THE COMPLETE CONTINUITY PROPERTY IN BANACH SPACES

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Abstract. Let X be a complex Banach space. We show that the following are equivalent: (i) X has the complete continuity property, (ii) for every (or equivalently for some) $1 < p < \infty$, for $f \in h^p(\mathbb{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy, where f_{r_n} is defined by $f_{r_n}(t) = f(r_n e^{it})$ for $t \in [0, 2\pi]$, (iii) for every (or equivalently for some) $1 < p < \infty$, for every $\mu \in V^p(X)$, the bounded linear operator $T : L^q(0, 2\pi) \to X$ defined by $T\phi = \int_0^{2\pi} \phi d\mu$ is compact, where $1/q + 1/p = 1$, (iv) for every (or equivalently for some) $1 < p < \infty$, each $\mu \in V^p(X)$ has a relatively compact range.

Through this note $(X, \|\cdot\|)$ denotes a complex Banach space, $\mathbb{D}$ denotes the open unit disc in the complex plane. λ will be the normalized Lebesgue measure on $[0, 2\pi]$. For a Banach space Y, we denote by B_Y the closed unit ball of Y. Given $1 < p < \infty$ we denote by $h^p(\mathbb{D}, X)$ the space of all X-valued harmonic functions f on $\mathbb{D}$ such that

$$\|f\|_p = \sup_{0<r<1} \Big(\int_0^{2\pi} \|f(re^{it})\|^p d\lambda(t)\Big)^{1/p} < \infty.$$

$h^\infty(\mathbb{D}, X)$ will be the space of all X-valued bounded harmonic functions on $\mathbb{D}$ equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} \|f(z)\|$. For $f \in h^p(\mathbb{D}, X)$ and $n \in \mathbb{Z}$, we define the Fourier coefficient $\hat{f}(n)$ by

$$\hat{f}(n) = r^{-|n|} \int_0^{2\pi} f(re^{it}) e^{-int} d\lambda(t).$$

for $0 < r < 1$. It is clear that $\hat{f}(n)$ is independent from the choice of $0 < r < 1$. We let for $\Lambda \subset \mathbb{Z}$ and $1 < p \le \infty$

$$h^p_\Lambda(\mathbb{D}, X) = \{f \in h^p(\mathbb{D}, X) : \hat{f}(n) = 0 \text{ for } n \notin \Lambda\}.$$

Let $\mathcal{B}$ be the collection of all Borel subsets of $[0, 2\pi]$. If μ is a countably additive X-valued measure on $[0, 2\pi]$, $1 < p < \infty$, we define the p-variation of μ by

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$$\|\mu\|_p = \sup(\sum_{E\in\pi} \frac{\|\mu(E)\|^p}{\lambda(E)^{p-1}})^{1/p}$$

where the supremum is taken over all finite partition of $[0, 2\pi]$, we use the usual convention: $\lambda/0$ is 0 or ∞ provided $\lambda = 0$ or $\lambda > 0$, respectively. For $p = \infty$, we let

$$\|\mu\|_\infty = \inf\{C \geq 0 : \|\mu(E)\| \leq C\lambda(E) \text{ for all } E \in \mathcal{B}\}.$$

We denote by $V^p(X)$ the space of all countably additive X-valued measures μ on $[0, 2\pi]$ such that $\|\mu\|_p < \infty$. For $\mu \in V^p(X)$, the range of μ is defined as the set $\{\mu(E) : E \in \mathcal{B}\}$. Let $\mu \in V^p(X)$ and let $n \in \mathbb{Z}$, the Fourier coefficient $\hat{\mu}(n)$ is defined by

$$\hat{\mu}(n) = \int_0^{2\pi} e^{-int} d\mu(t).$$

We let

$$V^p_\Lambda(X) = \{\mu \in V^p(X) : \hat{\mu}(n) = 0 \text{ for } n \notin \Lambda\}.$$

For $1 < p < \infty$ and $\mu \in V^p(X)$, one can give sense to $\int_0^{2\pi} \phi(t)d\mu(t)$ when $\phi \in L^q(0, 2\pi)$ (first for simple functions, then extend on $L^q(0, 2\pi)$ by density), where $1/p + 1/q = 1$. Furthermore we have $\|\int_0^{2\pi} \phi d\mu\| \leq \|\mu\|_p\|\phi\|_q$ (see [1]). In particular, for $z \in \mathbb{D}$, $z = re^{i\theta}$, we can define

$$P(\mu)(z) = \int_0^{2\pi} P_r(t - \theta)d\mu(t),$$

where $P_r(t) = \frac{1-r^2}{1-2r\cos(t)+r^2}$ is the Poisson kernel. It is known that $P(\mu) \in h^p(\mathbb{D}, X)$ and, moreover, the correspondence $\mu \mapsto P(\mu)$ yields an isomorphism between $V^p_\Lambda(X)$ and $h^p_\Lambda(\mathbb{D}, X)$ for $\Lambda \subset \mathbb{Z}$ (see [1, Theorem 1.1] and [4]).

If $1 \leq p < \infty$, $f \in L^p(0, 2\pi; X)$, the p-Pettis-norm of f is defined by

$$|||f|||_p = \sup_{\eta\in B_{X'}} (\int_0^{2\pi} | < \eta, f(t) > |^p d\lambda(t))^{1/p},$$

A sequence f_n in $L^p((0, 2\pi), X)$ is said to be p-Pettis-Cauchy, if f_n is a Cauchy sequence for the norm $|||\cdot|||_p$.

If Y is another Banach space, a bounded linear operator $T : X \to Y$ is said to be completely continuous (or Dunford-Pettis) if it maps weakly convergent sequences in X into norm convergent sequences in Y. Recall that X is said to have the complete continuity property (CCP, in short), if each bounded linear operator $T : L^1(0, 2\pi) \to X$ is completely continuous (the CCP was introduced in [8], see [6], [9] and [10] for more information about this property). It is known [8] that every space with the weak Radon-Nikodym property (see [7] for this notion) has the CCP. The simplest examples of Banach spaces with the CCP are separable dual spaces, since it is well known that they have the RNP.

Let $\Lambda \subset \mathbb{Z}$, X is said to have the type I-Λ-complete continuity property (I-Λ-CCP, in short), if every $\mu \in V_\Lambda^\infty(X)$ has a relatively compact range [10]. X is said to have the type II-Λ-complete continuity property (II-Λ-CCP, in short), if every $\mu \in V_\Lambda^1(X)$ which is λ-continuous, has a relatively compact range [10]. It is clear from the definitions that type II-Λ-CCP implies the type I-Λ-CCP, the type I-Λ-RNP (resp. II-Λ-RNP) implies the type I-Λ-CCP (II-Λ-CCP) [5] [3]. The following characterization of the type I-Λ-CCP has been given by M. A. Robdera and P. Saab [10, Theorem 3.3]. For $f \in h^p(\mathbb{D}, X)$ and $r_n \uparrow 1$, we denote by f_{r_n} the function in $L^p(0, 2\pi)$ defined by $f_{r_n}(t) = f(r_n e^{it})$ for $t \in [0, 2\pi]$.

Theorem 1. *Let $\Lambda \subset \mathbb{Z}$. Then X has the type I-Λ-CCP if and only if for every $f \in h_\Lambda^\infty(\mathbb{D}, X)$, $r_n \uparrow 1$, the sequence f_{r_n} in $L^\infty(0, 2\pi; X)$ is 1-Pettis-Cauchy.*

We will establish the following result which is the key point for our main result.

Theorem 2. *Let $\Lambda \subset \mathbb{Z}$ and assume that X has the type II-Λ-CCP. Then for $1 < p < \infty$, $f \in h_\Lambda^p(\mathbb{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy.*

It is well known that when $\Lambda = \mathbb{Z}$, the type I-Λ-CCP and type II-Λ-CCP coincide with the CCP [10]. This fact together with Theorem 1 and Theorem 2 gives the following characterization of the CCP which is one of the main result of this paper.

Theorem 3. *X has the CCP if and only if for every (or equivalently for some) $1 < p < \infty$, $f \in h^p(\mathbb{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy.*

Proof. The condition is clearly necessary by Theorem 2 as the type II-$\mathbb{Z}$-CCP and the CCP are equivalent. Now assume that for some $1 < p < \infty$, for every $f \in h^p(\mathbb{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy. Then in particular for every $f \in h^\infty(\mathbb{D}, X)$, $r_n \uparrow 1$, the sequence f_{r_n} is p-Pettis-Cauchy. Hence f_{r_n} is 1-Pettis-Cauchy. By Theorem 1 this implies that X has the type I-$\mathbb{Z}$-CCP, i.e. the CCP. This finishes the proof. $\square$

We should compare Theorem 3 with the following well known characterization of the RNP: a complex Banach space X has the RNP and only if for every (equivalently for some) $1 < p < \infty$, for every $f \in h^p(\mathbb{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} is convergent in $L^p(0, 2\pi; X)$.

In the proof of Theorem 2 we will use the following lemma.

Lemma 4. *Let $1 < p < \infty$ and $\mu \in V^p(X)$. Then the range of μ is relatively compact if and only if the operator $T : L^q(0, 2\pi) \to X$ defined by $T\phi = \int_0^{2\pi} \phi(t) d\mu(t)$ is compact, where $1/q + 1/p = 1$.*

Proof. Assume first that the range of $\mu \in V^p(X)$ is relatively compact. It is clear that the operator T is well defined and bounded on $L^q(0, 2\pi)$ [1, p.

349]. We claim that for any $\epsilon > 0$, there exists $\delta > 0$ such that for each $\phi \in L^q(0,2\pi)$ satisfying $\lambda(supp(\phi)) \le \delta$ with $\|\phi\|_q \le 1$ we have $\|T\phi\| \le \epsilon$. Here $supp(\phi) = \{x : \phi(x) \ne 0\}$ is the support of ϕ .

By [1, Proposition 1.1], there exists a positive function $g \in L^p(0,2\pi)$ such that for all $\phi \in L^q(0,2\pi)$, one has

$$\|\int_0^{2\pi} \phi(t) d\mu(t)\| \le \int_0^{2\pi} g(t)|\phi(t)| d\lambda(t).$$

Therefore

$$\|T\phi\| \le \int_0^{2\pi} g(t)|\phi(t)| d\lambda(t) \le \|\phi\|_q (\int_{supp(\phi)} |g(t)|^p d\lambda(t))^{1/p}.$$

Then the claim follows easily from the absolute continuity of the Lebesgue integrals.

Now since the range of μ is relatively compact, the set $T(B_{L^\infty(0,2\pi)})$ is also relatively compact as $B_{L^\infty(0,2\pi)}$ is the closed absolute convex hull of $\{\chi_A : A \in \mathcal{B}\}$, where we denote by χ_A the characteristic function of A.

Let $\epsilon > 0$ be fixed and let $0 < \delta < 1$ is the positive number according to the claim. Let $\phi \in L^q(0,2\pi)$ be such that $\|\phi\|_q \le 1$. We let $\phi = \phi_1 + \phi_2$, where $\phi_1(t) = \phi(t)$ if $|\phi(t)| \le 1/\delta$ and $\phi_1(t) = 0$ otherwise. Then

$$\lambda(supp(\phi_2))/\delta \le \int_{supp(\phi_2)} \frac{d\lambda(t)}{\delta^q} \le \int_{supp(\phi_2)} |\phi_2(t)|^q d\lambda(t) \le 1.$$

Therefore $\lambda(supp(\phi_2)) \le \delta$. One obtains that $\|T\phi_2\| \le \epsilon$ by the claim. Moreover $T\phi_1 \in M_\delta := T(\delta^{-1} B_{L^\infty(0,2\pi)})$. Hence $dist(T\phi, M_\delta) \le \epsilon$ for all $\phi \in L^q(0,2\pi)$ with $\|\phi\|_q \le 1$. This implies that the set $\{T\phi : \phi \in L^q(0,2\pi), \|\phi\|_q \le 1\}$ is relatively compact as M_δ is relatively compact and $\epsilon > 0$ is arbitrary.

Conversely, assume that the operator T is compact. Then $T(B_{L^\infty(0,2\pi)})$ is relatively compact as we have $B_{L^\infty(0,2\pi)} \subset B_{L^q(0,2\pi)}$. We deduce that the range of μ being a subset of $T(B_{L^\infty(0,2\pi)})$, is also relatively compact. This finishes the proof.

$\square$

PROOF OF THEOREM 2:

Assume that X has the type II-Λ-CCP, $1 < p < \infty$, $f \in h^p_\Lambda(\mathbb{D}, X)$ and $r_n \uparrow 1$. For any $\eta \in X'$, the function $< \eta, f >$ belongs to $h^p_\Lambda(\mathbb{D}, \mathbb{C})$. Therefore, by the classical result, there exists $f_\eta \in L^p(0,2\pi)$ with $\hat{f}_\eta(n) = 0$ for $n \notin \Lambda$ satisfying

$$< \eta, f(re^{i\theta}) >= \int_0^{2\pi} P_r(\theta - t) f_\eta(t) d\lambda(t),$$

for $\theta \in [0,2\pi]$ and $0 \le r < 1$. Now for $E \in \mathcal{B}$ we can define $\mu(E) \in X''$ by

$$< \mu(E), \eta >= \int_E f_\eta(t) d\lambda(t).$$

Since $f_\eta(t) = \lim_{r\uparrow 1} < \eta, f(re^{it}) >$ a.e. for $t \in [0, 2\pi]$, we get by Fatou's lemma

$$| < \mu(E), \eta > | \leq \int_E \lim_{r\uparrow 1} | < \eta, f(re^{it}) > | d\lambda(t) \leq \|\eta\| \liminf_{r\uparrow 1} \int_E \|f(re^{it})\| d\lambda(t).$$

It follows that

$$\|\mu(E)\| \leq \liminf_{r\uparrow 1} \int_E \|f(re^{it})\| d\lambda(t).$$

Now let π be a finite partition of $[0, 2\pi]$, so that we may estimate

$$\sum_{E\in\pi} \frac{\|\mu(E)\|^p}{\lambda(E)^p} \lambda(E) \leq \sum_{E\in\pi} \liminf_{r\uparrow 1} (\int_E \frac{\|f(re^{it})\| d\lambda(t)}{\lambda(E)})^p \lambda(E)$$

$$\leq \sum_{E\in\pi} \liminf_{r\uparrow 1} \int_E \|f(re^{it})\|^p d\lambda(t) \leq \liminf_{r\uparrow 1} \sum_{E\in\pi} \int_E \|f(re^{it})\|^p d\lambda(t)$$

$$= \lim_{r\uparrow 1} \int_0^{2\pi} \|f(re^{it})\|^p d\lambda(t) = \|f\|_p^p < \infty$$

by Jensen's inequality. Consequently $\mu \in V^p(X'')$. The same proof as [1, Theorem 1.1] shows that the range of μ is actually contained in X. It follows easily from the definition of μ that $\hat{\mu}(n) = 0$ whenever $n \notin \Lambda$, i.e. $\mu \in V^p_\Lambda(X)$.

The measure μ is λ-continuous as $\mu \in V^p(X)$. It follows from the definition of the type II-Λ-CCP that the range of μ is relatively compact. By Lemma 4 the operator $T : L^q(0, 2\pi) \to X$ defined by $T\phi = \int_0^{2\pi} \phi(t) d\mu(t)$ is compact. Hence the adjoint operator $T^* : X' \to L^p(0, 2\pi)$ is also compact.

For $\eta \in X'$ and $\phi \in L^q(0, 2\pi)$, one has

$$< T^*\eta, \phi > = < \eta, T\phi > = < \eta, \int_0^{2\pi} \phi(t) d\mu(t) > = \int_0^{2\pi} \phi(t) f_\eta(t) d\lambda(t).$$

Therefore $T^*\eta = f_\eta$. For each $\eta \in B_{X'}$, the function f_η belongs to $L^p(0, 2\pi)$, so we can identify f_η with its harmonic extension via the Poisson kernel in $\mathbb{D}$. By the classical result

$$\lim_{n,m\uparrow\infty} \|f_\eta(r_m \cdot) - f_\eta(r_n \cdot)\|_p = 0.$$

We deduce that

$$\lim_{n,m\uparrow\infty} \sup_{\eta\in B_{X'}} \|f_\eta(r_m \cdot) - f_\eta(r_n \cdot)\|_p = 0$$

as the set $\{f_\eta : \eta \in B_{X'}\} = T^*(B_{X'})$ is relatively compact in $L^p(0, 2\pi)$. This completes the proof.

From the proof of Theorem 3 and the isomorphism between $h^p(\mathbb{D}, X)$ and $V^p(X)$, it is clear that we have the following characterizations of the CCP.

Theorem 5. *The following statements are equivalent:*

(i) *X has the CCP.*
(ii) *For every $1 < p < \infty$, every $\mu \in V^p(X)$ has a relatively compact range.*
(iii) *For some $1 < p < \infty$, every $\mu \in V^p(X)$ has a relatively compact range.*
(iv) *For every $1 < p < \infty$, for every $\mu \in V^p(X)$, the corresponding operator T on $L^q(0, 2\pi)$ is compact, where $1/p + 1/q = 1$.*
(v) *For some $1 < p < \infty$, for every $\mu \in V^p(X)$, the corresponding operator T on $L^q(0, 2\pi)$ is compact, where $1/p + 1/q = 1$.*

Remarks: (i). It was shown in [2] that a complex Banach space X has the type I-$\mathbb{N}$-CCP (or equivalently the type II-$\mathbb{N}$-CCP, called the analytic CCP) if and only if for each $1 \leq p < \infty$, $f \in h^p_{\mathbb{N}}(\mathbb{D}, X)$ and $r_n \uparrow 1$, the sequence f_{r_n} in $L^p(0, 2\pi)$ is p-Pettis-Cauchy (see also [11]). One can easily use the argument used in the proof of Theorem 3 to give another proof of this result. We should also notice that the method used in [2] does not work in the CCP-case. The reason is that in [2], one uses the fact that for every $f \in h^p_{\mathbb{N}}(\mathbb{D}, X)$, there exist $g \in h^\infty_{\mathbb{N}}(\mathbb{D}, X)$ and $h \in h^\infty_{\mathbb{N}}(\mathbb{D}, \mathbb{C})$ such that $f = g/h$. This is no longer true for functions in $h^p(\mathbb{D}, X)$. One should also compare our Theorem 2 with [10, Theorem 3.4], which deals only with the case $p = 1$ and assumes that Λ is a Riesz-set.

(ii). We can also formulate a similar result as Theorem 5 for the analytic CCP, but in this case we use $\mu \in V^p_{\mathbb{N}}(X)$ for $1 \leq p < \infty$. $p = 1$ is allowed as for $f \in h^1_{\mathbb{N}}(\mathbb{D}, X)$, the corresponding measure μ in the proof of Theorem 2 is in $V^1_{\mathbb{N}}(X)$, hence μ is λ-continuous by the vector-valued Riesz Theorem.

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