Mappings of finite distortion: Injectivity radius of a local homeomorphism

Pekka Koskela, Jani Onninen and Kai Rajala*

Abstract

We establish a radius of injectivity for locally homeomorphic mappings of finite distortion in space, under minimal integrability assumptions on the distortion.

1 Introduction

We call a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ *a mapping of finite distortion* if it satisfies

$$|Df(x)|^n \leq K(x)J(x,f) \qquad \text{a.e.},$$

where $K(x) < \infty$, and if also $J(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$. We further say that f has bounded distortion if $K \in L^\infty(\Omega)$. Mappings of bounded distortion are also called quasiregular mappings. This class of mappings can be thought of as a generalization of analytic functions and they share many of the nice properties of analytic functions. Recently it has been observed (cf. [3], [4]) that topological conclusions such as discreteness and openness hold even when the boundedness of the distortion is replaced with a suitable integrability condition. Let us consider an increasing, differentiable function Φ. The following conditions have turned out to be useful. We require that

(Φ-1) $\displaystyle\int_1^\infty \frac{\Phi'(t)}{t}\,dt = \infty,$

(Φ-2) $\displaystyle\lim_{t\to\infty} t\,\Phi'(t) = \infty,$

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and assume that $\exp\big(\Phi(K)\big) \in L^1(\Omega)$.

In this note we establish a sharp result on the injectivity radius of a locally homeomorphic mapping of finite, suitably integrable distortion. Let us recall the 'classical' results. Zorich proved in [10] that each locally homeomorphic mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, of bounded distortion is in fact globally injective and onto. Martio, Rickman and Väisälä [6] (also see [1]) improved this to a local statement: if f is locally homeomorphic in $B(0,1)$, then f is injective in $B(0,\delta)$, where $\delta = \delta(n, ||K||_{L^\infty}) > 0$. The assumption $n \geq 3$ is essential, as is seen by considering the (rescaling of) locally injective analytic function $f(z) = e^z$.

Theorem 1.1. *Let $n \geq 3$. Let $f : B(0,1) \to \mathbb{R}^n$ be a mapping of finite distortion such that f is a local homeomorphism and*

$$\exp\big(\Phi(K)\big) \in L^1(B(0,1)),$$

where Φ satisfies (Φ-1) and (Φ-2). Then there exists a constant $\delta > 0$ that depends on $n, \Phi, ||\exp(\Phi(K))||_{L^1}$ such that f is injective in the ball $B(0,\delta)$. Moreover, given any increasing, differentiable Φ for which the integral at (Φ-1) converges, there is a constant M so that, for any $\delta > 0$, there is a locally homeomorphic mapping $f : B(0,1) \to \mathbb{R}^n$ of finite distortion that satisfies

$$\int_{B(0,1)} \exp\big(\Phi(K)\big) \leq M,$$

and that fails to be injective in the ball $B(0,\delta)$.

A simple rescaling argument and Theorem 1.1 show that, when $f : B(x,r) \to \mathbb{R}^n$, we obtain injectivity in $B(x,\delta r)$, where δ now depends on the integral average $|B(x,r)|^{-1} \int_{B(x,r)} \exp\big(\Phi(K)\big)$. Regarding the example referred to above, Perović [7] has shown that a bound on the integral averages of K^{n-1} is not sufficient for a bound on the injectivity radius even when K is bounded.

Our proof of Theorem 1.1 is based on the following ingredients. The method we use is a modification of the original proof by Zorich [10] and its improved version in [6]. We employ the recently established modulus inequalities (cf. [5]) as a substitute for the usual Poletsky inequality. Because our distortion is not bounded, the Zorich argument needs to be adapted to our situation and our adaptation in fact appears to be slightly simpler than the proofs given in [1], [6], [8]. The example referred to in Theorem 1.1 is obtained as follows. We first map $B(0,1)$ essentially to $B(0,2)$ so that the image of $B(0,\delta)$ is $B(0,1)$. Let us call this mapping f_1. Our actual mapping will be $f = f_3 \circ f_2 \circ f_1$, where f_3 is the Zorich exponential mapping and f_2 is a suitable quasiconformal mapping.

2 Properties of local homeomorphisms

In this short section, we list, for the convenience of the reader, the known properties of local homeomorphisms that we will need in the proof of our main theorem.

Let $A \subseteq \overline{\mathbb{R}}^n$. Then A is *relatively locally connected* if for every $x \in \overline{A}$ there exist arbitrarily small neighborhoods U of x such that $U \cap A$ is connected.

Lemma 2.1. *[8, III 3.1] Let $f : G \to \overline{\mathbb{R}}^n$ be a local homeomorphism. Let $A \subseteq \overline{\mathbb{R}}^n$ be simply connected and locally pathwise connected, and let P be a component of $f^{\{-1\}}A$ such that $\overline{P} \subseteq G$. Then f maps P homeomorphically onto A. If, in addition, A is relatively locally connected, then f maps $\overline{P}$ homeomorphically onto $\overline{A}$.*

We will also employ the following two lemmas.

Lemma 2.2. *[8, III 3.2] Let $f : G \to \overline{\mathbb{R}}^n$ be a local homeomorphism and let F be a compact set in G such that f is injective in F. Then F is injective in a neighborhood of F.*

Lemma 2.3. *[8, III 3.3] Let $f : G \to \overline{\mathbb{R}}^n$ be a local homeomorphism, let $A, B \subseteq G$ and let f be homeomorphic in both A and B. If $A \cap B \neq \emptyset$ and if $fA \cap fB$ is connected, then f is homeomorphic in $A \cup B$.*

3 Proof of the main theorem

The following simple geometric result turns out to be valuable; for a related argument see [9, Thm 11.7].

Lemma 3.1. *Let $n \geq 1$ and $r > 0$. Let $a \neq b$, $a, b \in S^{n-1}(0, r)$. Then there exists a point $p = p(a, b) \in B(0, r)$ such that for every $t \in \left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$ either*

$$0, b \in B(p, t) \text{ and } a \notin B(p, t)$$

or

$$a, b \in B(p, t) \text{ and } 0 \notin B(p, t).$$

Proof. Write $\angle(0, \frac{b}{2}, a)$ for the angle at the point $b/2$ formed by the line segments from 0 to $b/2$ and from $b/2$ to a. We have two cases.
1. $\angle(0, \frac{b}{2}, a) \in [0, \frac{\pi}{2})$. Then, for every $t > \frac{r}{2}$, 0 and $b \in B(\frac{b}{2}, t)$. Moreover, for every $t \leq \frac{\sqrt{3}r}{2}$, $a \notin B(\frac{b}{2}, t)$.
2. $\angle(0, \frac{b}{2}, a) \in [\frac{\pi}{2}, \pi)$. Then, for every $t > \frac{|a-b|}{2}$, a and $b \in B(\frac{a+b}{2}, t)$. Moreover, for every $t < \frac{|a+b|}{2}$, $0 \notin B(\frac{a+b}{2}, t)$. On the other hand, $\frac{|a-b|}{2} \leq \frac{r}{2}$ and $\frac{|a+b|}{2} \geq \frac{\sqrt{3}r}{2}$. $\square$

We will also employ two modulus estimates from [5]. The first one is a substitute for the Poletsky inequality for quasiregular mappings.

Lemma 3.2. *Let f be as in Theorem 1.1. Let Γ be a path family in $B(0,1)$. Then*

$$M(f\Gamma) \leq M_{K^{n-1}(\cdot,f)}(\Gamma). \tag{3.1}$$

Here M is the usual n-modulus defined by

$$\begin{aligned} M(\Gamma) &= \inf\Big\{ \int_{\mathbb{R}^n} \rho^n(x)\,dx : \rho : \mathbb{R}^n \to [0,\infty) \text{ is a Borel function such that} \\ &\int_\gamma \rho \geq 1 \text{ for each } \gamma \in \Gamma \Big\} \end{aligned}$$

and $M_{K^{n-1}(\cdot,f)}(\Gamma)$ is defined by

$$\begin{aligned} M_{K^{n-1}(\cdot,f)}(\Gamma) &= \inf\Big\{ \int_{\mathbb{R}^n} \rho^n(x)\,K^{n-1}(x)\,dx : \rho : \mathbb{R}^n \to [0,\infty) \text{ is a Borel} \\ &\text{function such that } \int_\gamma \rho \geq 1 \text{ for each } \gamma \in \Gamma \Big\}. \end{aligned}$$

Lemma 3.3. *Suppose that $I = \int_{B(0,1)} \exp(\Phi(K)) < \infty$. Let $0 < 4r < R < 1$. Then there exist $C_1, C_2 > 0$ depending on n, Φ and I such that*

$$M_{K^{n-1}(\cdot,f)}(\Gamma) \leq C_1 \left(\int_{2r}^{R/2} \frac{ds}{s\Phi^{-1}\left(\log(C_2 s^{-n})\right)} \right)^{1-n},$$

where Γ is the family of all curves connecting $\overline{B}(0,r)$ and $\mathbb{R}^n \setminus B(0,R)$.

Proof of Theorem 1.1. We may assume $f(0)=0$. Let $r_0 = \sup\{r : \overline{U}(0,r) \subseteq B(0,1)\}$ where $U(0,r)$ is the 0-component of $f^{\{-1\}}B(0,r)$. Fix $r < r_0$ and set $U = U(0,r)$,

$$\begin{aligned} l^* &= l^*(0,r) = \inf\{|z| : z \in \partial U\}, \text{ and} \\ L^* &= L^*(0,r) = \sup\{|z| : z \in \partial U\}. \end{aligned}$$

By Lemma 2.1, f maps $\overline{U}$ homeomorphically onto $\overline{B}(0,r)$. Thus f is injective in $B(0,l^*)$ and it suffices to find a lower bound for l^*. Note that $L^* \to 1$ as $r \to r_0$.
Pick x and $y \in \partial U$ such that $|x| = L^*$ and $|y| = l^*$. Note that, by the definition of U, $f(x), f(y) \in S^{n-1}(0,r)$. By Lemma 3.1 there exists a point $p \in B(0,r)$ such that, for every $t \in (\frac{r}{2}, \frac{\sqrt{3}r}{2})$, $f(x) \in B(p,t)$ and either 0 or $f(y) \in B(p,t)$ but not both. Fix such a t. Since $f(B(0,l^*))$ is connected, there exists a point $z_t \in S^{n-1}(p,t) \cap f(B(0,l^*))$. Let z_t^* be the unique point in $f^{\{-1\}}(z_t) \cap B(0,l^*)$. Let $C_t(\phi) \subseteq S^{n-1}(p,t)$ be the spherical cap with center z_t and opening angle ϕ. Let ϕ_t be the supremum of all ϕ:s for which the

z_t^*-component of $f^{\{-1\}}C_t(\phi)$ gets mapped homeomorphically onto $C_t(\phi)$. Let $C_t = C_t(\phi_t)$, and let C_t^* be the z_t^*-component of $f^{\{-1\}}C_t$. We claim that C_t^* meets $S^{n-1}(0, L^*)$. Suppose this is not true. Then $C_t^* \subseteq B(0, L^*)$. By Lemma 2.1, f maps $\overline{C}_t^*$ homeomorphically onto $\overline{C}_t$. By Lemma 2.2 f is injective in a neighborhood of $\overline{C}_t^*$. Thus $\phi_t = \pi$, $C_t = S^{n-1}(p,t)$ and C_t^* is a topological $(n-1)$-sphere. Now there is a unique point q in $f^{\{-1\}}(p) \cap U$. Clearly f maps the q-component of $\mathbb{R}^n \setminus C_t^*$ onto $B(p,t)$. By Lemmas 2.3 and 2.1, f is injective in the union of this q-component and $\overline{U}$. But $f(x) \in B(p,t)$, so the q-component must include a point of $f^{\{-1\}}(f(x))$, which contradicts the injectivity, because $x \in \overline{U}$. Hence the claim is true.

Let k_t^* be a point in $C_t^* \cap S^{n-1}(0, L^*)$ and $k_t = f(k_t^*)$. Let Γ_t' be the family of all curves connecting k_t and z_t in C_t. Moreover, let Γ' be the union of the curve families Γ_t', $t \in (\frac{r}{2}, \frac{\sqrt{3}r}{2})$. Denote by f_t the restriction of f to C_t^*. Then f_t maps C_t^* homeomorphically onto C_t. Furthermore, denote

$$\Gamma = \bigcup_{t \in (\frac{r}{2}, \frac{\sqrt{3}r}{2})} \left\{ f_t^{-1} \circ \gamma : \gamma \in \Gamma_t' \right\}.$$

Since for every $t \in (\frac{r}{2}, \frac{\sqrt{3}r}{2})$, $z_t^* \in B(0, l^*)$ and $k_t^* \in S^{n-1}(0, L^*)$, Lemma 3.3 gives

$$M_{K^{n-1}(\cdot, f)}(\Gamma) \leq C_1 \left(\int_{2l^*}^{L^*/2} \frac{ds}{s\Phi^{-1}\left(\log(C_2 s^{-n})\right)} \right)^{1-n}.$$

On the other hand, by [9], Theorem 10.2,

$$\int_{S^{n-1}(p,t)} \rho^n(x)\, dm_{n-1}(x) \geq \frac{C_n}{t}$$

for every ρ for which $\int_\gamma \rho \geq 1$ for every $\gamma \in \Gamma_t'$. Thus integration over t yields

$$M(\Gamma') \geq C_n.$$

So, using Lemma 3.2 and letting r tend to r_0, we have

$$C_n \leq C_1 \left(\int_{2l^*}^{1/2} \frac{ds}{s\Phi^{-1}\left(\log(C_2 s^{-n})\right)} \right)^{1-n}. \tag{3.2}$$

By change of variables and the condition (Φ-1), we have

$$\int_0^{1/2} \frac{ds}{s\Phi^{-1}\left(\log(C_2 s^{-n})\right)} = \frac{1}{n} \int_{C_2^{\frac{1}{n}} \exp(\frac{1}{n}\Phi(\frac{1}{2}))}^{\infty} \frac{\Phi'(t)}{t}\, dt = \infty,$$

and hence the right hand side of (3.2) tends to zero as $l^* \to 0$. So there is a $\delta = \delta(n, \Phi, I) > 0$ such that $l^* > \delta$. This proves the first part of the theorem.

Suppose then that we are given a function Φ for which the integral at (Φ-1) converges. Assume, without loss of generality, that Φ is unbounded. Let $0 < \delta < 1$ be small. Set

$$f_1(x) = \frac{x}{|x|}\,\rho(|x|)$$

for $x \in B(0,1)$, where

$$\rho(r) = \exp\Big(\lambda \int_{\Phi^{-1}(\log\frac{e}{r})}^{\infty} \frac{\Phi'(s)}{s}\,ds\Big)$$

when $\delta < r < 1$ and

$$\rho(r) = \frac{r}{\delta}\exp\Big(\lambda \int_{\Phi^{-1}(\log\frac{e}{\delta})}^{\infty} \frac{\Phi'(s)}{s}\,ds\Big)$$

when $0 \leq r \leq \delta$. Then f_1 is a homeomorphic mapping of finite distortion and, given $M > 1$, one can choose λ so that

$$\int_{B(0,1)} \exp\big(\Phi(MK_1)\big) \leq I < \infty,$$

where K_1 is the distortion of f_1 and I is independent of δ; for this computation see [4]. Moreover, f_1 maps $B(0,1) \setminus B(0,\delta)$ onto an annular region A_δ whose inner radius tends to 1 when δ tends to zero and whose outer radius is fixed. Furthermore, $f_1(B(0,1))$ is a ball. Thus there is a universal constant K_2 so that we can find a K_2-quasiconformal mapping f_2 so that $f_2(f_1(B(0,1)))$ is an ellipsoid that does not contain any point whose $n-1$ first coordinates are integers and so that both $(1/2,\cdots,1/2)$ and $(5/2,1/2,\cdots,1/2)$ are contained in $f_2(f_1(B(0,\delta)))$. Let f_3 be the Zorich exponential map that is of bounded distortion K_3, see [2], [8]. The branch set of f_3 then consists of those points whose $n-1$ first coordinates are integers. So $f_2(f_1(B(0,1)))$ does not contain any points of the branch set of f_3 and thus $f = f_3 \circ f_2 \circ f_1$ is a local homeomorphism. Because $f_3(1/2,\cdots,1/2) = f_3(5/2,1/2,\cdots,1/2)$, our mapping f is not injective in $B(0,\delta)$. However, f is of finite distortion MK_1, where $M = K_2K_3$, and the claim follows.

□

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University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35, Fin-40351 Jyväskylä
Finland
E-mail: **{pkoskela, jaonnine, kirajala}@maths.jyu.fi**