Weighted BMO and discrete time hedging within the Black-Scholes model

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Abstract

The paper combines two, at first glance rather different, objects: spaces of adapted stochastic processes of weighted bounded mean oscillation (weighted BMO) and approximations of certain stochastic integrals (driven by the geometric Brownian motion) by integrals over piece-wise constant integrands. The consideration of the approximation error with respect to weighted BMO implies L_p and uniform distributional estimates for the error by a John-Nirenberg type theorem. The general results about weighted BMO are given in the first part of the paper and applied to our approximation problem in the second one.

1 Introduction

The approximation of stochastic integrals by integrals over piece-wise constant integrands has for example in stochastic finance a natural interpretation: the pay-off of a continuously re-balanced portfolio is replaced by the pay-off of a portfolio, re-balanced at finitely many trading dates only. The approximation error between the stochastic integral, we were starting from, and its approximation can be interpreted as risk.

Usually, the approximation error is measured with respect to L_2, which has the following drawbacks: the resulting distributional tail-estimates are rather weak. Secondly, if one fixes the number of discretization points in the time-net and looks for the optimal discretization in order to minimize the L_2-error, then there are in general very different asymptotically optimal time nets and it may not be clear what net one should use.

The present paper approaches both problems. In order to replace the L_2-criteria by a stronger one, one would use the L_p-spaces with $2 < p < \infty$ in a first instance. However, in our situation it turns out that we can use much stronger spaces, spaces of weighted bounded mean oscillation (weighted BMO), while getting the same upper error bounds as in the case we would measure the error with respect to L_2. The used BMO-spaces have two advantages: estimates with respect to these spaces imply generally all L_p-estimates and secondly, by a weighted John-Nirenberg type theorem we obtain significantly better tail-estimates than we would get from L_2-estimates (see Section 3.4). Moreover, for a large class of situations we show in Section 3.3 that the asymptotically optimal time nets with respect to our weighted BMO-spaces are unique in some sense.

The paper is divided into two rather different parts. Section 2 deals with the weighted BMO-spaces in a more general context. In Section 3 we apply these results to our approximation problems of stochastic integrals.

2 Weighted BMO-spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $F = (\mathcal{F}_t)_{t\in[0,T]}$ be a right-continuous filtration with $\mathcal{F}_T \subseteq \mathcal{F}$ such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. By $\mathcal{CL}(F)$ we denote the set of all F-adapted processes $A = (A_t)_{t\in[0,T]}$ such that almost all paths are right-continuous and have finite left-hand side limits. The symbol $\mathcal{CL}^+(F)$ stands for the subclass of these processes such that $A_t(\omega) > 0$ on $[0,T] \times \Omega$, whereas $\mathcal{CL}_0(F)$ is standing for the $A \in \mathcal{CL}(F)$ with $A_0 \equiv 0$. Given a stopping time $\sigma : \Omega \to [0,T]$ and $A \in \mathcal{CL}_0(F)$ we let $A_{\sigma-} := \lim_{n\to\infty} \chi_{\Omega_0} A_{\left(\sigma - \frac{1}{n}\right)^+}$, where Ω_0 is a set of measure one on which A is right continuous and has finite left-hand side limits. For a stochastic process $X = (X_t)_{t\in[0,T]}$ we let, as usual, $X_t^* := \sup_{u\in[0,t]} |X_u|$. Given $B \in \mathcal{F}$, the normalized restriction of $\mathbb{P}$ to B is defined by $\mathbb{P}_B(\cdot) := \mathbb{P}(B \cap \cdot)/\mathbb{P}(B)$

if $\mathbb{P}(B) > 0$ and $\mathbb{P}_B \equiv 0$ if $\mathbb{P}(B) = 0$. Finally, given $A, B \geq 0$ and $c > 0$, the expression $A \sim_c B$ stands for $A/c \leq B \leq cA$. The BMO-spaces, we use, are defined as follows:

Definition 2.1 *Assume that $\theta \in (0,1)$, $p \in (0,\infty)$, $A \in \mathcal{CL}_0(F)$, and $\Phi \in \mathcal{CL}^+(F)$. Letting $\mathcal{S}$ be the set of all stopping times $\sigma : \Omega \to [0,T]$, we define*

$$\begin{aligned}
\|A\|_{\mathrm{BMO}_p^\Phi(\mathbb{P})} &:= \sup_{\sigma \in \mathcal{S}} \left\| \mathbb{E}\left[\frac{|A_T - A_{\sigma-}|^p}{\Phi_\sigma^p} \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty}^{\frac{1}{p}}, \\
\|A\|_{\mathrm{BMO}_p^{\Phi,*}(\mathbb{P})} &:= \sup_{\sigma \in \mathcal{S}} \left\| \mathbb{E}\left[\frac{\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}|^p}{\Phi_\sigma^p} \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty}^{\frac{1}{p}}, \\
\|A\|_{\mathrm{BMO}_{0,\theta}^\Phi(\mathbb{P})} &:= \inf\left\{ c > 0 : \left\| \mathbb{P}\left[\frac{|A_T - A_{\sigma-}|}{\Phi_\sigma} > c \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty} \leq \theta, \, \sigma \in \mathcal{S} \right\},
\end{aligned}$$

and $\|A\|_{\mathrm{BMO}_{0,\theta}^{\Phi,*}(\mathbb{P})} :=$

$$\inf\left\{ c > 0 : \left\| \mathbb{P}\left[\frac{\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}|}{\Phi_\sigma} > c \mid \mathcal{F}_\sigma \right] \right\|_{L_\infty} \leq \theta, \, \sigma \in \mathcal{S} \right\},$$

with $\inf \emptyset := \infty$.

The quantities $\|\cdot\|_{\mathrm{BMO}_p^\Phi}$ and $\|\cdot\|_{\mathrm{BMO}_p^{\Phi,*}}$ model a classical approach to weighted BMO in the probabilistic setting connected to the Garsia-Neveu lemma and related results of Garsia, Stroock, and many others (see for example [6], [14], [4], [1], and Remark 2.7 below). The approach to exploit $\|\cdot\|_{\mathrm{BMO}_{0,\theta}^\Phi}$ and $\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi,*}}$ has at least two sources: firstly, Strömberg [16] measured the mean-oscillation of complex-valued functions defined on $\mathbb{R}^n$ in a distributional way by the sharp function $M_{0,s}^\sharp f$. Secondly, in the non-weighted probabilistic setting, which means here $\Phi_t \equiv 1$, the distributional approach can be found in Emery [5] and in [8]. We also need

Definition 2.2 (i) Let Q be a probability measure on $[\Omega, \mathcal{F}]$ with $Q \sim \mathbb{P}$ and $L(\omega) = (dQ/d\mathbb{P})(\omega) > 0$ for all $\omega \in \Omega$. Given $v \in (1,\infty)$ and $c > 0$, we let $Q \in \mathcal{RH}_v(\mathbb{P}, c)$ provided that $L \in L_v(\Omega, \mathcal{F}, \mathbb{P})$ and

$$\sqrt[v]{\mathbb{E}\left(L^v \mid \mathcal{F}_\sigma\right)} \leq c\, \mathbb{E}\left(L \mid \mathcal{F}_\sigma\right) \quad \text{a.s.}$$

for all stopping times $\sigma : \Omega \to [0,T]$.

(ii) Given $q, d \in (0,\infty)$ and $\Phi \in \mathcal{CL}^+(F)$ with $\Phi_0 \in L_q(\Omega, \mathcal{F}, \mathbb{P})$, we let $\Phi \in \mathcal{SM}_q(\mathbb{P}, d)$ provided that for all stopping times $\sigma : \Omega \to [0,T]$ one has that

$$\mathbb{E}\left(\sup_{u\in[\sigma,T]} \Phi_u^q \mid \mathcal{F}_\sigma\right) \;\leq\; d^q\, \Phi_\sigma^q \;\text{ a.s.}$$

The class of weights $\mathcal{RH}_v$, satisfying a reverse Hölder inequality, is going back to Izumisawa and Kazamaki [12] in the probabilistic setting. To shorten the notation we agree that writing $\mathcal{RH}_v(\mathbb{P}, c)$ or $\mathcal{SM}_q(\mathbb{P}, d)$ means automatically that $v \in (1,\infty)$, $q \in (0,\infty)$, and $c, d > 0$. Our results about the spaces introduced in Definition 2.1 follow immediately from Theorem 2.3 below. This theorem transfers Emery [5] (Proposition 2) and [8] (Theorem 4.6) to the weighted case and Strömberg [16] (Lemma 3.4) from the classical setting of functions defined on $\mathbb{R}^n$ to the setting of stochastic processes.

Theorem 2.3 *Let $A \in \mathcal{CL}_0(\mathcal{F})$ and $\theta \in \left(0, \frac{1}{2}\right)$. Assume that $\Psi = (\Psi_t)_{t\in[0,T]}$ is a process of $\mathcal{F}$-measurable random variables $\Psi_t : \Omega \to [0,\infty)$ such that almost all trajectories have finite left-hand side limits, are right-continuous, and are non-increasing. Assume furthermore that there is a $\delta > 0$ such that*

$$\mathbb{P}\left(|A_T - A_{\sigma-}| > \nu \middle| \mathcal{F}_\sigma\right) \;\leq\; \theta + \delta \mathbb{P}\left(\Psi_\sigma > \nu \middle| \mathcal{F}_\sigma\right) \quad a.s. \tag{1}$$

for all $\nu > 0$ and stopping times $\sigma : \Omega \to [0,T]$. Then there are constants $\alpha, a > 0$, depending on θ and δ only, such that

$$\mathbb{P}\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda + \mu\nu \middle| \mathcal{F}_\sigma\right) \;\leq$$

$$\leq\; e^{1-\mu}\mathbb{P}\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda \middle| \mathcal{F}_\sigma\right) + \alpha \mathbb{P}\left(\Psi_\sigma > \frac{\nu}{a} \middle| \mathcal{F}_\sigma\right) \quad a.s.$$

for all $\lambda, \mu, \nu > 0$ and stopping times $\sigma : \Omega \to [0,T]$. Consequently, for all $p \in (0,\infty)$ and $B \in \mathcal{F}_\sigma$ of positive measure one has that

$$\left\| \sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| \right\|_{L_p(\mathbb{P}_B)} \;\leq\; c_p \left\|\Psi_\sigma\right\|_{L_p(\mathbb{P}_B)}$$

where the right-hand side or both sides may be infinite, $c_p > 0$ depends on (p, α, a) only, and $\sup_{1\leq p<\infty} \frac{c_p}{p} < \infty$.

Proof. We fix a stopping time σ and $B \in \mathcal{F}_\sigma$ of positive measure. For $\lambda > 0$ we let

$$\sigma_\lambda := \inf\left\{t \in [\sigma, T] \;\middle|\; |A_t - A_{\sigma-}| > \lambda\right\} \wedge T,$$

with the convention that $\inf \emptyset := \infty$. For $\lambda, \nu > 0$ we obtain

$$\begin{aligned}
&\mathbb{P}_B\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda + \nu\right) \\
\leq\; & \mathbb{P}_B\left(\left|A_{\sigma_{\lambda+\nu}} - A_{\sigma-}\right| \geq \lambda + \nu,\ \sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda\right) \\
\leq\; & \mathbb{P}_B\left(\left|A_{\sigma_{\lambda+\nu}} - A_{\sigma-}\right| \geq |A_{\sigma_\lambda -} - A_{\sigma-}| + \nu,\ \sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda\right) \\
\leq\; & \mathbb{P}_B\left(\left|A_{\sigma_{\lambda+\nu}} - A_T\right| + |A_T - A_{\sigma_\lambda -}| \geq \nu,\ \sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda\right) \\
=\; & \mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|>\lambda\right\}}\left(\left|A_{\sigma_{\lambda+\nu}} - A_T\right| + |A_T - A_{\sigma_\lambda -}| \geq \nu\right) \\
& \qquad\qquad \mathbb{P}_B\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda\right) \\
\leq\; & \left[\mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|>\lambda\right\}}\left(\left|A_{\sigma_{\lambda+\nu}} - A_T\right| > \frac{\nu}{2}\right) + \right. \\
& \qquad \left. + \mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|>\lambda\right\}}\left(|A_T - A_{\sigma_\lambda -}| \geq \frac{\nu}{2}\right)\right] \\
& \qquad\qquad \mathbb{P}_B\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda\right) \\
\leq\; & \left[\mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|>\lambda\right\}}\left(\liminf_{n\to\infty, n=1,2,\ldots} \left|A_T - A_{\left(\left(\sigma_{\lambda+\nu}+\frac{1}{n}\right)\wedge T\right)-}\right| > \frac{\nu}{2}\right)\right. \\
& \qquad \left. + \theta + \delta\mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|>\lambda\right\}}\left(\Psi_\sigma > \frac{\nu}{3}\right)\right] \\
& \qquad\qquad \mathbb{P}_B\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \lambda\right) \\
\leq\; & \left[\liminf_n \mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|>\lambda\right\}}\left(\left|A_T - A_{\left(\left(\sigma_{\lambda+\nu}+\frac{1}{n}\right)\wedge T\right)-}\right| > \frac{\nu}{2}\right) + \right. \\
& \qquad \left. + \theta + \delta\mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|>\lambda\right\}}\left(\Psi_\sigma > \frac{\nu}{3}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
& \mathbb{P}_B\left(\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}| > \lambda\right) \\
\le\ & \left[2\theta + 2\delta\mathbb{P}_{B\cap\left\{\sup_{u\in[\sigma,T]}|A_u-A_{\sigma-}|>\lambda\right\}}\left(\Psi_\sigma > \frac{\nu}{3}\right)\right] \\
& \mathbb{P}_B\left(\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}| > \lambda\right).
\end{aligned}
$$

Setting $g := \sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}|$, $s := 2\theta \in (0,1)$, and

$$
W(C,\nu) := 2\delta\mathbb{P}_{B\cap C}\left(\Psi_\sigma > \frac{\nu}{3}\right) \quad \text{for} \quad C \in \mathcal{F},
$$

we have shown that

$$
\mathbb{P}_B(g > \lambda + \nu) \ \le\ \Big[s + W(g>\lambda,\nu)\Big]\mathbb{P}_B(g>\lambda) \quad \text{for} \quad \lambda,\nu > 0.
$$

By induction over $N = 1, 2, ...$, where we use

$$
W(C_1,\nu)\mathbb{P}_B(C_1) \ \le\ W(C_2,\nu)\mathbb{P}_B(C_2) \quad \text{whenever} \quad C_1 \subseteq C_2,
$$

one easily sees that

$$
\begin{aligned}
\mathbb{P}_B(g > \lambda + N\nu) &\le \left[s^N + W(g>\lambda,\nu)\left(\sum_{k=1}^N s^{k-1}\right)\right]\mathbb{P}_B(g>\lambda) \\
&\le \left[s^N + \frac{1}{1-s}W(g>\lambda,\nu)\right]\mathbb{P}_B(g>\lambda) \\
&\le s^N\mathbb{P}_B(g>\lambda) + \frac{1}{1-s}W(\Omega,\nu).
\end{aligned}
$$

The rest follows by a computation for $\alpha := \frac{2\delta}{1-2\theta}$ and $a := 3\left(1 \vee \left(\frac{1}{\log\frac{1}{2\theta}}\right)\right)$.

The *consequently*-part follows from [3] (Lemma 7.1). $\square$

Remark 2.4 Looking at the proof of Theorem 2.3 one quickly checks that one can replace the assumption in Formula (1) by

$$
\mathbb{P}\left(\sup_{u\in[\sigma,T]}|A_u - A_{\sigma-}| > \nu \,\Big|\, \mathcal{F}_\sigma\right) \ \le\ \eta + \delta\mathbb{P}\left(\Psi_\sigma > \nu \,\Big|\, \mathcal{F}_\sigma\right) \ \text{ a.s.}
$$

for $\eta \in (0,1)$. To what extend the range of θ in Formula (1) can be enlarged is not investigated here. The range $\theta \in (0,1/2)$ is sufficient for our purpose.

Now we show that we can change the underlying measure in a moderate way. One should note: while changing the measure we keep the whole process. This is different from the usual setting (see for example [13] (Chapters 3.1 and 3.3)).

Corollary 2.5 *For $p, q \in (0, \infty)$, $Q \in \mathcal{RH}_v(\mathbb{P}, c)$, and $\Phi \in \mathcal{SM}_q(Q, d)$, there is a constant $C = C(p, q, v, c, d) > 0$ such that*

$$\|\cdot\|_{\mathrm{BMO}_q^{\Phi,*}(Q)} \leq C \, \|\cdot\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})} .$$

Proof. Assume that $\|A\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})} \leq 1$. For $\lambda > 0$ and $B \in \mathcal{F}_\sigma$ of positive measure we get

$$\mathbb{P}_B \left(\frac{|A_T - A_{\sigma-}|}{\Phi_\sigma} > \lambda \right) \leq \frac{1}{\lambda^p}.$$

Letting $1 < u < \infty$ with $1 = \frac{1}{u} + \frac{1}{v}$ we can use the arguments from [2] (in particular the lemma on p. 298) to conclude that

$$Q_B \left(\frac{|A_T - A_{\sigma-}|}{\Phi_\sigma} > \lambda \right) \leq c \sqrt[u]{\frac{1}{\lambda^p}} \quad \text{for} \quad B \in \mathcal{F}_\sigma \quad \text{and} \quad \lambda > 0.$$

Choosing $\lambda_0 > 0$ such that $\theta := c \sqrt[u]{\frac{1}{\lambda_0^p}} \in \left(0, \frac{1}{2}\right)$ we get that

$$\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}(Q)} \leq \lambda_0 \, \|\cdot\|_{\mathrm{BMO}_p^{\Phi}(\mathbb{P})} .$$

Hence $\|A\|_{\mathrm{BMO}_{0,\theta}^{\Phi}(Q)} \leq \lambda_0$ and the definition of $\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}}$ implies that

$$Q \left(|A_T - A_{\sigma-}| > \nu \middle| \mathcal{F}_\sigma \right) \leq \theta + Q \left(\lambda_0 \sup_{u \in [\sigma, T]} \Phi_u > \nu \middle| \mathcal{F}_\sigma \right) \text{ a.s.}$$

for $\nu > 0$. So we can finish by Theorem 2.3 and $\Phi \in \mathcal{SM}_q(Q, d)$. $\square$

Corollary 2.6 *Let $\theta \in \left(0, \frac{1}{2}\right)$, $\eta \in (0,1)$, $p \in (0,\infty)$, and $\Phi \in \mathcal{SM}_p(\mathbb{P}, d)$.*

(i) *There is a constant $c = c(\theta, \eta, p, d) > 0$ such that*

$$\|\cdot\|_{\mathrm{BMO}^{\Phi}_{0,\theta}(\mathbb{P})} \sim_c \|\cdot\|_{\mathrm{BMO}^{\Phi,*}_{0,\eta}(\mathbb{P})} \sim_c \|\cdot\|_{\mathrm{BMO}^{\Phi}_{p}(\mathbb{P})} \sim_c \|\cdot\|_{\mathrm{BMO}^{\Phi,*}_{p}(\mathbb{P})}. \tag{2}$$

(ii) *Moreover, the finiteness of the quantities in* Formula (2) *for a given $A \in \mathcal{CL}_0(\mathcal{F})$ is equivalent to:*

(a) *There are constants $\delta, d > 0$ such that*

$$\mathbb{P}\left(|A_T - A_{\sigma-}| > \nu \middle| \mathcal{F}_\sigma\right) \leq \theta + \delta \mathbb{P}\left(\sup_{u\in[\sigma,T]} \Phi_u > \frac{\nu}{d} \middle| \mathcal{F}_\sigma\right) \quad a.s.$$

for all $\nu > 0$ and stopping times $\sigma : \Omega \to [0,T]$.

(b) *There are constants $\alpha, a > 0$ such that, a.s.,*

$$\mathbb{P}\left(\sup_{u\in[\sigma,T]} |A_u - A_{\sigma-}| > \mu\nu \middle| \mathcal{F}_\sigma\right) \leq e^{1-\mu} + \alpha \mathbb{P}\left(\sup_{u\in[\sigma,T]} \Phi_u > \frac{\nu}{a} \middle| \mathcal{F}_\sigma\right)$$

for all $\mu, \nu > 0$ and all stopping times $\sigma : \Omega \to [0,T]$.

Proof. During the proof we drop the dependence on $\mathbb{P}$ in the notation of the BMO-spaces. (i) One easily has that

$$\theta^{\frac{1}{p}} \|\cdot\|_{\mathrm{BMO}^{\Phi}_{0,\theta}} \leq \|\cdot\|_{\mathrm{BMO}^{\Phi}_{p}} \leq \|\cdot\|_{\mathrm{BMO}^{\Phi,*}_{p}} \quad \text{and} \quad \eta^{\frac{1}{p}} \|\cdot\|_{\mathrm{BMO}^{\Phi}_{0,\eta}} \leq \|\cdot\|_{\mathrm{BMO}^{\Phi,*}_{p}}.$$

On the other hand, as in the proof of Corollary 2.5, Theorem 2.3 implies that

$$\|\cdot\|_{\mathrm{BMO}^{\Phi,*}_{p}} \leq c_{(2.3)}\, d\, \|\cdot\|_{\mathrm{BMO}^{\Phi}_{0,\theta}}$$

with $c_{(2.3)} = c_{(2.3)}(p,\theta) > 0$. By Remark 2.4 one also gets that

$$\|\cdot\|_{\mathrm{BMO}^{\Phi,*}_{p}} \leq c'_{(2.3)}\, d\, \|\cdot\|_{\mathrm{BMO}^{\Phi,*}_{0,\eta}}$$

with some $c'_{(2.3)} = c'_{(2.3)}(p,\eta) > 0$.

(ii) The implication $(a) \Rightarrow (b)$ follows from Theorem 2.3. Knowing (b) we can again use Theorem 2.3 and $\Phi \in \mathcal{SM}_p(\mathbb{P}, d)$ to get that $\|A\|_{\mathrm{BMO}^{\Phi,*}_{p}} < \infty$. Finally, $\|A\|_{\mathrm{BMO}^{\Phi}_{0,\theta}} \leq d$ with $d > 0$ implies (a) with $\delta = 1$. □

Part (ii) of the Corollary above can be viewed as a John-Nirenberg-type theorem for the introduced BMO-spaces.

Remark 2.7 (i) Assume that there is a constant $c > 0$ such that

$$\|\cdot\|_{\mathrm{BMO}_p^{\Phi'}} \leq c\,\|\cdot\|_{\mathrm{BMO}_p^{\Phi}} \quad \text{and} \quad \Phi \in \mathcal{SM}_p(\mathbb{P}, d).$$

Then

$$\Phi_t \leq cd\,\Phi_t' \text{ a.s.} \quad \text{for all} \quad t \in [0,T].$$

Because of the continuity properties of Φ and Φ' we have to check $t \in (0,T]$ only. Define $A = (A_u)_{u\in[0,T]}$ by $A_u := 0$ if $u \in [0,t)$ and $A_u := \Phi_t$ if $u \in [t,T]$. Then $\|A\|_{\mathrm{BMO}_p^{\Phi}} \leq d$ so that $\|A\|_{\mathrm{BMO}_p^{\Phi'}} \leq cd$. But this gives $\Phi_t \leq cd\Phi_t'$ a.s.

(ii) A preliminary version of Corollary 2.6 in the discrete time setting was announced in the seminar notes [9]. Instead of using Theorem 2.3 as fundamental tool, [9] refers to a more abstract version of Theorem 2.3 presented without proof in the seminar notes [7]. Hence Corollary 2.6 (and henceforth its version in [9]) are proved here for the first time.

(iii) In [9] there is also indicated an example for the discrete time setting showing that $\Phi \in \mathcal{SM}_p(\mathbb{P},\cdot)$ cannot be replaced by $\Phi \in \mathcal{SM}_q(\mathbb{P},\cdot)$ for $0 < q < p$ in Corollary 2.6.

(iv) Finally, the reader is referred to [9] for comments about relations of Theorem 2.3 to classical results of Garsia and Stroock: letting $\theta \in \left(0,\frac{1}{2}\right)$, $p \in (1,\infty)$, $\Phi_T \in L_p(\Omega,\mathcal{F},\mathbb{P})$, $\Phi_t := \mathbb{E}\left(\Phi_T \mid \mathcal{F}_t\right)$, the inequality

$$\|\cdot\|_{\mathrm{BMO}_p^{\Psi,*}} \leq c\,\|\cdot\|_{\mathrm{BMO}_{0,\theta}^{\Phi}} \quad \text{with} \quad \Psi_t := \mathbb{E}\left(\Phi_T^p \mid \mathcal{F}_t\right)^{\frac{1}{p}},$$

which can be deduced from Theorem 2.3 and Doob's inequality, can be considered as continuous time refinement of [6] (Lemma III.5.1) (see [4] (par. 107 and 108, Chapter VI)).

3 Approximation error for discretizations of certain stochastic integrals

3.1 Setting and motivation

We consider a Black-Scholes option pricing model with time-horizon $T > 0$ after discounting under the martingale measure. This means, we take $S = (S_t)_{t\in[0,T]}$ with

$$S_t := \exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right), \quad \sigma > 0,$$

as price-process, where $W = (W_t)_{t\in[0,T]}$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $W_0 \equiv 0$. As filtration $(\mathcal{F}_t)_{t\in[0,T]}$ the augmentation of the natural filtration of W is used. For simplicity, we also let $\mathcal{F} = \mathcal{F}_T$. By rescaling of time we may set $\sigma = 1$ in the following. Assume an European contingent claim with pay-off $f(S_T)$, where $f : (0,\infty) \to \mathbb{R}$ is a Borel-function with $\mathbb{E}f(S_T)^2 < \infty$. To obtain the stochastic integral representation of $f(S_T)$ we find an $\varepsilon > 0$ (see for example [11]) such that

$$F(t,y) := \mathbb{E}f(yS_{T-t}) \quad \text{gives} \quad F \in C^\infty((-\varepsilon, T) \times (0,\infty)) \tag{3}$$

and

$$\frac{\partial F}{\partial t}(t,y) + \frac{y^2}{2}\frac{\partial^2 F}{\partial y^2}(t,y) = 0. \tag{4}$$

Itô's formula implies that

$$F(t, S_t) = \mathbb{E}f(S_T) + \int_{(0,t]} \frac{\partial F}{\partial y}(u, S_u)dS_u \quad \text{for} \quad t \in [0,T) \text{ a.s.}$$

Because of $\lim_{t\uparrow T} \|F(t,S_t) - f(S_T)\|_{L_2} = 0$, and with $\frac{\partial F}{\partial y}(T,y) := 0$ for $y > 0$, we end up with the desired representation

$$f(S_T) = \mathbb{E}f(S_T) + \int_{(0,T]} \frac{\partial F}{\partial y}(u, S_u)dS_u \text{ a.s.}$$

To formulate the approximation problem, we are interested in, we introduce the error-processes $C(\tau)$ and $C(\tau, v)$.

Definition 3.1 (i) *Letting $N \geq 1$ we define*

$$\mathcal{T} := \left\{ (t_i)_{i=0}^n \;\middle|\; 0 = t_0 < \cdots < t_n = T \right\} \quad \textit{and} \quad \mathcal{T}_N := \left\{ (t_i)_{i=0}^n \;\middle|\; n \leq N \right\}.$$

(ii) *For $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ we let $\|\tau\|_\infty := \sup_i |t_i - t_{i-1}|$ and*

$$\mathcal{P}(\tau) := \left\{ v = (v_i)_{i=0}^{n-1} \;\middle|\; v_i : \Omega \to \mathbb{R} \quad \mathcal{F}_{t_i}\textit{-measurable, } \mathbb{E}\left|v_i S_{t_i}\right|^2 < \infty \right\}.$$

(iii) *Given $\tau \in \mathcal{T}$ and $v \in \mathcal{P}(\tau)$, we define $C(\tau, v) = (C_t(\tau, v))_{t \in [0,T]}$ by*

$$C_t(\tau, v) := \mathbb{E}\left(f(S_T) | \mathcal{F}_t \right) - \mathbb{E} f(S_T) - \sum_{i=1}^n v_{t_{i-1}} \left(S_{t_i \wedge t} - S_{t_{i-1} \wedge t} \right),$$

where all paths of $C(\tau, v)$ are assumed to be continuous, $C_0(\cdot, \cdot) \equiv 0$, and

$$C(\tau) := C(\tau, v^0) \quad \textit{with} \quad v^0 := \left(\frac{\partial F}{\partial y}(t_i, S_{t_i}) \right)_{i=0}^{n-1}.$$

In the definition above we used that $\mathbb{E}(S_t \frac{\partial F}{\partial y}(t, S_t))^2 < \infty$ for $t \in [0, T)$ (see for example [11]). The definition of $C(\tau)$ can be rephrased to be

$$C_t(\tau) = \int_0^t \frac{\partial F}{\partial y}(u, S_u) dS_u - \sum_{i=1}^n \frac{\partial F}{\partial y}(t_{i-1}, S_{t_{i-1}}) \left(S_{t_i \wedge t} - S_{t_{i-1} \wedge t} \right)$$

which is the error-process of a simple approximation of a stochastic integral. There are two results we would like to start from. The first one is due to Zhang and can be formulated in our setting as follows:

Theorem 3.2 ([17]) *Let $K : [0, \infty) \to \mathbb{R}$ be a Borel-function and assume that $\sup_{x \geq 0}(1 + |x|)^{-m} |K(x)| < \infty$ for some $m \geq 1$. If for*

$$f(y) := \int_0^y K(x) dx \quad \textit{for} \quad y \geq 0$$

does not exist a $b \in \mathbb{R}$ with $f(S_T) = b S_T$ and if $\tau_n := (in/T)_{i=0}^n$, then there is a $c = c(K, T) > 0$ such that

$$\frac{1}{c\sqrt{n}} \leq \inf_{v \in \mathcal{P}(v)} \|C_T(\tau_n, v)\|_{L_2} \leq \|C_T(\tau_n)\|_{L_2} \leq \frac{c}{\sqrt{n}} \quad \textit{for} \quad n = 1, 2, ...$$

The second result, we want to mention, is

Theorem 3.3 ([11]) *There is a constant $c = c(T) > 0$ such that for all Borel-functions $f : (0,\infty) \to \mathbb{R}$ with $\mathbb{E} f(S_T)^2 < \infty$ one has*

$$\frac{1}{c} a(f;\tau) \;\le\; \inf_{v \in \mathcal{P}(v)} \|C_T(\tau, v)\|_{L_2} \;\le\; \|C_T(\tau)\|_{L_2} \;\le\; c\, a(f;\tau)$$

where $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ and

$$a(f;\tau)^2 := \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u) \mathbb{E} \left| S_u^2 \frac{\partial F^2}{\partial y^2}(u, S_u) \right|^2 du.$$

For example, applying Theorem 3.3 to $f(y) := (y - K_0)^+$, $K_0 > 0$, gives

$$\frac{1}{c} a(f;\tau) \;\le\; \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{t_i - u}{\sqrt{T-u}} du \right)^{\frac{1}{2}} \;\le\; c\, a(f;\tau), \tag{5}$$

where $c = c(T, K_0) > 0$ (see [11]). Formula (5) shows that there are time nets, quite different from the equidistant one, realizing the optimal rate $1/\sqrt{n}$ in Theorem 3.2. For instance, exploiting [11] (proof of Theorem 6.2) one could use

$$\tau_n^\varepsilon := \left(T - T \left(\frac{n-i}{n} \right)^\varepsilon \right)_{i=0}^n \quad \text{for} \quad \frac{2}{3} < \varepsilon < 1. \tag{6}$$

From the above, two problems naturally arises: firstly, is it possible to replace in Theorem 3.2 the L_2-norm by a stronger quantity? This is of particular interest if one wants to control uniformly the distribution of $C_T(\tau)$. Up to now, Theorem 3.2 only implies that

$$\mathbb{P}\left(|C_T(\tau_n)| > \frac{\lambda}{\sqrt{n}} \right) \le \left(\frac{c_{(3.2)}}{\lambda} \right)^2, \;\; \lambda > 0, \tag{7}$$

which is far from being optimal in general, as we shall see later. Secondly, it is not clear which of the time-nets, giving the order $1/\sqrt{n}$ in Formula (5), is the best one in a certain sense. Both problems are connected to each other and will be approached now.

3.2 The basic technical estimate

We provide the main technical upper estimate for the expected quadratic error in Theorem 3.4 below, which is used later on. For the rest of the paper it is assumed that $K : [0, \infty) \to \mathbb{R}$ is integrable over compact intervals,

$$f(y) := \int_0^y K(x)dx, \quad \text{and} \quad \Psi(y) := yK(y) \;\; \text{for} \;\; y \geq 0. \tag{8}$$

Theorem 3.4 *Assume that*

$$\mathbb{E}\left|\int_0^{S_T} |K(x)|dx\right|^2 + \mathbb{E}\Psi(S_T)^2 < \infty. \tag{9}$$

Given $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ *and* $a \in [t_{i-1}, t_i)$*, one has*

$$\begin{aligned}
&\mathbb{E}\left(|C_T(\tau) - C_a(\tau)|^2 \;\middle|\; \mathcal{F}_a\right) \\
&\quad \leq c^2 \, \|\tau\|_\infty \left[\mathbb{E}\left(\Psi(S_T)^2 \;\middle|\; \mathcal{F}_a\right) + \left(S_a/S_{t_{i-1}}\right)^2 \left(\mathbb{E}\left(\Psi(S_T) \;\middle|\; \mathcal{F}_{t_{i-1}}\right)\right)^2\right] \; a.s.
\end{aligned}$$

where $c > 0$ *depends on* T *only.*

Let us turn to the proof. Through the whole sub-section we assume that the conditions in Formulas (8) and (9) are satisfied.

Lemma 3.5 *Let* $0 \leq a < b \leq T$ *and let* $F \in C^{\infty,\infty}([0,T) \times (0,\infty))$ *be given as in* Formula (3)*. Then, for* $v_a \in \mathbb{R}$ *and* $(a, y_0) = (0, 1)$ *or* $(a, y_0) \in (0,T) \times (0, \infty)$*,*

$$\begin{aligned}
&\mathbb{E}\int_a^b \left(\frac{\partial F}{\partial y}(t, y_0 S_{t-a}) - v_a\right)^2 (y_0 S_{t-a})^2 dt \\
&\qquad \sim_{C(b-a)} \mathbb{E}\int_a^b \left(\left(\frac{\partial F}{\partial y}(a, y_0) - v_a\right)^2 y_0^2 + \right. \\
&\qquad\qquad \left. (b-u)\left((y_0 S_{u-a})^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 S_{u-a})\right)^2\right) du,
\end{aligned}$$

where $C : (0, \infty) \to [1, \infty)$ *is an absolute function which is increasing and satisfies* $\lim_{\delta \downarrow 0} C(\delta) = 1$.

Proof. We define $\widetilde{f}(y) := f(y_0 y)$ for $y > 0$ and $\widetilde{T} := T - a$. Applying [11] (Corollary 3.3) to $\widetilde{f}$, $\widetilde{T}$, and the time interval $[\widetilde{a}, \widetilde{b}] = [0, b-a]$ gives our assumption. $\square$

In the following we use the Gaussian measure

$$d\gamma(x) := (2\pi)^{-1/2} \exp\left(-x^2/2\right) dx$$

and, for $t \geq 0$ and $x \in \mathbb{R}$, the function

$$S_t(x) := e^{\sqrt{t}x - \frac{t}{2}}.$$

Lemma 3.6 *For $u \in [0, T)$ and $y = S_u(x)$, $x \in \mathbb{R}$, one has*

(i) $y \frac{\partial F}{\partial y}(u, y) = \int_{\mathbb{R}} \Psi \left(S_T \left(\sqrt{\frac{u}{T}} x + \sqrt{\frac{T-u}{T}} \eta \right) \right) d\gamma(\eta)$,

(ii) $y^2 \frac{\partial^2 F}{\partial y^2}(u, y) = \int_{\mathbb{R}} \Psi \left(S_T \left(\sqrt{\frac{u}{T}} x + \sqrt{\frac{T-u}{T}} \eta \right) \right) \left[\frac{\eta}{\sqrt{T-u}} - 1 \right] d\gamma(\eta)$.

Proof. For $u^* := T - u \in (0, T]$ we have (see for example [11] (Lemma A.2))

$$\begin{aligned} y \frac{\partial F}{\partial y}(u, y) &= \mathbb{E} f(y S_{u^*}) \frac{W_{u^*}}{u^*}, \\ y^2 \frac{\partial^2 F}{\partial y^2}(u, y) &= \mathbb{E} f(y S_{u^*}) \left(\frac{W_{u^*}^2}{u^{*2}} - \frac{W_{u^*}}{u^*} - \frac{1}{u^*} \right). \end{aligned}$$

We restrict ourself now to assertion (ii), the first one can be verified in the same way. We get

$$\begin{aligned} y^2 \frac{\partial^2 F}{\partial y^2}(u, y) &= \mathbb{E} f(y S_{u^*}) \left(\frac{W_{u^*}^2}{u^{*2}} - \frac{W_{u^*}}{u^*} - \frac{1}{u^*} \right) \\ &= \mathbb{E} \int_0^\infty \chi_{[\eta, \infty)}(y S_{u^*}) K(\eta) d\eta \left(\frac{W_{u^*}^2}{u^{*2}} - \frac{W_{u^*}}{u^*} - \frac{1}{u^*} \right) \\ &= \int_0^\infty K(\eta) \mathbb{E} \left[\chi_{[\eta, \infty)}(y S_{u^*}) \left(\frac{W_{u^*}^2}{u^{*2}} - \frac{W_{u^*}}{u^*} - \frac{1}{u^*} \right) \right] d\eta \end{aligned}$$

$$= -\int_{(0,\infty)} K(\eta) N' \left(\frac{\log(y/\eta) - u^*/2}{\sqrt{u^*}} \right) \left(1 + \frac{\log(y/\eta) - u^*/2}{u^*} \right) \frac{d\eta}{\sqrt{u^*}}$$

$$= -\int_{(0,\infty)} K(\eta) N' \left(\frac{\sqrt{u}x - \log\eta - T/2}{\sqrt{u^*}} \right) \left(1 + \frac{\sqrt{u}x - \log\eta - T/2}{u^*} \right) \frac{d\eta}{\sqrt{u^*}}.$$

Letting $\eta' := -\frac{\sqrt{u}x - \log\eta - T/2}{\sqrt{u^*}}$, assertion (ii) follows. □

Given $\sigma > 0$, we use the operator

$$A_\sigma : L_2(\mathbb{R}, \gamma) \to L_2\left((0,1) \times \mathbb{R}, \lambda \times \gamma\right),$$

where λ is the Lebesgue measure, given by

$$(A_\sigma \psi)(v,x) := \int_{\mathbb{R}} \psi\left(\sqrt{v}x + \sqrt{1-v}\eta\right) \left[\frac{\eta}{\sqrt{1-v}} - \sqrt{\sigma} \right] d\gamma(\eta).$$

The operator is related to the Ornstein-Uhlenbeck semi-group and satisfies

Lemma 3.7 $\|A_\sigma \psi\|_{L_2((0,1)\times\mathbb{R},\lambda\times\gamma)} \leq (1+\sqrt{\sigma}) \|\psi\|_{L_2(\mathbb{R},\gamma)}$.

Proof. Let $(h_m)_{m=0}^\infty$ be the orthonormal basis of Hermite polynomials in $L_2(\mathbb{R}, \gamma)$. It is easy to check that

$$(A_\sigma h_m)(u,x) = u^{\frac{m-1}{2}} \sqrt{m}\, h_{m-1}(x) - \sqrt{\sigma} u^{\frac{m}{2}} h_m(x)$$

for $m = 1, 2, \ldots$ and $(A_\sigma h_0)(u,x) = -\sqrt{\sigma} h_0(x)$ so that the claim follows. □

Lemma 3.8 *Let $0 \leq a \leq u < T$, $v := \frac{u-a}{T-a}$, $y = S_{u-a}(x)$, $x \in \mathbb{R}$, and $y_0 > 0$, where $y_0 = 1$ in case of $a = 0$. Then*

$$(y_0 y)^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 y) = \frac{(A_{T-a}\psi)(v,x)}{\sqrt{T-a}} \quad \textit{with} \quad \psi(x) := \Psi(y_0 S_{T-a}(x)).$$

Proof. We let $\widetilde{f}(y) := f(y_0 y)$ and $\widetilde{T} := T - a$. The corresponding functions $\widetilde{K}$ and $\widetilde{\Psi}$ compute as $\widetilde{K}(x) = y_0 K(y_0 x)$ and $\widetilde{\Psi}(x) = \Psi(y_0 x)$. Moreover, $\widetilde{F}(v,y) = F(a+v, y_0 y)$ for $(v,y) \in [0,\widetilde{T}) \times (0,\infty)$, so that

$$y_0^2 \frac{\partial^2 F}{\partial y^2}(a+v, y_0 y) = \frac{\partial^2 \widetilde{F}}{\partial y^2}(v,y).$$

Applying Lemma 3.6 gives, for $u \in [a,T)$ and $y > 0$,

$$\begin{aligned}
(y_0 y)^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 y) &= y^2 \frac{\partial^2 \widetilde{F}}{\partial y^2}(u-a, y) \\
&= \int_{\mathbb{R}} \widetilde{\Psi}\left(S_{\widetilde{T}} \left(\sqrt{\frac{u-a}{\widetilde{T}}} x + \sqrt{\frac{\widetilde{T}-(u-a)}{\widetilde{T}}} \eta \right) \right) \\
&\qquad \left[\frac{\eta}{\sqrt{\widetilde{T}-(u-a)}} - 1 \right] d\gamma(\eta) \\
&= \frac{1}{\sqrt{T-a}} \int_{\mathbb{R}} \Psi\left(y_0 S_{T-a} \left(\sqrt{\frac{u-a}{T-a}} x + \sqrt{\frac{T-u}{T-a}} \eta \right) \right) \\
&\qquad \left[\frac{\eta}{\sqrt{(T-u)/(T-a)}} - \sqrt{T-a} \right] d\gamma(\eta) \\
&= \frac{1}{\sqrt{T-a}} \int_{\mathbb{R}} \psi\left(\sqrt{v} x + \sqrt{1-v} \eta \right) \\
&\qquad \left[\frac{\eta}{\sqrt{1-v}} - \sqrt{T-a} \right] d\gamma(\eta).
\end{aligned}$$

□

Proof of Theorem 3.4. We recall that $t_{i-1} \le a < t_i$ for some $1 \le i \le n$. Let $s_{i-1} := a$ and $s_k := t_k$ for $i \le k \le n$. We obtain, a.s.,

$$\begin{aligned}
&|C_T(\tau) - C_a(\tau)| \\
&\quad = \left| \int_{(a,T]} \left(\varphi(u, S_u) - \sum_{k=1}^{n} \chi_{(t_{k-1},t_k]}(u) \frac{\partial F}{\partial y} \left(t_{k-1}, S_{t_{k-1}} \right) \right) dS_u \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\int_{(a,t_i]}\left(\frac{\partial F}{\partial y}(a,S_a)-\frac{\partial F}{\partial y}\left(t_{i-1},S_{t_{i-1}}\right)\right)dS_u\right| \\
&\qquad +\left|\int_{(a,t_i]}\left(\frac{\partial F}{\partial y}(u,S_u)-\frac{\partial F}{\partial y}(a,S_a)\right)dS_u\right. \\
&\qquad \left.+\sum_{k=i+1}^{n}\int_{(t_{k-1},t_k]}\left(\frac{\partial F}{\partial y}(u,S_u)-\frac{\partial F}{\partial y}\left(t_{k-1},S_{t_{k-1}}\right)\right)dS_u\right| \\
&\leq \left|\frac{\partial F}{\partial y}(a,S_a)-\frac{\partial F}{\partial y}\left(t_{i-1},S_{t_{i-1}}\right)\right|\left|S_{t_i}-S_a\right| \\
&\qquad +\left|\sum_{k=i}^{n}\int_{(s_{k-1},s_k]}\left(\frac{\partial F}{\partial y}(u,S_u)-\frac{\partial F}{\partial y}\left(s_{k-1},S_{s_{k-1}}\right)\right)dS_u\right|.
\end{aligned}
$$

Hence by Lemma 3.5, a.s.,

$$
\begin{aligned}
&\mathbb{E}\left(\left|C_T(\tau)-C_a(\tau)\right|^2 \;\middle|\; \mathcal{F}_a\right) \\
\leq\;& 2\mathbb{E}\left(\left|\frac{\partial F}{\partial y}(a,S_a)-\frac{\partial F}{\partial y}\left(t_{i-1},S_{t_{i-1}}\right)\right|^2\left|(S_{t_i}-S_a)\right|^2 \;\middle|\; \mathcal{F}_a\right) \\
&\qquad +2C_{(3.5)}(T)\mathbb{E}\left(\sum_{k=1}^{n}\int_{s_{k-1}}^{s_k}(s_k-u)\left[S_u^2\frac{\partial^2 F}{\partial y^2}(u,S_u)\right]^2 du \;\middle|\; \mathcal{F}_a\right) \\
\leq\;& 2\left[\overline{\mathbb{E}}\left|\overline{S}_{t_i-a}-1\right|^2\right]S_a^2\left[\frac{\partial F}{\partial y}(a,S_a)-\frac{\partial F}{\partial y}\left(t_{i-1},S_{t_{i-1}}\right)\right]^2 \\
&\qquad +2C_{(3.5)}(T)\left\|\tau\right\|_\infty\overline{\mathbb{E}}\left(\int_a^T\left[\left(S_a\overline{S}_{u-a}\right)^2\frac{\partial^2 F}{\partial y^2}(u,S_a\overline{S}_{u-a})\right]^2 du\right).
\end{aligned}
$$

With respect to the first term we get, a.s.,

$$
\begin{aligned}
&2\left[\overline{\mathbb{E}}\left|\overline{S}_{t_i-a}-1\right|^2\right]S_a^2\left[\frac{\partial F}{\partial y}(a,S_a)-\frac{\partial F}{\partial y}\left(t_{i-1},S_{t_{i-1}}\right)\right]^2 \\
\leq\;& 4e^T\left\|\tau\right\|_\infty\left[\left[S_a\frac{\partial F}{\partial y}(a,S_a)\right]^2+(S_a/S_{t_{i-1}})^2\left[S_{t_{i-1}}\frac{\partial F}{\partial y}\left(t_{i-1},S_{t_{i-1}}\right)\right]^2\right] \\
=\;& 4e^T\left\|\tau\right\|_\infty\left[\mathbb{E}\left(\Psi(S_T)\;\middle|\;\mathcal{F}_a\right)^2+(S_a/S_{t_{i-1}})^2\mathbb{E}\left(\Psi(S_T)\;\middle|\;\mathcal{F}_{t_{i-1}}\right)^2\right]
\end{aligned}
$$

where we have used Lemma 3.6 (i). Let us turn to the second term and fix $y_0 = S_a(\omega)$. We have that

$$
\begin{aligned}
& \overline{\mathbb{E}} \int_a^T \left[(y_0 \overline{S}_{u-a})^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 \overline{S}_{u-a}) \right]^2 du \\
= & \int_{\mathbb{R}} \int_a^T \left[(y_0 S_{u-a}(x))^2 \frac{\partial^2 F}{\partial y^2}(u, y_0 S_{u-a}(x)) \right]^2 du d\gamma(x) \\
= & \int_{\mathbb{R}} \int_a^T \left[\frac{1}{\sqrt{T-a}} \Big(A_{T-a} \Psi(y_0 S_{T-a}(\cdot)) \Big) \left(\frac{u-a}{T-a}, x \right) \right]^2 du d\gamma(x) \\
= & \int_{\mathbb{R}} \int_0^1 \left[\Big(A_{T-a} \Psi(y_0 S_{T-a}(\cdot)) \Big) (v, x) \right]^2 dv d\gamma(x) \\
\leq & \ \|A_{T-a}\|^2 \, \|\Psi(y_0 S_{T-a}(\cdot))\|_{L_2(\gamma)}^2 \\
\leq & \ (1+\sqrt{T})^2 \, \|\Psi(y_0 S_{T-a}(\cdot))\|_{L_2(\gamma)}^2
\end{aligned}
$$

where we have used Lemmas 3.8 and 3.7. Hence

$$
\overline{\mathbb{E}} \int_a^T \left[(S_a \overline{S}_{u-a})^2 \frac{\partial^2 F}{\partial y^2}(u, S_a \overline{S}_{u-a}) \right]^2 du \leq (1+\sqrt{T})^2 \overline{\mathbb{E}} \Psi(S_a \overline{S}_{T-a})^2.
$$

□

3.3 The case $K(S_T) \in L_\infty$

Our aim is to consider the approximation error with respect to the weighted BMO-spaces BMO_p^Φ. One problem consists in the variety of parameters $p \in (0, \infty)$ and processes Φ one can take. With respect to p we take $p = 2$, which comes naturally from the techniques we use (and may look then what other p lead to equivalent spaces according to the results of Section 2). With respect to the processes Φ our policy is as follows: given $f(S_T)$, we are searching for Φ such that, on the one hand-side,

$$
\inf_{\tau \in \mathcal{T}_n} \|C(\tau)\|_{\mathrm{BMO}_2^\Phi} \sim \frac{1}{\sqrt{n}},
$$

which is the optimal rate in case we would measure $\|C_T(\tau)\|_{L_2}$ (see [11]), and that on the other hand-side the tail distributions of Φ, for example

$\mathbb{P}\left(\sup_{t\in[0,T]}\Phi_t > \lambda\right)$, are small. Considering the example $f(y) := (K_0 - y)^+$, $K_0 > 0$, and taking $\tau = (0,T) \in \mathcal{T}$ gives that

$$C_T(\tau) = f(S_T) - \mathbb{E}f(S_T) - \frac{\partial F}{\partial y}(0,1)(S_T - 1)$$

with $\frac{\partial F}{\partial y}(0,1) \neq 0$, so that $C_T(\tau)$ cannot behave better than S_T in a certain distributional sense. This leads us to our first case $\Phi = S$.

Given $Q \in \mathcal{RH}_v(\mathbb{P}, c)$, it is easy to check that for all $q \in (0,\infty)$ there is some $d > 0$ such that $S \in \mathcal{SM}_q(\mathbb{P}, d) \cap \mathcal{SM}_q(Q, d)$. Given $p \in (0,\infty)$, Corollary 2.5 implies

$$\|\cdot\|_{\mathrm{BMO}_p^S(Q)} \sim_C \|\cdot\|_{\mathrm{BMO}_2^S(\mathbb{P})} \tag{10}$$

for all measures $Q \sim \mathbb{P}$ satisfying $Q \in \mathcal{RH}_v(\mathbb{P}, c)$ and $\mathbb{P} \in \mathcal{RH}_v(Q, c)$ for some $v \in (1,\infty)$ and $c > 0$, where $C = C(p, v, c) > 0$.

The following result should be compared with the equivalence proved for the L_2-error in Theorem 3.3.

Theorem 3.9 *Let* $\sup_{x\geq 0}|K(x)| < \infty$ *and assume that there is no* $b \in \mathbb{R}$ *such that* $f(S_T) = bS_T$*. Then there is a constant* $c = c(K,T) > 0$ *such that for all* $\tau \in \mathcal{T}$*:*

$$\frac{1}{c}\sqrt{\|\tau\|_\infty} \leq \inf_{v\in\mathcal{P}(\tau)} \|C(\tau, v)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \leq \|C(\tau)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \leq c\sqrt{\|\tau\|_\infty}.$$

It turns out that the asymptotically optimal time nets are unique: one has to take the equidistant nets in order to minimize $\inf_{\tau\in\mathcal{T}_n}\sqrt{\|\tau\|_\infty}$. In particular this holds for the pay-off function of the European Call Option $f(y) = (y - K_0)^+$ with $K_0 > 0$ and is in contrast to the situation in which we would measure $\|C_T(\tau)\|_{L_2}$ (see Formulas (5) and (6)). Moreover, this example shows nicely that $\|C_T(\tau)\|_{L_2(\mathbb{P})}$ and $\|C(\tau)\|_{\mathrm{BMO}_2^S(\mathbb{P})}$ behave differently, however taking the infimum over $\tau \in \mathcal{T}_n$ yields to the same rate $1/\sqrt{n}$, since

$$\inf_{v\in\mathcal{P}(\tau)} \|C_T(\tau, v)\|_{L_2(\mathbb{P})} \sim \|C_T(\tau)\|_{L_2(\mathbb{P})} \sim \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{t_i - u}{\sqrt{T-u}}du\right)^{\frac{1}{2}}$$

according to Formula (5), but

$$\inf_{v\in\mathcal{P}(\tau)} \|C(\tau,v)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \sim \|C(\tau)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \sim \sqrt{\max_{i=1,\dots,n} |t_i - t_{i-1}|}.$$

Finally, as remarked by Formula (10), with respect to the BMO-estimates we are allowed to change the measure in a moderate way.

Remark 3.10 Originally, the paper was only considering the pay-off of the European Call Option $f(y) = (y - K_0)^+$, $K_0 > 0$, which can be proved in a much more direct way. As an application of the paper in this earlier form, [10] was showing that: given one knows the estimate for the European Call Option, one can deduce estimates when K is non-negative, bounded, and *monotone*. However, this argument cannot be applied to Theorem 3.9 in the meanwhile general form, since K is not assumed to be monotone.

Proof of Theorem 3.9. (a) To get the upper estimate for $\|C(\tau)\|_{\mathrm{BMO}_2^S(\mathbb{P})}$, we remark that Theorem 3.4 implies that

$$\mathbb{E}\left(|C_T(\tau) - C_a(\tau)|^2 \;\middle|\; \mathcal{F}_a\right) \le c_{(3.4)}^2 \|\tau\|_\infty \sup_{x\ge 0} |K(x)|^2 (1 + \mathbb{E}S_T^2) S_a^2 \text{ a.s.}$$

and that we can replace in this inequality a by a stopping time $\sigma : \Omega \to [0,T]$.

(b) To get the lower bound for $\|C(\tau,v)\|_{\mathrm{BMO}_2^S(\mathbb{P})}$, we fix $i_0 \in \{1,\dots,n\}$ and let $a \in (t_{i_0-1}, t_{i_0})$. For $B \in \mathcal{F}_a$ of positive measure we obtain, by Lemma 3.5,

$$\begin{aligned}
&\int_B |C_T(\tau,v) - C_a(\tau,v)|^2 \, d\mathbb{P}_B \\
&\ge \int_B \int_a^{t_{i_0}} \left(\frac{\partial F}{\partial y}(u,S_u) - v_{t_{i_0-1}}\right)^2 S_u^2 du d\mathbb{P}_B \\
&= \int_B \overline{\mathbb{E}} \int_a^{t_{i_0}} \left(\frac{\partial F}{\partial y}(u,\overline{S}_{u-a}S_a) - v_{t_{i_0-1}}\right)^2 \left(\overline{S}_{u-a}S_a\right)^2 du d\mathbb{P}_B \\
&\ge \frac{1}{c_{(3.5)}(T)} \int_B (t_{i_0} - a) \left(\frac{\partial F}{\partial y}(a,S_a) - v_{t_{i_0-1}}\right)^2 S_a^2 d\mathbb{P}_B.
\end{aligned}$$

Hence

$$\|C(\tau,v)\|_{\mathrm{BMO}_2^S(\mathbb{P})}^2 \ge \frac{1}{c_{(3.5)}(T)} (t_{i_0} - a) \left(\frac{\partial F}{\partial y}(a,S_a) - v_{t_{i_0-1}}\right)^2 \text{ a.s.}$$

and

$$\|C(\tau,v)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \;\geq\; \frac{1}{2\sqrt{c_{(3.5)}(T)}}\sqrt{t_{i_0}-a}\sup_{y_1,y_2>0}\left|\frac{\partial F}{\partial y}(a,y_1)-\frac{\partial F}{\partial y}(a,y_2)\right|. \tag{11}$$

In order to prove our lower estimate, it is sufficient to prove that

$$\inf_{t\in(0,T)}\sup_{y_1,y_2>0}\left|\frac{\partial F}{\partial y}(t,y_1)-\frac{\partial F}{\partial y}(t,y_2)\right| \geq \varepsilon_0 > 0, \tag{12}$$

because this would imply that

$$\|C(\tau,v)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \;\geq\; \frac{\varepsilon_0}{2\sqrt{c_{(3.5)}(T)}}\sqrt{\|\tau\|_\infty}.$$

To check Formula (12) let us fix $t\in(0,T)$. By Lemma 3.6 we get

$$\begin{aligned}
&\sup_{y_1,y_2>0}\left|\frac{\partial F}{\partial y}(t,y_1)-\frac{\partial F}{\partial y}(t,y_2)\right| \\
&= \sup_{y_1,y_2>0}\left(\frac{\partial F}{\partial y}(t,y_1)-\frac{\partial F}{\partial y}(t,y_2)\right) \\
&= \sup_{y_1,y_2>0}\mathbb{E}S_{T-t}\left(K(y_1S_{T-t})-K(y_2S_{T-t})\right) \\
&= \overline{\mathbb{E}}\,\overline{S}_t\sup_{y_1,y_2>0}\mathbb{E}S_{T-t}\left(K(y_1\overline{S}_tS_{T-t})-K(y_2\overline{S}_tS_{T-t})\right) \\
&\geq \sup_{y_1,y_2>0}\overline{\mathbb{E}}\,\overline{S}_t\mathbb{E}S_{T-t}\left(K(y_1\overline{S}_tS_{T-t})-K(y_2\overline{S}_tS_{T-t})\right) \\
&= \sup_{y_1,y_2>0}\mathbb{E}S_T\left(K(y_1S_T)-K(y_2S_T)\right).
\end{aligned}$$

So it remains to verify that

$$\sup_{y_1,y_2>0}\mathbb{E}S_T\left(K(y_1S_T)-K(y_2S_T)\right) > 0. \tag{13}$$

Assuming the contrary implies

$$\mathbb{E}\Psi(yS_T) = y\mathbb{E}S_TK(yS_T) = ay \tag{14}$$

for some $a\in\mathbb{R}$. Let $h(x) := \Psi\left(e^{\sqrt{T}x-\frac{T}{2}}\right) = \sum_{m=0}^{\infty}\alpha_m h_m(x)$ in $L_2(\gamma)$ so that, for $x\in\mathbb{R}$,

$$ae^{\sqrt{T}x} \;=\; \mathbb{E}\Psi\left(e^{\sqrt{T}x}S_T\right)$$

$$
\begin{aligned}
&= \mathbb{E}h\left(x+W_1\right) \\
&= \int_{\mathbb{R}} h(\xi) e^{-\frac{(x-\xi)^2}{2}} \frac{d\xi}{\sqrt{2\pi}} \\
&= \int_{\mathbb{R}} h(\xi) e^{x\xi-\frac{x^2}{2}} e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}} \\
&= \int_{\mathbb{R}} h(\xi) \left(\sum_{m=0}^{\infty} \frac{x^m}{\sqrt{m!}} h_m(\xi) \right) e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}} \\
&= \sum_{m=0}^{\infty} \alpha_m \frac{x^m}{\sqrt{m!}}.
\end{aligned}
$$

Hence $\alpha_m = a\frac{T^{\frac{m}{2}}}{\sqrt{m!}}$ and

$$
h(x) = a \sum_{m=0}^{\infty} \frac{T^{\frac{m}{2}}}{\sqrt{m!}} h_m(x) = a e^{\sqrt{T}x - \frac{T}{2}}
$$

in $L_2(\gamma)$. This implies that $\Psi(y) = ay$ a.s. on $(0,\infty)$ and that K is a.s. constant which is a contradiction to our assumption. Hence Formula (14) cannot be true and we get Formula (13). $\square$

The next theorem shows that the boundedness of K is necessary.

Theorem 3.11 *Assume that* Formula (9) *is satisfied and assume that there is some $c > 0$ such that*

$$
\inf_{v \in \mathcal{P}(\tau)} \|C(\tau, v)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \; \le \; c\sqrt{\|\tau\|_\infty}
$$

for all $\tau \in \mathcal{T}$. Then $\|K(S_T)\|_{L_\infty(\mathbb{P})} < \infty$.

Proof. Assuming that $a \in (t_{i_0-1}, t_{i_0})$, $i_0 \in \{1, ..., n\}$ we get from Formula (11) and our assumption that

$$
\begin{aligned}
c\sqrt{\|\tau\|_\infty} \; &\ge \; \inf_{v \in \mathcal{P}(\tau)} \|C\left(\tau, v\right)\|_{\mathrm{BMO}_2^S(\mathbb{P})} \\
&\ge \; \frac{1}{2\sqrt{c_{(3.5)}(T)}} \sqrt{t_{i_0} - a} \sup_{y_1, y_2 > 0} \left| \frac{\partial F}{\partial y}(a, y_1) - \frac{\partial F}{\partial y}(a, y_2) \right|.
\end{aligned}
$$

By arranging the time nets in an appropriate way and by Lemma 3.6 we may conclude

$$c \;\geq\; \frac{1}{2\sqrt{c_{(3.5)}(T)}} \sup_{y_1,y_2>0} \left| \frac{\partial F}{\partial y}(a,y_1) - \frac{\partial F}{\partial y}(a,y_2) \right|$$

and

$$\sup_{a\in(0,T)} \sup_{y>0} |\mathbb{E} S_{T-a} K(y S_{T-a})| = \sup_{a\in(0,T)} \sup_{y>0} \left| \frac{\partial F}{\partial y}(a,y) \right| \;\leq\; 4c\sqrt{c_{(3.5)}(T)}.$$

With $d := 4c\sqrt{c_{(3.5)}(T)}$ this gives

$$\sup_{a\in(0,T)} |\mathbb{E}\Psi(y S_{T-a})| \;\leq\; d\, y$$

for all $y > 0$. Letting

$$G(a,y) := \mathbb{E}\Psi(y S_{T-a})$$

for $a \in [0,T)$ and $y > 0$ we get an L_2-martingale $(G(a,S_a))_{a\in[0,T)}$ with

$$\lim_{a\uparrow T} G(a,S_a) = \Psi(S_T) \text{ a.s.}$$

and in L_2 so that $|\Psi(y)| \leq dy$ a.s. because of $|G(a,y)| \leq dy$ on $(0,T)\times(0,\infty)$. Hence $|K(y)| \leq d$ for almost all $y \geq 0$. □

3.4 The case $K(S_T) \in L_q$

In this section we show that we can extend to error estimates from Theorem 3.2 from L_2 to L_p in Theorem 3.12 and that we can improve Formula (7) by Theorem 3.14. We shall only work under the measure $\mathbb{P}$. Letting $E \in \{L_p, \mathrm{BMO}_2^\Phi\}$ we define

$$\begin{aligned} a_n^{\mathrm{sim}}(f(S_T) \,|\, E) &:= \inf_{\tau\in\mathcal{T}_n} \|C_T(\tau)\|_E, \\ a_n^{\mathrm{opt}}(f(S_T) \,|\, E) &:= \inf_{\tau\in\mathcal{T}_n} \inf_{v\in\mathcal{P}(\tau)} \|C_T(\tau,v)\|_E \end{aligned}$$

with the convention $\|(X_t)_{t\in[0,T]}\|_{L_p} := \|X_T\|_{L_p}$.

Theorem 3.12 *Let $2 \le p < r < q < \infty$,*

$$\mathbb{E}\left[\left|\int_0^{S_T} |K(x)|dx\right|^2 + |K(S_T)|^q\right] < \infty,$$

and

$$\Phi_a := S_a \sup_{u \le a} \left[\mathbb{E}\left(|K(S_T)|^r \mid \mathcal{F}_u\right)\right]^{\frac{1}{r}} \quad \textit{for} \quad a \in [0,T].$$

Assume that there is no $b \in \mathbb{R}$ such that $f(S_T) = b\, S_T$ and assume that $E \in \left\{L_p, \mathrm{BMO}_2^\Phi\right\}$. Then one has that

$$\frac{1}{c\sqrt{n}} \;\le\; a_n^{\mathrm{opt}}(f(S_T) \mid E) \;\le\; a_n^{\mathrm{sim}}(f(S_T) \mid E) \;\le\; \frac{c}{\sqrt{n}}$$

for $n = 1, 2, \ldots$, where $c > 0$ depends at most on (K, T, p, r, q). In order to obtain the asymptotic optimal approximation rate $1/\sqrt{n}$, equidistant nets can be used.

Proof. For $\Psi(y) = yK(y)$ Hölder's inequality gives

$$\mathbb{E}\left(\Psi(S_T)^2 \mid \mathcal{F}_a\right) + \left(S_a/S_{t_{i-1}}\right)^2 \left(\mathbb{E}\left(\Psi(S_T) \mid \mathcal{F}_{t_{i-1}}\right)\right)^2 \le c_1^2 \Phi_a^2 \text{ a.s.},$$

where $c_1 = c_1(T, r) > 0$, so that by Theorem 3.4 we get

$$\mathbb{E}\left(|C_T(\tau) - C_a(\tau)|^2 \mid \mathcal{F}_a\right) \;\le\; c_{(3.4)}^2 \, \|\tau\|_\infty \, c_1^2 \Phi_a^2.$$

Again we can replace in this inequality a by a stopping time $\sigma : \Omega \to [0,T]$ and deduce that

$$a_n^{\mathrm{sim}}(f(S_T) \mid \mathrm{BMO}_2^\Phi) \;\le\; c_{(3.4)} c_1 \frac{\sqrt{T}}{\sqrt{n}}.$$

On the other hand, since there are no $a, b \in \mathbb{R}$ such that $f(S_T) = a + b\, S_T$ a.s. [11] (Theorem 4.6, Lemma A.3) implies that $u \to \mathbb{E}\left|S_u^2 (\partial^2 F/\partial y^2)(u, S_u)\right|^2$ is a continuous function on $[0,T)$ which is not identically zero, so that Theorem 3.3 gives that

$$\frac{1}{c_2\sqrt{n}} \;\le\; a_n^{\mathrm{opt}}(f(S_T) \mid L_2)$$

for some $c_2 = c_2(f,T) > 0$ (in [11] it was assumed that $f(y) \geq 0$ which is irrelevant in this respect). To conclude the proof we have to check that

$$\|A_T\|_{L_p} \;\leq\; c_3 \,\|A\|_{\mathrm{BMO}_2^\Phi}$$

for $A \in \mathcal{CL}_0(F)$ with $c_3 = c_3(K,T,p,r,q) > 0$ following from $\|\Phi_T^*\|_{L_p} < \infty$ and Theorem 2.3. $\square$

Remark 3.13 *Letting* $2 < q < \infty$, $T = 1$, *and*

$$f(y) := \begin{cases} e^{\frac{1}{2q}\left(\log y + \frac{1}{2}\right)^2} & : \; y \geq e^{-\frac{1}{2}} \\ 1 & : \; y < e^{-\frac{1}{2}} \end{cases}$$

yields to $K(S_1) \in L_q$, $f(S_1) \in L_2$, *but* $f(S_1) \notin L_q$. *The latter implies for instance that* $a_1^{\mathrm{opt}}(f(S_1) \mid L_q) = \infty$. *Hence in Theorem 3.12 one cannot take* $p = q$.

Theorem 3.14 *Let* $\alpha \geq 0$ *and* $c > 0$. *Assume that* $|K(x)| \leq c[1 + |x|^\alpha]$ *for* $x \geq 0$. *Then*

$$\limsup_{\lambda\to\infty,\lambda\geq e} \frac{\log\left[\sup_\tau \mathbb{P}\left(C_T^*(\tau) > \|\tau\|_\infty^{\frac{1}{2}}\,\lambda\right)\right]}{[\log\lambda]^2} \leq -\frac{1}{2T(\alpha+1)^2}.$$

For all $\alpha \geq 0$ *there are* K *such that one has equality.*

Proof. Let Φ be defined as in Theorem 3.12. We get that

$$\Phi_a^* \leq c_1 (S_a^*)^{\alpha+1} \;\text{ a.s.}$$

for $a \in [0,T]$ and some $c_1 = c_1(T,r,c,\alpha) > 0$. Theorem 3.4 gives that

$$\mathbb{E}\left(|C_T(\tau) - C_\sigma(\tau)|^2 \;\middle|\; \mathcal{F}_\sigma\right) \;\leq\; c_{(3.4)}^2 \;\|\tau\|_\infty \, c_2^2 (S_a^*)^{2(\alpha+1)} \;\text{ a.s.}$$

for all stopping times $\sigma : \Omega \to [0,T]$ and some $c_2 = c_2(T,r,c,\alpha) > 0$. Using Theorem 2.3 with

$$\Psi_a := \sup_{u\in[a,T]} (S_u^*)^{\alpha+1}$$

implies that

$$\begin{aligned} \mathbb{P}\left(C_T^*(\tau) > \|\tau\|_\infty^{\frac{1}{2}} \mu\nu\right) &\le e^{1-\mu} + \beta \mathbb{P}\left(\Psi_0 > \frac{\nu}{b}\right) \\ &= e^{1-\mu} + \beta \mathbb{P}\left((S_T^*)^{\alpha+1} > \frac{\nu}{b}\right) \end{aligned}$$

for all $\mu, \nu > 0$ with some $\beta, b > 0$ depending at most on (T, r, c, α). Finally, given $\varepsilon \in (0,1)$ we find $\lambda_0, \gamma > 0$ depending at most on (β, b, T, α) such that

$$\inf_{\lambda=\mu\nu, \mu>0, \nu>0} \left\{ e^{1-\mu} + \beta \mathbb{P}\left((S_T^*)^{\alpha+1} > \frac{\nu}{b}\right)\right\} \le \gamma \mathbb{P}\left((S_T^*)^{\alpha+1} > \lambda^{1-\varepsilon}\right)$$

for $\lambda \ge \lambda_0$ so that

$$\mathbb{P}\left(C_T^*(\tau) > \|\tau\|_\infty^{\frac{1}{2}} \lambda\right) \le \gamma \mathbb{P}\left((S_T^*)^{\alpha+1} > \lambda^{1-\varepsilon}\right)$$

for $\lambda \ge \lambda_0$ and the upper estimate follows by a standard computation. The examples for equality are given by $f(y) = (K_0 - y)^+$ with $K_0 > 0$ for $\alpha = 0$ and $f(y) = y^{\alpha+1}$ for $\alpha > 0$. $\square$

Remark 3.15 There are limit theorems for the error process $C_T(\tau)$ of a different nature: in [15] one considers limits of type $\lim_{n\to\infty} \sqrt{n} C_T(\tau_n)$ for $\tau_n := (iT/n)_{i=0}^n$ in *distribution*. Our problem is different: We are going to improve Theorem 3.2 form L_2 to L_p and even to stronger spaces as shown in Theorem 3.14. So, in a sense, we are working with *uniform* distributions holding for any time-net, and not with limit distributions.

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