# ON COMPACTNESS OF THE DIFFERENCE OF COMPOSITION OPERATORS

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ABSTRACT. We give a negative answer to a conjecture of J. E. Shapiro concerning compactness of the difference of two composition operators acting on the Hardy space $H^2$. This is done by providing an example of composition operators $C_{\phi_1}$ and $C_{\phi_2}$ such that their difference is non-compact on $H^2$, but the singular parts of the Aleksandrov measures of the inducing maps $\phi_1$ and $\phi_2$ agree at each point $\alpha \in \partial\mathbb{D}$. Here $\mathbb{D}$ stands for the unit disc of the complex plane and $\phi_1, \phi_2 : \mathbb{D} \to \mathbb{D}$ are holomorphic.

## 1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc of the complex plane and $\phi : \mathbb{D} \to \mathbb{D}$ a holomorphic map. The classical Littlewood subordination principle guarantees that the equation $C_\phi f = f \circ \phi$ always defines a bounded linear operator $C_\phi$ on the Hardy spaces $H^p$ for $0 < p < \infty$. During the past few decades a considerable amount of research has been done with the goal of explaining the operator-theoretic properties of $C_\phi$ in terms of the function-theoretic properties of the inducing map $\phi$.

The compactness of $C_\phi$ was first described by J. H. Shapiro [S]. He considered the *Nevanlinna counting function* for $\phi$, which is defined as $N_\phi(w) = \sum_{\phi(z)=w} \log(1/|z|)$, and obtained the general expression

$$\|C_\phi\|_e^2 = \limsup_{|w|\to 1-} \frac{N_\phi(w)}{\log(1/|w|)}$$

for the essential norm (i.e. the distance, in the operator norm, from the space of compact operators) of $C_\phi$ acting on $H^2$. In particular, $C_\phi$ is compact on $H^2$ if and only if $N_\phi(w) = o(\log(1/|w|))$ as $|w| \to 1-$. It was previously known (see [ST]) that if $C_\phi$ is compact on $H^p$ for some $p$, then it is compact on $H^p$ for all $p$.

Another solution to the compactness problem can be given by means of the positive measures $\tau_\alpha$ that are defined on the unit circle $\partial\mathbb{D}$ by the Poisson representation

$$\operatorname{Re} \frac{\alpha + \phi(z)}{\alpha - \phi(z)} = \int_{\partial\mathbb{D}} P(z,\zeta)\, d\tau_\alpha(\zeta) \tag{1}$$

for each $\alpha \in \partial\mathbb{D}$. These measures are often called the *Aleksandrov measures* of $\phi$ because A. B. Aleksandrov [A] used them to analyse the boundary values of inner functions. In [CM] Cima and Matheson showed that the essential norm of $C_\phi$ on $H^2$ can also be expressed as

$$\|C_\phi\|_e^2 = \sup_{\alpha\in\partial\mathbb{D}} \|\sigma_\alpha\|,$$

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where $\sigma_\alpha$ is the singular part of $\tau_\alpha$. In particular, it follows that $C_\phi$ is compact on the Hardy spaces $H^p$ if and only if all the measures $\tau_\alpha$ are absolutely continuous. An implicit form of this last statement was already obtained by Sarason [Sa] and Shapiro and Sundberg [SS1], who considered $C_\phi$ as an operator acting on the spaces $M$ and $L^1$ of complex Borel measures and integrable functions on $\partial\mathbb{D}$ and showed that the compactness and weak compactness of $C_\phi$ on these spaces are equivalent to compactness on $H^p$.

The natural problem of characterizing the compactness of the *difference* of two composition operators has also been studied. This question first arose in the works of MacCluer [M] and Shapiro and Sundberg [SS2], which deal with connectivity properties of the set of composition operators. It was later studied by J. E. Shapiro [Sh] in terms of the Aleksandrov measures of the inducing maps. To be specific, let $\phi_1, \phi_2 : \mathbb{D} \to \mathbb{D}$ be two holomorphic maps and let $\sigma_\alpha^i$ stand for the singular part of the Aleksandrov measure of $\phi_i$ at point $\alpha \in \partial\mathbb{D}$. Shapiro proved that a necessary condition for the difference $C_{\phi_1} - C_{\phi_2}$ to be compact on $H^2$ is that

$$\sigma_\alpha^1 = \sigma_\alpha^2 \qquad \text{for all } \alpha \in \partial\mathbb{D}, \tag{2}$$

and he conjectured [Sh, Conjecture 5.4] that this condition is also sufficient.

The main result of this note is a negative answer to Shapiro's conjecture.

**Theorem 1.** *There exist two holomorphic maps $\phi_1, \phi_2 : \mathbb{D} \to \mathbb{D}$ such that their Aleksandrov measures satisfy* (2)*, but the difference $C_{\phi_1} - C_{\phi_2}$ is non-compact on $H^p$ for all $p \in [1, \infty)$.*

It may be of interest to note that our proof shows that the maps $\phi_1$ and $\phi_2$ can actually be chosen univalent. In addition, it turns out that the corresponding composition operators have equally interesting properties on the non-reflexive spaces $H^1$, $L^1$ and $M$.

**Theorem 2.** *Let $\phi_1, \phi_2$ be the maps provided by the proof of Theorem 1 so that they satisfy condition* (2)*. Then the operator $C_{\phi_1} - C_{\phi_2}$ is not weakly compact on any of the spaces $H^1$, $L^1$ and $M$.*

## 2. Proof of Theorems 1 and 2

We will make use of the following well-known estimate for the norm of a function $f \in H^2$:

$$\|f\|_{H^2}^2 - |f(0)|^2 \sim \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|)\, dA(z),$$

where $A$ denotes the planar Lebesgue measure, normalized so that the area of the unit disc is 1. The abbreviation $a \sim b$ entails the existence of positive constants $C_1$ and $C_2$ so that always $C_1 a \le b \le C_2 a$. In fact, a precise identity rather than just an equivalent expression for the $H^2$ norm can be obtained by replacing the weight $1 - |z|$ by $2\log(1/|z|)$. The resulting identity is known as the Littlewood-Paley identity.

If $\phi : \mathbb{D} \to \mathbb{D}$ is a univalent map, we may perform a change of variables in the above integral to get

$$\|C_\phi f\|_{H^2}^2 - |f(\phi(0))|^2 \sim \int_{\phi(\mathbb{D})} |f'(w)|^2 (1 - |\phi^{-1}(w)|)\, dA(w). \tag{3}$$

The following simple estimate will be instrumental to our work.

**Lemma 1.** *Let $\phi : \mathbb{D} \to \mathbb{D}$ be univalent with $\phi(0) = 0$, and assume that $B$ is an open disc of radius $\frac{3}{4}$ contained in $\phi(\mathbb{D})$. Then, for all $f \in H^2$,*

$$\|C_\phi f\|_{H^2}^2 \geq c \int_B |f'(w)|^2 \operatorname{dist}(w, \partial B)\, dA(w),$$

*where $c > 0$ is a constant independent of $\phi$, $B$ and $f$.*

*Proof.* Let $\psi$ be a conformal map taking $\mathbb{D}$ onto $B$ with $\psi(0) = 0$. Applying the Schwarz lemma to the map $\phi^{-1} \circ \psi$ one sees that $|\phi^{-1}(w)| \leq |\psi^{-1}(w)|$ for $w \in B$. Moreover, since $\psi$ is a Möbius transformation and $\operatorname{dist}(0, \partial B) \geq \frac{1}{4}$, it is not difficult to show that $1 - |\psi^{-1}(w)| \geq c' \operatorname{dist}(w, \partial B)$ where $c' > 0$ is an absolute constant. Thus $1 - |\phi^{-1}(w)| \geq c' \operatorname{dist}(w, \partial B)$ for $w \in B$, and the lemma follows from (3). $\square$

We will next define the maps $\phi_1, \phi_2$ we need for the proof of Theorems 1 and 2. Let us first introduce for $s \in [0, \frac{1}{4})$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the notation

$$A(\theta, s) = B((\tfrac{1}{4} - s)e^{i\theta}, \tfrac{3}{4}),$$

so that $A(\theta, s)$ is an open disc contained in $\mathbb{D}$ with radius $\frac{3}{4}$. Its distance to $\partial\mathbb{D}$ equals $s$, the nearest point on $\partial\mathbb{D}$ being $e^{i\theta}$. We also let

$$\Omega_1 = A(0,0) \cup \bigcup_{k=2}^{\infty} A(\tfrac{1}{k}, \tfrac{1}{k^9}),$$

and

$$\Omega_2 = A(0,0) \cup \bigcup_{k=2}^{\infty} A(-\tfrac{1}{k}, \tfrac{1}{k^9}).$$

It is now easy to check that $\Omega_1$ and $\Omega_2$ are simply connected Jordan domains (in fact, they are starlike with respect to all points near origin as unions of domains having the same property) and touch the unit circle only at the point 1. Moreover, they are obtained from each other by reflection with respect to the real axis. Finally we let $\phi_i$ be the conformal map taking $\mathbb{D}$ onto $\Omega_i$ and satisfying $\phi_i(0) = 0$ and $\phi_i(1) = 1$ for $i = 1, 2$. Here and elsewhere we consider $\phi_i$ as extended to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega_i}$.

We start the analysis of the maps $\phi_1$ and $\phi_2$ by determining the singular parts of the corresponding Aleksandrov measures.

**Lemma 2.** *Let $\sigma_\alpha^i$ stand for the singular part of the Aleksandrov measure of $\phi_i$ at the boundary point $\alpha \in \partial\mathbb{D}$ for $i = 1, 2$. Then $\sigma_\alpha^1 = \sigma_\alpha^2 = 0$ for all $\alpha \neq 1$ and $\sigma_1^1 = \sigma_1^2 = \gamma\delta_1$ with $\gamma > 0$. In particular, $\sigma_\alpha^1 = \sigma_\alpha^2$ for all $\alpha \in \partial\mathbb{D}$.*

*Proof.* Observe first that $\sigma_\alpha^i = 0$ for all $\alpha \neq 1$ because $\overline{\Omega_i} \cap \partial\mathbb{D} = \{1\}$ and hence for these values of $\alpha$ the function (1) is bounded on $\mathbb{D}$. Moreover, $\sigma_1^i$ must be a multiple of $\delta_1$ since in the case $\alpha = 1$ the function (1) is continuous on $\overline{\mathbb{D}} \setminus \{1\}$. At this stage the symmetry of $\phi_1$ and $\phi_2$ (observe that $\phi_1(z) = \overline{\phi_2(\overline{z})}$) guarantees that $\sigma_1^1 = \sigma_1^2$. That $\sigma_1^1 \neq 0$ can now be deduced from the fact that the composition operator $C_{\phi_1}$ is non-compact (which is a consequence of the next lemma). $\square$

In the following we will work with the normalized reproducing kernel functions $f_a \in H^2$, defined for $a \in \mathbb{D}$ by

$$f_a(z) = \frac{\sqrt{1 - |a|^2}}{1 - \overline{a}z}.$$

An easy computation with power series shows that $\|f_a\|_{H^2} = 1$. The importance of the functions $f_a$ to the compactness problem stems from the fact that $f_a \to 0$ weakly in $H^2$ as $|a| \to 1-$.

**Lemma 3.** *Let $\epsilon_k = \frac{1}{k^9}$ and $a_k = (1-\epsilon_k)e^{i/k}$ for $k = 2, 3, \dots$. Then $\|C_{\phi_1} f_{a_k}\|_{H^2} \geq c_1$ for all $k$ and some constant $c_1 > 0$, whereas $\lim_{k\to\infty} \|C_{\phi_2} f_{a_k}\|_{H^2} = 0$.*

*Proof.* We first apply Lemma 1 to estimate

$$\begin{aligned}\|C_{\phi_1} f_{a_k}\|_{H^2}^2 &\geq c \int_{A(1/k,\epsilon_k)} |f'_{a_k}(w)|^2 \operatorname{dist}(w, \partial A(\tfrac{1}{k}, \epsilon_k))\, dA(w) \\ &= c \int_{A(0,\epsilon_k)} |f'_{r_k}(w)|^2 \operatorname{dist}(w, \partial A(0, \epsilon_k))\, dA(w),\end{aligned}$$

where $r_k = |a_k| = 1 - \epsilon_k$. Write $G_k = B(1 - 3\epsilon_k, \epsilon_k) \subset A(0, \epsilon_k)$. Easy estimates show that $|1 - r_k w| \leq 5\epsilon_k$ and $\operatorname{dist}(w, \partial A(0, \epsilon_k)) \geq \epsilon_k$ for $w \in G_k$. Consequently, for $w \in G_k$,

$$|f'_{r_k}(w)|^2 = \frac{r_k^2(1 - r_k^2)}{|1 - r_k w|^4} \geq \frac{r_k^2(1 - r_k^2)}{(5\epsilon_k)^4} \geq \frac{c'}{\epsilon_k^3},$$

where $c' > 0$ is a constant. Thus, by restricting the integration above over the set $G_k$ only, we obtain

$$\|C_{\phi_1} f_{a_k}\|_{H^2}^2 \geq c\frac{c'}{\epsilon_k^3}\epsilon_k A(G_k) = cc'.$$

We next deduce from (3) the estimate

$$\|C_{\phi_2} f_{a_k}\|_{H^2}^2 \leq |f_{a_k}(0)|^2 + c \int_{\Omega_2} |f'_{a_k}(w)|^2\, dA(w).$$

Clearly $|f_{a_k}(0)|^2 = 1 - |a_k|^2 \to 0$ as $k \to \infty$. To estimate the above integral, we first observe that by the definition of the domain $\Omega_2$ one obtains

$$\operatorname{dist}(1/\overline{a_k}, \partial\Omega_2) \geq \operatorname{dist}(e^{i/k}, \partial A(0, 0)) = |e^{i/k} - \tfrac{1}{4}| - \tfrac{3}{4} \geq \frac{1}{16k^2}.$$

Hence, if $w \in \Omega_2$, we see that

$$|f'_{a_k}(w)|^2 = \frac{1 - |a_k|^2}{|a_k|^2 |1/\overline{a_k} - w|^4} \leq \frac{2\epsilon_k}{(\frac{1}{2})^2(1/16k^2)^4} = 2^{19}k^8\epsilon_k = 2^{19}/k,$$

and it follows that $\int_{\Omega_2} |f'_{a_k}|^2\, dA \to 0$ as $k \to \infty$. This completes the proof. □

**Proof of Theorem 1.** Let $\phi_1, \phi_2$ and $f_{a_k}$ be as in the above lemma. By Lemma 2 the singular parts of the Aleksandrov measures of $\phi_1$ and $\phi_2$ coincide at each point of $\partial\mathbb{D}$. We first show that $C_{\phi_1} - C_{\phi_2}$ is non-compact on $H^2$. For that end, observe that the sequence $(f_{a_k})$ tends to zero weakly in $H^2$ since $|a_k| \to 1$ as $k \to \infty$. Hence it is enough to verify that the sequence $(C_{\phi_1} f_{a_k} - C_{\phi_2} f_{a_k})$ is not null in norm. This is however an immediate consequence of Lemma 3. This yields the claim for $p = 2$.

In order to treat the other values of $p$ we use interpolation. Note that we may assume $p > 1$ since the case $p = 1$ is contained in Theorem 2. Assume first that $p \in (1, 2)$ and put $T = C_{\phi_1} - C_{\phi_2}$. Also assume, to reach a contradiction, that $T : H^p \to H^p$ is compact. Recall that for $q \in (1, \infty)$ the Riesz projection, defined in terms of Fourier coefficients as $Rf(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, yields a bounded linear projection $R : L^q \to H^q$, where $L^q = L^q(\partial\mathbb{D})$. Let us consider the map $\widetilde{T} = TR$, and observe that $T : H^q \to H^q$ is compact if and only if $\widetilde{T} : L^q \to L^q$ is. In the present situation $\widetilde{T} : L^p \to L^p$ is compact and $\widetilde{T} : L^{p'} \to L^{p'}$ is bounded when

$p' = p/(p-1)$ is the dual exponent. As $p < 2 < p'$, we obtain by interpolation (see e.g. [BS, Theorem 2.9, Chapter 3]) that $\widetilde{T} : L^2 \to L^2$ is compact, which contradicts the first part of the proof and hence shows that $T : H^p \to H^p$ is non-compact. The case $p \in (2, \infty)$ is analogous.

**Proof of Theorem 2.** Since $H^1$ can be regarded as a subspace of $L^1$ and $M$, it is enough to show that $C_{\phi_1} - C_{\phi_2}$ is not weakly compact as an operator on $H^1$. For this purpose we consider the functions $g_k = f_{a_k}^2$ where $a_k$ are as in Lemma 3. Then $\|g_k\|_{H^1} = 1$ for all $k$, and since $\|C_{\phi_2} g_k\|_{H^1} = \|C_{\phi_2} f_{a_k}\|_{H^2}^2 \to 0$ as $k \to \infty$, it is enough to show that the set $\{C_{\phi_1} g_k\}$ is not relatively weakly compact in $H^1$. By the classical Dunford-Pettis theorem (see e.g. [W, III.C.12]) this amounts to showing that this set is not uniformly integrable on $\partial\mathbb{D}$.

For $k \geq 2$ define $B_k = B(1/\overline{a_k}, 1/k^4)$. A simple computation shows that the discs $B_k$ are disjoint, and for $w \notin B_k$ we also have the estimate

$$|g_k(w)| = \frac{1 - |a_k|^2}{|a_k|^2 |1/\overline{a_k} - w|^2} \leq \frac{2/k^9}{(\frac{1}{2})^2 (1/k^4)^2} = \frac{8}{k}.$$

Now let $S_k = \{\zeta \in \partial\mathbb{D} : \phi_1(\zeta) \in B_k\}$ and denote by $m$ the normalized Lebesgue measure on $\partial\mathbb{D}$. The preceding estimate yields that

$$\int_{S_k} |C_{\phi_1} g_k| \, dm = \|C_{\phi_1} f_{a_k}\|_{H^2}^2 - \int_{\partial\mathbb{D} \setminus S_k} |C_{\phi_1} g_k| \, dm \geq c_1^2 - 8/k,$$

where $c_1 > 0$ is the constant obtained in Lemma 3. Since the sets $S_k$ are disjoint, it follows that $\{C_{\phi_1} g_k\}$ is not uniformly integrable, and the proof is complete.

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