FRACTIONAL BROWNIAN MOTION AND MARTINGALE-DIFFERENCES

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Abstract. We generalize a result of Sottinen [6] by proving an approximation theorem for the fractional Brownian motion, with $H > 1/2$, using martingale-differences.

1. Introduction

The fractional Brownian motion is a continuous Gaussian process with stationary increments. It is perhaps the simplest process with a long-range dependency property: when the so-called Hurst index, $H > 1/2$ the increments of the process are positively correlated, when $H < 1/2$ the increments are negatively correlated and when $H = 1/2$ they are uncorrelated and we have a Brownian motion. Some studies of financial time series and telecommunication networks have shown that this kind of process with long-range dependency - memory - might be a better model in some cases than the traditional standard Brownian motion.

T. Sottinen [6] has shown how to approximate the fractional Brownian motion, in case $H > 1/2$ by a "disturbed" random walk. In section 2 we show how to do this by martingale-differences.

2. Fractional Brownian motion and martingale-difference

2.1. **Fractional Brownian motion.** We denote by $(Z_t)_{t\geq 0}$ a normalized fractional Brownian motion (FBM) with self-similarity parameter $H \in (0,1)$. Fractional Brownian motion is a continuous zero mean Gaussian process which has stationary increments and the following covariance function

$$EZ_tZ_s = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}).$$

We assume that Z is defined on a probability space (Ω, F, P). If $H = \frac{1}{2}$ we have the standard Brownian motion, which is denoted by W, with independent increments. We assume that $H > \frac{1}{2}$, so that the increments are positively correlated and we have the following kernel representation of Z with respect to the standard Brownian motion (see [2] and [4])

$$Z_t = \int_0^t z(t,s)dW_s, \tag{2.1}$$

with the deterministic kernel

$$z(t,s) := \begin{cases} (H-1/2)c_H s^{1/2-H}\int_s^t u^{H-1/2}(u-s)^{H-3/2}du, & 0 < s \leq t \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

where

$$c_H := \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}.$$

Remark 2.1. The function z is square integrable. When $H > 1/2$, there exists constant M_H such that $z(t,s) \leq M_H s^{1/2-H}$ for all $t,s \in \mathbb{R}$.

2.2. **Convergence to the FBM.** We state here two results that are needed to prove the Theorem 2.8. We denote by $D = D(0,T)$ the Skorohod space of right continuous functions on the interval $[0,T]$, that have left-hand limits and equip D with the following metric.

Definition 2.2. Let $\Lambda := \{\lambda : [0,1] \to [0,1] : \lambda$ a strictly increasing and continuous mapping of $[0,1]$ onto itself$\}$. We define

$$d(x,y) := \inf\{\varepsilon > 0 : \exists \lambda \in \Lambda \text{ such that } ||\lambda|| \leq \varepsilon \text{ and } \sup_t |x(t) - y(\lambda(t))| \leq \varepsilon\},$$

where $||\lambda|| := \sup_{s \neq t} |\log \frac{\lambda(t)-\lambda(s)}{t-s}|$.

Under this metric D is a separable and complete metric space. For details we refer to [1]. Let X be a random function of D, i.e. $X : \Omega \to D$ and $(X^n)_{n=1}^{\infty}$ be a sequence of random functions of D and
$T_X := \{t \in (0,1) : P(X_t \neq X_{t-}) = 0\} \cup \{0,1\}$. We denote by $\xrightarrow{d}$ the convergence in distribution and by $\xrightarrow{P}$ the convergence in probability. By convergence in distribution we mean, that a sequence X^n converges in distribution to X, if for every bounded, continuous real function ϕ on D

$$E\phi(X^n) \xrightarrow{n\to\infty} E\phi(X).$$

For details of the convergence in distribution in D we refer to [1]. The proof of the first theorem we need can be found in [1], p.129.

Theorem 2.3. *Suppose that*

$$(X^n_{t_1}, \ldots, X^n_{t_k}) \xrightarrow{d} (X_{t_1}, \ldots, X_{t_k})$$

holds whenever $t_1, \ldots, t_k \in T_X$. *Assume further that* $P\{X_1 \neq X_{1-}\} = 0$ *and that*

$$E\{|X^n_t - X^n_{t_1}|^C |X^n_{t_2} - X^n_t|^C\} \leq [F(t_2) - F(t_1)]^{2\alpha},$$

for $t_1 \leq t \leq t_2$ *and* $n \geq 1$, *where* $C \geq 0$, $\alpha > \frac{1}{2}$ *and* F *is a nondecreasing, continuous function on* $[0,1]$. *Then*

$$X^n \xrightarrow{d} X.$$

The second theorem we need can be found for example in [5], p. 511. Let's suppose that stochastic sequences are given on the probability space (Ω, F, P). We denote a sequence of martingale-differences by $\xi^n := (\xi^n_i, F^n_i)$, $1 \leq i \leq n$, meaning that the process ξ^n is adapted to the filtration F^n, where $F^n_0 = (\emptyset, \Omega)$ and $F^n_i \subset F^n_{i+1} \subset F$. Let

$$X^n_t = \sum_{i=1}^{\lfloor nt \rfloor} \xi^n_i, \quad 0 \leq t \leq T,$$

where $X^n_t = 0$, when $\lfloor nt \rfloor = 0$.

Theorem 2.4. *Let* $t \in (0,T]$, $\sigma^2_t \geq 0$ *and let the square-integrable martingale-differences* ξ^n, $n \geq 1$, *satisfy the Lindeberg condition: for* $\varepsilon > 0$

$$\sum_{i=1}^{\lfloor nt \rfloor} E((\xi^n_i)^2 I_{\{|\xi^n_i| > \varepsilon\}} | F^n_{i-1}) \xrightarrow{P} 0.$$

Then

$$\sum_{i=1}^{\lfloor nt \rfloor} (\xi^n_i)^2 \xrightarrow{P} \sigma^2_t \quad \Rightarrow \quad X^n_t \xrightarrow{d} N(0, \sigma^2_t).$$

Now we can prove, by using representation (2.1), that one can approximate the fractional Brownian motion with martingale-differences, when $H > 1/2$. Let $(\xi^n)_{n\geq 1} = (\xi_i^n, F_i^n)_{n\geq 1}$, $i \leq n$, be a sequence of square integrable martingale-differences such that

$$\lim_{n\to\infty} \frac{(\xi_i^n)^2}{1/n} = 1 \quad a.s. \tag{2.3}$$

for all $1 \leq i \leq n$ and

$$\max_{1\leq i\leq n} |\xi_i^n| \leq \frac{C}{\sqrt{n}} \quad a.s. \quad \text{for some } C \geq 1. \tag{2.4}$$

This kind of sequences are pretty easy to construct as we can see in the next example.

Example 2.5. Let ε_i^n, $i = 1, \ldots, n$, be for all $n \geq 1$, independent random variables satisfying $P(\varepsilon_i^n = 1) = P(\varepsilon_i^n = -1) = 1/2$ and define the filtration $F_i^n := \sigma(\varepsilon_1^n, \ldots, \varepsilon_i^n)$. Let $(V_i^n)_{i=1}^n$ be a sequence of predictable functions such that $|V_i^n| \leq C$ a.s for all $i \leq n$, and $(V_i^n)^2 \xrightarrow{n\to\infty} 1$ a.s. for all $1 \leq i \leq n$. Denote $\xi_i^n := V_i^n \varepsilon_i^n / \sqrt{n}$. Then the sequence $(\xi_i^n)_{n\geq 1}$, $i \leq n$, is a sequence of martingale-differences and satisfies the conditions (2.3) and (2.4).

Denote

$$W_t^n := \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^n.$$

The following result is from [3], p. 437. This theorem states, that if we change the condition (2.3) a bit, then the process W_t^n converges in distribution to a fractional Brownian motion with $H = 1/2$. Thus this result is kind of an extension of the Theorem 2.8.

Theorem 2.6. *If we replace the condition* (2.3) *by a condition*

$$\sum_{i=1}^{\lfloor nt \rfloor} (\xi_i^n)^2 \xrightarrow{n\to\infty} t \quad a.s.,$$

then the process W_t^n converges in distribution to a Brownian motion W.

Let's finally state a lemma that is needed for the proof of the Theorem 2.8.

Lemma 2.7. *Let z and $(\xi^n)_{n\geq 1}$ be as above, $H \in (1/2, 1)$ and $t_k, t_l \in [0,1]$. Then*

$$\sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k \rfloor}{n}, s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_l \rfloor}{n}, s)ds(\xi_i^n)^2 \xrightarrow{n\to\infty} \int_0^1 z(t_k, s)z(t_l, s)ds \quad a.s.$$

Proof. Let's take $t_l = t_k$ for simplicity and prove first the case

$$\sum_{i=1}^n n^2 \Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(s)ds \Big)^2 (\xi_i^n)^2 \xrightarrow{n\to\infty} \int_0^1 f^2(s)ds \quad \text{a.s.}, \tag{2.5}$$

where f is a square-integrable function such that $0 \leq f(s) \leq Ms^{1/2-H}$. Notice that $z(t_k, s) \leq M_H s^{1/2-H}$.

Denote now

$$g_n(t) := n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(s)ds \Big(\frac{\xi_i^n}{1/\sqrt{n}} \Big),$$

where $t \in (\frac{i-1}{n}, \frac{i}{n}]$. Now

$$g_n^2(t) = \Big(n\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(s)ds\Big)^2\Big(\frac{\xi_i^n}{1/\sqrt{n}}\Big)^2,$$

and so we need to prove that

$$\int_0^1 g_n^2(s)ds = \sum_{i=1}^n n^2\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(s)ds\Big)^2(\xi_i^n)^2 \xrightarrow{n\to\infty} \int_0^1 f^2(s)ds \quad \text{a.s.}$$

Now we have, for $t \in (\frac{i-1}{n}, \frac{i}{n}]$

$$g_n^2(t) \le M^2 n^3\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{1/2-H}ds\Big)^2(\xi_i^n)^2 =: h_n(t)$$

and, for all $n \ge 1$, and for $0 < \varepsilon < \frac{1-H}{H-1/2}$ we get by using Hölder's inequality

$$\begin{aligned}\int_0^1 h_n^{1+\varepsilon}(s)ds &= M^2\sum_{i=1}^n n^{3(1+\varepsilon)-1}\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{1/2-H}ds\Big)^{2(1+\varepsilon)}(\xi_i^n)^{2(1+\varepsilon)}\\ &\le M^2\sum_{i=1}^n n^{3(1+\varepsilon)-1}n^{-(1+2\varepsilon)}\int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{(1/2-H)(2(1+\varepsilon))}ds(\xi_i^n)^{2(1+\varepsilon)}\\ &\le M^2C^{2(1+\varepsilon)}\sum_{i=1}^n\int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{(1-2H)(1+\varepsilon)}ds \quad \text{a.s.}\\ &\le M^2C^{2(1+\varepsilon)}\int_0^1 s^{(1-2H)(1+\varepsilon)}ds < \infty,\end{aligned}$$

since $(1-2H)(1+\varepsilon) > -1$. Thus $(h_n)_{n\ge1}$ is a.s. uniformly integrable and so is $(g_n^2)_{n\ge1}$. Furthermore, for all $t \in (0,1]$

$$g_n^2(t) = \Big(n\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(s)ds\Big)^2\Big(\frac{\xi_i^n}{1/\sqrt{n}}\Big)^2 \xrightarrow{n\to\infty} f^2(t) \quad \text{a.s.},$$

where $i = i(t,n)$, because

$$\Big(n\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(s)ds\Big)^2 \xrightarrow{n\to\infty} f^2(t) \quad \text{and} \quad \Big(\frac{\xi_i^n}{1/\sqrt{n}}\Big)^2 \xrightarrow{n\to\infty} 1 \quad \text{a.s.}$$

Thus

$$\int_0^1 g_n^2(s)ds \xrightarrow{n\to\infty} \int_0^1 f^2(s)ds \quad \text{a.s.}$$

and we have proved (2.5). Next we will prove the original claim. Let us first denote $t_k^n := \frac{\lfloor nt_k\rfloor}{n}$ and $t_l^n := \frac{\lfloor nt_l\rfloor}{n}$. We need to show that the difference

$$\sum_{i=1}^n n^2\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_l,s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k^n,s)ds\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_l^n,s)ds\Big)(\xi_i^n)^2$$

tends to zero almost surely as n tends to infinity. Notice that $z(t_k,s) \geq z(t_k^n,s)$ because z is increasing with respect to the first argument. Now we obtain

$$0 \leq \sum_{i=1}^n n^2 \Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_l,s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k^n,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_l^n,s)ds \Big) (\xi_i^n)^2$$

$$= \sum_{i=1}^n n^2 \Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(t_l,s) - z(t_l^n,s))ds$$

$$- \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(t_l^n,s) - z(t_l,s) + z(t_l,s))ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(t_k^n,s) - z(t_k,s))ds \Big) (\xi_i^n)^2$$

$$\leq \sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(t_l,s) - z(t_l^n,s))ds (\xi_i^n)^2$$

$$+ \sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(t_l^n,s) - z(t_l,s) + z(t_l,s))ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} |z(t_k^n,s) - z(t_k,s)| ds (\xi_i^n)^2$$

$$\leq \sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(t_l,s) - z(t_l^n,s))ds (\xi_i^n)^2$$

$$+ \sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} |z(t_l^n,s) - z(t_l,s)| ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} |z(t_k^n,s) - z(t_k,s)| ds (\xi_i^n)^2$$

$$+ \sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_l,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} |z(t_k^n,s) - z(t_k,s)| ds (\xi_i^n)^2.$$

From (2.2) we see that

$$|z(t_l,s) - z(t_l^n,s)| \leq M'_H s^{1/2-H} n^{-(H-1/2)}$$

and the same is, of course, true for $|z(t_k,s) - z(t_k^n,s)|$. Now we get that

$$\sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(t_l,s) - z(t_l^n))ds (\xi_i^n)^2$$

$$\leq M'_H n^{-(H-1/2)} \sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{1/2-H} ds (\xi_i^n)^2 \xrightarrow{n\to\infty} 0,$$

because $M'_H n^{-(H-1/2)} \xrightarrow{n\to\infty} 0$ and

$$\sum_{i=1}^n n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{1/2-H} ds (\xi_i^n)^2 \xrightarrow{n\to\infty} \int_0^1 z(t_k,s) s^{1/2-H} ds < \infty,$$

a.s by the first part of this proof. Similarly the last two summands tend to zero and we have the claim. □

Denote now

$$Z_t^n := \int_0^t z^n(t,s)dW_s^n$$

$$= \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt \rfloor}{n}, s)ds\xi_i^n,$$

where function $z^n(t,\cdot)$ is an approximation of function $z(t,\cdot)$: for $s \geq \frac{1}{n}$ and for all $t \in [0,T]$

$$z^n(t,s) := n \int_{s-\frac{1}{n}}^{s} z(\frac{\lfloor nt \rfloor}{n}, u)du.$$

Theorem 2.8. *If $H > 1/2$, then the process Z^n converges in distribution to the fractional Brownian motion Z.*

*Proof.*We prove the theorem by using the Theorem 2.3. First we show that the finite-dimensional distributions of Z^n converge to those of Z. We use Cramer-Wold device and the Theorem 2.4 for that. Let $a_1, \ldots, a_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0,T]$ be arbitrary. We want to show, that the linear combination

$$Y^n := \sum_{k=1}^{d} a_k Z_{t_k}^n$$

converges in distribution to a normally distributed random variable with expectation zero and variance

$$E\Big(\sum_{k=1}^{d} a_k Z_{t_k}\Big)^2.$$

The fact that the expectation is zero is trivial.

Let us write Y^n as

$$Y^n = \sum_{k=1}^{d} a_k Z_{t_k}^n$$

$$= \sum_{k=1}^{d} a_k \sum_{i=1}^{\lfloor nt_k \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k \rfloor}{n}, s)ds\xi_i^n$$

$$= \sum_{i=1}^{\lfloor nT \rfloor} n\xi_i^n \sum_{k=1}^{d} a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k \rfloor}{n}, s)ds$$

$$=: \sum_{i=1}^{\lfloor nT \rfloor} Y_i^n.$$

The Lindeberg condition is satisfied if for all $\varepsilon > 0$ we have that

$$\sum_{i=1}^{\lfloor nT \rfloor} E((Y_i^n)^2 I_{\{|Y_i^n| > \varepsilon\}} | F_{i-1}^n) \xrightarrow{P} 0.$$

Let us consider the set

$$\{|Y_i^n| > \varepsilon\} = \{(Y_i^n)^2 > \varepsilon^2\}.$$

By using the Cauchy-Schwartz inequality and the facts that z is an increasing function with respect to its first argument and decreasing with respect to the second argument(see [2]) we have

$$\begin{aligned}
(Y_i^n)^2 &= n^2(\xi_i^n)^2\Big(\sum_{k=1}^d a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k \rfloor}{n}, s)ds\Big)^2 \\
&\le n^2(\xi_i^n)^2\Big(\sum_{k=1}^d a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(T,s)ds\Big)^2 \\
&\le n^2(\xi_i^n)^2\Big(\sum_{k=1}^d a_k\Big)^2 \frac{1}{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} z^2(T,s)ds \\
&\le n(\xi_i^n)^2\Big(\sum_{k=1}^d a_k\Big)^2 \int_0^{\frac{1}{n}} z^2(T,s)ds \\
&=: n(\xi_i^n)^2 A\delta^n,
\end{aligned}$$

where $A := (\sum_{k=1}^d a_k)^2$ and $\delta^n := \int_0^{\frac{1}{n}} z^2(T,s)ds$.
So we obtain

$$\{|Y_i^n| > \varepsilon\} \subset \{n(\xi_i^n)^2 A\delta^n > \varepsilon^2\}. \tag{2.6}$$

Using the inclusion (2.6) and the Cauchy-Schwartz inequality we get

$$\begin{aligned}
&E((Y_i^n)^2 I_{\{|Y_i^n|>\varepsilon\}}|F_{i-1}^n) \\
&= \Big(n\sum_{k=1}^d a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k \rfloor}{n}, s)ds\Big)^2 E((\xi_i^n)^2 I_{\{|Y_i^n|>\varepsilon\}}|F_{i-1}^n) \\
&\le \Big(n\sum_{k=1}^d a_k \int_0^{\frac{1}{n}} z(T,s)ds\Big)^2 \frac{C^2}{n} E(I_{\{|Y_i^n|>\varepsilon\}}|F_{i-1}^n) \\
&\le C^2 A\delta^n E(I_{\{n(\xi_i^n)^2 A\delta^n>\varepsilon^2\}}|F_{i-1}^n),
\end{aligned}$$

where the first inequality is true almost surely. Now we are able to show that the Lindeberg condition is satisfied:

$$\sum_{i=1}^{\lfloor nT\rfloor} E((Y_i^n)^2 I_{\{|Y_i^n|>\varepsilon\}}|F_{i-1}^n)$$

$$\leq \sum_{i=1}^{\lfloor nT\rfloor} C^2A\delta^n E(I_{\{n(\xi_i^n)^2A\delta^n>\varepsilon^2\}}|F_{i-1}^n) \quad \text{a.s.}$$

$$\leq C^2A\delta^n \sum_{i=1}^{\lfloor nT\rfloor} I_{\{C^2A\delta^n>\varepsilon^2\}} \xrightarrow{n\to\infty} 0$$

Big because $\delta^n \to 0$, so we have $I_{\{C^2A\delta^n>\varepsilon^2\}} = 0$ for large n.
Furthermore,

$$\sum_{i=1}^{\lfloor nT\rfloor}(Y_i^n)^2 = \sum_{i=1}^{\lfloor nT\rfloor}\Big(n\xi_i^n\sum_{k=1}^{d}a_k\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k\rfloor}{n},s)ds\Big)^2$$

$$= \sum_{i=1}^{\lfloor nT\rfloor} n^2(\xi_i^n)^2\sum_{k,l=1}^{d}a_ka_l\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k\rfloor}{n},s)ds\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_l\rfloor}{n},s)ds$$

$$= \sum_{k,l=1}^{d}a_ka_ln^2\sum_{i=1}^{\lfloor nT\rfloor}\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k\rfloor}{n},s)ds\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_l\rfloor}{n},s)ds(\xi_i^n)^2.$$

By Lemma 2.7 and the Itô-isometry

$$\lim_{n\to\infty}\sum_{k,l=1}^{d}a_ka_ln^2\sum_{i=1}^{\lfloor nT\rfloor}\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_k\rfloor}{n},s)ds\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt_l\rfloor}{n},s)ds(\xi_i^n)^2$$

$$= \sum_{k,l=1}^{d}a_ka_l\int_0^T z(t_k,s)z(t_l,s)ds$$

$$= E\Big(\sum_{k,l=1}^{d}a_ka_l\int_0^T z(t_k,s)dW_s\int_0^T z(t_l,s)dW_s\Big)$$

$$= E\Big(\sum_{k,l=1}^{d}a_ka_l\int_0^{t_k} z(t_k,s)dW_s\int_0^{t_l} z(t_l,s)dW_s\Big)$$

$$= E\sum_{k,l=1}^{d}a_ka_lZ_{t_k}Z_{t_l}$$

$$= E\Big(\sum_{k=1}^{d}a_kZ_{t_k}\Big)^2.$$

Now by the Theorem 2.4, the finite-dimensional distributions of Z^n converge to those of Z.

Let $s,t \in [0,T]$, $s < t$, be arbitrary. By the Cauchy-Schwarz inequality, the Itô-isometry and the fact that $E\xi_i^n\xi_j^n = 0$, when $i \neq j$, we get

$$
\begin{aligned}
E(Z_t^n - Z_s^n)^2 &= E\Big(\sum_{i=1}^{\lfloor nt\rfloor} n\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt\rfloor}{n},u) - z(\frac{\lfloor ns\rfloor}{n},u)du\xi_i^n\Big)^2 \\
&= \sum_{i=1}^{\lfloor nt\rfloor}\Big(n\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt\rfloor}{n},u) - z(\frac{\lfloor ns\rfloor}{n},u)du\Big)^2 E(\xi_i^n)^2 \\
&\leq \sum_{i=1}^{\lfloor nt\rfloor} n\int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(\frac{\lfloor nt\rfloor}{n},u) - z(\frac{\lfloor ns\rfloor}{n},u))^2 du E(\xi_i^n)^2 \\
&\leq C^2\sum_{i=1}^{\lfloor nt\rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(\frac{\lfloor nt\rfloor}{n},u) - z(\frac{\lfloor ns\rfloor}{n},u))^2 du \\
&\leq C^2\int_0^t (z(\frac{\lfloor nt\rfloor}{n},u) - z(\frac{\lfloor ns\rfloor}{n},u))^2 du \\
&= C^2\left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}.
\end{aligned}
$$

Now for arbitrary $s < t < u$

$$
\begin{aligned}
E|Z_t^n - Z_s^n||Z_u^n - Z_t^n| &\leq (E(Z_t^n - Z_s^n)^2)^{1/2}(E(Z_u^n - Z_t^n)^2)^{1/2} \\
&\leq C^2\left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^H\left|\frac{\lfloor nu\rfloor}{n} - \frac{\lfloor nt\rfloor}{n}\right|^H \\
&\leq C^2\left|\frac{\lfloor nu\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}.
\end{aligned}
$$

If $u - s \geq \frac{1}{n}$, then

$$E|Z_t^n - Z_s^n||Z_u^n - Z_t^n| \leq |2C^{1/H}(u-s)|^{2H}. \tag{2.7}$$

If on the other hand $u - s < \frac{1}{n}$, then either s and t or t and u lie in a same subinterval $\left[\frac{m}{n}, \frac{m+1}{n}\right)$ for some integer m. Thus the left-hand side of (2.7) is zero. Therefore (2.7) holds for all $s < t < u$ and by the Theorem 2.3 $Z^n \xrightarrow{d} Z$. □

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