A lower bound for the Bloch radius of K-quasiregular mappings

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Abstract

We give a quantitative proof to Eremenko's theorem [6], which extends the classical Bloch's theorem to the class of n-dimensional K-quasiregular mappings.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. We call a mapping $f : \Omega \to \mathbb{R}^n$ K-quasiregular, $1 \leq K < \infty$, if the coordinate functions of f belong to $W^{1,n}_{loc}(\Omega)$ and if

$$||Df(x)||^n \leq K J_f(x)$$

for almost every $x \in \Omega$. Here $||\cdot||$ is the operator norm and J_f the Jacobian determinant of f. The definition of quasiregular mappings easily extends to mappings $f : \Omega \to \overline{\mathbb{R}}^n$, where $\overline{\mathbb{R}}^n$ is the one-point compactification of $\mathbb{R}^n$, equipped with the spherical metric, that is the metric that makes the stereographic projection $\pi : \overline{\mathbb{R}}^n \to S^n(e_{n+1}/2, 1/2)$ an isometry. For calculations in $\overline{\mathbb{R}}^n$ it is more convenient to use the chordal metric q, defined by

$$\begin{aligned} q(x,y) &= \frac{|x-y|}{\sqrt{(1+|x|^2)(1+|y|^2)}}, \qquad x,y \in \mathbb{R}^n \\ q(x,\infty) &= \frac{1}{\sqrt{1+|x|^2}}. \end{aligned}$$

The chordal metric is $\pi/2$-biLipschitz equivalent with the spherical metric. Quasiregular mappings into $\overline{\mathbb{R}}^n$ are often called quasimeromorphic.

It has turned out that (non-constant) quasiregular mappings form a natural generalization of analytic functions to higher dimensions. In the late 1960's Reshetnyak showed that they are continuous, open and discrete, and that they preserve sets of measure zero. Reshetnyak's work initiated a systematic study of quasiregular mappings, see the monographs [15] and [16].

In [6] Eremenko generalized the classical Bloch's Theorem [17], [2], [1] of analytic functions, as well as a corresponding theorem for meromorphic functions by Minda [12] (see also [3]), to the class of K-quasiregular mappings.

Let $f : \Omega \to \mathbb{R}^n$ be a quasiregular mapping. For every $x \in \Omega$ define $d_f(x)$ as the radius of the maximal open ball $B \subseteq \mathbb{R}^n$ centered at $f(x)$ such that a continuous right inverse ϕ with the property $\phi(f(x)) = x$ exists in B. The Bloch radius of f is defined as

$$\mathcal{B}_e(f) = \sup_{x \in \Omega} d_f(x).$$

Similarly, we define the Bloch radius $\mathcal{B}_s(f)$ for quasimeromorphic mappings f by looking at maximal open balls in the spherical metric. Notice that $\mathcal{B}_s(f) \leq \mathcal{B}_e(f)$ if f is considered as the same mapping into $\mathbb{R}^n$ with either Euclidean or spherical metric. Recall that a family $\mathcal{F}$ of continuous mappings from an open and connected subset Ω of $\mathbb{R}^n$ into a metric space Y is normal, if every sequence in $\mathcal{F}$ contains a subsequence converging uniformly on compact subsets of Ω. If Y is compact, $\mathcal{F}$ is normal if and only if its restriction to any compact subset of Ω is equicontinuous.

Eremenko's theorem now reads as follows:

Theorem 1.1 ([6], Theorem 1).

(i) There exists a constant $b(n, K) > 0$ such that the family of all K-quasimeromorphic mappings $\mathbb{R}^n \supseteq B(0, 1) \to \overline{\mathbb{R}}^n$, whose spherical Bloch radii are at most $b(n, K) - \epsilon$, is normal for every $\epsilon \in (0, b(n, K))$.

(ii) For every non-constant K-quasiregular mapping $f : \mathbb{R}^n \to \overline{\mathbb{R}}^n$ we have $\mathcal{B}_s(f) \geq b(n, K)$.

(iii) Every family of K-quasiregular mappings $\mathbb{R}^n \supseteq B(0, 1) \to \mathbb{R}^n$ with bounded Euclidean Bloch constant is equicontinuous with respect to the Euclidean metric.

(iv) Every non-constant K-quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\mathcal{B}_e(f) = \infty$.

Eremenko also gives an example showing that in any dimension $n \geq 3$, $b(n, K)$ depends on K in Theorem 1.1. In dimension two $b(2, K)$ does not depend on K, thanks to the Stoilow factorization theorem, see [9], page 241. It should also be noted that there exists a constant $K_p > 1$ so that any non-constant K-quasiregular mapping $f : \Omega \to \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, with $K < K_p$ is a local homeomorphism, see [16], VI Theorem 8.14. Moreover, every locally homeomorphic quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$ is homeomorphic by Zorich's theorem, see [16], III.3. Theorem 1.1 relates to the difficult question of describing the branching of quasiregular mappings, see [7] for discussion on this topic.

Eremenko gives a normal family argument to prove Theorem 1.1 (see [10] for related results for normal families of quasiregular mappings). The proof is short and elegant, but it only gives the existence of the constant $b(n,K)$ without any estimates. The purpose of this paper is to give a quantitative proof for Theorem 1.1.

Theorem 1.2. *One can choose $b(n,K)$ in Theorem 1.1 as follows: set $b(n,K)=\phi_M$ and define ϕ_M iteratively by*

$$\phi_1 = \frac{\eta}{2}\exp\left(-\left(\frac{2mK\omega_{n-1}}{C_1\log(1+\eta^{1/\alpha}/(8^{1/\alpha}\epsilon))}\right)^{1/(n-1)}\right)$$

and

$$\phi_{i+1} = \frac{\phi_i}{2}\exp\left(-\left(\frac{2mK\omega_{n-1}}{C_1\log(1+\phi_i^{1/\alpha}/(8^{1/\alpha}\epsilon))}\right)^{1/(n-1)}\right),$$

where

$$\eta = 2^{-\beta-1}C_2^{\beta/\alpha}\epsilon^{\beta}, \quad \beta = \frac{(4K^n\omega_{n-1}C_1^{-1})^{1/(n-1)}}{\log 2} - 1,$$

$$m = \left(\frac{K\log\frac{4}{\eta}}{\log 2}\right)^{n-1}, \quad \alpha = K^{1/(1-n)},$$

$$\epsilon = 2^{-6(1+1/\alpha)}\exp(-4K^n\omega_{n-1}/(C_1\log\sqrt{3})^{1/(n-1)}),$$

$\log_2 m - 1 < M \le \log_2 m$ *and C_1, C_2 are the constants in Lemma 2.1 and Inequality* (2.5) *below, depending only on n.*

The estimate in Theorem 1.2 is far from being optimal. Still, the proof of Theorem 1.2 is constructive, at least after we have used a normal family method to restrict the consideration to mappings with desirable properties. Some of the ideas of the proof are from the proof of Rickman's theorem on omitted values of entire quasiregular mappings, see [16], Chapter IV.

2 Preliminary results

We shall follow [16] as our basic reference for the theory of quasiregular mappings. We shall use notation $M(\Gamma)$ for the usual n-modulus of a curve family Γ. Chordal balls in $\overline{\mathbb{R}}^n$ are denoted by $Q(x,r)$, and Euclidean balls by $B(x,r)$. The diameter of a set $E \subseteq \overline{\mathbb{R}}^n$ in the chordal metric is denoted by $q(E)$, while the corresponding notation in the Euclidean metric is $\operatorname{diam}(E)$.

Let us recall a useful modulus estimate.

Lemma 2.1 ([19], Theorem 10.12). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Suppose that $0<a<b$ and that $E, F \subseteq \Omega$ are disjoint sets such that every sphere $S^{n-1}(0,t)$, $a<t<b$, meets both E and F. If Ω contains the spherical ring*

$B(0,b) \setminus B(0,a)$ and if Γ is the family of all curves joining E and F in Ω, then

$$M(\Gamma) \geq C_1 \log \frac{b}{a}, \tag{2.1}$$

where C_1 only depends on n (see [19], pages 28 and 31 for the precise value of C_1).

Recall that if $f : \Omega \to \overline{\mathbb{R}}^n$ is an open and discrete mapping and if $U \subset\subset \Omega$ is a domain, then U is called a normal domain (for f) if $f\partial U = \partial f U$. Now we have the following:

Lemma 2.2 ([16], I Lemma 4.7). *If $V \subseteq \overline{\mathbb{R}}^n$ is a domain and if U is a component of $f^{-1}V$ such that $U \subset\subset \Omega$, then U is a normal domain and $fU = V \subset\subset f\Omega$.*

As in [16], we will use notation $U(x, f, s)$ for the x-component of the preimage $f^{-1}(Q(f(x), s))$. Notation $\mu(y, f, U)$ will be used for the topological degree of the mapping f at $y \notin f(\partial U)$ with respect to the domain U. If U is a normal domain, then $\mu(y, f, U)$ is constant for all $y \in fU$. We will denote this constant by $\mu(f, U)$.

For a quasiregular mapping $f : \Omega \to \mathbb{R}^n$ and a Borel set $E \subseteq \Omega$, define the counting function $n(E, y)$ by

$$n(E, y) = \sum_{x \in f^{-1}(y) \cap E} i(x, f),$$

where $i(x, f)$ is the local index. See [16], I.4 for information on the topological degree and the local index. The following modulus inequality will be useful for our purposes. Recall that a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is said to be admissible for a curve family Γ if it qualifies as a test function for the modulus $M(\Gamma)$, i.e. if

$$\int_\gamma \rho \, ds \geq 1$$

for all locally rectifiable curves $\gamma \in \Gamma$.

Lemma 2.3 ([16], II Theorem 2.4 and Remark 2.5). *Let $f : \Omega \to \mathbb{R}^n$ be a quasiregular mapping, and suppose $E \subset\subset \Omega$ is a Borel set. If Γ is a family of curves in E and if ρ is an admissible function for $f\Gamma$, then*

$$M(\Gamma) \leq K \int_{\mathbb{R}^n} \rho^n(y) \, n(E, y) \, dy. \tag{2.2}$$

We will also use a comparison inequality for the counting function. For an $n-1$-dimensional sphere $Y = S^{n-1}(0, t) \subseteq \mathbb{R}^n$, define the average of the counting function $n(E, \cdot)$ over the sphere Y, $\nu(E, Y)$, by

$$\nu(E, Y) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} n(E, ty) \, dy,$$

where S^{n-1} is the unit sphere. The definition extends to all spheres in $\mathbb{R}^n$ (as well as in $\overline{\mathbb{R}}^n$) by translation. We shall use the notation $\nu(r,t)$ if $E = B(x,r)$ and if the center points need not be emphasized. Now we have the following comparison result in terms of the Euclidean metric. Here $K_I(f) = K_I \leq K^{n-1}$ is the inner dilatation of f, see [16], page 11.

Lemma 2.4 ([16], IV Lemma 1.1). *Let $f : \Omega \to \mathbb{R}^n$ be a K-quasiregular mapping. If $\theta > 1$, if $r, s, t > 0$, and if $\overline{B}(\theta r) \subseteq \Omega$, then*

$$\nu(\theta r, t) \geq \nu(r, s) - \frac{K_I |\log(t/s)|^{n-1}}{(\log\theta)^{n-1}}, \tag{2.3}$$

whenever the t- and s-spheres have the same centerpoint.

The normal family method, introduced by Zalcman [20], has turned out to be an important tool in complex function theory. This method was generalized to the class of quasimeromorphic mappings in higher dimensions by Miniowitz [13]. We shall formulate Miniowitz's result as in [6].

Lemma 2.5 ([13], Lemma 1). *Let $\mathcal{F}$ be a family of K-quasimeromorphic mappings $\mathbb{R}^n \supseteq B(0,1) \to \overline{\mathbb{R}}^n$ which is not normal. Then there exist $r \in (0,1)$ and sequences $(f_m) \subseteq \mathcal{F}$, $(x_m) \subseteq B(0,r)$ and (ρ_m), $0 < \rho_m \to 0$, such that $g_m(x) := f_m(x_m + \rho_m x) \to f(x) \neq const$ uniformly on compacta in $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \overline{\mathbb{R}}^n$ is K-quasimeromorphic. Moreover, we have for every $x_1, x_2 \in B(0,R)$,*

$$q(f(x_1), f(x_2)) \leq 2(1+R^2)^\alpha |x_1 - x_2|^\alpha, \tag{2.4}$$

where $\alpha = K^{1/(1-n)}$, and

$$q(f(B(0,1))) \geq C_2^{1/\alpha} > 0, \tag{2.5}$$

where $0 < C_2 < 1$ is a constant only depending on the dimension.

What makes normal families useful here is the fact that the Bloch radius is lower semicontinuous with respect to locally uniform convergence.

Lemma 2.6 ([6], page 559). *Let (g_m) be a family of K-quasimeromorphic mappings into $\overline{\mathbb{R}}^n$, and suppose that $g_m \to f$ uniformly on compacta. Then*

$$\mathcal{B}_s(f) \leq \liminf_m \mathcal{B}_s(g_m).$$

3 Proof of Theorem 1.2

We first show that mappings as f in Lemma 2.5 behave well in some normal domains U.

Proposition 3.1. *Let f be as in Lemma 2.5. Then there exist $x \in B(0,1)$ and constants η, ϵ, $m > 0$ depending only on n and K, so that*

$$U = U(x, f, \eta) \subseteq B(x, \epsilon) \subseteq B(0,1)$$

is a normal domain and $\mu(f, U) \leq m$. The values of the constants η, ϵ and m are given in the statement of Theorem 1.2.

Proof. Set $\theta = \exp((4KK_I\omega_{n-1}/(C_1 \log \sqrt{3}))^{1/(n-1)})$, where C_1 is as in Lemma 2.1, and $\epsilon = 2^{-6(1+1/\alpha)}/\theta$, where α is as in Lemma 2.5. Let x and y be such that

$$q(f(\overline{B}(0,1))) = q(f(x), f(y)).$$

We cover the line segment joining x and y by pairwise disjoint ϵ-balls B_i so that $B_i \subseteq B(0,1)$ for every i. For this at most ϵ^{-1} balls are needed. By the lower bound (2.5),

$$C_2^{\frac{1}{\alpha}} \leq q(f(B(0,1))) \leq \sum_{i=1}^{\epsilon^{-1}} q(f(B_i)) \leq \epsilon^{-1} \max_i q(f(B_i)),$$

and thus for some $B(x, \epsilon) \subseteq B(0,1)$,

$$q(f(B(x,\epsilon))) \geq C_2^{\frac{1}{\alpha}} \epsilon. \tag{3.1}$$

Denote

$$r = \sup\{t : U(x, f, t) \subseteq B(x, \epsilon)\}$$

and

$$R = \inf\{t : f(B(x,\epsilon)) \subseteq Q(f(x), t)\}.$$

Then the x-component U_r of $f^{-1}(Q(f(x), r))$ has the properties $x \in U_r$ and $\overline{U}_r \cap S^{n-1}(x, \epsilon) \neq \emptyset$. Also, we claim that $f^{-1}(\overline{\mathbb{R}}^n \setminus \overline{B}(f(x), R)) =: W$ has, for arbitrarily large i, components U_i with

$$S^{n-1}(x, 2\epsilon) \cap U_i \neq \emptyset, \quad d(U_i, S^{n-1}(x, \epsilon)) < \frac{1}{i}.$$

To see this we take a point $z \in W$ so that $d(z, S^{n-1}(x,\epsilon)) < \frac{1}{i}$. This is possible by the definition of R and by the fact that for all $x \in \mathbb{R}^n$ there exists a radius r_x such that $U(x, f, s)$ is a normal domain for all $s \leq r_x$, see [16], I Lemma 4.9. Then assume that the z-component U_i of W lies inside $B(x, 2\epsilon)$. By Lemma 2.2, $f(U_i) = \overline{\mathbb{R}}^n \setminus Q(f(x), R)$, and in particular $q(f(B(x, 2\epsilon))) = q(\overline{\mathbb{R}}^n) = 1$. This is a contradiction by the continuity estimate (2.4) and our choice of ϵ. Thus the claim holds.

Consider the family Γ of all curves joining $B(x, 2\epsilon) \cap (\cup_i U_i)$ and $U_r \cap B(x, 2\epsilon)$ in $B(x, 2\epsilon)$. Then a simple geometric argument (see [8], Lemma 3.1) shows that we can use Lemma 2.1 with $a = \epsilon/2$, $b = \sqrt{3}\epsilon/2$ and $\Omega = B(x, 2\epsilon)$ in order to have a lower bound for the modulus of Γ. By composing f with a Möbius transformation, we may assume $f(x) = 0$. Now, since by (2.4) $R \leq q(f(B(x,\epsilon))) \leq 2^{-5}$,

$$\Lambda = \mathrm{Id} : (Q(0,R), q) \to (Q(0,R), |\cdot|)$$

is (at least) 2-biLipschitz. Thus using the Euclidean metric in calculating $M(f\Gamma)$ only changes the value by a multiplicative constant smaller than 2. Let $\Lambda(Q(0,R)) = B(0,R')$ and $\Lambda(Q(0,r)) = B(0,r')$.

Lemma 2.3 can now be used with the function

$$\rho(y) = \begin{cases} \frac{1}{(\log \frac{R'}{r'})|y|}, & r' < |y| < R', \\ 0 & \text{elsewhere} \end{cases}$$

(compare [16], page 82). Combining the estimates and using Lemma 2.4, we have

$$\begin{aligned} C_1 \log \sqrt{3} &\leq M(\Gamma) \leq 2K\Big(\log\frac{R'}{r'}\Big)^{-n} \int_{B(0,R')\setminus \overline{B}(0,r')} n(2\epsilon, y)|y|^{-n}\, dy \\ &= 2\omega_{n-1}K\Big(\log\frac{R'}{r'}\Big)^{-n}\int_{r'}^{R'} \frac{\nu(2\epsilon, s)}{s}\, ds \leq 2\omega_{n-1}K\Big(\log\frac{R'}{r'}\Big)^{-n} \\ &\quad \int_{r'}^{R'} s^{-1}\Big(\nu(2^{-6(1+1/\alpha)}, R') + K_I\Big(\log\frac{R'}{r'}\Big)^{n-1}(\log\theta)^{1-n}\Big)\, ds \\ &\leq 2\omega_{n-1}K\Big(\log\frac{R'}{r'}\Big)^{1-n}\nu(2^{-6(1+1/\alpha)}, R') + 2K_I K\omega_{n-1}(\log\theta)^{1-n}. \end{aligned}$$

By our choice of θ, the second term on the previous line is smaller than $\frac{1}{2}C_1 \log\sqrt{3}$, and so we further have

$$\Big(\log\frac{R'}{r'}\Big)^{n-1} \leq 4K\omega_{n-1}C_1^{-1}\nu(2^{-6(1+1/\alpha)}, R'). \tag{3.2}$$

Since $B(x, 2^{-6(1+1/\alpha)+1}) \subseteq B(0,2)$, (2.4) gives

$$\operatorname{diam}((\Lambda \circ f)(B(x, 2^{-6(1+1/\alpha)+1}))) << 1. \tag{3.3}$$

Applying Lemma 2.4 again, with $t = 1$ and $s = R'$ gives

$$\begin{aligned} \nu(2^{-6(1+1/\alpha)}, R') &\leq \nu(2^{-6(1+1/\alpha)+1}, 1) + K_I(\log 2)^{1-n}\Big(\log\frac{1}{R'}\Big)^{n-1} \\ &= K_I(\log 2)^{1-n}\Big(\log\frac{1}{R'}\Big)^{n-1}. \end{aligned} \tag{3.4}$$

Since $R' \geq R \geq C_2^{\frac{1}{\alpha}}\epsilon/2$, combining Inequalities (3.2) and (3.4) gives

$$\Big(\log\frac{C_2^{1/\alpha}\epsilon}{2r'}\Big)^{n-1} \leq 4K\omega_{n-1}C_1^{-1}K_I(\log 2)^{1-n}\Big(\log\frac{2}{C_2^{1/\alpha}\epsilon}\Big)^{n-1},$$

i.e.

$$(3.5)\quad r \geq \frac{1}{2}r' \geq \frac{1}{2}\Big(\frac{C_2^{1/\alpha}\epsilon}{2}\Big)^{\beta} =: \eta, \text{ where } \beta = \frac{(4K\omega_{n-1}C_1^{-1}K_I)^{1/(n-1)}}{\log 2} - 1.$$

Inequality (3.5) tells us that whenever $t < \eta$, $\overline{U}(x, f, t)$ does not intersect $S^{n-1}(B(x, \epsilon))$, and thus by Lemma 2.2 $U = U(x, f, t)$ is a normal domain in this case. This proves the first claim of the proposition.

For the second claim we need to show that the topological degree is bounded in U, with a bound depending only on n and K. By (3.3), also

$$\operatorname{diam}((\Lambda \circ f)(B(x, 2\epsilon))) << 1,$$

and using Lemma 2.4 as above gives

$$(3.6)\qquad \begin{aligned} \nu(\epsilon, \eta/4) &\leq \nu(2\epsilon, 1) + \frac{K_I}{(\log 2)^{n-1}}\Big(\log\frac{4}{\eta}\Big)^{n-1} \\ &= \frac{K_I}{(\log 2)^{n-1}}\Big(\log\frac{4}{\eta}\Big)^{n-1} =: m. \end{aligned}$$

By the definition of the average ν and the correspondance between the topological degree and the local index (see [16], I.4), Inequality (3.6) shows in particular that there exists a point $y \in Q(0, \eta)$ such that $\mu(y, f, U) \leq m$. On the other hand, since the topological degree is constant in a normal domain, we have $\mu(f, U) \leq m$. This proves the second claim of the proposition. □

We need a topological lemma from [11]. This lemma is closely related to Newman's theorem [14] on transformation groups. See [5], [4], [18] for related results. By a proper mapping we mean a mapping with the property that the preimage of an arbitrary compact set is compact.

Lemma 3.2 ([11], Theorem 2). *Let U be an open, connected, relatively compact subset of $\mathbb{R}^n$, $f : \overline{U} \to Y$ a proper finite-to-one open mapping which is not a homeomorphism where Y is an n-dimensional manifold (possibly with boundary),*

$$f^{-1}(\partial Y) \subseteq (\overline{U} \setminus U), \qquad D = \max\{d(x, \overline{U} \setminus U) : x \in U\},$$

and

$$C = \max\{\operatorname{diam}(f^{-1}(f(x))) : x \in \overline{U} \setminus U\}.$$

Then $D \leq C$.

Proof of Theorem 1.2. First, we argue as in [6] to show that it suffices to consider mappings with the properties of f in Lemma 2.5. For (i) in Theorem 1.1 this is clear by Lemma 2.6. For (ii) it suffices to notice that if $f : \mathbb{R}^n \to \overline{\mathbb{R}}^n$ is K-quasimeromorphic and non-constant, then the family $\{x \to f(2^n x) : n \in \mathbb{N}\}$ cannot be a normal family.

Let f, x, η, ϵ, U and m be as in Proposition 3.1. Recall that we have a uniform continuity estimate (2.4) for f, and thus

$$D \geq \Big(\frac{\eta}{8}\Big)^{1/\alpha} =: T,$$

where D is as in Lemma 3.2. Now by Lemma 3.2, we have a point $z_1 \in \partial Q(f(x), \eta/2)$ such that

$$\mathrm{diam}(f^{-1}(z_1) \cap \overline{U}(x, f, \eta/2)) \geq T \tag{3.7}$$

(unless $f : U(x, f, \eta/2) \to Q(f(x), \eta/2)$ is a homeomorphism, which would complete the proof). We claim that there exists a constant $\phi_1 = \phi_1(n, K) > 0$ such that $f^{-1}(z_1) \cap \partial U(x, f, \eta/2)$ cannot be contained in a single component of $f^{-1}(Q(z_1, r))$ for $r \leq \phi_1$.

Suppose $f^{-1}(z_1) \cap \partial U(x, f, \eta/2)$ is contained in a single component V of $f^{-1}(Q(z_1, r))$ for $r < \eta/4$. Then by (3.7), $\mathrm{diam}\, V \geq T$. Denote by Γ the family of all curves joining ∂V and ∂U in $B(x, \epsilon)$. Since $U \subseteq B(x, \epsilon)$, $M(\Gamma) \geq M(\Gamma')$, where Γ' is the family of all curves joining $S^{n-1}(x, \epsilon)$ and V. For $M(\Gamma')$ we can apply 2.1 with $a = \epsilon$ and $b = T + \epsilon$. On the other hand, U and V are normal domains, and so $M(f\Gamma) \leq M(\Gamma^*)$, where Γ^* is the family of all curves joining $Q(z, r)$ and $Q(z, 2\eta)$. As in the proof of Proposition 3.1, we can use the 2-biLipschitz property of Λ in order to estimate the modulus by calculating in the Euclidean metric. Combining with Lemma 2.3 (notice again the correspondance between the topological degree and the local index), we have

$$\begin{aligned} C_1 \log\Big(1 + \frac{T}{\epsilon}\Big) &\leq M(\Gamma') \leq M(\Gamma) \leq mKM(f\Gamma) \leq mKM(\Gamma^*) \\ &\leq 2\omega_{n-1} mK\Big(\log\frac{\eta}{2r}\Big)^{1-n}. \end{aligned}$$

This is a contradiction for

$$r \leq \frac{\eta}{2}\exp\Big(-\Big(\frac{2mK\omega_{n-1}}{C_1\log(1+T/\epsilon)}\Big)^{1/(n-1)}\Big) =: \phi_1$$

and thus the claim holds.

By the claim, there exists a component U_1 of $f^{-1}(Q(z_1, \phi_1)) \subseteq B(x, \epsilon)$, which is a normal domain, so that $\mu(f, U_1) \leq m/2$. By repeating the previous arguments at most $\log_2 m$ times, we get a chordal ball $Q = Q(z_M, \phi_M)$ so that $f^{-1}(Q)$ has a component U_M with $\mu(f, U_M) = 1$. Hence f restricted to U_M is a homeomorphism onto Q.. □

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