic (or dyadic) cubes, see [C][Lemma 2.3]. The general case of k-adic cubes follows similarly.)

The lower Hausdorff and packing dimensions are defined in terms of local dimensions in the usual way:

(2.1)

$$\underline{\dim}_{\text{H}}\, \mu = \mu\text{-}\operatorname*{ess\,inf}_{x \in \mathbb{R}^n} \underline{\dim}_{\text{loc}}\, \mu(x) \text{ and } \underline{\dim}_{\text{p}}\, \mu = \mu\text{-}\operatorname*{ess\,inf}_{x \in \mathbb{R}^n} \overline{\dim}_{\text{loc}}\, \mu(x).$$

The following equalities relate dimensions of measures to those of sets: (see [C] or [F][Proposition 10.2])

$$(2.2)\quad \begin{aligned} &\underline{\dim}_{\rm H}\,\mu=\inf\{\dim_{\rm H} A\mid A \text{ is a Borel set with } \mu(A)>0\},\ \text{and}\\ &\underline{\dim}_{\rm p}\,\mu=\inf\{\dim_{\rm p} A\mid A \text{ is a Borel set with } \mu(A)>0\}.\end{aligned}$$

Replacing μ- ess inf by μ- ess sup in (2.1) gives the upper Hausdorff and packing dimensions:

$$\overline{\dim}_{\rm H}\,\mu=\mu\text{- ess}\sup_{x\in\mathbb{R}^n}\underline{\dim}_{\rm loc}\,\mu(x)\ \text{and}\ \overline{\dim}_{\rm p}\,\mu=\mu\text{- ess}\sup_{x\in\mathbb{R}^n}\overline{\dim}_{\rm loc}\,\mu(x).$$

Analogously to (2.2) we have (see [C] or [F][Proposition 10.3])

$$\overline{\dim}_{\rm H}\,\mu=\inf\{\dim_{\rm H} A\mid A \text{ is a Borel set with } \mu(\mathbb{R}^n\setminus A)=0\},$$

$$\overline{\dim}_{\rm p}\,\mu=\inf\{\dim_{\rm p} A\mid A \text{ is a Borel set with } \mu(\mathbb{R}^n\setminus A)=0\}.$$

Clearly

$$\underline{\dim}_{\rm H}\,\mu\le\overline{\dim}_{\rm H}\,\mu\ \text{and}\ \underline{\dim}_{\rm p}\,\mu\le\overline{\dim}_{\rm p}\,\mu.$$

We continue by introducing the notation needed for defining homogeneity in Definition 2.1. Given a positive integer k, let $I=\{1,\dots,k^n\}$. For all positive integers l, let

$$\mathbf{I}^l=\{(i_1,\dots,i_l)\mid i_j\in I \text{ for all } j=1,\dots,l\}.$$

Define

$$\mathbf{I}^\infty=\{(i_1,i_2,\dots)\mid i_j\in I \text{ for all } j=1,2,\dots\}.$$

For $\mathbf{i}=(i_1,i_2,\dots)\in\mathbf{I}^\infty$, let $\mathbf{i}|_l=(i_1,\dots,i_l)\in\mathbf{I}^l$ be the sequence of the first l digits of $\mathbf{i}$. Moreover, for all $j\in I$, let $\mathrm{n}_j(i_1,\dots,i_l)$ be the number of j's in $(i_1,\dots,i_l)\in\mathbf{I}^l$.

Any finite measure μ on the half-open unit cube $[0,1)^n$ induces an enumeration of the k-adic subcubes of $[0,1)^n$ in order of μ-magnitude as follows: Let $K_1^\mu,\ \dots,K_{k^n}^\mu\in\mathcal{K}_1$ be the k-adic subcubes of $[0,1)^n$ enumerated in such a way that

$$\mu(K_j^\mu)\le\mu(K_{j+1}^\mu)$$

for all $j=1,\dots,k^n-1$. Given any positive integer l and $(i_1,\dots,i_l)\in\mathbf{I}^l$, we continue inductively by dividing $K^\mu_{i_1,\dots,i_l}\in\mathcal{K}_l$ into k^n k-adic subcubes $K^\mu_{i_1,\dots,i_l,j}\in\mathcal{K}_{l+1}$ enumerated such that for all $j=1,\dots,k^n-1$

$$\mu(K^\mu_{i_1,\dots,i_l,j})\le\mu(K^\mu_{i_1,\dots,i_l,j+1}).$$

The enumeration in order of μ-magnitude is the basis of the definition of homogeneity. Intuitively, the k-average homogeneity of μ of order j describes the behaviour of the average of the μ-measure of the k-adic cubes having jth smallest proportions of μ.

Definition 2.1. Let μ be a Radon probability measure on $[0,1)^n$. For all positive integers k and for all $j \in I$, the lower and upper k-average homogeneities of μ of order j are defined as follows:

$$\underline{\mathrm{Hom}}_{k\text{-aver}}^{\,j}(\mu) = \liminf_{l\to\infty} \frac{k^n}{l} \sum_{m=1}^{l} \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m \\ i_m=j}} \mu(K^\mu_{i_1,\dots,i_m}), \tag{2.3}$$

and

$$\overline{\mathrm{Hom}}_{k\text{-aver}}^{\,j}(\mu) = \limsup_{l\to\infty} \frac{k^n}{l} \sum_{m=1}^{l} \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m \\ i_m=j}} \mu(K^\mu_{i_1,\dots,i_m}). \tag{2.4}$$

Remark 2.2. (1) Homogeneity is a weaker concept than weak porosity defined by Beliaev and Smirnov in [BS]. According to their definition, a measure μ is $(1/k,\varepsilon)$-weak porous if for all i and for all $K \in \mathcal{K}_i$ there is a k-adic subcube $K' \subset K$ such that $K' \in \mathcal{K}_{i+1}$ and $\mu(K') \le \varepsilon k^{-n}\mu(K)$. Obviously, if μ is a $(1/k,\varepsilon)$-weak porous Radon probability measure on $[0,1)^n$, then $\overline{\mathrm{Hom}}_{k\text{-aver}}^{\,1}(\mu) \le \varepsilon$. On the other hand, the condition $\overline{\mathrm{Hom}}_{k\text{-aver}}^{\,1}(\mu) = 0$ does not guarantee that μ is $(1/k,\varepsilon)$-weak porous for any small $\varepsilon > 0$ (see Example 2.3 (5)).

(2) The definition 2.1 extends naturally to Radon measures having compact support.

(3) Note that always

$$\underline{\mathrm{Hom}}_{k\text{-aver}}^{\,j}(\mu) \le \underline{\mathrm{Hom}}_{k\text{-aver}}^{\,j+1}(\mu) \text{ and } \overline{\mathrm{Hom}}_{k\text{-aver}}^{\,j}(\mu) \le \overline{\mathrm{Hom}}_{k\text{-aver}}^{\,j+1}(\mu).$$

(4) Since $\mu(K^\mu_{i_1,\dots,i_{l-1},1}) \le k^{-n}\mu(K^\mu_{i_1,\dots,i_{l-1}})$ for all $(i_1,\dots,i_{l-1}) \in \mathbf{I}^{l-1}$, one obtains immediately that

$$0 \le \underline{\mathrm{Hom}}_{k\text{-aver}}^{\,1}(\mu) \le \overline{\mathrm{Hom}}_{k\text{-aver}}^{\,1}(\mu) \le 1, \tag{2.5}$$

and furthermore, these upper and lower bounds are the best possible ones (see Example 2.3 (1)). However, for higher orders of homogeneity the natural bounds are different from those in (2.5). In fact, using $\mu(K^\mu_{i_1,\dots,i_{l-1},j}) \le \frac{1}{k^n-(j-1)}\mu(K^\mu_{i_1,\dots,i_{l-1}})$ gives the following bounds which are again the best possible ones (see Example 2.3 (2)):

$$0 \le \underline{\mathrm{Hom}}_{k\text{-aver}}^{\,j}(\mu) \le \overline{\mathrm{Hom}}_{k\text{-aver}}^{\,j}(\mu) \le \frac{k^n}{k^n-(j-1)}. \tag{2.6}$$

Of course, the difference between the upper bounds in (2.5) and (2.6) is due to the choice of the normalizing constant in (2.3) and (2.4). Replacing the constant k^n in (2.3) and (2.4) by $k^n-(j-1)$ one would have the upper bound 1 also in (2.6). However, this is artificial for several reasons. First of all, the measure maximizing the homogeneity of order j is different for each j (see Example 2.3 (2)). Furthermore, our normalization guarantees that we have a simple formulation for the dimension estimates in Theorem 3.1. Finally, for the n-dimensional Lebesgue measure the average homogeneities of all orders have the same value (see Example 2.3 (1)).

(5) Homogeneity of a measure μ may be regarded as an inverse of the notion of porosity of μ defined in [EJJ] as follows:

$$\operatorname{por}(\mu) = \mu\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \operatorname{por}(\mu, x)$$

where

$$\operatorname{por}(\mu, x) = \lim_{\varepsilon\to 0} \liminf_{r\to 0} \operatorname{por}(\mu, x, r, \varepsilon)$$

and

$$\begin{aligned}\operatorname{por}(\mu, x, r, \varepsilon) = \sup\{p \geq 0 \mid & \text{there is } z \in \mathbb{R}^n \text{ with } B(z, pr) \subset B(x, r) \\ & \text{and } \mu(B(z, pr)) \leq \varepsilon\mu(B(x, r))\}.\end{aligned}$$

In [JJ][Theorem 2.8] it was shown that large porosity implies small homogeneity in the sense that if the porosity of a Borel probability measure μ is large enough, then for all $\varepsilon > 0$ there is a restriction μ_ε of μ and a positive integer j depending on porosity such that the 2^k-average upper homogeneity of μ_ε of order j is bounded above by a constant multiple of ε.

Example 2.3. (1) Let $\mu = \mathcal{L}^n|_{[0,1)^n}$ be the restriction of the n-dimensional Lebesgue measure to $[0, 1)^n$. Clearly, for all positive integers k and j, we have $\underline{\operatorname{Hom}}^{\,j}_{k\text{-aver}}(\mu) = \overline{\operatorname{Hom}}^{\,j}_{k\text{-aver}}(\mu) = 1$.

(2) For a fixed positive integer k, divide the unit cube $[0, 1)^n$ into k^n k-acid subcubes. Given any integer $1 \leq l \leq k^n$, attach the weight 0 to any $l - 1$ subcubes, and the weight $1/(k^n - (l - 1))$ to the remaining ones. Enumerate the subcubes so that the first $l - 1$ of them have weight 0, and iterate the construction. For the limiting measure μ we have $\underline{\operatorname{Hom}}^{\,j}_{k\text{-aver}}(\mu) = \overline{\operatorname{Hom}}^{\,j}_{k\text{-aver}}(\mu) = 0$ for all $j = 1, \dots, l - 1$ and $\underline{\operatorname{Hom}}^{\,j}_{k\text{-aver}}(\mu) = \overline{\operatorname{Hom}}^{\,j}_{k\text{-aver}}(\mu) = k^n/(k^n - (l - 1))$ for all $j = l, \dots, k^n$. This construction shows that the bounds in (2.6) are the best possible ones.

(3) Defining $\mu = \sum_{i=1}^\infty 2^{-i}\delta_{2^{-i}}$, where $\delta_{2^{-i}}$ is the Dirac measure at the point 2^{-i}, we get $\underline{\operatorname{Hom}}^{\,1}_{2\text{-aver}}(\mu) = \overline{\operatorname{Hom}}^{\,1}_{2\text{-aver}}(\mu) = 0$ and $\underline{\operatorname{Hom}}^{\,2}_{2\text{-aver}}(\mu) = \overline{\operatorname{Hom}}^{\,2}_{2\text{-aver}}(\mu) = 2$.

(4) For any $0 < p < 1/2$, let μ be the Bernoulli measure on $[0, 1)$ obtained by attaching first the weights p and $1 - p$ to the dyadic intervals $[0, 1/2)$ and $[1/2, 1)$, respectively, and iterating this process. Then

$\underline{\mathrm{Hom}}_{2\text{-aver}}^{\,1}(\mu) = \overline{\mathrm{Hom}}_{2\text{-aver}}^{\,1}(\mu) = 2p$ and $\underline{\mathrm{Hom}}_{2\text{-aver}}^{\,2}(\mu) = \overline{\mathrm{Hom}}_{2\text{-aver}}^{\,2}(\mu) = 2(1-p)$. Bernoulli measures play a crucial role in the proof of Theorem 3.1.

(5) We will construct a measure μ on $[0,1)$ such that $\underline{\mathrm{Hom}}_{2\text{-aver}}^{\,1}(\mu) = \overline{\mathrm{Hom}}_{2\text{-aver}}^{\,1}(\mu) = 0$ and μ is $(1/2,1/2)$-weak porous in the sense of Beliaev and Smirnov (see Remark 2.2 (1)). In particular, μ is not $(1/2,\varepsilon)$-weak porous for any $0 \le \varepsilon < 1/2$.

The construction is based on varying uniform and singular distributions of weights between dyadic intervals in a suitable way at different stages of the construction. Let $\mathcal{D}_i$ be the family of dyadic intervals having length 2^{-i}. By a uniform distribution of weights at stage k we mean that if D_1, $D_2 \in \mathcal{D}_k$ are subsets of the same dyadic interval belonging to $\mathcal{D}_{i-1}$ and having weight w, then the weight w is equally distributed between D_1 and D_2, that is, both intervals have the same weight $w/2$. On the other hand, a singular distribution means that one interval has weight 0, and the other one has weight w. The measure μ with the desired properties is the limiting measure of the process where, starting from the weight 1 on the interval $[0,1)$, we distribute the weights uniformly between the dyadic subintervals of $[0,1)$ belonging to $\mathcal{D}_k$, if $k = 2^p$ for some positive integer p, and otherwise singularly.

Remark 2.4. As observed in [BS], one may define a "mean" version of the weak porosity (recall Remark 2.2 (1)): A measure μ is mean $(1/k,\varepsilon,\kappa)$-weak porous if for μ-almost all x there exist N and n_0 such that for all $n \ge n_0$ at least for κN integers between $n+1,\dots,n+N$ there exists $K' \in \mathcal{K}_{j+1}$ such that $K' \subset K_j(x)$ and $\mu(K') \le \varepsilon k^{-n}\mu(K_j(x))$. Even this concept is stronger than average homogeneity since it demands that there is a fixed proportion of cubes with small measure at all small distances around almost every point, whereas homogeneity allows both arbitrarily long blocks without cubes having small measure, and positively many exceptional points. Mandelbrot percolation is a system that presents clearly the difference between blocks of fixed size and arbitrarily long blocks (see [BJ]). The role of positively many exceptional points is exhibited in Examples 3.4 and 4.3.

3. The effect of homogeneity on dimensions

In this section we discuss relations between homogeneities and dimensions. First we consider self-similar measures which will be our main tool in the study of general measures. The main result of this section (Theorem 3.1) gives the best possible upper bound for lower

Hausdorff dimension of a measure as a function of lower average homogeneity and its order. We also give examples illustrating the sharpness of our results.

The following theorem is similar to [H] [Proposition 5.3]. However, both our assumptions and conclusions are weaker than those of Heurteaux's. In [JJ] it was erroneously stated that the claim of Theorem 3.1 holds for packing dimension which is not the case as shown by Example 3.5. The following correct proof has not been published earlier.

Theorem 3.1. *Let k be a positive integer and let $p \leq k^{-n}$. Assume that μ is a Radon probability measure on $[0,1)^n$ such that*

$$\underline{\mathrm{Hom}}_{k\text{-}aver}^{\;L}(\mu) \leq k^n p$$

for some $L \in \{1, \dots, k^n - 1\}$. Then

$$\underline{\dim}_{\mathrm{H}}\, \mu \leq -\frac{1}{\log k}\Big(Lp \log p + (1 - Lp) \log\Big(\frac{1 - Lp}{k^n - L}\Big)\Big) =: c(p, L).$$

For the proof we need the following lemma dealing with self-similar measures. Let k and l be positive integers. Given a family $\{\alpha_{i_1,\dots,i_l} \mid (i_1, \dots, i_l) \in \mathbf{I}^l\}$ of positive real numbers such that

$$\sum_{(i_1,\dots,i_l)\in\mathbf{I}^l} \alpha_{i_1,\dots,i_l} = 1,$$

and an enumeration of the k-adic subcubes $K_{i_1,\dots,i_l} \in \mathcal{K}_l$ of $[0,1)^n$ by indices $(i_1, \dots, i_l) \in \mathbf{I}^l$, we attach the weight $\alpha_{i_1,\dots,i_l}$ to the corresponding cube $K_{i_1,\dots,i_l}$. After dividing each $K_{i_1,\dots,i_l}$ into k^{ln} subcubes $K_{i_1,\dots,i_l,j_1,\dots,j_l}$, where $(j_1, \dots, j_l) \in \mathbf{I}^l$, labeled by the natural enumeration induced by the given enumeration of the cubes $K_{i_1,\dots,i_l} \in \mathcal{K}_l$, we attach the weight $\alpha_{i_1,\dots,i_l}\alpha_{j_1,\dots,j_l}$ to the cube $K_{i_1,\dots,i_l,j_1,\dots,j_l}$. The resulting measure, obtained by iterating this process, is called the self-similar measure determined by the weights $\{\alpha_{i_1,\dots,i_l} \mid (i_1, \dots, i_l) \in \mathbf{I}^l\}$ and the fixed enumeration of the cubes $K_{i_1,\dots,i_l} \in \mathcal{K}_l$.

According to the following lemma, the lower Hausdorff dimension of the self-similar measure determined by given weights and enumeration is bounded above by the constant $c(p, L)$ under a certain growth condition for the weights (see (3.1) below). The proof is based on solving an extreme value problem for the Hausdorff dimension formulas of self-similar measures.

Lemma 3.2. *Given positive integers k and l, let $\{\alpha_{i_1,\dots,i_l} \mid (i_1,\dots,i_l) \in \mathbf{I}^l\}$ be a family of positive real numbers such that*

$$\sum_{(i_1,\dots,i_l)\in\mathbf{I}^l} \alpha_{i_1,\dots,i_l} = 1.$$

Assuming that ν is the self-similar Radon probability measure on $[0,1)^n$, determined by $\{\alpha_{i_1,\dots,i_l} \mid (i_1,\dots,i_l) \in \mathbf{I}^l\}$ and a given enumeration of the k-adic subcubes $K_{i_1,\dots,i_l} \in \mathcal{K}_l$ of $[0,1)^n$ by the indices $(i_1,\dots,i_l) \in \mathbf{I}^l$, the following properties hold:

(1) $\underline{\dim}_{\mathrm{H}}\,\nu = -\frac{1}{\log k^l}\sum_{(i_1,\dots,i_l)\in\mathbf{I}^l} \alpha_{i_1,\dots,i_l}\log\alpha_{i_1,\dots,i_l}$

(2) *If there are $p \le k^{-n}$ and $L \in \{1,\dots,k^n-1\}$ such that*

$$\frac{1}{l}\sum_{m=1}^{l}\sum_{\substack{(i_1,\dots,i_l)\in\mathbf{I}^l\\ i_m=j}} \alpha_{i_1,\dots,i_l} \le p \tag{3.1}$$

for all $j = 1,\dots,L$, then $\underline{\dim}_{\mathrm{H}}\,\nu \le c(p,L)$.

Proof. The lower Hausdorff dimension formula (1) follows similarly as [B][(14.4) p. 139].

We use the notation $x(\beta_{i_1,\dots,i_l})$ for the point in the unit cube of $\mathbb{R}^{k^{ln}}$ having coordinates $\{\beta_{i_1,\dots,i_l} \in [0,1] \mid (i_1,\dots,i_l) \in \mathbf{I}^l\}$. For the purpose of proving claim (2), we will determine the maximum of the concave function

$$f(x(\beta_{i_1,\dots,i_l})) = -\sum_{(i_1,\dots,i_l)\in\mathbf{I}^l} \beta_{i_1,\dots,i_l}\log\beta_{i_1,\dots,i_l}$$

on the unit cube of $\mathbb{R}^{k^{ln}}$ given the linear restrictions

$$h_1(x(\beta_{i_1,\dots,i_l})) = \sum_{(i_1,\dots,i_l)\in\mathbf{I}^l} \beta_{i_1,\dots,i_l} = 1$$

and

$$0 \le h_{j+1}(x(\beta_{i_1,\dots,i_l})) = \frac{1}{l}\sum_{m=1}^{l}\sum_{\substack{(i_1,\dots,i_l)\in\mathbf{I}^l\\ i_m=j}} \beta_{i_1,\dots,i_l} \le p$$

for all $j = 1,\dots,L$. Since $\bigtriangledown f = 0$ if and only if $\beta_{i_1,\dots,i_l} = e^{-1}$ for all $(i_1,\dots,i_l) \in \mathbf{I}^l$ the maximum is obtained on the boundary. Using the method of Lagrange's multipliers, we will end up solving for all $(i_1,\dots,i_l) \in \mathbf{I}^l$ the equations

$$-\log\beta_{i_1,\dots,i_l} - 1 = \lambda_1 + \frac{1}{l}\sum_{j=1}^{L}\lambda_{j+1}\,\mathrm{n}_j(i_1,\dots,i_l)$$

with $h_1 = 1$ and $h_{j+1} = p$ for all $j = 1, \dots, L$. Recall that $\mathrm{n}_j(i_1, \dots, i_l)$ is the number of j's in $(i_1, \dots, i_l) \in \mathbf{I}^l$.

Since f is concave and the restrictions are linear there is only one solution for these equations. Choosing $e^{-\lambda_1 - 1} = (\frac{1-Lp}{k^n-L})^l$ and $\lambda_{j+1} = -l\log(\frac{p(k^n-L)}{1-Lp})$ for $j = 1, \dots, L$, it is easily checked that

$$\beta_{i_1,\dots,i_l} = p^{\sum_{j=1}^{L} n_j(i_1,\dots,i_l)} \Big(\frac{1-Lp}{k^n-L}\Big)^{\sum_{j=L+1}^{k^n} n_j(i_1,\dots,i_l)}$$

is the unique solution. Note that these $\beta_{i_1,\dots,i_l}$'s are the lth iterates of the self-similar measure determined by attaching L times the weight p and $k^n - L$ times the weight $\frac{1-Lp}{k^n-L}$ to the k-adic subcubes of $[0,1)^n$ having side-length $1/k$. Since this measure has Hausdorff dimension $c(p, L)$ (see (1)), the claim follows. □

Applying Lemma 3.2 to certain self-similar approximations of arbitrary measures, gives Theorem 3.1.

Proof of Theorem 3.1. Assume that μ is a Radon probability measure on $[0,1)^n$ such that $\underline{\mathrm{Hom}}^{\,L}_{k\text{-aver}}(\mu) \le k^n p$ for some $L \in \{1, \dots, k^n - 1\}$. Given $\varepsilon > 0$, there exists a positive integer l such that

$$\frac{1}{l}\sum_{m=1}^{l} \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m \\ i_m=j}} \mu(K^\mu_{i_1,\dots,i_m}) \le p + \varepsilon =: p'$$

for all $j = 1, \dots, L$, and furthermore,

$$\underline{\dim}_{\mathrm{loc}}\,\mu(x) - \varepsilon \le \frac{\log \mu(K_l(x))}{\log k^{-l}} \tag{3.2}$$

for all $x \in B$ with $\mu(B) \ge 1 - \varepsilon$. Recall that $K_l(x) \in \mathcal{K}_l$ is the unique k-adic subcube of $[0,1)^n$ containing x. We may assume that $\underline{\dim}_{\mathrm{H}}\,\mu \le \underline{\dim}_{\mathrm{loc}}\,\mu(x)$ for all $x \in B$.

Let ν be the self-similar measure determined by the weights

$$\{\alpha_{i_1,\dots,i_l} = \mu(K^\mu_{i_1,\dots,i_l}) \mid (i_1, \dots, i_l) \in \mathbf{I}^l\},$$

and the enumeration in order of μ-magnitude for the k-adic cubes in $\mathcal{K}_l$. Defining

$$G = \{(i_1, \dots, i_l) \in \mathbf{I}^l \mid K^\mu_{i_1,\dots,i_l} \cap B \ne \emptyset\}$$

and noting that

$$\sum_{m=1}^{l} \sum_{\substack{(i_1,\dots,i_l)\in\mathbf{I}^l \\ i_m=j}} \alpha_{i_1,\dots,i_l} = \sum_{m=1}^{l} \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m \\ i_m=j}} \mu(K^\mu_{i_1,\dots,i_m})$$

for all $j = 1, \dots, L$, we obtain, by applying Lemma 3.2 to ν and using (3.2),

$$\begin{aligned}
-\frac{1}{\log k}\Big(Lp'\log p' + (1-Lp')\log\Big(\frac{1-Lp'}{k^n-L}\Big)\Big) \\
&\geq -\frac{1}{\log k^l}\sum_{(i_1,\dots,i_l)\in\mathbf{I}^l}\alpha_{i_1,\dots,i_l}\log\alpha_{i_1,\dots,i_l} \\
&\geq (\underline{\dim}_{\mathrm{H}}\,\mu - \varepsilon)\sum_{(i_1,\dots,i_l)\in G}\mu(K^{\mu}_{i_1,\dots,i_l}) \\
&\geq (\underline{\dim}_{\mathrm{H}}\,\mu - \varepsilon)\mu(B) \geq (\underline{\dim}_{\mathrm{H}}\,\mu - \varepsilon)(1-\varepsilon).
\end{aligned}$$

Letting ε tend to zero, gives the claim. □

Remark 3.3. (1) As indicated by the self-similar measure constructed by means of the weights which are the extreme points of the extreme value problem in the proof of Lemma 3.2, the upper bound for the lower Hausdorff dimension in Theorem 3.1 is the best possible one. More precisely, given $p \leq k^{-n}$ and $L = \{1, \dots, k^n - 1\}$, let μ be the self-similar measure determined by attaching L times the weight p and $k^n - L$ times the weight $\frac{1-Lp}{k^n-L}$ to the subcubes of $[0,1)^n$ belonging to $\mathcal{K}_1$. Clearly, $\underline{\dim}_{\mathrm{H}}\,\mu = c(p, L)$ and $\underline{\mathrm{Hom}}^{\,L}_{k\text{-aver}}(\mu) = k^n p$.

(2) Lemma 3.2 is clearly valid for upper Hausdorff, and lower and upper packing dimensions as well, since all these dimensions agree for self-similar measures. However, Theorem 3.1 fails for upper Hausdorff, and lower and upper packing dimensions as indicated by Examples 3.4 and 3.5 below.

Example 3.4. Let k be a positive integer and $0 < \varepsilon < 1$. There exists a Radon probability measure μ on $[0,1)^n$ such that $\overline{\mathrm{Hom}}^{\,k^n-1}_{k\text{-aver}}(\mu) = \frac{k^n\varepsilon}{k^n-\varepsilon}$ and $\overline{\dim}_{\mathrm{H}}\,\mu = n$. In particular, Theorem 3.1 does not hold for upper Hausdorff dimension. Note that $c(p, k^n - 1) \to 0$ as $p \to 0$ in Theorem 3.1.

Proof. Letting $A = [0,1)^n \setminus [0,1/k)^n$, define the measure μ as the normalized sum of the weighted Dirac measure at 0 and the weighted restriction of the n-dimensional Lebesgue measure to the set A, that is, $\mu = \frac{k^n}{k^n-\varepsilon}((1-\varepsilon)\delta_0 + \varepsilon\mathcal{L}^n|_A)$. Then $\overline{\mathrm{Hom}}^{\,k^n-1}_{k\text{-aver}}\,\mu = \frac{k^n\varepsilon}{k^n-\varepsilon}$ and $\overline{\dim}_{\mathrm{H}}\,\mu = n$. □

Next we give an example illustrating that Theorem 3.1 is not valid for packing dimension. For simplicity, we consider the case $n = 1$ and $k = 2$.

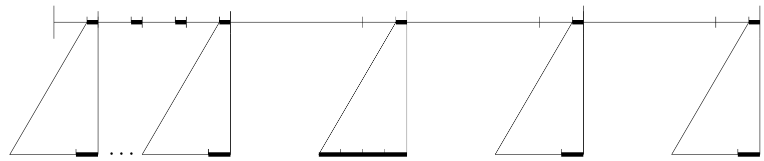

FIGURE 1. First four steps of the construction of Example 3.5. Here $k_0 = k_1 = k_2 = k_3 = 2$. The mass is concentrated on the thick intervals.

Example 3.5. Let $0 < \varepsilon < 1/2$. There is a Radon probability measure μ on $[0,1)$ satisfying $\overline{\mathrm{Hom}}^{1}_{2\text{-aver}}(\mu) \le 2\varepsilon$ such that $\underline{\dim}_{\mathrm{H}}\,\mu = \overline{\dim}_{\mathrm{H}}\,\mu = 0$ and $\underline{\dim}_{\mathrm{p}}\,\mu = \overline{\dim}_{\mathrm{p}}\,\mu = 1$.

Proof. For all positive integers i, we use the notation $\mathcal{D}_i$ for the dyadic subintervals of $[0,1)$ having side-length 2^{-i}. Furthermore, given $x \in [0,1)$, let $D_i(x) \in \mathcal{D}_i$ be the unique interval containing x.

We will construct a measure μ with the following properties:

(1) There exists a sequence $(l_k) \to \infty$ of positive integers such that for all $x \in \operatorname{spt}\mu$

$$\lim_{k\to\infty} \frac{\log \mu(D_{l_k}(x))}{\log 2^{-l_k}} = 0.$$

(2) For all $x \in \operatorname{spt}\mu$ there is a sequence $(i_k) \to \infty$ of positive integers such that

$$\lim_{k\to\infty} \frac{\log \mu(D_{i_k}(x))}{\log 2^{-i_k}} = 1.$$

(3) For all positive integers m

$$\sum_{\substack{(i_1,\dots,i_m)\in \mathbf{I}^m \\ i_m=1}} \mu(D^{\mu}_{i_1,\dots,i_m}) \le \varepsilon.$$

In (1) and (2) the support of μ is denoted by $\operatorname{spt}\mu$, and in (3) the intervals $D_{i_1,\dots,i_m} \in D_m$ are determined by the enumeration in order of μ-magnitude. Note that (1), (2), and (3) obviously imply the desired properties for μ.

We first define a sequence (μ_k) of measures; the measure μ is the weak limit of this sequence. The first four steps of the construction are shown in Figure 1. Let k_0 be a positive integer such that $(1-\varepsilon)^{k_0} \le \varepsilon$. The measure μ_1 is obtained by attaching the weights ε and $1-\varepsilon$ to dyadic intervals $[0,1/2)$ and $[1/2,1)$, respectively, that is, $\mu_1 = 2(\varepsilon\mathcal{L}^1|_{[0,1/2)} + (1-\varepsilon)\mathcal{L}^1|_{[1/2,1)})$. Iterating this process k_0-1 times gives the measures μ_k for all $k = 1,\dots,k_0$.

Let $D_1, \dots, D_{2^{k_0}} \in \mathcal{D}_{k_0}$ be the dyadic subintervals of $[0,1)$ at level k_0. The choice of k_0 guarantees that

$$\mu_{k_0}(D_j) \le \varepsilon \tag{3.3}$$

for all $j = 1, \dots, 2^{k_0}$. Furthermore, for all $k = 1, \dots, k_0$

$$\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_k(D^{\mu_k}_{i_1,\dots,i_m}) \le \varepsilon \tag{3.4}$$

for all $m = 1, \dots, k$. This follows directly from the construction, since $\mu_k(D) = \mu_m(D)$ for all $D \in \mathcal{D}_m$. In particular, $D^{\mu_k}_{i_1,\dots,i_m} = D^{\mu_m}_{i_1,\dots,i_m}$.

Given a positive integer k, we define the measure μ_{k_0+k} as follows: First divide each interval D_j into 2^k dyadic subintervals. The weight is distributed uniformly inside D_1, that is, the weight $2^{-k}\mu_{k_0}(D_1)$ is attached to each dyadic subintervals of D_1 having length 2^{-k_0-k}. Inside every other interval D_j, $j \ne 1$, we use singular distribution, meaning that the weight $\mu_{k_0}(D_j)$ is given to the rightmost dyadic subinterval of D_j belonging to $\mathcal{D}_{k_0+k}$, whilst every other one will have weight 0. Noting that as the result of this process we have for all k and $x \in D_1$

$$\frac{\log \mu_{k_0+k}(D_{k_0+k}(x))}{\log 2^{-k_0-k}} = \frac{\log(2^{-k}\mu_{k_0}(D_1))}{\log 2^{-k_0-k}},$$

we may choose a positive integer k_1 such that

$$\frac{\log \mu_{k_0+k_1}(D_{k_0+k_1}(x))}{\log 2^{-k_0-k_1}} \ge 1 - \frac{1}{3} \tag{3.5}$$

for $x \in D_1$. Similarly as in (3.4), we obtain, using (3.3), that for all $k = k_0+1, \dots, k_0+k_1$

$$\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_k(D^{\mu_k}_{i_1,\dots,i_m}) \le \begin{cases} \varepsilon & \text{for } m = 1, \dots, k_0 \\ \frac{1}{2}\mu_{k_0}(D_1) \le \frac{\varepsilon}{2} & \text{for } m = k_0+1, \dots, k. \end{cases} \tag{3.6}$$

For $k = 1, 2, \dots$, the measure $\mu_{k_0+k_1+k}$ is defined by dividing each construction interval at level $k_0 + k_1$ into 2^k dyadic subintervals, and giving all the weight to the rightmost subinterval. Clearly,

$$\frac{\log \mu_{k_0+k_1+k}(D_{k_0+k_1+k}(x))}{\log 2^{-k_0-k_1-k}} \le \frac{\log \mu_{k_0+k_1}(D_{k_0+k_1}(x))}{\log 2^{-k_0-k_1-k}}$$

provided that $\mu_{k_0+k_1+k}(D_{k_0+k_1+k}(x)) > 0$. Hence we may choose a positive integer k_2 such that

$$\frac{\log \mu_{k_0+k_1+k_2}(D_{k_0+k_1+k_2}(x))}{\log 2^{-k_0-k_1-k_2}} \le \frac{1}{3} \tag{3.7}$$

for all $x \in [0,1)$ with $\mu_{k_0+k_1+k_2}(D_{k_0+k_1+k_2}(x)) > 0$. As before, for all $k = k_0+k_1+1,\dots,k_0+k_1+k_2$ we have

$$\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_k(D^{\mu_k}_{i_1,\dots,i_m}) \le \begin{cases} \varepsilon & \text{for } m=1,\dots,k_0\\ \frac{\varepsilon}{2} & \text{for } m=k_0+1,\dots,k_0+k_1\\ 0 & \text{for } m=k_0+k_1+1,\dots,k. \end{cases} \tag{3.8}$$

We continue the construction by repeating the above process in the following sense: If $D \in \mathcal{D}_{k_0+k_1+k_2}$ is a subset of D_2, we distribute the weight $\mu_{k_0+k_1+k_2}(D)$ uniformly between its dyadic subintervals in $\mathcal{D}_{k_0+k_1+k_2+k}$. On the other hand, if $D \in \mathcal{D}_{k_0+k_1+k_2}$ is not a subset of D_2, we attach the weight $\mu_{k_0+k_1+k_2}(D)$ singularly to the rightmost subinterval in $\mathcal{D}_{k_0+k_1+k_2+k}$. Similarly as before, we find a positive integer k_3 such that

$$\frac{\log \mu_{k_0+k_1+k_2+k_3}(D_{k_0+k_1+k_2+k_3}(x))}{\log 2^{-k_0-k_1-k_2-k_3}} \ge 1 - \frac{1}{4} \tag{3.9}$$

for all $x \in D_2$ such that $\mu_{k_0+k_1+k_2}(D_{k_0+k_1+k_2}(x)) > 0$. Proceeding in terms of the singular distribution inside all the remaining dyadic intervals with positive weight, we find a positive integer k_4 such that

$$\frac{\log \mu_{k_0+k_1+k_2+k_3+k_4}(D_{k_0+k_1+k_2+k_3+k_4}(x))}{\log 2^{-k_0-k_1-k_2-k_3-k_4}} \le \frac{1}{4} \tag{3.10}$$

for all $x \in [0,1)$ such that $\mu_{k_0+k_1+k_2+k_3+k_4}(D_{k_0+k_1+k_2+k_3+k_4}(x)) > 0$. Moreover, from (3.4), (3.6), and (3.8) we get for all $k = 1,\dots,k_0+k_1+k_2+k_3+k_4$

$$\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_k(D^{\mu_k}_{i_1,\dots,i_m}) \le \varepsilon \tag{3.11}$$

for all $m = 1,\dots,k$. Continue in this way until all the intervals $D_1,\dots,D_{2^{k_0}}$ have been handled. Then start the process again from D_1.

The resulting measure, defined as the weak limit of the measures μ_k, is a Radon probability measure which satisfies properties (1), (2), (3). Clearly, (1) and (2) are immediate consequences of (3.5), (3.9), (3.7), and (3.10), and the analogous choices made later in the construction. To see that (3) holds, fix a positive integer m. Combining [Ke][The Portmanteau Theorem 17.20] with the fact that, as the result of the construction, the analogue of (3.11) holds for all k, we get

$$\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu(D^{\mu}_{i_1,\dots,i_m}) = \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \lim_{k\to\infty} \mu_{m+k}(D^{\mu}_{i_1,\dots,i_m})$$
$$= \lim_{k\to\infty} \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_{m+k}(D^{\mu_{m+k}}_{i_1,\dots,i_m}) \le \varepsilon.$$

This completes the proof. □

4. The effect of dimensions on homogeneity

In this section we continue discussing relations between dimensions and homogeneities. The emphasis is given to opposite implications of those in Section 3. To avoid some technical complications, we concentrate on the first order homogeneities; the higher orders are discussed in Remark 4.8. Combining Theorem 4.2 to the examples constructed in this section, gives a complete answer to the question of how dimensions effect homogeneities. We begin by proving a technical lemma.

Lemma 4.1. *Let p be a positive integer. Given a finite measure μ on $[0,1)^n$, and disjoint k-adic subcubes $Q_1,\dots,Q_M \in \mathcal{K}_p$ of $[0,1)^n$, define*

$$\mu_M = \sum_{i=1}^{M} \mu|_{Q_i}.$$

Then

$$\sum_{m=1}^{p} \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_M(K^{\mu_M}_{i_1,\dots,i_m}) \le k^{-n}\mu_M([0,1)^n)\log_{k^n}(M). \tag{4.1}$$

Proof. We proceed by induction on M. For $M=1$, both sides of (4.1) equal 0. Assume that (4.1) holds for any finite measure ν and for all $C_1,\dots,C_m \in \mathcal{K}_p$ where $m \le M-1$. Fix $Q_1,\dots,Q_M \in \mathcal{K}_p$ and a finite measure μ. Let $l \ge 0$ be the largest integer such that for all $j=0,\dots,l$ all cubes $Q_1,\dots,Q_M$ belong to the same k-adic cube in $\mathcal{K}_j$. Denote by C the k-adic cube in $\mathcal{K}_l$ such that $Q_i \subset C$ for all $i=1,\dots,M$. Letting $\widetilde{Q}_1,\dots,\widetilde{Q}_{k^n} \in \mathcal{K}_{l+1}$ be the subcubes of C, we use for all $j=1,\dots,k^n$ the notation M_j for the number of cubes Q_i that are subsets of $\widetilde{Q}_j$. The choice of l implies that $M_j \le M-1$ for all $j=1,\dots,k^n$.

Defining for all $j=1,\dots,k^n$

$$\nu^j_{M_j} = \sum_{\{i \mid Q_i\subset\widetilde{Q}_j\}} \mu_M|_{Q_i},$$

and applying the induction hypothesis to each $\nu^j_{M_j}$, one obtains

$$\begin{aligned}\sum_{m=1}^{p}&\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}}\mu_M(K^{\mu_M}_{i_1,\dots,i_m})\\ &=\sum_{m=1}^{l}\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}}\mu_M(K^{\mu_M}_{i_1,\dots,i_m})+\sum_{\substack{(i_1,\dots,i_{l+1})\in\mathbf{I}^{l+1}\\ i_{l+1}=1}}\mu_M(K^{\mu_M}_{i_1,\dots,i_{l+1}})\\ &\qquad+\sum_{j=1}^{k^n}\sum_{m=l+2}^{p}\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}}\nu^j_{M_j}(K^{\nu^j_{M_j}}_{i_1,\dots,i_m})\\ &\le 0+\min_{j=1,\dots,k^n}\mu_M(\widetilde{Q}_j)+k^{-n}\sum_{j=1}^{k^n}\mu_M(\widetilde{Q}_j)\log_{k^n}(M_j).\end{aligned}$$

Here we use the natural interpretation $0\log_{k^n}0=0$.

Let $\mu_M(\widetilde{Q}_i)=\min_{j=1,\dots,k^n}\mu_M(\widetilde{Q}_j)$. Setting $K=k^n$ and $a_j=K^{-1}\log_K(M_j)$ for all $j=1,\dots,K$, after normalization claim (4.1) reduces to proving that

$$x_i+\sum_{j=1}^{K}a_jx_j\le K^{-1}\log_K(M) \tag{4.2}$$

with the linear restrictions $0\le x_i\le K^{-1}$, $x_i\le x_j\le 1$ for $j\ne i$, and $\sum_{j=1}^K x_j=1$. The function $(x_1,\dots,x_K)\mapsto x_i+\sum_{j=1}^K a_jx_j$ attains its maximum at the vertices of the polyhedron determined by the linear restrictions. At the vertices, where $x_i=0$ and $x_j=1$ for some $j\ne i$, inequality (4.2) is obvious. On the other hand, for the remaining vertex $x_j=K^{-1}$ for all $j=1,\dots,K$, the left-hand side of (4.2) is equal to $K^{-1}(1+\sum_{j=1}^K a_j)=K^{-2}\log_K(\prod_{j=1}^K KM_j)$, and therefore, (4.2) follows from the fact that

$$\prod_{j=1}^{K}Ky_j\le M^K \tag{4.3}$$

with the restriction $\sum_{j=1}^K y_j=M$. □

Using Lemma 4.1 we obtain:

Theorem 4.2. *Assume that μ is a Radon probability measure on $[0,1)^n$. Then*

$$\overline{\mathrm{Hom}}_{\frac{1}{k}\text{-aver}}(\mu)\le\frac{1}{n}\,\overline{\dim}_{\mathrm{p}}\,\mu$$

for all positive integers k.

Proof. Let $\overline{\dim}_{\mathrm{p}}\,\mu < d$. Consider $\delta > 0$. There exists a positive integer P such that $\mu(A_P) \geq 1 - \delta$ for the Borel set

$$A_P = \{x \in [0,1)^n \mid \mu(K_i(x)) \geq k^{-di} \text{ for all } i \geq P\}.$$

Letting $p \geq P$, define

$$\mathcal{M}_p = \{Q \in \mathcal{K}_p \mid Q \subset [0,1)^n \text{ and } \mu(Q) \geq k^{-dp}\},$$

and denote by M_p the number of cubes in $\mathcal{M}_p$. Clearly, $M_p \leq k^{dp}$ and $A_P \subset \cup_{Q \in \mathcal{M}_p} Q$. Defining

$$\mu_p = \sum_{Q \in \mathcal{M}_p} \mu|_Q,$$

we have

$$\begin{aligned}\sum_{\substack{(i_1,\ldots,i_m)\in \mathbf{I}^m \\ i_m=1}} \mu(K^{\mu}_{i_1,\ldots,i_m}) &\leq \mu\Big(\Big(\bigcup_{\substack{(i_1,\ldots,i_m)\in \mathbf{I}^m \\ i_m=1}} K^{\mu_p}_{i_1,\ldots,i_m}\Big) \cap A_P\Big) + \delta \\ &\leq \sum_{\substack{(i_1,\ldots,i_m)\in \mathbf{I}^m \\ i_m=1}} \mu_p(K^{\mu_p}_{i_1,\ldots,i_m}) + \delta,\end{aligned}$$

and therefore Lemma 4.1 gives

$$\frac{k^n}{p}\sum_{m=1}^{p} \sum_{\substack{(i_1,\ldots,i_m)\in \mathbf{I}^m \\ i_m=1}} \mu(K^{\mu}_{i_1,\ldots,i_m}) \leq \frac{1}{n} d + k^n \delta.$$

Letting first $p \to \infty$, and then $\delta \to 0$, gives the claim. □

We conclude this section by constructing examples which illustrate the sharpness of Theorem 4.2 in the following two senses: First of all, both the lower packing dimension, and the lower and the upper Hausdorff dimensions may be equal to 0 even if both the lower and the upper homogeneities are close to their maximum value. Secondly, the upper bound in Theorem 4.2 is the best possible one.

Example 4.3. Let $0 < \varepsilon < 1/2$. There is a Radon probability measure μ on $[0,1)^n$ such that $\underline{\dim}_{\mathrm{H}}\,\mu = \underline{\dim}_{\mathrm{p}}\,\mu = 0$ and $\underline{\mathrm{Hom}}^{\,1}_{2\text{-aver}}(\mu) = \overline{\mathrm{Hom}}^{\,1}_{2\text{-aver}}(\mu) = 1 - \varepsilon$.

Proof. The measure $\mu = \varepsilon\delta_0 + 2^n(1-\varepsilon)\mathcal{L}^n|_{[1/2,1)^n}$ satisfies the desired properties. □

For simplicity, we consider the case $n = 1$ in the next example.

Example 4.4. For any $0 < \varepsilon < 1$, there exists a Radon probability measure μ on $[0,1)$ with $\overline{\dim}_{\mathrm{H}}\,\mu = 0$ and $\underline{\mathrm{Hom}}_{\frac{1}{2}\text{-aver}} \geq 1-\varepsilon$.

Proof. We use the same notation as in Example 3.5 and construct a Radon probability measure μ on $[0,1)$ with the following properties:

(1) For all $x \in \operatorname{spt}\mu$ there is a sequence $(i_k) \to \infty$ of positive integers such that

$$\lim_{k\to\infty} \frac{\log \mu(D_{i_k}(x))}{\log 2^{-i_k}} = 0.$$

(2) For all positive integers p

$$\sum_{m=1}^{p} \sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu(D^{\mu}_{i_1,\dots,i_m}) \geq \frac{p}{2}(1-\varepsilon).$$

The measure μ is constructed as a weak limit of a sequence (μ_k) of measures defined as follows: Choosing a positive integer k_0 such that $2^{-k_0} \leq \varepsilon$, define $\mu_k = \mathcal{L}^1|_{[0,1)}$ for all $k = 1,\dots,k_0$. Let $D_1,\dots,D_{2^{k_0}} \in \mathcal{D}_{k_0}$ be the dyadic subintervals of $[0,1)$. At stage k_0+k the weight $\mu_{k_0}(D_1) = 2^{-k_0}$ is distributed uniformly between the dyadic subintervals of each D_j, $j \neq 1$, having length 2^{-k_0-k}. Inside D_1 the weight 2^{-k_0} is distributed singularly to the rightmost subinterval of D_1 having length 2^{-k_0-k}. Continuing in this way, we find a positive integer k_1 such that for all $x \in D_1$ with $\mu_{k_0+k_1}(D_{k_0+k_1}(x)) > 0$

$$\frac{\log \mu_{k_0+k_1}(D_{k_0+k_1}(x))}{\log 2^{-k_0-k_1}} \leq \frac{\log 2^{-k_0}}{\log 2^{-k_0-k_1}} \leq \frac{1}{2}. \tag{4.4}$$

Note that for all $k = 1,\dots,k_0+k_1$

$$\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_k(D^{\mu_k}_{i_1,\dots,i_m}) \geq \begin{cases} 1/2 & \text{for } m = 1,\dots,\min\{k_0,k\} \\ (1-\varepsilon)/2 & \text{for } m = \min\{k_0,k\}+1,\dots,k. \end{cases}$$

We continue the construction by repeating the above procedure inside all intervals $D_2,\dots,D_{2^{k_0}}$, respectively. When handling D_2, we simply distribute the weight of the dyadic interval $D \in \mathcal{D}_{k_0+k_1+k}$ to its rightmost subinterval at the next level, provided that $D \subset D_2$. If $D \cap D_2 = \emptyset$, we use the uniform weight distribution between the dyadic subintervals of D at level k_0+k_1+k+1. Choose k_2 such that the analogue of the local dimension estimate (4.4) holds for all $x \in D_2$ at level $k_0+k_1+k_2$. Then for all $k = k_0+k_1,\dots,k_0+k_1+k_2$ and $m = 1,\dots,k$

$$\sum_{\substack{(i_1,\dots,i_m)\in\mathbf{I}^m\\ i_m=1}} \mu_k(D^{\mu_k}_{i_1,\dots,i_m}) \geq (1-\varepsilon)/2.$$

After having gone through all intervals $D_3, \dots, D_{2^{k_0}}$, we start the process again from D_1, and replace the constant $1/2$ by a smaller one in the local dimension estimate. Setting $\mu = \lim_{k\to\infty} \mu_k$, properties (1) and (2) follow similarly as in Example 3.5. $\square$

Next we prove that the upper bound in Theorem 4.2 is attained provided that the upper packing dimension is a rational number. As a corollary it is shown that the upper bound in Theorem 4.2 cannot be replaced by a smaller bound depending on upper packing dimension.

Example 4.5. For all rational numbers $0 \le q \le n$ and for all positive integers k, there exists a Radon probability measure μ on $[0,1)^n$ with $\overline{\dim}_{\mathrm{p}}\, \mu = q$ such that $\overline{\mathrm{Hom}}^{\,1}_{k\text{-aver}}(\mu) = q/n$.

Proof. Fix positive integers p and l such that $p \le l$ and $q = np/l$. Divide the unit cube $[0,1)^n$ into k^{ln} k-adic cubes, and choose k^{pn} of them by taking every k^{l-p}th one of the k^l k-adic cubes in every coordinate direction starting from the origin. We attach the weight k^{-pn} to the chosen cubes and iterate this process. Letting μ be the limiting measure, we have for all $x \in \operatorname{spt}\mu$

$$\mu(K_{rl}(x)) = k^{-rlq}$$

for all $r = 1, 2, \dots$, and moreover,

$$\mu(K_i(x)) \ge k^{-q} k^{-qi}$$

for all $i = 1, 2, \dots$. This gives $\overline{\dim}_{\mathrm{p}}\, \mu = q$.

Furthermore, it follows from the construction that for all $r = 1, 2, \dots$

$$\sum_{\substack{(i_1,\dots,i_m)\in \mathbf{I}^m \\ i_m=1}} \mu(D^\mu_{i_1,\dots,i_m}) = \begin{cases} k^{-n} & \text{for } m = (r-1)l+1, \dots, (r-1)l+p \\ 0 & \text{for } m = (r-1)l+p+1, \dots, rl \end{cases}$$

implying that

$$\frac{k^n}{rl} \sum_{m=1}^{rl} \sum_{\substack{(i_1,\dots,i_m)\in \mathbf{I}^m \\ i_m=1}} \mu(D^\mu_{i_1,\dots,i_m}) = \frac{p}{l}.$$

Combined with Theorem 4.2 this, in turn, completes the proof. $\square$

Corollary 4.6. *Let $0 \le d \le n$ and let k be a positive integer. Given $0 < \varepsilon < 1$, there exists a Radon probability measure on $[0,1)^n$ such that $\overline{\dim}_{\mathrm{p}}\, \mu = d$ and $\overline{\mathrm{Hom}}^{\,1}_{k\text{-aver}}(\mu) \ge d/n - \varepsilon$.*

Proof. Using Example 4.5, we find a Radon probability measure μ_1 on $[0,1)^n$ such that $\overline{\dim}_{\mathrm{p}}\,\mu_1 < d$ and $\overline{\mathrm{Hom}}^{\,1}_{k\text{-aver}}(\mu_1) > (1-\varepsilon)^{-1}\,(\frac{d}{n}-\varepsilon)$. Taking

$$\mu = (1-\varepsilon)\mu_1 + \varepsilon\mu_2,$$

where μ_2 is a Radon probability measure on $[0,1)^n$ with $\overline{\dim}_{\mathrm{p}}\,\mu_2 = d$, we have $\overline{\dim}_{\mathrm{p}}\,\mu = \max_{i=1,2}\overline{\dim}_{\mathrm{p}}\,\mu_i = d$. Moreover,

$$\overline{\mathrm{Hom}}^{\,1}_{k\text{-aver}}(\mu) \geq (1-\varepsilon)\,\overline{\mathrm{Hom}}^{\,1}_{k\text{-aver}}(\mu_1) > \frac{d}{n} - \varepsilon.$$

□

Theorem 4.2 and Corollary 4.6 combine to give:

Corollary 4.7. *For all $0 < d < n$ and positive integers k we have*

$$\sup\{\overline{\mathrm{Hom}}^{\,1}_{k\text{-}aver}(\mu) \mid \mu \text{ is a Radon probability measure on } [0,1)^n \text{ such that } \overline{\dim}_{\mathrm{p}}\,\mu \leq d\} = \frac{d}{n}.$$

Remark 4.8. Using similar methods as above, it can be proven that

$$\overline{\mathrm{Hom}}^{\,j}_{k\text{-aver}}(\mu) \leq \frac{k^n}{(k^n-j+1)\log_k(k^n-j+1)}\,\overline{\dim}_{\mathrm{p}}\,\mu$$

provided that $\overline{\dim}_{\mathrm{p}}\,\mu \leq \log_k(k^n-j+1)$. Here the upper bound is again the best possible one. Note that if $\overline{\dim}_{\mathrm{p}}\,\mu$ exceeds $\log_k(k^n-j+1)$, the best possible upper bound for $\overline{\mathrm{Hom}}^{\,j}_{k\text{-aver}}$ decreases. This is due to the fact that, in order to increase the dimension, one has to give a proportion of the weight also to some $(j-1)$th subcube. It is not difficult to see that, under the assumption $\log_k(k^n-j+l) < \overline{\dim}_{\mathrm{p}}\,\mu \leq \log_k(k^n-j+l+1)$, the best possible upper bound is as follows:

$$\overline{\mathrm{Hom}}^{\,j}_{k\text{-aver}}(\mu) \leq \frac{k^n(1-p)}{k^n-j+l}$$

where p is obtained by solving the equation

$$\overline{\dim}_{\mathrm{p}}\,\mu = -p\log_k p - (1-p)\log_k\frac{1-p}{k^n-j+l}.$$

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