Characterizations of uniform approximation rates for a class of stochastic integrals

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Abstract

The approximation of stochastic integrals appears in Stochastic Finance while replacing continuously adjusted portfolios by discretely adjusted ones. We describe uniform approximation properties of stochastic integrals of this type by asymptotic properties of its integrands and by the real interpolation method with parameters (θ, ∞). Moreover, connections to the Malliavin Sobolev spaces $D_{1,2}$ and the corresponding Besov type spaces are given.

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Introduction

The approximation of stochastic integrals has a natural interpretation in Stochastic Finance: one replaces a continuously adjusted portfolio by a discretely adjusted one. To do this optimal in the quadratic mean sense, one has to consider an L_2-approximation problem, in which a stochastic integral (the pay-off of the continuously adjusted portfolio) is approximated by an integral over an integrand, piece-wise constant in certain time intervals. Zhang [13] considered this approximation with equidistant time nets in case the pay-off $g(Y_T)$, $(Y)_{t\in[0,T]}$ is the discounted price process, is formed by an absolutely continuous function g. Gobet and Temam [7] extended this to some irregular pay-offs including the Binary-option with pay-off $g(Y_T) = \chi_{[K,\infty)}(Y_T)$, $K > 0$. On the other hand, in [6] the L_2-approximation is considered by means of deterministic time-nets, which are not necessarily equidistant. An approximation with respect to weighted bounded mean oscillation instead of L_2, which gives all L_p-estimates, is described in [5].

In the above papers, except in [6], uniform error-estimates of type

$$\left\| [g(Y_T) - \mathbb{E}g(Y_T)] - \sum_{i=1}^{n} v_{t_{i-1}}(Y_{t_i} - Y_{t_{i-1}}) \right\|_{L_2} \leq c \max_i |t_i - t_{i-1}|^{\eta}, \qquad (1)$$

where the v_{t_i} are certain $\mathcal{F}_{t_i}$-measurable random variables, play a central role: for $t_i := \frac{i}{n}T$ Zhang [13] obtains $\eta = \frac{1}{2}$ in his situation and Gobet and Temam [7] give examples for $\frac{1}{4} \leq \eta < \frac{1}{2}$.

The present paper relates the exponent η from Formula (1) to equivalent conditions on the stochastic integral representation of $g(Y_T)$, to the Sobolev spaces $D_{1,2}$ in the Malliavin sense, and to the corresponding Besov type spaces $[D_{1,2}, L_2]_{\theta,\infty}$. The results can be found in Section 2 and are proved in Section 3. Examples are given in Section 4.

1 Preliminaries

Let us start with some notation. Given an open non-empty set $A \subseteq \mathbb{R}^n$, we let $C^\infty(A)$ be the space of all $h : A \to \mathbb{R}$ such that the partial derivatives ∂h of all orders (also mixed) exist on A. By $C_b^\infty(A)$ we denote the subspace of $h \in C^\infty(A)$ such that $\sup_{x\in A} |\partial h(x)| < \infty$ for all ∂. Moreover, $C^\infty([0,T) \times \mathbb{R})$ consists of those $h \in C^\infty((0,T) \times (0,\infty))$ such that all ∂h

can be continuously extended to $[0,T) \times \mathbb{R}$. The space $C^\infty([0,T) \times (0,\infty))$ is defined analogously.

In this paper we assume a standard Brownian motion $W = (W_t)_{t\in[0,T]}$ with $W_0 \equiv 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From W we generate the right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ such that $\mathcal{F}_0$ contains all null-sets of $\mathcal{F}$, where we also assume that $\mathcal{F} = \mathcal{F}_T$. We let $\sigma : (0,\infty) \to \mathbb{R} \in C^\infty((0,\infty))$ and $\widehat{b}, \widehat{\sigma} : \mathbb{R} \to \mathbb{R}$ be functions with

$$0 < \varepsilon_0 \le \widehat{\sigma}(x) := \frac{\sigma(e^x)}{e^x} \in C_b^\infty(\mathbb{R}) \quad \text{and} \quad \widehat{b}(x) := -\frac{1}{2}\widehat{\sigma}(x)^2. \tag{2}$$

The process $X = (X_t)_{t\in[0,T]}$ is the unique strong solution of

$$dX_t = \widehat{\sigma}(X_t)dW_t + \widehat{b}(X_t)dt \tag{3}$$

with $X_0 \equiv x_0 \in \mathbb{R}$, where we assume that all paths are continuous. Letting

$$Y_t := e^{X_t} \quad \text{and} \quad y_0 := e^{x_0},$$

we get

$$dY_t = \sigma(Y_t)dW_t \quad \text{with} \quad Y_0 \equiv y_0 > 0.$$

The condition $\widehat{\sigma} \in C_b^\infty(\mathbb{R})$ generates a behavior of $Y = (Y_t)_{t\in[0,T]}$ similar to that of the geometric Brownian motion. The process Y is generated via the process X since later we shall exploit the transition density of X. Our condition on $\widehat{\sigma}$ implies that there are $\kappa, \kappa' \in (0,\infty)$ with

$$|\sigma(y)\sigma''(y)| \le \kappa \quad \text{and} \quad |\sigma'(y)| \le \kappa'.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function such that condition (i) of Definition 1.1 below is satisfied and fix a transition probability Γ of $X = (X_t)_{t\in[0,T]}$ like in Theorem 5.1 below. Define

$$F(t,x) := \int_{\mathbb{R}} \Gamma(T-t,x,\xi)f(\xi)d\xi \quad \text{for} \quad (t,x) \in [0,T) \times \mathbb{R} \tag{4}$$

so that $F \in C^\infty([0,T) \times \mathbb{R})$ and

$$\frac{\partial F}{\partial t} + \frac{\widehat{\sigma}^2}{2}\frac{\partial^2 F}{\partial x^2} + \widehat{b}\frac{\partial F}{\partial x} = 0 \quad \text{on} \quad [0,T) \times \mathbb{R}. \tag{5}$$

The transformation into the exponential setting is done by $g : (0, \infty) \to \mathbb{R}$ and $G \in C^{\infty}([0,T) \times (0,\infty))$ with

$$g(y) := f(\log y) \quad \text{and} \quad G(t,y) := F(t, \log y) \tag{6}$$

so that

$$\frac{\partial G}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 G}{\partial y^2} = 0 \quad \text{and} \quad g(Y_T) = \mathbb{E} g(Y_T) + \int_0^T \frac{\partial G}{\partial y}(u, Y_u) dY_u \ \text{a.s.} \tag{7}$$

by Itô's formula (where we may set $(\partial G / \partial y)(T, y) := 0$). The set of terminal value functions f, we finally are going to use, is given by

Definition 1.1 *Let $\mathcal{C}_\gamma$ be the set of all Borel measurable $f : \mathbb{R} \to \mathbb{R}$ such that the following is satisfied:*

(i) *There is some $m \in \{0, 1, 2, ...\}$ such that for all $t > 0$ one has*

$$\sup_{x \in \mathbb{R}} (1 + |x|^m)^{-1} (|f|^2 * \gamma_t)(x) < \infty \quad \textit{with} \quad \gamma_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

(ii) $\sup_{t \in [0,T)} \mathbb{E} \left| \int_{y_0}^{Y_t} \sigma''(\eta) G(t, \eta) d\eta \right|^2 < \infty.$

Remark 1.2 In Definition 1.1 part (i) is the essential part. Part (ii) is a convenient technical condition for us. It could be of interest to check to what extend it can be weakened or even removed. Relevant situations, in whose condition (ii) is satisfied, are the following.

(i) One has $\sigma(\eta) = \eta$ so that $\sigma''(\eta) = 0$.

(ii) In case, no additional assumptions on σ are imposed: the map $f : \mathbb{R} \to \mathbb{R}$ is locally integrable and

$$\mathbb{E}(If)(x + W_t)^2 < \infty \quad \text{with} \quad (If)(\xi) := \int_{x_0}^{\xi} |f(\eta)| d\eta$$

for all $t > 0$ and $x \in \mathbb{R}$. The proof can be found in Lemma 5.2 below. In particular, this condition is satisfied, if for all $\beta > 0$ there is some $\alpha > 0$ such that $|f(x)| \leq \alpha e^{\beta x^2}$ for all $x \in \mathbb{R}$.

Next, motivated by Formula (1), we define classes A_η^Y of random variables having the same approximation rate η with respect to the process Y.

Definition 1.3 (i) *The set of deterministic time nets* $\tau = (t_i)_{i=0}^n$ *with* $0 = t_0 < t_1 < \cdots < t_n = T$ *is denoted by* $\mathcal{T}$ *and equipped with* $\|\tau\|_\infty := \max_{0<i\le n} |t_i - t_{i-1}|$.

(ii) *Given* $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\tau = (t_i)_{i=0}^n \in \mathcal{T}$ *we let*

$$a^{\mathrm{opt}}(Z;\tau) := \inf \left\| Z - \mathbb{E}Z - \sum_{i=1}^n v_{i-1}\left(Y_{t_i} - Y_{t_{i-1}}\right) \right\|_{L_2}$$

where the infimum is taken over all $\mathcal{F}_{t_i}$*-measurable step-functions* v_i.

(iii) *Let* A_η^Y, $\eta \in \left[0, \frac{1}{2}\right]$ *be the space of all* $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ *such that there exists a* $c > 0$ *such that for all* $\tau \in \mathcal{T}$ *one has*

$$a^{\mathrm{opt}}(Z;\tau) \le c\|\tau\|_\infty^\eta.$$

We let $|Z|_{A_\eta^Y} := \inf c$*, where the infimum is taken over all* $c > 0$ *such that the above condition is satisfied. The space* A_η^Y *will be equipped with the norm*

$$\|Z\|_{A_\eta^Y} := \|Z\|_{L_2} + |Z|_{A_\eta^Y}.$$

Remark 1.4 In case $Z = g(Y_T)$ with $g(y) = f(\log y)$, $f \in C_\gamma$, we have

$$a^{\mathrm{opt}}(g(Y_T);\tau) \sim_c \inf \left\| Z - \mathbb{E}Z - \sum_{i=1}^n \frac{\partial G}{\partial y}(t_{i-1}, Y_{t_{i-1}})\left(Y_{t_i} - Y_{t_{i-1}}\right) \right\|_{L_2} \tag{8}$$

with $c = c(\kappa, \kappa', T)$ according to Lemma 3.3 below. So, if one would restrict all considerations to random variables $g(Y_T)$ the right-hand side expression of Formula (8) could replace $a^{\mathrm{opt}}(\cdot,\tau)$. However, since we consider the general interpolation space $[A_{\frac{1}{2}}^Y, L_2]_{1-2\eta,\infty}$ in Theorem 2.1 below we have to use $a^{\mathrm{opt}}(\cdot,\tau)$ as defined in Definition 1.3.

Later the approximation spaces A_η^Y are described by interpolation spaces whose definition we recall now.

Definition 1.5 *Let* X_0 *and* X_1 *be Banach spaces embedded in some Hausdorff topological vector space.*

(i) *For $x \in X_0 + X_1$ and $\lambda \geq 0$ the K-functional is given by*

$$K(x,\lambda;X_0,X_1) := \inf\left\{\|x_0\|_{X_0} + \lambda\|x_1\|_{X_1} \;\middle|\; x = x_0 + x_1\right\}.$$

(ii) *Given $\theta \in (0,1)$, we let $[X_0,X_1]_{\theta,\infty}$ be the space of all $x \in X_0 + X_1$ such that*

$$\|x\|_{[X_0,X_1]_{\theta,\infty}} := \sup_{\lambda>0} \lambda^{-\theta} K(x,\lambda;X_0,X_1) < \infty.$$

It follows from the definition that, for $x \in X_1$ and $\lambda_0 \in [0,\infty)$, one has that $x \in [X_0,X_1]_{\theta,\infty}$ if and only if $\sup_{\lambda\geq\lambda_0} \lambda^{-\theta} K(x,\lambda;X_0,X_1) < \infty$. This is the situation we are later in. The method described in Definition 1.5 (ii) is called *real interpolation method* with parameters (θ,∞). For more information the reader is referred to [2] or [1].

Finally, we shortly want to recall the definition of the Malliavin Sobolev spaces $D_{1,2}$. Let $\mathcal{H} = L_2[0,T]$ and consider a centered Gaussian family $(W(h))_{h\in\mathcal{H}}$ with covariance structure $\mathbb{E}W(g)W(h) = \langle g,h\rangle_{\mathcal{H}}$. We assume that the Brownian motion $W = (W_t)_{t\in[0,T]}$, used so far, is a modification of $W(\chi_{[0,t]})_{t\in[0,T]}$. The Sobolev space $\mathrm{D}_{1,2}$ is then the closure of all

$$Z = \alpha(W(h_1),...,W(h_n)) \in L_2,$$

where all partial derivatives of $\alpha \in C^\infty(\mathbb{R}^n)$ have at most polynomial growth, with respect to the norm

$$\|Z\|_{\mathrm{D}_{1,2}} := \sqrt{\|Z\|_{L_2}^2 + \|DZ\|_{L_2^{\mathcal{H}}}^2}$$

with

$$DZ = \sum_{i=1}^{n} \frac{\partial \alpha}{\partial x_i}(W(h_1),...,W(h_n))h_i$$

being the Malliavin differential operator.

We conclude with some notation: given $A,B,C \geq 0$ and $c > 0$, the expression $A \sim_c B$ stands for $\frac{1}{c}B \leq A \leq cB$ and $A \sim B \pm C$ for $B - C \leq A \leq B + C$. For $\xi \in \mathbb{R}$ we let $\xi^+ := \max\{\xi, 0\}$ and for a real-valued function α defined on a subset of $\mathbb{R}$, $\|\alpha\|_\infty := \sup_x |\alpha(x)|$.

2 Results

Before we formulate the characterizations of Formula (1) we need

$$K(t,y) := \sigma(y)\frac{\partial G}{\partial y}(t,y) \quad \text{and} \quad H(t) := \left\| \sigma^2(Y_t)\frac{\partial^2 G}{\partial y^2}(t,Y_t) \right\|_{L_2} \tag{9}$$

for $t \in [0,T)$ and $y > 0$. In particular, one has

$$g(Y_T) = \mathbb{E}g(Y_T) + \int_0^T K(t,Y_t)dW_t \text{ a.s.} \tag{10}$$

According to Lemma 5.4 below the function $H : [0,T) \to [0,\infty)$ is increasing and continuous.

Theorem 2.1 *For $\eta \in (0,1/2]$ and $g(y) = f(\log y)$ with $f \in \mathcal{C}_\gamma$ the following assertions are equivalent:*

(C1) $g(Y_T) \in A_\eta^Y$.

(C2) $\sup_{t\in[0,T)} (T-t)^{\frac{1}{2}-\eta} \left(\int_0^t H(u)^2 du \right)^{\frac{1}{2}} < \infty$.

(C3) $\sup_{t\in[0,T)} (T-t)^{\frac{1}{2}-\eta} \| K(t,Y_t) \|_{L_2} < \infty$.

(C4) $\sup_{t\in[0,T)} (T-t)^{\frac{1}{2}-\eta} \left\| D\big(\mathbb{E}\left(g(Y_T) \mid \mathcal{F}_t\right)\big) \right\|_{L_2^{\mathcal{H}}} < \infty$

with D being the Malliavin derivative. If $\eta \in (0,1/2)$, then the above assertions are equivalent to

(C5) $g(Y_T) \in \left[A_{\frac{1}{2}}^Y, L_2 \right]_{1-2\eta,\infty}$.

Moreover, the decomposition $g(Y_T) = Z_0^\lambda + Z_1^\lambda$ with

$$Z_0^\lambda := \mathbb{E}\left(g(Y_T)|\mathcal{F}_t\right) + \frac{\partial G}{\partial y}(t,Y_t)(Y_T - Y_t) \quad \textit{for} \quad \lambda \geq T^{-\frac{1}{2}},$$

where $t := T - \lambda^{-2}$, satisfies

$$\sup_{\lambda \geq T^{-1/2}} \lambda^{2\eta-1} \left[\left\| Z_0^\lambda \right\|_{A_{\frac{1}{2}}^Y} + \lambda \| Z_1^\lambda \|_{L_2} \right] < \infty.$$

The theorem will be proved in Section 3. Let us first shortly discuss the equivalences above.

Interpretation and further equivalent conditions 2.2 (i) Condition (C1) is the approximation quantity we are interested in. For example, $(Y_T - K)^+ \in A^Y_{\frac{1}{2}}$ and $\chi_{[K,\infty)}(Y_T) \in A^Y_{\frac{1}{4}}$, $K > 0$, in the situation of Section 4. As shown in the proof of Theorem 2.1, condition (C1) is also equivalent to the following. There is some $c > 0$ such that

(C1') $a(g(Y_T);\tau_n) \leq cn^{-\eta}$ for $n = 1, 2, ...$ and $\tau_n := (iT/n)_{i=0}^n$.

That means, it is enough to check (C1) for equidistant time nets.

(ii) As shown in [6] (Remark 4.2(2)), in case Y is the geometric Brownian motion, one has, for $u \in [0, T)$,

$$H(u)^2 = \frac{d^2}{dt^2}\mathbb{E}\left|\int_u^t \frac{\partial G}{\partial y}(v, Y_v)dY_v - \frac{\partial G}{\partial y}(u, Y_u)(Y_t - Y_u)\right|^2$$

where the derivative on the right-hand side is taken in the right-hand side sense in $t = u$. So the expression $H(u)^2$ in (C2) corresponds to a certain 'L_2-convexity' of the martingale $(\mathbb{E}(g(Y_T)|\mathcal{F}_t))_{t\in[0,T]}$ with respect to the underlying diffusion $(Y_t)_{t\in[0,T]}$. Large values of $H(u)^2$ correspond to a large convexity and henceforth to a bad approximation.

(iii) Conditions (C3) shows that the approximation properties of $g(Y_T)$ are completely characterized by the asymptotics of the integrand K from the integral representation in Formula (10). Given $\delta \in (0, T)$, condition (C3) is also equivalent to

(C3') $\sup_{t\in[0,T)}(T-t)^{\frac{1}{2}-\eta}\left\|K(t, Y_t) - \mathbb{E}\left[K(t, Y_t)|\mathcal{F}_{(t-\delta)\vee 0}\right]\right\|_{L_2} < \infty$,

which follows from Lemma 5.3 below, and to

(C3") $\sup_{t\in[0,T)}(T-t)^{\frac{1}{2}-\eta}\left\|K(t, Y_t) - \mathbb{E}\left[K(t, Y_t)\right]\right\|_{L_2} < \infty$,

because of $(C3) \Rightarrow (C3'') \Rightarrow (C3')$.

(iv) As recalled in Proposition 3.5 below, the inclusion

$$\left[A^Y_{\frac{1}{2}}, L_2\right]_{1-2\eta,\infty} \subseteq A^Y_\eta \tag{11}$$

is true in general and demonstrates that the real interpolation method with parameters (θ, ∞) from Definition 1.5 is well adapted to the approximation spaces A_η^Y. Condition (C5) shows that one has the converse to inclusion (11) in case one restricts itself to random variables of type $g(Y_T)$.

Now we show that $g(Y_T) \in \bigcup_{0<\eta\le\frac{1}{2}} A_\eta^Y$ implies an approximation rate of $1/\sqrt{n}$ after optimizing over time nets of cardinality n. Hence Theorem 2.1 is also saying how far we are from the optimal rate if we are using equidistant time nets instead of non-equidistant ones.

Corollary 2.3 *For all $g(Y_T) \in A_\eta^Y$, $g(\exp(\cdot)) \in C_\gamma$, with $\eta \in (0, 1/2]$ there exist time nets $\tau_n = (t_i^{(n)})_{i=1}^n \in \mathcal{T}$ such that*

$$\sup_{n=1,2,\ldots} \sqrt{n}\, a^{\rm opt}(g(Y_T); \tau_n) < \infty.$$

Proof. From Theorem 2.1 we know that $g(Y_T) \in A_\eta^Y$ implies that

$$\int_0^t H(u)^2 du \;\le\; c\,(T-t)^{2\eta-1}$$

for $t \in [0, T)$ and some $c > 0$. Let $s_n := \left(1 - \frac{1}{2^n}\right) T$, $n = 0, 1, 2, \ldots$ For $n \ge 1$ we conclude that

$$(T-s_n)H(s_{n-1})^2 = (s_n - s_{n-1})H(s_{n-1})^2 \le \int_{s_{n-1}}^{s_n} H(u)^2 du \le c\,(T-s_n)^{2\eta-1}$$

and $H(s_{n-1}) \le \sqrt{c}\,(T-s_n)^{\eta-1}$. Now assume that $s_{n-1} \le s \le s_n$. We get

$$H(s) \le H(s_n) \le \sqrt{c}\,(T-s_{n+1})^{\eta-1} \le \sqrt{c}\,4^{1-\eta}(T-s)^{\eta-1}$$

and can finish with the arguments of [6](Theorem 6.2) and Lemma 3.3 below. □

Remark 2.4 (i) In general, that means in case that $\sup_{t\in[0,T)} H(t) > 0$, the approximation rate of $1/\sqrt{n}$ as described in Corollary 2.3 is optimal. To see this, one can use Lemma 5.4 below to check that

$$\inf_{0\le t_1\le\cdots\le t_n\le T} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u)H(u)^2 du \ge \frac{c}{n}$$

for $n = 1, 2, \ldots$ and some $c > 0$ depending on H only, and can combine this with Lemma 3.3 below.

(ii) Recently it was shown in [8] that there is an $g(Y_T) \in L_2$, where $\sigma(y) = y$ and $y_0 = 1$, such that the conclusion of Corollary 2.3 does not hold true.

To formulate relations to the Sobolev spaces we let $L_2(Y_T)$ be the subspace of all $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ such that there is a $g : (0,\infty) \to \mathbb{R}$ with

$$g(\exp(\cdot)) \in C_\gamma \quad \text{and} \quad g(Y_T) = Z \text{ a.s.}$$

Moreover, we need

$$A_{\frac{1}{2}}(Y_T) := A^Y_{\frac{1}{2}} \cap L_2(Y_T) \quad \text{and} \quad D_{1,2}(Y_T) := D_{1,2} \cap L_2(Y_T).$$

All three subspaces are equipped with the corresponding induced norms.

Theorem 2.5 $A_{\frac{1}{2}}(Y_T) = D_{1,2}(Y_T)$.

Theorem 2.6 *For* $\eta \in (0, 1/2)$ *one has*

$$\left[\mathrm{D}_{1,2}(Y_T), L_2(Y_T)\right]_{1-2\eta,\infty} \subseteq A_\eta(Y_T) \subseteq \left[\mathrm{D}_{1,2}, L_2\right]_{1-2\eta,\infty}. \tag{12}$$

Like Theorem 2.1, the statements are verified in Section 3. Comments about finding a converse to Theorem 2.6 can be found in Remark 3.9.

3 Proofs

We start with some lemmas before proving Theorem 2.1 itself. Throughout this section we assume that $f \in C_\gamma$ and $f(x) = g(e^x)$.

Lemma 3.1 *Fix* $t \in (0,T)$. *Let* $h, \Phi : (0,\infty) \to \mathbb{R}$ *be continuous such that*

$$\Phi(y) := \int_{y_0}^{y} \frac{h(\eta)}{\sigma(\eta)} d\eta \quad \textit{and} \quad \mathbb{E}\left(h(Y_t)^2 + \Phi(Y_t)^2\right) < \infty,$$

and define $M_u := \mathbb{E}\left(h(Y_t) \mid \mathcal{F}_u\right)$ *for* $u \in [0,t]$. *Then, for* $a \in [0,t)$ *with* $t - a < \frac{2}{\kappa}$ *and* $c := \frac{2}{2-\kappa(t-a)}$ *one has*

$$\left(\int_a^t \mathbb{E} M_u^2 du\right)^{\frac{1}{2}} \sim_c \left\|\Phi(Y_t) - \mathbb{E}\left(\Phi(Y_t) \mid \mathcal{F}_a\right)\right\|_{L_2}. \tag{13}$$

Proof. (a) First we show that we can find

$$0 < \cdots \le a_2 \le a_1 \le y_0 \le b_1 \le b_2 \le \cdots < \infty$$

with $\lim_n a_n = 0$ and $\lim_n b_n = \infty$ such that for

$$h_n^0 := h(\eta)\chi_{\{a_n \le \eta \le b_n\}} \quad \text{and} \quad \Phi_n^0(y) := \int_{y_0}^{y} \frac{h_n^0(\eta)}{\sigma(\eta)} d\eta$$

one has that

$$\lim_n \left\| h_n^0(Y_t) - h(Y_t) \right\|_{L_2} = 0 \quad \text{and} \quad \lim_n \left\| \Phi_n^0(Y_t) - \Phi(Y_t) \right\|_{L_2} = 0. \tag{14}$$

In fact, from $\lim_n a_n = 0$ and $\lim_n b_n = \infty$ the first limit follows by monotone convergence. The particular choice of the a_n and b_n is done as follows: if there are $y_0 \le b_n \uparrow \infty$ such that $\sup_n |\Phi(b_n)| < \infty$ we take this sequence, otherwise

$$b_n := \sup \{ y > 0 | \ |\Phi(y)| = n \} .$$

And analogously, we take $y_0 \ge a_n \downarrow 0$ with $\sup_n |\Phi(a_n)| < \infty$ or, otherwise,

$$a_n := \inf \{ y > 0 | \ |\Phi(y)| = n \} .$$

Since

$$\left\| \Phi_n^0(Y_t) - \Phi(Y_t) \right\|_{L_2}^2 = \int_{Y_t < a_n} |\Phi(Y_t) - \Phi(a_n)|^2 \, d\mathbb{P} + \int_{Y_t > b_n} |\Phi(Y_t) - \Phi(b_n)|^2 \, d\mathbb{P},$$

by bounded or monotone convergence the second part of Formula (14) follows. Replacing the h_n^0 by appropriate compactly supported $h_n \in C^\infty((0,\infty))$ such that

$$\lim_n \left\| h_n(Y_t) - h_n^0(Y_t) \right\|_{L_2} = \lim_n \left\| \Phi_n(Y_t) - \Phi_n^0(Y_t) \right\|_{L_2} = 0$$

for

$$\Phi_n(y) := \int_{y_0}^{y} \frac{h_n(\eta)}{\sigma(\eta)} d\eta,$$

where we also use the distributional properties of Y_t, we see that it is sufficient to show our assertion for h_n and Φ_n.

(b) We fix $n \geq 1$ and, applying [12](Theorem 3.2.5), find a function $A \in C_b^{1,2}([0,t] \times \mathbb{R}) \cap C^\infty([0,t) \times \mathbb{R})$ such that

$$\frac{\partial A}{\partial u} + \frac{\widehat{\sigma}^2}{2}\frac{\partial^2 A}{\partial x^2} + \left(\widehat{\sigma}'\widehat{\sigma} + \frac{\widehat{\sigma}^2}{2}\right)\frac{\partial A}{\partial x} = 0 \quad \text{and} \quad A(t,x) = \Phi_n'(e^x) = \frac{h_n(e^x)}{\sigma(e^x)}.$$

Letting $B(u,y) := \sigma(y)A(u,\log y)$, $(u,y) \in [0,t] \times (0,\infty)$ and

$$\mathcal{A} := \frac{\partial}{\partial u} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}$$

we obtain

$$\frac{\partial B}{\partial u} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial y^2} - \frac{\sigma''\sigma}{2}B = 0 \quad \text{with} \quad B(t,y) = h_n(y)$$

and, by Itô's formula,

$$\begin{aligned} B(u,Y_u) &= B(a,Y_a) + \int_a^u \frac{\partial B}{\partial y}(v,Y_v)dY_v + \int_a^u (\mathcal{A}B)(v,Y_v)dv \\ &= B(a,Y_a) + \int_a^u \frac{\partial B}{\partial y}(v,Y_v)dY_v + \int_a^u \frac{(\sigma''\sigma)(Y_v)}{2}B(v,Y_v)dv \ \text{ a.s.} \end{aligned}$$

for $u \in [a,t]$. In particular,

$$h_n(Y_t) = B(a,Y_a) + \int_a^t \frac{\partial B}{\partial y}(v,Y_v)dY_v + \int_a^t \frac{(\sigma''\sigma)(Y_v)}{2}B(v,Y_v)dv \ \text{ a.s.}$$

For $u \in [a,t]$ and $M_u^n := \mathbb{E}\left(h_n(Y_t) \mid \mathcal{F}_u\right)$ we conclude ($(\partial B/\partial y)(v,y) = \sigma'(y)A(v,\log y) + (\sigma(y)/y)(\partial A/\partial x)(v,\log y)$ is bounded) that

$$M_u^n = B(u,Y_u) + \mathbb{E}\left(\int_u^t \frac{(\sigma\sigma'')(Y_v)}{2}B(v,Y_v)dv \mid \mathcal{F}_u\right) \ \text{ a.s.}$$

and

$$\left(\int_a^t \mathbb{E}(M_u^n)^2 du\right)^{\frac{1}{2}} \sim \left(\int_a^t \mathbb{E}B(u,Y_u)^2 du\right)^{\frac{1}{2}} \pm \left(\int_a^t \mathbb{E}\left[\mathbb{E}\left(\int_u^t \frac{(\sigma\sigma'')(Y_v)}{2}B(v,Y_v)dv \mid \mathcal{F}_u\right)\right]^2 du\right)^{\frac{1}{2}}.$$

Moreover,

$$
\begin{aligned}
&\left(\int_a^t \mathbb{E}\left[\mathbb{E}\left(\int_u^t \frac{(\sigma\sigma'')(Y_v)}{2} B(v,Y_v)dv \mid \mathcal{F}_u\right)\right]^2 du\right)^{\frac{1}{2}} \\
&\qquad \leq \left(\int_a^t \mathbb{E}\left[\int_u^t \frac{(\sigma\sigma'')(Y_v)}{2} B(v,Y_v)dv\right]^2 du\right)^{\frac{1}{2}} \\
&\qquad \leq \left(\int_a^t \mathbb{E}(t-u)\int_u^t \left[\frac{(\sigma\sigma'')(Y_v)}{2} B(v,Y_v)\right]^2 dvdu\right)^{\frac{1}{2}} \\
&\qquad \leq \sqrt{t-a}\,\frac{\kappa}{2}\left(\int_a^t \mathbb{E}\int_u^t B(v,Y_v)^2 dvdu\right)^{\frac{1}{2}} \\
&\qquad \leq (t-a)\,\frac{\kappa}{2}\left(\int_a^t \mathbb{E}B(u,Y_u)^2 du\right)^{\frac{1}{2}},
\end{aligned}
$$

so that

$$
\left(\int_a^t \mathbb{E}(M_u^n)^2 du\right)^{\frac{1}{2}} \sim_c \left(\int_a^t \mathbb{E}B(u,Y_u)^2 du\right)^{\frac{1}{2}}.
$$

(c) Finally, letting

$$
C(u,y) := \int_{y_0}^y \frac{B(u,\eta)}{\sigma(\eta)} d\eta + \frac{1}{2}\int_u^t \left[\sigma(y_0)\frac{\partial B}{\partial y}(v,y_0) - \sigma'(y_0)B(v,y_0)\right] dv
$$

for $u \in [0,t]$ and $y \in (0,\infty)$ we obtain

$$
\frac{\partial C}{\partial u} + \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial y^2} = 0 \quad \text{with} \quad C(t,y) = \Phi_n(y)
$$

so that, by Itô's formula and $dY_t = \sigma(Y_t)dW_t$,

$$
\begin{aligned}
\mathbb{E}\left(\Phi_n(Y_t) - \mathbb{E}\left(\Phi_n(Y_t) \mid \mathcal{F}_a\right)\right)^2 &= \mathbb{E}\left|\int_a^t \frac{\partial C}{\partial y}(u,Y_u)dY_u\right|^2 \\
&= \mathbb{E}\left|\int_a^t \frac{B(u,Y_u)}{\sigma(Y_u)}dY_u\right|^2 \\
&= \mathbb{E}\int_a^t B(u,Y_u)^2 du.
\end{aligned}
$$

□

Lemma 3.2 *Let $D : [0,T) \to [0,\infty)$ be a non-decreasing and continuous function and $1 < r \leq \infty$. Then the following assertions are equivalent:*

(i) *There exists a constant $c_1 \geq 0$ such that, for all $t \in [0,T)$,*

$$\int_0^t D(u)du \;\leq\; \frac{c_1}{(T-t)^{\frac{1}{r}}}.$$

(ii) *There exists a constant $c_2 \geq 0$ such that, for all $\tau = (t_i)_{i=0}^n \in \mathcal{T}$, one has*

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u)D(u)du \;\leq\; c_2 \;\|\tau\|_\infty^{1-\frac{1}{r}}.$$

(ii') *There exists a constant $c_2' \geq 0$ such that, for all $n = 1,2,...$, one has*

$$\sum_{i=1}^n \int_{\frac{i-1}{n}T}^{\frac{i}{n}T} \left(\frac{i}{n}T - u\right) D(u)du \;\leq\; c_2' \left(\frac{T}{n}\right)^{1-\frac{1}{r}}.$$

In particular, one can take $c_2 = \frac{6r}{r-1}c_1$, $c_2' = c_2$, and $c_1 = 4\,2^{\frac{1}{r}}\,c_2'$ if $1 < r < \infty$ and $c_2 = c_1$, $c_2' = c_2$, and $c_1 = 4c_2'$ if $r = \infty$.

Proof. We only check the case $1 < r < \infty$. $(ii) \Rightarrow (ii')$ is trivial, so let us start with $(ii') \Rightarrow (i)$. Let $t \in (0,T)$ and $n \geq 2$ such that $\frac{1}{n} \leq \frac{T-t}{T} < \frac{1}{n-1} \leq \frac{2}{n}$. Then

$$\int_0^t D(u)du \leq \sum_{i=1}^{n-1} \int_{\frac{i-1}{n}T}^{\frac{i}{n}T} D(u)du \leq \frac{4n}{T} \sum_{i=1}^{n-1} \int_{\frac{i}{n}T}^{\frac{i+1}{n}T} \left(\frac{i+1}{n}T - u\right) D(u)du$$
$$\leq \frac{4n}{T}c_2' \left(\frac{T}{n}\right)^{1-\frac{1}{r}} = 4c_2' \left(\frac{n}{T}\right)^{\frac{1}{r}} \leq \frac{4\,2^{\frac{1}{r}}\,c_2'}{(T-t)^{\frac{1}{r}}}.$$

$(i) \Rightarrow (ii)$. Assume now that $0 = s_0 < s_1 < \cdots < s_m = T$. We get

$$\sum_{j=1}^m \int_{s_{j-1}}^{s_j} (s_j - u)D(u)du$$
$$\leq \sup_{1 \leq j < m} |s_j - s_{j-1}| \int_0^{s_{m-1}} D(u)du + \int_{s_{m-1}}^T (T-u)D(u)du$$

$$
\begin{aligned}
&= \sup_{1\le j<m} |s_j - s_{j-1}| \int_0^{s_{m-1}} D(u)du + \left[(T-u)\int_0^u D(v)dv\right] \Big|_{s_{m-1}}^{T} \\
&\qquad\qquad + \int_{s_{m-1}}^T \int_0^u D(v)dvdu \\
&\le \sup_{1\le j<m} |s_j - s_{j-1}| \int_0^{s_{m-1}} D(u)du - (T-s_{m-1})\int_0^{s_{m-1}} D(v)dv \\
&\qquad\qquad + \int_{s_{m-1}}^T \frac{c_1}{(T-u)^{\frac{1}{r}}}du \\
&\le \sup_{1\le j<m} |s_j - s_{j-1}| \int_0^{s_{m-1}} D(u)du + \frac{c_1\, r}{r-1}(T-s_{m-1})^{1-\frac{1}{r}} \\
&\le c_1 \left[\sup_{1\le j<m} |s_j - s_{j-1}| \; (T-s_{m-1})^{-\frac{1}{r}} + \frac{r}{r-1}(T-s_{m-1})^{1-\frac{1}{r}}\right].
\end{aligned}
$$

Now consider $\tau = (t_i)_{i=0}^n \in \mathcal{T}$. We find a sub-net $(s_j)_{j=0}^m$, $0 = s_0 < \cdots < s_m = T$ such that

$$|s_0 - s_1| \le 3\delta \quad \text{and} \quad \delta \;\le\; |s_j - s_{j-1}| \le 3\delta$$

for $j = 2, ..., m$, where $\delta := \|\tau\|_\infty$. In case of $m \ge 2$ we get

$$
\begin{aligned}
&\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - u)D(u)du \\
&\quad\le \sum_{j=1}^m \int_{s_{j-1}}^{s_j} (s_j - u)D(u)du \\
&\quad\le c_1 \left[\sup_{1\le j<m} |s_j - s_{j-1}| \; (T-s_{m-1})^{-\frac{1}{r}} + \frac{r}{r-1}(T-s_{m-1})^{1-\frac{1}{r}}\right] \\
&\quad\le c_1 \left[3\delta\, \delta^{-\frac{1}{r}} + \frac{r}{r-1}(3\delta)^{1-\frac{1}{r}}\right] \le 3\, c_1 \frac{2r-1}{r-1} \|\tau\|_\infty^{1-\frac{1}{r}}.
\end{aligned}
$$

For $m = 1$ we get $T \le 3\delta$ and

$$
\begin{aligned}
\sum_{j=1}^m \int_{s_{j-1}}^{s_j} (s_j - u)D(u)du &= \int_0^T (T-u)D(u)du = \int_0^T \int_0^u D(v)dvdu \\
&\le c_1 \int_0^T \frac{1}{(T-u)^{\frac{1}{r}}}du = c_1 \frac{r}{r-1} T^{1-\frac{1}{r}} \le c_1 \frac{3r}{r-1} \|\tau\|_\infty^{1-\frac{1}{r}}
\end{aligned}
$$

and we are done. □

Lemma 3.3 [6] *Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a deterministic time net. Then, for all $1 \le K \le L \le n$,*

$$\inf \left\| \int_{t_{K-1}}^{t_L} \frac{\partial G}{\partial y}(t, Y_t) dY_t - \sum_{i=K}^{L} v_{i-1} \left(Y_{t_i} - Y_{t_{i-1}} \right) \right\|_{L_2}^2 \sim_{c^2} \sum_{i=K}^{L} \int_{t_{i-1}}^{t_i} (t_i - u) H(u)^2 du$$

where the infimum is taken over all $\mathcal{F}_{t_i}$-measurable step-functions v_i and where $c \ge 1$ depends on T, κ, and κ' only.

Proof. We have to verify that we can apply [6] (Theorem 3.1). Let us fix $0 \le a < b < T$ and take an $\mathcal{F}_a$-measurable step-function v. For $\Phi(s, y) := \sigma(y)((\partial G/\partial y)(s, y) - v)$ we have that

$$\mathbb{E} \sup_{u \in [a,b]} \Phi(u, Y_u)^2 \le 2 \left[\mathbb{E} \sup_{u \in [a,b]} \left(\sigma \frac{\partial G}{\partial y} \right)^2 (u, Y_u) + \mathbb{E} \sup_{u \in [a,b]} [\sigma(Y_u) v]^2 \right] < \infty$$

according to Lemma 5.3 below and $\mathbb{E} \sup_{u \in [0,T]} Y_u^2 < \infty$. Moreover,

$$\left| \frac{\sigma^2}{2} \sigma'' \right| + |\sigma \sigma'| \le \left[\frac{\kappa}{2} + \kappa' \right] \sigma.$$

Finally, Formula (7) implies

$$\frac{\partial}{\partial t} \left[\frac{\partial G}{\partial y} \right] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} \left[\frac{\partial G}{\partial y} \right] + \sigma \sigma' \frac{\partial}{\partial y} \left[\frac{\partial G}{\partial y} \right] = 0$$

on $[0, T) \times (0, \infty)$ ([6] (Theorem 3.1) uses as range of definition $[0, T) \times \mathbb{R}$, but from its proof it follows that the range can be replaced by $[0, T) \times (0, \infty)$). Now [6] (Theorem 3.1) implies

$$\begin{aligned} \mathbb{E} \left(\int_a^b \frac{\partial G}{\partial y}(t, Y_t) dY_t - v(Y_b - Y_a) \right)^2 &= \int_a^b \mathbb{E} \sigma(Y_t)^2 \left(\frac{\partial G}{\partial y}(t, Y_t) - v \right)^2 dt \\ &\sim_{c^2} (b - a) \mathbb{E} \left(\sigma(Y_a) \left(\frac{\partial G}{\partial y}(a, Y_a) - v \right) \right)^2 + \int_a^b (b - t) H(t)^2 dt. \quad (15) \end{aligned}$$

This can be extended to $0 \le a < b \le T$ by letting $b \uparrow T$. Consequently,

$$\mathbb{E}\left(\int_0^T \frac{\partial G}{\partial y}(t,Y_t)dY_t - \sum_{i=1}^n v_{i-1}(Y_{t_i}-Y_{t_{i-1}})\right)^2 \sim_{c^2}$$
$$\sum_{i=1}^n (t_i - t_{i-1})\mathbb{E}\left[\sigma(Y_{t_{i-1}})\left[\frac{\partial G}{\partial y}(t_{i-1},Y_{t_{i-1}}) - v_{i-1}\right]\right]^2 + \sum_{i=1}^n \int_{t_{i-1}}^{t_i}(t_i - t)H(t)^2 dt.$$

Taking the infimum over the v_i, we arrive at the desired result. □

Proposition 3.4 *For $\rho \in [0,1/2)$ and $\delta \in (0, T \wedge (2/\kappa))$ the following assertions are equivalent:*

(i) $\sup_{t\in[0,T)}(T-t)^\rho \left(\int_0^t H(u)^2 du\right)^{\frac{1}{2}} < \infty$.

(ii) $\sup_{t\in[0,T)}(T-t)^\rho \left\|K(t,Y_t) - \mathbb{E}\left(K(t,Y_t) \mid \mathcal{F}_{(t-\delta)\vee 0}\right)\right\|_{L_2} < \infty$.

(iii) $\sup_{t\in[0,T)}(T-t)^\rho \left\|K(t,Y_t)\right\|_{L_2} < \infty$.

Proof. $(ii) \Leftrightarrow (iii)$ follows by Lemma 5.3 below and $(i) \Leftrightarrow (ii)$ is proved as follows: Fix $t \in (0,T)$ and let $a := (t-\delta)\vee 0$ so that $t - a \le \delta$. Define

$$h(y) := \sigma^2(y)\frac{\partial^2 G}{\partial y^2}(t,y) \quad \text{and} \quad \Phi(y) := \int_{y_0}^y \frac{h(\eta)}{\sigma(\eta)}d\eta$$

so that

$$\Phi(y) = \sigma(\eta)\frac{\partial G}{\partial y}(t,\eta) \mid_{y_0}^{y} - \sigma'(\eta)G(t,\eta) \mid_{y_0}^{y} + \int_{y_0}^y \sigma''(\eta)G(t,\eta)d\eta.$$

We get $\mathbb{E}h(Y_t)^2 < \infty$ and $\mathbb{E}\Phi(Y_t)^2 < \infty$ as a consequence of Lemmas 5.4 and 5.3, $\|G(t,Y_t)\|_{L_2} \le \|g(Y_T)\|_{L_2}$, and Definition 1.1. Moreover, we have

$$\left\|K(t,Y_t) - \mathbb{E}\left(K(t,Y_t) \mid \mathcal{F}_a\right)\right\|_{L_2} \sim \left\|\Phi(Y_t) - \mathbb{E}\left(\Phi(Y_t) \mid \mathcal{F}_a\right)\right\|_{L_2} \pm c \quad (16)$$

with

$$c := \|\sigma'\|_\infty \|g(Y_T)\|_{L_2} + \sup_{u\in[0,T)}\left\|\int_{y_0}^{Y_u} \sigma''(\eta)G(u,\eta)d\eta\right\|_{L_2} < \infty$$

again by Definition 1.1. For the right-hand side of Formula (16) we obtain

$$\left\|\Phi(Y_t) - \mathbb{E}\left(\Phi(Y_t) \mid \mathcal{F}_a\right)\right\|_{L_2}^2 \sim_{c^2_{(3.1)}} \int_a^t \mathbb{E}\left(\mathbb{E}\left(h(Y_t)|\mathcal{F}_u\right)\right)^2 du$$

by Lemma 3.1 with $c_{(3.1)} = \frac{2}{2-\kappa\delta}$. Moreover, Lemma 5.4 yields

$$\mathbb{E}\left(h(Y_t)|\mathcal{F}_u\right) = \sigma(Y_u)^2 \frac{\partial^2 G}{\partial y^2}(u, Y_u)$$

so that

$$\left\|\Phi(Y_t) - \mathbb{E}\left(\Phi(Y_t) \mid \mathcal{F}_a\right)\right\|_{L_2}^2 \sim_{c_{(3.1)}^2} \int_a^t H(u)^2 du$$

and

$$\frac{1}{c_{(3.1)}}\left(\int_{(t-\delta)^+}^t H(u)^2 du\right)^{\frac{1}{2}} - c \ \leq\ \left\|K(t,Y_t) - \mathbb{E}\left(K(t,Y_t) \mid \mathcal{F}_{(t-\delta)\vee 0}\right)\right\|_{L_2}$$

$$\leq\ c_{(3.1)}\left(\int_{(t-\delta)^+}^t H(u)^2 du\right)^{\frac{1}{2}} + c$$

which implies our assertion. □

The next proposition is standard in approximation theory. For convenience of the reader we include its proof.

Proposition 3.5 *For $\eta \in (0,1/2)$ one has $[A_{1/2}^Y, L_2]_{1-2\eta,\infty} \subseteq A_\eta^Y$.*

Proof. Assuming $Z \in [A_{1/2}^Y, L_2]_{1-2\eta,\infty}$ there is an $c > 0$ and, for all $\lambda \geq T^{-\frac{1}{2}}$, a decomposition $Z = Z_0^\lambda + Z_1^\lambda$ with

$$|Z_0^\lambda|_{A_{\frac{1}{2}}^Y} + \lambda \left\|Z_1^\lambda\right\|_{L_2} \ \leq\ c\,\lambda^{1-2\eta}.$$

For $\varepsilon > 0$ and $\tau = (t_i)_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_n = T$ and $\|\tau\|_\infty = \lambda^{-2}$ we find $v_0^\lambda, ..., v_{n-1}^\lambda$ such that

$$\left\|Z_0^\lambda - \mathbb{E}Z_0^\lambda - \sum_{i=1}^n v_{i-1}^\lambda\left(Y_{t_i} - Y_{t_{i-1}}\right)\right\|_{L_2} \ \leq\ \left(|Z_0^\lambda|_{A_{\frac{1}{2}}^Y} + \varepsilon\right)\ \|\tau\|_\infty^{\frac{1}{2}}.$$

Hence

$$\left\|Z - \mathbb{E}Z - \sum_{i=1}^n v_{i-1}^\lambda\left(Y_{t_i} - Y_{t_{i-1}}\right)\right\|_{L_2}$$

$$
\begin{aligned}
&\leq \left\| Z_0^\lambda - \mathbb{E} Z_0^\lambda - \sum_{i=1}^{n} v_{i-1}^\lambda \left(Y_{t_i} - Y_{t_{i-1}} \right) \right\|_{L_2} + \left\| Z_1^\lambda - \mathbb{E} Z_1^\lambda \right\|_{L_2} \\
&\leq \left(|Z_0^\lambda|_{A^Y_{\frac{1}{2}}} + \varepsilon \right) \|\tau\|_\infty^{\frac{1}{2}} + \|Z_1^\lambda\|_{L_2} \\
&\leq \frac{1}{\lambda} |Z_0^\lambda|_{A^Y_{\frac{1}{2}}} + \|Z_1^\lambda\|_{L_2} + \varepsilon \sqrt{T} \\
&\leq c\lambda^{-2\eta} + \varepsilon \sqrt{T} \\
&= c \|\tau\|_\infty^\eta + \varepsilon \sqrt{T},
\end{aligned}
$$

so that $|Z|_{A^Y_\eta} \leq c$. Since $[A^Y_{1/2}, L_2]_{1-2\eta,\infty} \subseteq L_2$ is true as well, we are done. $\square$

Proposition 3.6 *Let $\rho \in (0, 1/2)$ and*

$$
c := \sup_{t \in [0,T)} (T-t)^\rho \left(\int_0^t H(u)^2 du \right)^{\frac{1}{2}} < \infty.
$$

Then

$$
\sup_{\lambda > 0} \lambda^{-2\rho} K \left(g(Y_T), \lambda; A^Y_{\frac{1}{2}}, L_2 \right) < \infty.
$$

Proof. Fix $\lambda \geq T^{-\frac{1}{2}}$ and let $t := T - \lambda^{-2}$ so that $t \in [0, T)$. Define a decomposition $g(Y_T) = Z_0^\lambda + Z_1^\lambda$ by

$$
Z_0^\lambda := \mathbb{E} g(Y_T) + \int_0^t \frac{\partial G}{\partial y}(u, Y_u) dY_u + \frac{\partial G}{\partial y}(t, Y_t)(Y_T - Y_t)
$$

and

$$
Z_1^\lambda := \int_t^T \frac{\partial G}{\partial y}(u, Y_u) dY_u - \frac{\partial G}{\partial y}(t, Y_t)(Y_T - Y_t).
$$

For Z_1^λ we get that

$$
\begin{aligned}
\|Z_1^\lambda\|_{L_2} &\leq c_{(3.3)} \sqrt{\int_t^T (T-u) H(u)^2 du} \\
&= c_{(3.3)} \sqrt{\int_t^T \int_t^u H(v)^2 dv du}
\end{aligned}
$$

$$
\begin{aligned}
&\le c_{(3.3)} c \sqrt{\int_t^T \frac{du}{(T-u)^{2\rho}}} \\
&= c_{(3.3)} c \sqrt{\frac{1}{1-2\rho}(T-t)^{1-2\rho}} \\
&= c_{(3.3)}\, c \sqrt{\frac{1}{1-2\rho}\lambda^{2\rho-1}},
\end{aligned}
$$

where $c_{(3.3)} > 0$ is taken from Lemma 3.3.

Now we consider Z_0^λ. Let $\tau = (t_i)_{i=0}^n$ be a net $0 = t_0 < t_1 < \cdots < t_n = T$ and let $i_0 \in \{1, ..., n\}$ be such that

$$t_{i_0-1} \le t < t_{i_0}.$$

Then one has that

$$
\begin{aligned}
&\Bigg\| Z_0^\lambda - \mathbb{E} Z_0^\lambda - \sum_{i=1}^{i_0-1} \frac{\partial G}{\partial y}(t_{i-1}, Y_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) \\
&\qquad\qquad - \sum_{i=i_0+1}^{n} \frac{\partial G}{\partial y}(t, Y_t)(Y_{t_i} - Y_{t_{i-1}}) \Bigg\|_{L_2} \\
&\le \Bigg\| \int_0^t \frac{\partial G}{\partial y}(u, Y_u) dY_u - \sum_{i=1}^{i_0-1} \frac{\partial G}{\partial y}(t_{i-1}, Y_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) \\
&\qquad\qquad - \frac{\partial G}{\partial y}(t_{i_0-1}, Y_{t_{i_0-1}})(Y_t - Y_{t_{i_0-1}}) \Bigg\|_{L_2} \\
&\quad + \left\| \frac{\partial G}{\partial y}(t_{i_0-1}, Y_{t_{i_0-1}})(Y_t - Y_{t_{i_0-1}}) + \frac{\partial G}{\partial y}(t, Y_t)(Y_{t_{i_0}} - Y_t) \right\|_{L_2} \\
&\le c_{(3.3)} \sqrt{\sum_{i=1}^{i_0-1} \int_{t_{i-1}}^{t_i} (t_i - u) H(u)^2 du + \int_{t_{i_0-1}}^{t} (t-u) H(u)^2 du} \\
&\qquad\qquad +2 \sup_{a \le t \le b \le T, b-a \le \|\tau\|_\infty} \left\| \frac{\partial G}{\partial y}(a, Y_a)(Y_b - Y_a) \right\|_{L_2} \\
&\le c_{(3.3)} \|\tau\|_\infty^{\frac{1}{2}}\, c (T-t)^{-\rho} \\
&\quad +2 \sup_{0 \le a \le t \le b \le T, b-a \le \|\tau\|_\infty} \sqrt{\mathbb{E}\left(\left[\frac{\partial G}{\partial y}(a, Y_a) Y_a \right]^2 \mathbb{E}\left(\left(\frac{Y_b}{Y_a} - 1 \right)^2 \,\Big|\, \mathcal{F}_a \right) \right)}.
\end{aligned}
$$

Since $(T-t)^{-\rho} = \lambda^{2\rho}$ it is sufficient to bound the second term from above by $\sqrt{b-a}\lambda^{2\rho}$. In fact, applying Lemma 5.5, Formula (2), and Proposition 3.4 we have

$$\begin{aligned}
&\mathbb{E}\left(\left[\frac{\partial G}{\partial y}(a,Y_a)Y_a\right]^2 \mathbb{E}\left(\left(\frac{Y_b}{Y_a}-1\right)^2 \mid \mathcal{F}_a\right)\right) \\
&\qquad \leq c_{(5.5)}\ \mathbb{E}\left[\frac{\partial G}{\partial y}(a,Y_a)Y_a\right]^2 \ (b-a) \\
&\qquad \leq c_{(5.5)}\ \varepsilon_0^{-2}\mathbb{E}K(a,Y_a)^2\ (b-a) \\
&\qquad \leq c_{(5.5)}\ \varepsilon_0^{-2}c_{(3.4)}^2(T-t)^{-2\rho}\ (b-a) \\
&\qquad \leq c_{(5.5)}\ \varepsilon_0^{-2}c_{(3.4)}^2\lambda^{4\rho}\ (b-a).
\end{aligned}$$

Since, for $\lambda \geq T^{-\frac{1}{2}}$,

$$\begin{aligned}
&\lambda^{-2\rho}\left(\left\|Z_0^\lambda\right\|_{A^Y_{\frac{1}{2}}} + \lambda\left\|Z_1^\lambda\right\|_{L_2}\right) \\
&\quad \leq \lambda^{-2\rho}\left(|Z_0^\lambda|_{A^Y_{\frac{1}{2}}} + \lambda\left\|Z_1^\lambda\right\|_{L_2} + \|g(Y_T)\|_{L_2} + \left\|Z_1^\lambda\right\|_{L_2}\right) \\
&\quad \leq T^\rho\|g(Y_T)\|_{L_2} + (T^{\frac{1}{2}}+1)\lambda^{-2\rho}\left(|Z_0^\lambda|_{A^Y_{\frac{1}{2}}} + \lambda\left\|Z_1^\lambda\right\|_{L_2}\right)
\end{aligned}$$

we have that

$$\sup_{\lambda \geq T^{-\frac{1}{2}}} \lambda^{-2\rho}K\left(g(Y_T),\lambda;A^Y_{\frac{1}{2}},L_2\right) < \infty.$$

On the other hand,

$$\sup_{0<\lambda<T^{-1/2}} \lambda^{-2\rho}K\left(g(Y_T),\lambda;A^Y_{\frac{1}{2}},L_2\right) \leq \lambda^{-2\rho}\lambda\|g(Y_T)\|_{L_2} < \infty.$$

$\square$

Lemma 3.7 *For* $c := [\|\widehat{\sigma}\|_\infty + (1/\varepsilon_0)]e^{(T/2)\kappa}$ *one has*

$$\|DX_t\|_{\mathcal{H}} \sim_c \sqrt{t} \ \text{ a.s.} \quad \text{for} \quad t \in [0,T].$$

Proof. Fix $t \in [0,T]$. It is known (cf. [10], p. 107) that we can write $DX_t = (D_s X_t)_{s\in[0,T]}$ in $\mathcal{H}$ with

$$D_s X_t := \widehat{\sigma}(X_s)\exp\left(\int_s^t \widehat{\sigma}'(X_u)dW_u + \int_s^t \left[\widehat{b}' - \frac{1}{2}(\widehat{\sigma}')^2\right](X_u)du\right)$$

for $0 \le s \le t \le T$ and $D_s X_t := 0$ for $0 \le t < s \le T$. Since

$$\begin{aligned}\int_s^t \widehat{\sigma}'(X_u)dW_u &= \int_s^t (\ln\widehat{\sigma})'(X_u)\widehat{\sigma}(X_u)dW_u \\ &= \int_s^t (\ln\widehat{\sigma})'(X_u)dX_u - \int_s^t (\ln\widehat{\sigma})'(X_u)\widehat{b}(X_u)du\end{aligned}$$

for $s \in [0,t]$ a.s., Itô's formula gives that

$$\begin{aligned}&\int_s^t \widehat{\sigma}'(X_u)dW_u + \int_s^t \left[\widehat{b}' - \frac{1}{2}(\widehat{\sigma}')^2\right](X_u)du \\ &\quad= (\ln\widehat{\sigma})(X_t) - (\ln\widehat{\sigma})(X_s) - \frac{1}{2}\int_s^t \left[\widehat{\sigma}''\widehat{\sigma} + \widehat{\sigma}'\widehat{\sigma}\right](X_u)du \\ &\quad= (\ln\widehat{\sigma})(X_t) - (\ln\widehat{\sigma})(X_s) - \frac{1}{2}\int_s^t \left[\sigma\sigma''\right](Y_u)du\end{aligned}$$

and

$$D_s X_t = \widehat{\sigma}(X_t)e^{-\frac{1}{2}\int_s^t [\sigma\sigma''](Y_u)du}$$

for $s \in [0,t]$ a.s. The assertion follows from $|\sigma\sigma''| \le \kappa$ and $\widehat{\sigma} \ge \varepsilon_0 > 0$. □

Lemma 3.8 *For $t \in [0,T)$ one has that*

$$\frac{1}{A}\int_0^t uH(u)^2du - C \le \|DG(t,Y_t)\|^2_{L_2^{\mathcal{H}}} \le A\int_0^t uH(u)^2du + C$$

for some $A, C > 0$ depending at most on g, σ, and T. Moreover,

$$g(Y_T) \in D_{1,2} \quad \text{if and only if} \quad \int_0^T H(u)^2du < \infty.$$

Proof. Applying [10] (Lemma 1.3.4) (cf. [11]) to

$$G(t, Y_t) = \mathbb{E} g(Y_T) + \int_0^t \left(\sigma \frac{\partial G}{\partial y} \right) (u, Y_u) dW_u \text{ a.s.}$$

gives

$$\|DG(t, Y_t)\|_{L_2^{\mathcal{H}}}^2 = \|G(t, Y_t) - \mathbb{E} g(Y_T)\|_{L_2}^2 + \int_0^t \left\| D \left(\left(\sigma \frac{\partial G}{\partial y} \right) (u, Y_u) \right) \right\|_{L_2^{\mathcal{H}}}^2 du$$

with

$$D \left(\left(\sigma \frac{\partial G}{\partial y} \right) (u, Y_u) \right) = (DX_u) Y_u \left[\left(\sigma' \frac{\partial G}{\partial y} \right) (u, Y_u) + \left(\sigma \frac{\partial^2 G}{\partial y^2} \right) (u, Y_u) \right].$$

Lemma 3.7 implies

$$\left\| D \left(\left(\sigma \frac{\partial G}{\partial y} \right) (u, Y_u) \right) \right\|_{L_2^{\mathcal{H}}} \sim_{c_{(3.7)}} \sqrt{u} \left\| Y_u \left[\left(\sigma' \frac{\partial G}{\partial y} \right) (u, Y_u) + \left(\sigma \frac{\partial^2 G}{\partial y^2} \right) (u, Y_u) \right] \right\|_{L_2} \tag{17}$$

Moreover, $|\sigma'| \le \kappa'$, $\sigma(y) \sim_d y$ for some $d = d(\sigma) > 0$, and

$$\int_0^T \mathbb{E} Y_u^2 \left[\frac{\partial G}{\partial y} (u, Y_u) \right]^2 du < \infty \text{ since } \int_0^T \mathbb{E} \sigma(Y_u)^2 \left[\frac{\partial G}{\partial y} (u, Y_u) \right]^2 du < \infty,$$

so that the first assertion is proved. Applying [10] (Lemma 1.3.4) directly to Formula (7) gives the *moreover*-part. □

Proof of Theorem 2.1. $(C1) \Leftrightarrow (C1') \Leftrightarrow (C2)$ Letting $D(u) := H(u)^2$ and $r \in (1, \infty]$ such that $\frac{1}{2} - \frac{1}{2r} = \eta$ the equivalences follow from Lemmas 3.2 and 3.3.

$(C2) \Leftrightarrow (C3)$ follows from Proposition 3.4.

$(C2) \Leftrightarrow (C4)$ follows from Lemma 3.8.

$(C5) \Rightarrow (C1)$ follows from Proposition 3.5.

$(C2) \Rightarrow (C5)$ follows from Proposition 3.6 where $\rho + \eta = \frac{1}{2}$. The *moreover* part is a consequence of the proof of Proposition 3.6. □

Proof of Theorem 2.5. One has to combine Lemma 3.8 and $(C1) \Leftrightarrow (C2)$ of Theorem 2.1. □

Proof of Theorem 2.6. (a) We start with the right-hand side inclusion. Since for $\lambda \in (0, T^{-1/2})$ we can use $\|0\|_{\mathrm{D}_{1,2}} + \lambda \|g(Y_T)\|_{L_2} \le c\lambda^{1-2\eta}$ with $c := T^{-\eta} \|g(Y_T)\|_{L_2}$, we may assume $\lambda \ge T^{-1/2}$. Taking $g(Y_T) = A_0^\lambda + A_1^\lambda$ with

$$A_0^\lambda := \mathbb{E}g(Y_T) + \int_0^t \frac{\partial G}{\partial y}(s, Y_s) dY_s, \qquad A_1^\lambda := \int_t^T \frac{\partial G}{\partial y}(s, Y_s) dY_s,$$

and $t := T - \lambda^{-2} \in [0, T)$, we have

$$\begin{aligned} &\sup_{\lambda \ge T^{-1/2}} \lambda^{2\eta-1} K(g(Y_T), \lambda; \mathrm{D}_{1,2}, L_2) \\ &\le \sup_{\lambda \ge T^{-1/2}} \lambda^{2\eta-1} \left(\|A_0^\lambda\|_{D_{1,2}} + \lambda \|A_1^\lambda\|_{L_2} \right) \\ &\le \sup_{\lambda \ge T^{-1/2}} \lambda^{2\eta-1} \left(\|DA_0^\lambda\|_{L_2^{\mathcal{H}}} + \|g(Y_T)\|_{L_2} + (1+\lambda)\|A_1^\lambda\|_{L_2} \right). \end{aligned}$$

So it remains to show that

$$\sup_{\lambda \ge T^{-1/2}} \lambda^{2\eta-1} \|DA_0^\lambda\|_{L_2^{\mathcal{H}}} < \infty \quad \text{and} \quad \sup_{\lambda \ge T^{-1/2}} \lambda^{2\eta} \|A_1^\lambda\|_{L_2} < \infty. \tag{18}$$

We start with the left-hand side inequality. Exploiting again [10] (Lemma 1.3.4), we conclude

$$\begin{aligned} \|DA_0^\lambda\|_{L_2^{\mathcal{H}}}^2 &= \mathbb{E}|A_0^\lambda - \mathbb{E}A_0^\lambda|^2 + \int_0^t \mathbb{E} \|DK(s, Y_s)\|_{\mathcal{H}}^2 ds \\ &\le \mathbb{E}|g(Y_T)|^2 + \int_0^t \mathbb{E} \|DK(s, Y_s)\|_{\mathcal{H}}^2 ds \end{aligned}$$

so that it remains to check that

$$\sup_{\lambda \ge T^{-1/2}} \lambda^{4\eta-2} \int_0^t \mathbb{E} \|DK(s, Y_s)\|_{\mathcal{H}}^2 ds < \infty.$$

Because of Formula (17) and $\sigma(y) \sim y$, this is equivalent to

$$\sup_{t \in [0,T)} (T - t)^{1-2\eta} \int_0^t s\mathbb{E} \left[\left(\sigma\sigma' \frac{\partial G}{\partial y} \right)(s, Y_s) + \left(\sigma^2 \frac{\partial^2 G}{\partial y^2} \right)(s, Y_s) \right]^2 ds < \infty$$

which follows from Theorem 2.1 since $|\sigma'| \le \kappa'$. So the left-hand side inequality of Formula (18) is verified. For the right-hand side Theorem 2.1 gives

$$\|A_1^\lambda\|_{L_2} = \left(\int_t^T \|K(u,Y_u)\|_{L_2}^2 du \right)^{\frac{1}{2}} \le c_{(2.1)}(2\eta)^{-\frac{1}{2}}(T-t)^\eta = c_{(2.1)}(2\eta)^{-\frac{1}{2}}\lambda^{-2\eta}.$$

(b) The left-hand side inclusion of the theorem follows from Theorem 2.5 and the proof of Proposition 3.5. □

Remark 3.9 It would be of interest to know to what extend we have equality for the left-hand side in Formula (12). The proof of the right-hand side inclusion gives a decomposition $g(Y_T) = A_0^\lambda + A_1^\lambda$, however A_0^λ is not of form $g_0(Y_T)$ as it would be required to get a converse to the left-hand side inclusion of Formula (12).

4 Examples

For simplicity we restrict our-self to $\sigma(y) = y$ and $x_0 = 1$. Let $K > 0$,

$$\theta \in \left(0, \frac{1}{2}\right), \quad \text{and} \quad h_\theta(x) := \begin{cases} x^{-\theta} & : \ x > 0 \\ 0 & : \ x \le 0 \end{cases}.$$

Then the following table shows that the parameter η from Theorem 2.1 may range through the whole interval $(0, 1/2]$:

Case	$g(y)$	Optimal η in Theorem 2.1
A	$(y-K)^+$	$1/2$
B	$((y-K)^+)^\theta$	$(1/4) + (\theta/2)$
C	$\chi_{[K,\infty)}(y)$	$1/4$
D	$h_\theta\left((T/2) + \log y\right)$	$(1/4) - (\theta/2)$

Case A: Using the hedging formula for the European Call Option in the Black-Scholes model (see for example [9]), we immediately get

$$K(t,y) = yN\left(\frac{\log \frac{y}{K} + \frac{1}{2}(T-t)}{\sqrt{T-t}} \right) \quad \text{with} \quad N(\xi) := \frac{1}{\sqrt{2\pi}} \int_\infty^\xi e^{-\frac{\eta^2}{2}} d\eta$$

so that $\sup_{0 \le t < T} \|K(t,Y_t)\|_{L_2} \le \|Y_T\|_{L_2} < \infty$. □

Cases B and C follow from Gobet-Temam [7] (Theorems 1 and 2) in combination with property (C1') from Remark 2.2.

Case D: It is sufficient to show that

$$\lim_{t\to T}(T-t)^{\frac{\theta}{2}+\frac{1}{4}}\left\|K(t,Y_t)\right\|_{L_2}\in(0,\infty).$$

Using $\Gamma(t,x,\xi)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\xi-x+\frac{t}{2})^2}{2t}}$ and $K(t,Y_t)=\frac{\partial F}{\partial x}(t,X_t)$ on can compute

$$K(t,Y_t)=\int_{\mathbb{R}}h_\theta\left(W_t+\eta\sqrt{T-t}\right)\frac{\eta}{\sqrt{T-t}}e^{-\frac{\eta^2}{2}}\frac{d\eta}{\sqrt{2\pi}}$$

for $t\in[0,T)$, so that

$$\begin{aligned}&\mathbb{E}K(t,Y_t)^2\\ &=\frac{1}{\sqrt{2\pi t}}(T-t)^{-\theta-\frac{1}{2}}\int_{\mathbb{R}}e^{-\frac{y^2}{2(t/(T-t))}}\left[\int_{\mathbb{R}}h_\theta(\eta+y)\eta e^{-\frac{\eta^2}{2}}\frac{d\eta}{\sqrt{2\pi}}\right]^2dy.\end{aligned}$$

The assertion follows by majorized convergence since

$$0<\int_{\mathbb{R}}\left[\int_{\mathbb{R}}h_\theta(\eta+y)\eta e^{-\frac{\eta^2}{2}}\frac{d\eta}{\sqrt{2\pi}}\right]^2dy<\infty.$$

□

5 Appendix

For convenience of the reader we summarize some classical or technical facts used throughout this paper. If nothing stated on the contrary, we shall use the notation from Section 1. In particular, the functions F, G, and K are defined as in formulas (4), (6), and (9) so that

$$K(u,e^x)=\left(\sigma\frac{\partial G}{\partial y}\right)(u,e^x)=\left(\widehat{\sigma}\frac{\partial F}{\partial x}\right)(u,x). \tag{19}$$

Theorem 5.1 ([3], [4]) *There exists a function* $\Gamma:(0,T]\times\mathbb{R}\times\mathbb{R}\to[0,\infty)\in C^\infty$ *such that the following is satisfied:*

(i) *For* $(s,x,\xi)\in(0,T]\times\mathbb{R}\times\mathbb{R}$ *one has*

$$\frac{\partial\Gamma}{\partial s}(s,x,\xi)=\frac{\widehat{\sigma}(x)^2}{2}\frac{\partial^2\Gamma}{\partial x^2}(s,x,\xi)+\widehat{b}(x)\frac{\partial\Gamma}{\partial x}(s,x,\xi).$$

(ii) *If* $(\widetilde{X}_t)_{t\in[0,T]}$ *is the unique strong solution of* Formula (3) *with* $\widetilde{X}_0=x$ *a.s., then for* $s\in(0,T]$ *and* $B\in\mathcal{B}(\mathbb{R})$ *one has*

$$\mathbb{P}\left(\widetilde{X}_s\in B\right)=\int_B\Gamma(s,x,\xi)d\xi.$$

(iii) *Let* $k\in\{0,1\}$ *and* $l,m\in\{0,1,2,...\}$. *There exists a constant* $c>0$ *such that for* $(s,x,\xi)\in(0,T]\times\mathbb{R}\times\mathbb{R}$ *one has that*

$$\left|\frac{\partial^{k+l+m}\Gamma}{\partial s^k\partial x^l\partial\xi^m}(s,x,\xi)\right|\le c\,s^{-k-\frac{l+m}{2}}\gamma_{cs}(x-\xi)\quad\textit{where}\quad\gamma_t(\eta)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{\eta^2}{2t}}.$$

Moreover, assuming that $f:\mathbb{R}\to\mathbb{R}$ *satisfies Definition 1.1 (i), we have*

$$\frac{\partial^{k+l+m}}{\partial x^k\partial s^l\partial x^m}\int_{\mathbb{R}}\Gamma(s,x,\xi)f(\xi)d\xi=\int_{\mathbb{R}}\frac{\partial^{k+l+m}\Gamma}{\partial x^k\partial s^l\partial x^m}(s,x,\xi)f(\xi)d\xi$$

on $(0,T]\times\mathbb{R}$ *for* $k,m\in\{0,1,2,...\}$ *and* $l\in\{0,1\}$.

Lemma 5.2 *Assume that* $f:\mathbb{R}\to\mathbb{R}$ *is locally integrable, satisfies Definition 1.1 (i), and that*

$$\mathbb{E}(If)(x+W_t)^2<\infty\quad\textit{with}\quad(If)(\xi):=\int_{x_0}^{\xi}|f(\eta)|d\eta$$

for all $t>0$ *and* $x\in\mathbb{R}$. *Then* $f\in C_\gamma$.

Proof. Without loss of generality we can assume that $f\ge 0$.

(a) Observing that for all $t,c>0$ there is an $t_0\in(0,t)$ such that

$$\frac{1}{\sqrt{s}}e^{-\frac{x^2}{2s}}\le\frac{1}{\sqrt{t}}e^{-\frac{x^2}{2t}}\quad\text{for all}\quad s\in(0,t_0]\quad\text{and}\quad x\notin[-c,c]$$

we get $\lim_{t\downarrow 0} \mathbb{E}h(W_t) = h(0)$ for a continuous $h : \mathbb{R} \to [0,\infty)$ with $\mathbb{E}h(W_t) < \infty$ for all $t > 0$. Since

$$\sup_{s\in[\varepsilon,t]} \mathbb{E}(If)(x+W_s)^2 \le \sqrt{t/\varepsilon}\,\mathbb{E}(If)(x+W_t)^2$$

for $0 < \varepsilon \le t$, this gives

$$\sup_{s\in(0,t]} \mathbb{E}(If)(x+W_s)^2 < \infty \quad \text{for all} \quad t > 0 \text{ and } x \in \mathbb{R}.$$

(b) Using that $\widehat{\sigma}(x) = e^{-x}\sigma(e^x)$, $\widehat{\sigma} \ge \varepsilon_0 > 0$, and $|\sigma\sigma''| \le \kappa$ we get that

$$\begin{aligned}
&\sup_{t\in[0,T)} \mathbb{E}\left|\int_{y_0}^{Y_t} \sigma''(\eta)G(t,\eta)d\eta\right|^2 \\
&\le \left(\frac{\kappa}{\varepsilon_0}\right)^2 \sup_{t\in[0,T)} \mathbb{E}\left|\int_{y_0}^{Y_t} \frac{G(t,\eta)}{\eta}d\eta\right|^2 \\
&= \left(\frac{\kappa}{\varepsilon_0}\right)^2 \sup_{t\in[0,T)} \mathbb{E}\left|\int_{x_0}^{X_t} F(t,\xi)d\xi\right|^2 \\
&\le \left(\frac{\kappa}{\varepsilon_0}\right)^2 \sup_{t\in[0,T)} \mathbb{E}\left|\int_{x_0}^{X_t}\int_{\mathbb{R}} c_{(5.1)}\gamma_{c_{(5.1)}(T-t)}(\xi')f(\xi'+\xi)d\xi'd\xi\right|^2 \\
&\le \left(\frac{\kappa}{\varepsilon_0}\right)^2 c_{(5.1)}^2 \sup_{t\in[0,T)} \mathbb{E}\int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t)}(\xi')\left|\int_{x_0}^{X_t} f(\xi'+\xi)d\xi\right|^2 d\xi' \\
&= \left(\frac{\kappa}{\varepsilon_0}\right)^2 c_{(5.1)}^2 \sup_{t\in[0,T)} \mathbb{E}\int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t)}(\xi')\left|\int_{x_0+\xi'}^{X_t+\xi'} f(\xi)d\xi\right|^2 d\xi' \\
&\le 2\left(\frac{\kappa}{\varepsilon_0}\right)^2 c_{(5.1)}^2 \sup_{t\in[0,T)} \left[\mathbb{E}\int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t)}(\xi')(If)(X_t+\xi')^2 d\xi'\right] \\
&\quad +2\left(\frac{\kappa}{\varepsilon_0}\right)^2 c_{(5.1)}^2 \sup_{t\in[0,T)} \left[\int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t)}(\xi')(If)(x_0+\xi')^2 d\xi'\right] \\
&\le 2\left(\frac{\kappa}{\varepsilon_0}\right)^2 c_{(5.1)}^3 \mathbb{E}(If)(x_0+W_{c_{(5.1)}T})^2 \\
&\qquad +2\left(\frac{\kappa}{\varepsilon_0}\right)^2 c_{(5.1)}^2 \sup_{t\in[0,T)} \mathbb{E}(If)(x_0+W_{c_{(5.1)}(T-t)})^2
\end{aligned}$$

so that we are done because of step (a). □

Lemma 5.3 *For $f \in C_\gamma$ one has*

(i) $\mathbb{E}\sup_{u\in[0,b]} K(u,Y_u)^2 < \infty$ *for* $b \in [0,T)$,

(ii) $\sup_{t\in[0,T)} \mathbb{E}\left|\mathbb{E}\left(K(t,Y_t) \mid \mathcal{F}_{(t-\delta)\vee 0}\right)\right|^2 < \infty$ *for* $\delta \in (0,T)$.

Proof. (i) We have

$$\begin{aligned}
&\mathbb{E}\sup_{u\in[0,b]} K(u,Y_u)^2 \\
&\qquad \le \|\widehat{\sigma}\|_\infty^2\, \mathbb{E}\sup_{u\in[0,b]}\left|\frac{\partial F}{\partial x}(u,X_u)\right|^2 \\
&\qquad = \|\widehat{\sigma}\|_\infty^2\, \mathbb{E}\sup_{u\in[0,b]}\left|\int_{\mathbb{R}}\frac{\partial \Gamma}{\partial x}(T-u,X_u,\xi)f(\xi)d\xi\right|^2 \\
&\qquad \le \|\widehat{\sigma}\|_\infty^2\, \frac{c_{(5.1)}^2}{T-b}\mathbb{E}\sup_{u\in[0,b]}\left|\int_{\mathbb{R}}\gamma_{c_{(5.1)}(T-u)}(X_u-\xi)|f(\xi)|d\xi\right|^2 \\
&\qquad \le \|\widehat{\sigma}\|_\infty^2\, \frac{c_{(5.1)}^2}{T-b}\mathbb{E}\sup_{u\in[0,b]}|c(1+|X_u|^m)|^2 \\
&\qquad < \infty
\end{aligned}$$

where we have used Definition 1.1 (i).

(ii) For $0 \le t \le \delta$ item (i) gives

$$\mathbb{E}\left|\mathbb{E}\left(K(t,Y_t)|\mathcal{F}_{(t-\delta)\vee 0}\right)\right|^2 \le \sup_{u\in[0,\delta]}\mathbb{E}K(u,Y_u)^2 < \infty.$$

Hence we assume $0 < \delta \le t < T$ and $s := t-\delta$. We get, a.s.,

$$\begin{aligned}
\left|\mathbb{E}\left(\widehat{\sigma}(X_t)\frac{\partial F}{\partial x}(t,X_t) \mid \mathcal{F}_s\right)\right| &= \left|\int_{\mathbb{R}}\widehat{\sigma}(\xi)\frac{\partial F}{\partial \xi}(t,\xi)\Gamma(\delta,X_s,\xi)d\xi\right| \\
&= \left|\int_{\mathbb{R}}F(t,\xi)\frac{\partial}{\partial \xi}\left(\widehat{\sigma}(\xi)\Gamma(\delta,X_s,\xi)\right)d\xi\right| \\
&\le \|\widehat{\sigma}'\|_\infty\int_{\mathbb{R}}|F(t,\xi)|\,|\Gamma(\delta,X_s,\xi)|\,d\xi \\
&\quad + \|\widehat{\sigma}\|_\infty\int_{\mathbb{R}}|F(t,\xi)|\left|\frac{\partial \Gamma}{\partial \xi}(\delta,X_s,\xi)\right|d\xi.
\end{aligned}$$

Since for $m = 0, 1$

$$\left| \frac{\partial^m \Gamma}{\partial \xi^m}(\delta, \eta, \xi) \right| \;\leq\; c_{(5.1)}\, \delta^{-\frac{m}{2}}\, \gamma_{c_{(5.1)}\delta}(\eta - \xi),$$

we get, with $d = c_{(5.1)} \left[\|\widehat{\sigma}'\|_\infty + \sqrt{\frac{1}{\delta}}\, \|\widehat{\sigma}\|_\infty \right] < \infty$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \widehat{\sigma}(\xi) \frac{\partial F}{\partial \xi}(t, \xi) \Gamma(\delta, X_s, \xi) d\xi \right| \\
\leq\;& d \int_{\mathbb{R}} |F(t, \xi)| \gamma_{c_{(5.1)}\delta}(X_s - \xi) d\xi \\
\leq\;& d \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma(T - t, \xi, \xi') |f(\xi')| d\xi'\; \gamma_{c_{(5.1)}\delta}(X_s - \xi) d\xi \\
\leq\;& c_{(5.1)}\, d \int_{\mathbb{R}} \int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t)}(\xi - \xi') |f(\xi')| d\xi'\; \gamma_{c_{(5.1)}\delta}(X_s - \xi) d\xi \\
=\;& c_{(5.1)}\, d \int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t+\delta)}(\xi - X_s) |f(\xi)| d\xi.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\mathbb{E} \left| \int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t+\delta)}(\xi - X_s) |f(\xi)| d\xi \right|^2 \\
\leq\;& \mathbb{E} \int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t+\delta)}(\xi - X_s) f(\xi)^2 d\xi \\
\leq\;& c_{(5.1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \gamma_{c_{(5.1)}s}(x_0 - \xi') \gamma_{c_{(5.1)}(T-t+\delta)}(\xi - \xi') f(\xi)^2 d\xi d\xi' \\
=\;& c_{(5.1)} \int_{\mathbb{R}} \gamma_{c_{(5.1)}(T-t+\delta+s)}(x_0 - \xi) f(\xi)^2 d\xi \\
\leq\;& c_{(5.1)} \sup_{u \in [\delta, 2T]} \int_{\mathbb{R}} \gamma_{c_{(5.1)}u}(x_0 - \xi) f(\xi)^2 d\xi \\
<\;& \infty.
\end{aligned}$$

□

Lemma 5.4 *The process* $\left(\sigma^2(Y_t) \frac{\partial^2 G}{\partial y^2}(t, Y_t) \right)_{t \in [0,T)} \subseteq L_2(\Omega, \mathcal{F}, \mathbb{P})$ *is a martingale with respect to* $(\mathcal{F}_t)_{t \in [0,T)}$*, so that the function* $H : [0, T) \to [0, \infty)$ *defined in* Formula (9) *is continuous and increasing.*

Proof. By Formula (7) we have

$$\frac{\partial}{\partial t}\left(\sigma^2\frac{\partial^2 G}{\partial y^2}\right)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2\frac{\partial^2 G}{\partial y^2}\right)=0$$

on $[0,T)\times(0,\infty)$ so that $\left(\sigma^2(Y_t)\frac{\partial^2 G}{\partial y^2}(t,Y_t)\right)_{t\in[0,T)}$ is a local martingale. To conclude, we show that

$$\mathbb{E}\sup_{t\in[0,b]}\left[\sigma^2(Y_t)\frac{\partial^2 G}{\partial y^2}(t,Y_t)\right]^2<\infty$$

for all $b\in[0,T)$. Because of Formula (6) one quickly checks that

$$\sigma^2(y)\frac{\partial^2 G}{\partial y^2}(t,y)=\frac{\sigma(y)^2}{y^2}\left(\frac{\partial^2 F}{\partial x^2}(t,\log y)-\frac{\partial F}{\partial x}(t,\log y)\right).$$

In view of Lemma 5.3 it remains to verify that

$$\mathbb{E}\sup_{t\in[0,b]}\left|\frac{\partial^2 F}{\partial x^2}(t,X_t)\right|^2<\infty$$

which follows from Theorem 5.1 and Definition 1.1. □

Lemma 5.5 *For $0\le a<b\le T$ and $c:=2\,\|\widehat{\sigma}\|_\infty^2\,e^{2\|\widehat{\sigma}\|_\infty^2 T}$ one has that*

$$\mathbb{E}\left(\left(\frac{Y_b}{Y_a}-1\right)^2\,\Big|\,\mathcal{F}_a\right)\;\le\;c\,(b-a)\ a.s.$$

The lemma follows by a standard computation using Gronwall's lemma.

References

[1] C. Bennet and R. Sharpley. *Interpolation of operators.* Academic Press, 1988.

[2] J. Bergh and J. Lőfströ̋m. *Interpolation spaces. An introduction.* Springer, 1976.

[3] A. Friedman. *Partial Differential Equations of Parabolic Type.* 1964.

[4] A. Friedman. *Stochastic Differential Equations and Applications*, volume 1. Academic Press, 1975.

[5] S. Geiss. On the approximation of stochastic integrals and weighted BMO. In R. Buckdahn, H. J. Engelbert, and M. Yor, editors, *Stochastic Processes and Related Topics*, Stochastics Monographs. Gordon & Breach, 2000.

[6] S. Geiss. Quantitative approximation of certain stochastic integrals. *Stochastics and Stochastics Reports*, 73:241–270, 2002.

[7] E. Gobet and E. Temam. Discrete time hedging errors for options with irregular payoffs. *Finance and Stochastics*, 5(357-367), 2001.

[8] M. Hujo. *Licentiate Thesis, University of Jyvaeskylae*, In preparation.

[9] D. Lamberton and B. Lapeyre. *Stochastic Calculus Applied to Finance.* Chapman & Hall, London, 1996.

[10] D. Nualart. *The Malliavin Calculus and Related Topics.* Springer, 1995.

[11] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential system. *Systems and Control Letters*, 14:55–61, 1990.

[12] D.W. Strook and S.R.S. Varadhan. *Multidimensional Diffusion Processes*, volume 233 of *Grundlehren der mathematischen Wissenschaften.* Springer, 1979.

[13] R. Zhang. *Couverture approchée des options Européennes.* PhD thesis, Ecole Nationale des Ponts et Chaussées, Paris, 1998.