LOCAL DIMENSIONS OF SLICED MEASURES AND STABILITY OF PACKING DIMENSIONS OF SECTIONS OF SETS

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Abstract. Let m and n be integers with $0 < m < n$. We relate the absolutely continuous and singular parts of a measure μ on $\mathbb{R}^n$ to certain properties of plane sections of μ. This leads us to prove, among other things, that the lower local dimension of $(n-m)$-plane sections of μ is typically constant provided that the Hausdorff dimension of μ is greater than m. The analogous result holds for the upper local dimension if μ has finite t-energy for some $t > m$. We also give a sufficient condition for stability of packing dimensions of section of sets.

1. Introduction

The geometry of different concepts of a dimension has been an object of intensive study for several years. The emphasis is given to questions like the constancy of dimensions of projections, plane sections, general intersections etc. For projections geometrical results of this type are well-known for both Hausdorff dimension and many other dimensions introduced in the literature quite recently (for Hausdorff dimension see [HT], [Ka], [Mar], [Mat1], for packing and box counting dimensions see [FH1], [FH2], [FM], [H], [J], for q-dimensions see [FO], [HK1], [JJ], [SY], for average dimension see [Z], for infinite dimensional setting see [HK2], and for Hausdorff and Fourier dimension in a very general setting see [PS]). In particular, whilst the lower local dimension, $\underline{\dim}_{\text{loc}}\,\mu(x)$, of a finite Radon measure μ at a point $x \in \mathbb{R}^n$ (see (2.1)) is typically preserved under projections, the behaviour of the upper local dimension, $\overline{\dim}_{\text{loc}}\,\mu(x)$ (see (2.2)), is less regular than that. More precisely, letting $(\mathrm{P}_V)_*\mu$ be the image of μ under the orthogonal projection $\mathrm{P}_V : \mathbb{R}^n \to V$ onto an $(n-m)$-plane V, we have for almost all $(n-m)$-planes V

$$\underline{\dim}_{\text{loc}}(\mathrm{P}_V)_*\mu(\mathrm{P}_V(x)) = \min\{\underline{\dim}_{\text{loc}}\,\mu(x), n-m\} \tag{1.1}$$

$$\overline{\dim}_{\text{loc}}(\mathrm{P}_V)_*\mu(\mathrm{P}_V(x)) = \overline{\mathrm{c}}_\mu(x) \tag{1.2}$$

for μ-almost all $x \in \mathbb{R}^n$. In (1.2) the quantity $\overline{\mathrm{c}}_\mu(x)$, defined in terms of a convolution of μ with a certain kernel, may be strictly less than $\min\{\overline{\dim}_{\text{loc}}\,\mu(x), n-m\}$ for all $x \in \mathbb{R}^n$. (For (1.1) see [FH1], [FO], [HK1], and [Z], and for (1.2) see [FO].) Note that (1.1) and (1.2) obey the general rule of dimension results of projections:

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in all the above mentioned cases dimension is either preserved in the sense of (1.1) or it may decrease but it is a constant as in (1.2).

The slice $\mu_{V,x}$ of a measure μ by the translate V_x of an $(n-m)$-plane V going through $x \in \mathbb{R}^n$ may be regarded as a natural measure on V_x (for the definition see (2.9)). Continuing the work by Falconer and Mattila [FM], Järvenpää and Mattila proved in [JM] that assuming $\dim_{\mathrm{H}} \mu > m$ ($\dim_{\mathrm{H}}$ is the Hausdorff dimension), we have for almost all $(n-m)$-planes V

$$\mathcal{H}^m\text{-}\operatorname{ess\,inf}\{\dim_{\mathrm{H}} \mu_{V,a} \mid a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^n) > 0\} = \dim_{\mathrm{H}} \mu - m. \tag{1.3}$$

Here $\mathcal{H}^s$ is the s-dimensional Hausdorff measure and $V^\perp$ is the orthogonal complement of V. Furthermore, provided that the $(m+d)$-energy of μ is finite for some $d > 0$, equality (1.3) extends to packing dimension, $\dim_{\mathrm{p}}$, that is, for almost all $(n-m)$-planes V

$$\mathcal{H}^m\text{-}\operatorname{ess\,inf}\{\dim_{\mathrm{p}} \mu_{V,a} \mid a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^n) > 0\} = \mu\text{-}\operatorname*{ess\,inf}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) - m \tag{1.4}$$

where $\overline{\mathrm{d}}_\mu(x)$ is as in (2.5).

For both Hausdorff and packing dimensions of sections of sets we have the following natural upper bounds: For all $A \subset \mathbb{R}^n$ and for any $(n-m)$-plane V

$$\dim_{\mathrm{H}}(A \cap V_a) \le \max\{\dim_{\mathrm{H}} A - m, 0\} \tag{1.5}$$

$$\dim_{\mathrm{p}}(A \cap V_a) \le \max\{\dim_{\mathrm{p}} A - m, 0\} \tag{1.6}$$

for $\mathcal{H}^m$-almost all $a \in V^\perp$. However, the differences between these dimensions become crucial as far as the validity of the opposite inequalities is concerned. It is well-known that if $m < s < n$ and $A \subset \mathbb{R}^n$ is a Borel set with $0 < \mathcal{H}^s(A) < \infty$, then for almost all $(n-m)$-planes V

$$\mathcal{H}^m(\{a \in V^\perp \mid \dim_{\mathrm{H}}(A \cap V_a) = \dim_{\mathrm{H}} A - m\}) > 0. \tag{1.7}$$

This was first proved by Marstrand [Mar] in the plane and later generalized by Mattila [Mat1] to higher dimensions. In general constancy results of this type are not valid for packing dimension. In [FJM] Falconer, Järvenpää, and Mattila constructed a compact set $F \subset \mathbb{R}^n$ such that for positively many $(n-m)$-planes V we have $\dim_{\mathrm{p}}(F \cap V_a) = 0$ for $\mathcal{H}^m$-almost all $a \in V^\perp$, and on the other hand $\mathcal{H}^m\text{-}\operatorname{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}(F \cap V_a) = n - m$ for positively many $(n-m)$-planes V. For further generalizations see [Cs] and for related results on sliced measures see [FJ] and [L].

In this paper we consider the decomposition of a measure μ into absolutely continuous and singular parts and relate these parts to certain properties of sliced measures. This leads, among other things, to the following analogues of (1.1) and (1.2) for sections of measures extending an earlier result by Falconer and O'Neil [FO]: For almost all $(n-m)$-planes V and for μ-almost all $x \in \mathbb{R}^n$

$$\underline{\dim}_{\mathrm{loc}}\, \mu_{V,x}(x) = \underline{\dim}_{\mathrm{loc}}\, \mu(x) - m \tag{1.8}$$

$$\overline{\dim}_{\mathrm{loc}}\, \mu_{V,x}(x) = \overline{\mathrm{d}}_\mu(x) - m \tag{1.9}$$

(see Theorem 2.11). Equality (1.8) holds under the assumption $\dim_{\mathrm{H}} \mu > m$, and in (1.9) it is assumed that μ has finite $(m+d)$-energy for some $d > 0$.

In section 3 we analyze the structure of sets $A \subset \mathbb{R}^n$ for which the quantity $\mathcal{H}^m$- ess $\sup_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a)$ is typically constant. It follows from the constructions in [FJM] and [Cs] that some kind of uniformity must be imposed on A for stability results of this type (see the discussion at the end of Section 3). This leads us to define $(n-m)$-thick sets (Definition 3.1) and to prove that for such sets A

$$\mathcal{H}^m\text{- ess}\sup_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a) = \sup\{\mu\text{- ess}\sup_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} \tag{1.10}$$

for almost all $(n-m)$-planes V (Theorem 3.10). Here $\mathcal{M}_*(A)$ is as in (3.8). When proving (1.10), our main tools are the results on upper packing dimensions in [JM], the effect of the absolutely continuous and singular parts of a measure on the existence of sliced measures, dimensional properties of certain measures defined in terms of Riesz representation theorem (Lemma 3.4), and measurability properties obtained using Jankov-von Neuman theorem.

Finally, in Section 4 we use the methods developed in this paper to indicate another difference between Hausdorff and packing dimensions of sliced measures by considering an analogue of a projection result by Martsrand in [Mar].

2. Local dimensions and sliced measures

Throughout this paper m and n will be integers with $0 < m < n$. For any $s \geq 0$ we denote the s-dimensional Hausdorff and packing measures by $\mathcal{H}^s$ and $\mathcal{P}^s$, respectively. We denote the lower and upper local dimensions of a finite Radon measure μ on $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ by $\underline{\dim}_{\mathrm{loc}}\, \mu(x)$ and $\overline{\dim}_{\mathrm{loc}}\, \mu(x)$, that is,

$$\underline{\dim}_{\mathrm{loc}}\, \mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{2.1}$$

and

$$\overline{\dim}_{\mathrm{loc}}\, \mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{2.2}$$

where $B(x,r)$ is the closed ball of radius r and with centre at x. The following characterization of the lower local dimension follows easily from the definition (2.1)

$$\underline{\dim}_{\mathrm{loc}}\, \mu(x) = \sup\{s \geq 0 \mid \int |x-y|^{-s} d\mu(y) < \infty\}. \tag{2.3}$$

Replacing in (2.1) and (2.2) the convolution of μ and the characteristic function of the ball $B(0,r)$ by that of the function

$$\psi_r(x) = \begin{cases} r^m |x|^{-m} & , \text{ if } |x| \leq r \\ 0 & , \text{ if } |x| > r, \end{cases}$$

we define

$$\underline{\mathrm{d}}_\mu(x) = \liminf_{r \to 0} \frac{\log(r^m \int_{B(x,r)} |x-y|^{-m} d\mu(y))}{\log r} \tag{2.4}$$

and

$$\overline{\mathrm{d}}_\mu(x) = \limsup_{r\to 0} \frac{\log(r^m \int_{B(x,r)} |x-y|^{-m} d\mu(y))}{\log r}. \tag{2.5}$$

Clearly

$$\underline{\mathrm{d}}_\mu(x) \le \underline{\dim}_{\mathrm{loc}}\, \mu(x) \text{ and } \overline{\mathrm{d}}_\mu(x) \le \overline{\dim}_{\mathrm{loc}}\, \mu(x) \tag{2.6}$$

for all $x \in \mathbb{R}^n$. Note that definition (2.5) gives

$$\overline{\mathrm{d}}_\mu(x) = \sup\{s \ge 0 \mid \liminf_{r\to 0} r^{-s} \int_{B(x,r)} |x-y|^{-m} d\mu(y) = 0\} + m. \tag{2.7}$$

(If $A = \emptyset$, we define $\sup A = -\infty$.) Moreover, for μ-almost all $x \in \mathbb{R}^n$

$$\underline{\mathrm{d}}_\mu(x) = \underline{\dim}_{\mathrm{loc}}\, \mu(x) \ge m \tag{2.8}$$

provided that $\int |x-y|^{-m}\, d\mu(y) < \infty$ for μ-almost all $x \in \mathbb{R}^n$ [FO, (4.12)].

The quantities $\underline{\mathrm{d}}_\mu$ and $\overline{\mathrm{d}}_\mu$ were used in [FM] and [FO] for the purpose of studying dimensional properties of sliced measures. To define slices of measures by affine planes, we introduce the following setting. We equip the Grassmann manifold $G_{n,n-m}$ of $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$ with the Haar measure $\gamma_{n,n-m}$. Given $V \in G_{n,n-m}$, we use the notation $V^\perp$ for the orthogonal complement of V, and $\mathrm{P}_V : \mathbb{R}^n \to V$ for the orthogonal projection onto V. Furthermore, let $V_a = \{v + a \mid v \in V\}$ be the affine $(n-m)$-plane which is parallel to V and goes through $a \in V^\perp$, and let

$$V_a(\delta) = \{y \in \mathbb{R}^n \mid \mathrm{dist}(y, V_a) \le \delta\}$$

be the closed δ-neighbourhood of V_a. The restriction of a measure μ on $\mathbb{R}^n$ to a set $E \subset \mathbb{R}^n$ is denoted by $\mu|_E$, that is,

$$\mu|_E(A) = \mu(E \cap A)$$

for all $A \subset \mathbb{R}^n$.

For $V \in G_{n,n-m}$, the slices of a finite Radon measure μ on $\mathbb{R}^n$ by affine $n-m$-planes V_a are defined as the weak limits of the normalized restriction measures $(2\delta)^{-m}\mu|_{V_a(\delta)}$ as δ goes to zero. It turns out that for any $V \in G_{n,n-m}$ and for $\mathcal{H}^m$-almost all $a \in V^\perp$ there exists a Radon measure $\mu_{V,a}$ on V_a such that

$$\int \varphi\, d\mu_{V,a} = \lim_{\delta\to 0} (2\delta)^{-m} \int_{V_a(\delta)} \varphi\, d\mu < \infty \tag{2.9}$$

for all continuous functions $\varphi : \mathbb{R}^n \to [0,\infty)$ with compact support. The measure $\mu_{V,a}$ is the slice of μ by the plane V_a. Note that the limit on the right hand side of (2.9) is finite. For the process of defining measures $\mu_{V,a}$ and proving their basic properties, see [Mat3, Chapter 10].

The following disintegration formula gives an important relation between the original measure and its slices [Mat2, Lemma 3.4]: For any Borel function $f : \mathbb{R}^n \to [0,\infty)$ with $\int f\,d\mu < \infty$ we have

$$\int_{V^\perp} \int f\,d\mu_{V,a}\,d\mathcal{H}^m(a) \le \int f\,d\mu. \tag{2.10}$$

Furthermore, the opposite inequality holds in (2.10) provided that the projected measure $(\mathrm{P}_{V^\perp})_*\mu$, defined for all $A \subset V^\perp$ as

$$(\mathrm{P}_{V^\perp})_*\mu(A) = \mu(\mathrm{P}_{V^\perp}^{-1}(A)),$$

is absolutely continuous with respect to $\mathcal{H}^m$. In this case we use the notation $(\mathrm{P}_{V^\perp})_*\mu \ll \mathcal{H}^m$.

In order to introduce measures $\mu_{V,x}$ on affine $(n-m)$-planes V_x parallel to $V \in G_{n,n-m}$ and going through $x \in \mathbb{R}^n$ we simply set

$$\mu_{V,x} = \mu_{V,a}$$

for any $x \in \mathrm{P}_{V^\perp}^{-1}(\{a\})$ whenever $a \in V^\perp$ is such that $\mu_{V,a}$ is defined.

In [FO] Falconer and O'Neil proved the following relations between local dimensions of sliced measures and the quantities $\underline{\mathrm{d}}_\mu$ and $\overline{\mathrm{d}}_\mu$ (see (2.8)).

2.1. Theorem (Falconer, O'Neil). *Let μ be a Radon measure on $\mathbb{R}^n$ with compact support such that $\int |x-y|^{-m}d\mu(y) < \infty$ for μ-almost all $x \in \mathbb{R}^n$. Then for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ and for μ-almost all $x \in \mathbb{R}^n$*

$$\underline{\dim}_{\mathrm{loc}}\,\mu_{V,x}(x) \ge \underline{\dim}_{\mathrm{loc}}\,\mu(x) - m = \underline{\mathrm{d}}_\mu(x) - m \quad \textit{and}$$
$$\overline{\dim}_{\mathrm{loc}}\,\mu_{V,x}(x) \ge \overline{\mathrm{d}}_\mu(x) - m.$$

Proof. See [FO, Proposition 4.1]. □

2.2. Remarks. *(a) The assumption of the previous theorem rules out only the case where*

$$\mu(\{x \in \mathbb{R}^n \mid \underline{\dim}_{\mathrm{loc}}\,\mu(x) = m\}) > 0.$$

In fact, by (2.3) the assumption of Theorem 2.1 is satisfied provided that

$$\underline{\dim}_{\mathrm{loc}}\,\mu(x) > m$$

for μ-almost all $x \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ such that $\underline{\dim}_{\mathrm{loc}}\,\mu(x) < m$, it follows easily from (2.3), (2.4), and (2.5) that $\underline{\mathrm{d}}_\mu(x) = \overline{\mathrm{d}}_\mu(x) = -\infty$, and therefore the lower bounds in Theorem 2.1 are trivial for all such points x.

(b) Under the condition that $\underline{\dim}_{\mathrm{loc}}\,\mu(x) < m$ for some $x \in \mathbb{R}^n$ the sliced measure $\mu_{V,x}$ is not defined for any $V \in G_{n,n-m}$. In fact, given such $x \in \mathbb{R}^n$, we find a sequence (r_i) tending to zero such that $\lim_{i\to\infty} r_i^{-m}\mu(B(x,r_i)) = \infty$. Choosing a continuous function $\varphi : \mathbb{R}^n \to [0,\infty)$ having compact support and satisfying $\varphi(y) \ge 1$ for all $y \in B(x,r_1)$ we have

$$\int \varphi\,d\mu_{V,x} = \lim_{\delta\to 0}(2\delta)^{-m}\int_{V_x(\delta)} \varphi\,d\mu \ge \lim_{i\to\infty}(2r_i)^{-m}\mu(B(x,r_i)) = \infty$$

for all $V \in G_{n,n-m}$. This implies that the measure $\mu_{V,x}$ is not defined (see (2.9)).

The main purpose of this paper is to achieve a better understanding of the irregular behaviour of the packing dimensions of sections of sets described in [C], [FJM], and [FJ]. The first step into this direction is to prove that Theorem 2.1 holds without the assumption that $\int |x-y|^{-m}d\mu(y) < \infty$ for μ-almost all $x \in \mathbb{R}^n$. The following remarks and Proposition 2.5 are needed for this purpose.

2.3. Remarks. *(a) Let μ be a finite Radon measure on $\mathbb{R}^n$ and let $B \subset \mathbb{R}^n$ be a Borel set. Using* [Mat3, Corollary 2.14] *we have*

$$\underline{\dim}_{\mathrm{loc}}\, \mu|_B(x) = \underline{\dim}_{\mathrm{loc}}\, \mu(x) \text{ and } \overline{\dim}_{\mathrm{loc}}\, \mu|_B(x) = \overline{\dim}_{\mathrm{loc}}\, \mu(x)$$

and

$$\underline{\mathrm{d}}_{\mu|_B}(x) = \underline{\mathrm{d}}_\mu(x) \text{ and } \overline{\mathrm{d}}_{\mu|_B}(x) = \overline{\mathrm{d}}_\mu(x)$$

for μ-almost all $x \in B$.

(b) Note that

$$\underline{D}(\mu,\nu,x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} = \infty$$

for μ-almost all $x \in \mathbb{R}^n$ if μ and ν are mutually singular finite Radon measures on $\mathbb{R}^n$. In this case we use the notation $\mu \perp \nu$. In fact, assuming that the Borel set

$$E = \{x \in \mathbb{R}^n \mid \underline{D}(\mu,\nu,x) < \infty\}$$

has positive μ-measure and considering the Radon measure $\rho = \mu|_E$, the density point theorem [Mat3, Corollary 2.14] *gives*

$$\underline{D}(\rho,\nu,x) = \underline{D}(\mu,\nu,x)$$

for ρ-almost all $x \in \mathbb{R}^n$. From this we get the contradiction $\rho \ll \nu$ by [Mat3, Theorem 2.12].

(c) Let μ be a finite Radon measure on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a Borel set. It is shown in [JM, Lemma 3.2] *that if $V \in G_{n,n-m}$ such that $(\mathrm{P}_{V^\perp})_*\mu \ll \mathcal{H}^m$, then*

$$\mu_{V,a}|_A = (\mu|_A)_{V,a}$$

for $\mathcal{H}^m$-almost all $a \in V^\perp$.

2.4. Lemma. *Let $V \in G_{n,n-m}$. Assume that μ is a finite Radon measure on $\mathbb{R}^n$. Then there are finite Radon measures $\mu^{\mathrm{V}}_{\mathrm{sing}}$ and $\mu^{\mathrm{V}}_{\mathrm{abs}}$ on $\mathbb{R}^n$ such that*

(1) $(\mathrm{P}_{V^\perp})_*\, \mu^{\mathrm{V}}_{\mathrm{sing}} \perp \mathcal{H}^m$,

(2) $(\mathrm{P}_{V^\perp})_*\, \mu^{\mathrm{V}}_{\mathrm{abs}} \ll \mathcal{H}^m$, *and*

(3) $\mu(B) = \mu^{\mathrm{V}}_{\mathrm{sing}}(B) + \mu^{\mathrm{V}}_{\mathrm{abs}}(B)$

for all Borel sets $B \subset \mathbb{R}^n$.

Proof. For $V \in G_{n,n-m}$, we consider the following decomposition of μ [Mat3, Theorem 2.17]: Let m^1 and m^2 be Radon measures on $V^\perp$ such that $m^1 \perp \mathcal{H}^m$, $m^2 \ll \mathcal{H}^m$, and

$$(\mathrm{P}_{V^\perp})_*\mu(E) = m^1(E) + m^2(E)$$

for all Borel sets $E \subset V^\perp$. Taking a Borel set $A \subset V^\perp$ such that $m^1(V^\perp \setminus A) = 0 = m^2(A)$, define compactly supported Radon measures

$$\mu^{\mathrm{V}}_{\mathrm{sing}} = \mu|_{\mathrm{P}^{-1}_{V^\perp}(A)} \text{ and } \mu^{\mathrm{V}}_{\mathrm{abs}} = \mu|_{\mathrm{P}^{-1}_{V^\perp}(V^\perp\setminus A)}.$$

The claim follows, since $(\mathrm{P}_{V^\perp})_* \,\mu^{\mathrm{V}}_{\mathrm{sing}} = m^1$ and $(\mathrm{P}_{V^\perp})_* \,\mu^{\mathrm{V}}_{\mathrm{abs}} = m^2$. $\square$

Using Lemma 2.4, we are able to relate absolute continuity and singularity of the projected measures to the existence of sliced measures.

Let μ be a finite Radon measure on $\mathbb{R}^n$. For $V \in G_{n,n-m}$, set

$$E(\mu, V) = \{x \in \mathbb{R}^n \mid \mu_{V,x} \text{ is defined}\}.$$

Then

$$E(\mu, V) = \mathrm{P}_{V^\perp}^{-1}(\{a \in V^\perp \mid \mu_{V,a} \text{ is defined}\})$$

is a Borel set (see [Mat2, proof of Lemma 3.3]). Recall that [Mat3, Chapter 10]

$$\mathcal{H}^m(\{a \in V^\perp \mid \mu_{V,a} \text{ is not defined}\}) = 0. \tag{2.11}$$

2.5. Proposition. *Let μ be a finite Radon measure on $\mathbb{R}^n$. For all $V \in G_{n,n-m}$ we have*

(1) $(\mathrm{P}_{V^\perp})_*\mu \perp \mathcal{H}^m \iff \mu_{V,a} = 0$ *for $\mathcal{H}^m$-almost all $a \in V^\perp$*
(2) $(\mathrm{P}_{V^\perp})_*\mu \perp \mathcal{H}^m \iff \mu(E(\mu, V)) = 0$
(3) $(\mathrm{P}_{V^\perp})_*\mu \ll \mathcal{H}^m \iff \mu(\mathbb{R}^n \setminus E(\mu, V)) = 0.$

Proof. (1) Assuming that $(\mathrm{P}_{V^\perp})_*\mu \perp \mathcal{H}^m$ and letting $E \subset V^\perp$ be a Borel set with $\mathcal{H}^m(E) = 0$ and $(\mathrm{P}_{V^\perp})_*\mu(V^\perp \setminus E) = 0$, the disintegration inequality (2.10) gives

$$0 = \int_{V^\perp \setminus E} \mu_{V,a}(\mathrm{P}_{V^\perp}^{-1}(V^\perp \setminus E))\, d\mathcal{H}^m(a) = \int_{V^\perp} \mu_{V,a}(V_a)\, d\mathcal{H}^m(a),$$

as required for the claim.

To prove that the opposite implication holds in (1), we consider the decomposition $\mu = \mu^{\mathrm{V}}_{\mathrm{sing}} + \mu^{\mathrm{V}}_{\mathrm{abs}}$ given by Lemma 2.4. Then $(\mu^{\mathrm{V}}_{\mathrm{abs}})_{V,a} = 0$ for $\mathcal{H}^m$-almost all $a \in V^\perp$, and therefore, the disintegration formula (2.10) gives $\mu^{\mathrm{V}}_{\mathrm{abs}} = 0$. This completes the proof of (1).

(2) Assume first that $(\mathrm{P}_{V^\perp})_*\mu \perp \mathcal{H}^m$. Now Remark 2.3 (b) gives

$$\underline{D}((\mathrm{P}_{V^\perp})_*\mu, \mathcal{H}^m, a) = \infty$$

for $(\mathrm{P}_{V^\perp})_*\mu$-almost all $a \in V^\perp$. This in turn gives $\mu(E(\mu, V)) = 0$ since

$$\underline{D}((\mathrm{P}_{V^\perp})_*\mu, \mathcal{H}^m, \mathrm{P}_{V^\perp}(x)) = \liminf_{\delta \to 0} (2\delta)^{-m} \mu(V_{\mathrm{P}_{V^\perp}(x)}(\delta)) < \infty$$

if $\mu_{V,x}$ is defined (see (2.9)). (Recall that $\mathcal{H}^m(B(,r)) = (2r)^m$.)

On the other hand, if $\mu(E(\mu, V)) = 0$, then $(\mathrm{P}_{V^\perp})_*\mu(\mathrm{P}_{V^\perp}(E(\mu, V))) = 0$. Now the claim follows since $\mathcal{H}^m(V^\perp \setminus \mathrm{P}_{V^\perp}(E(\mu, V))) = 0$ by (2.11).

(3) Supposing that $\mu(\mathbb{R}^n \setminus E(\mu, V)) = 0$ we have $\underline{D}((\mathrm{P}_{V^\perp})_*\mu, \mathcal{H}^m, a) < \infty$ for $(\mathrm{P}_{V^\perp})_*\mu$-almost $a \in V^\perp$. This implies the absolute continuity by [Mat3, Theorem 2.12]. The opposite implication is clear from (2.11). $\square$

2.6. Corollary. *Assume that μ is a finite Radon measure on $\mathbb{R}^n$. For $V \in G_{n,n-m}$, let $\mu = \mu^{\mathrm{V}}_{\mathrm{sing}} + \mu^{\mathrm{V}}_{\mathrm{abs}}$ be the decomposition given by Lemma 2.4. Then*

$$\mu(E(\mu, V)) = \mu^{\mathrm{V}}_{\mathrm{abs}}(\mathbb{R}^n) = \mu^{\mathrm{V}}_{\mathrm{abs}}(E(\mu^{\mathrm{V}}_{\mathrm{abs}}, V))$$

and

$$\mu(\mathbb{R}^n \setminus E(\mu, V)) = \mu^{\mathrm{V}}_{\mathrm{sing}}(\mathbb{R}^n) = \mu^{\mathrm{V}}_{\mathrm{sing}}(\mathbb{R}^n \setminus E(\mu^{\mathrm{V}}_{\mathrm{sing}}, V)).$$

Proof. Since $(\mathrm{P}_{V^\perp})_* \, \mu^{\mathrm{V}}_{\mathrm{sing}} \perp \mathcal{H}^m$, Remark 2.3 (b) implies that

$$\underline{D}((\mathrm{P}_{V^\perp})_* \mu, \mathcal{H}^m, \mathrm{P}_{V^\perp}(x)) = \infty$$

for $\mu^{\mathrm{V}}_{\mathrm{sing}}$-almost all $x \in \mathbb{R}^n$, and therefore $\mu^{\mathrm{V}}_{\mathrm{sing}}(E(\mu, V)) = 0$. Moreover, from (2.11) we get $\mu^{\mathrm{V}}_{\mathrm{abs}}(\mathbb{R}^n \setminus E(\mu, V)) = 0$. With Proposition 2.5 one has

$$\mu(E(\mu, V)) = \mu^{\mathrm{V}}_{\mathrm{abs}}(E(\mu, V)) = \mu^{\mathrm{V}}_{\mathrm{abs}}(\mathbb{R}^n) = \mu^{\mathrm{V}}_{\mathrm{abs}}(E(\mu^{\mathrm{V}}_{\mathrm{abs}}, V))$$

as claimed. The second equality follows similarly. □

2.7. Remarks. *(a) Let μ be a Radon measure on $\mathbb{R}^n$ with compact support. By (2.3) the local lower dimension condition $\underline{\dim}_{\mathrm{loc}} \, \mu(x) > m$ implies the local energy condition $\int |x - y|^{-m} d\mu(y) < \infty$. If the latter is true for μ-almost all $x \in \mathbb{R}^n$, then*

$$(\mathrm{P}_{V^\perp})_* \mu \ll \mathcal{H}^m$$

for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$. This follows directly from [Mat3, Theorem 9.7] *by decomposing μ into a countable sum of measures with finite m-energy. (The details of the proof are similar to those in* [FM, proof of Lemma 4.1]*.)*

(b) By Proposition 2.5 $\mu(\mathbb{R}^n \setminus E(\mu, V)) = 0$ provided that $(\mathrm{P}_{V^\perp})_* \mu \ll \mathcal{H}^m$. This condition is guaranteed for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ by the assumption on Theorem 2.1 (see remark (a)).

Now we are ready to generalize Theorem 2.1.

2.8. Theorem. *Let μ be a finite Radon measure on $\mathbb{R}^n$. Then for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$*

$$\begin{aligned} &\underline{\dim}_{\mathrm{loc}} \, \mu_{V,x}(x) \geq \underline{\dim}_{\mathrm{loc}} \, \mu(x) - m \geq \underline{\mathrm{d}}_\mu(x) - m \quad \textit{and} \\ &\overline{\dim}_{\mathrm{loc}} \, \mu_{V,x}(x) \geq \overline{\mathrm{d}}_\mu(x) - m \end{aligned}$$

for μ-almost all $x \in E(\mu, V)$. Furthermore, $\overline{\mathrm{d}}_\mu(x) = -\infty$ for μ-almost all $x \in \mathbb{R}^n \setminus E(\mu, V)$.

Proof. We will prove the second inequality. The first one may be treated similarly (see also (2.6)).

Since clearly $\mu_{V,x}(B(x,r)) = (\mu|_{B(x,2)})_{V,x}(B(x,r))$ for all $V \in G_{n,n-m}$, $x \in \mathbb{R}^n$, and $r < 1$, we may assume that μ has a compact support. Introducing the Borel set

$$B = \{x \in \mathbb{R}^n \mid \int |x - y|^{-m} d\mu(y) < \infty\},$$

Theorem 2.1 implies that for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ and for μ-almost all $x \in B$ we have

$$\overline{\dim}_{\mathrm{loc}}(\mu|_B)_{V,x}(x) \geq \overline{\mathrm{d}}_{\mu|_B}(x) - m \geq \overline{\mathrm{d}}_{\mu}(x) - m. \tag{2.12}$$

Moreover, from Remark 2.7 (a) we get

$$(\mathrm{P}_{V^\perp})_*\mu|_B \ll \mathcal{H}^m \tag{2.13}$$

for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$.

For fixed $V \in G_{n,n-m}$ for which (2.12) and the absolute continuity condition (2.13) are satisfied, we consider the decomposition of $\mu = \mu^{\mathrm{V}}_{\mathrm{sing}} + \mu^{\mathrm{V}}_{\mathrm{abs}}$ given by Lemma 2.4. Note that by (2.13)

$$\mu^{\mathrm{V}}_{\mathrm{sing}}(B) = 0. \tag{2.14}$$

Since by Proposition 2.5 (1) $(\mu^{\mathrm{V}}_{\mathrm{sing}})_{V,a} = 0$ for $\mathcal{H}^m$-almost all $a \in V^\perp$, Remark 2.3 (c) allows us to conclude that

$$\begin{aligned}(\mu|_B)_{V,a} &= (\mu^{\mathrm{V}}_{\mathrm{abs}}|_B)_{V,a} = (\mu^{\mathrm{V}}_{\mathrm{sing}})_{V,a}|_B + (\mu^{\mathrm{V}}_{\mathrm{abs}})_{V,a}|_B \\ &= (\mu^{\mathrm{V}}_{\mathrm{sing}} + \mu^{\mathrm{V}}_{\mathrm{abs}})_{V,a}|_B = \mu_{V,a}|_B\end{aligned} \tag{2.15}$$

for $\mathcal{H}^m$-almost all $a \in V^\perp$. This together with (2.12) and (2.13) combine to give

$$\overline{\dim}_{\mathrm{loc}}\, \mu_{V,x}|_B(x) \geq \overline{\mathrm{d}}_{\mu}(x) - m$$

for μ-almost all $x \in B$. Note that by Remark 2.3 (a) we have for $\mathcal{H}^m$-almost all $a \in V^\perp$ that

$$\overline{\dim}_{\mathrm{loc}}\, \mu_{V,a}|_B(x) = \overline{\dim}_{\mathrm{loc}}\, \mu_{V,a}(x) \tag{2.16}$$

for $\mu_{V,a}$-almost all $x \in B$. Deducing from (2.10), Lemma 2.4 (2), and (2.15) that

$$\mu(F) = \int \mu_{V,a}(F)\, d\mathcal{H}^m(a) \tag{2.17}$$

for all Borel sets $F \subset B$ and using (2.16), we get

$$\overline{\dim}_{\mathrm{loc}}\, \mu_{V,x}(x) \geq \overline{\mathrm{d}}_{\mu}(x) - m$$

for μ-almost all $x \in B$. (When applying (2.17) we use the facts that the functions $x \mapsto \overline{\mathrm{d}}_{\mu}(x)$, $x \mapsto \overline{\dim}_{\mathrm{loc}}\, \mu_{V,x}(x)$, and $x \mapsto \overline{\dim}_{\mathrm{loc}}\, \mu_{V,x}|_B(x)$ are Borel measurable. The proofs of these facts reduce to similar arguments as in [Mat2, Lemma 4.2].) To complete the proof, the final observation is to combine (2.14) and Corollary 2.6 with the fact $\overline{\mathrm{d}}_{\mu}(x) = -\infty$ for all $x \in \mathbb{R}^n \setminus B$. $\square$

2.9. Remarks. *(a) Theorem 2.8 is genuinely an almost all result, that is, the lower bounds in Theorem 2.8 are not necessarily valid for all subspaces and all points as illustrated by the following construction: Let* $\mu = g(x,y) \cdot \mathcal{L}^2|_Q$ *where* $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$ *and*

$$g(x,y) = \begin{cases} 1, & \text{for } x \le y^{1/2} \\ 0, & \text{for } y^{1/2} < x < y^{1/3} \\ \frac{7}{6} x^{-1/2}, & \text{for } x \ge y^{1/3}. \end{cases}$$

Then it is not difficult to see that $\mu(\mathbb{R}^2) = 1$, $\underline{\mathrm{d}}_\mu(0) = \overline{\mathrm{d}}_\mu(0) = \underline{\dim}_{\mathrm{loc}}\, \mu(0) = \overline{\dim}_{\mathrm{loc}}\, \mu(0) = 2$, *and* $\underline{\dim}_{\mathrm{loc}}\, \mu_{L,0}(0) = \overline{\dim}_{\mathrm{loc}}\, \mu_{L,0}(0) = \frac{1}{2}$ *for* $L = \{(x,0) \mid x \in \mathbb{R}\} \in G_{2,1}$.

(b) Theorem 2.8 is the local counterpart of [JM, Theorem 3.3].

(c) It is possible that $\mu(E(\mu,V)) > 0$ *although the assumption* $\int |x-y|^{-m}\, d\mu(y) < \infty$ *in Theorem 2.1 is not valid. To see this let* $\mu = \mathcal{H}^1|_I$ *where* I *is the unit interval embedded in* $\mathbb{R}^2$.

To verify the validity of the opposite inequalities in Theorem 2.8, we must make further assumptions about the measure μ. In Theorem 2.11 we will prove that for the lower local dimension the equality holds in Theorem 2.8 for measures having Hausdorff dimension strictly larger than m and for the upper one provided that the measure has finite $m+d$-energy for some $d > 0$. The s-energy $I_s(\mu)$ of a finite Radon measure on $\mathbb{R}^n$ is defined by

$$I_s(\mu) = \iint |x-y|^{-s} d\mu(x) d\mu(y).$$

Note that both of the above assumptions are stronger than the local energy condition in Theorem 2.1 (see (2.3)), and therefore Theorem 2.1 gives the lower bounds in Theorem 2.11.

Our approach is based on the results in [JM] concerning upper Hausdorff and packing dimensions. For a finite (non-zero) Radon measure μ on $\mathbb{R}^n$ we define the upper Hausdorff and packing dimensions as follows (see [F2, Proposition 10.3])

$$\begin{aligned} \dim_{\mathrm{H}}^* \mu &= \inf\{\dim_{\mathrm{H}} A \mid A \text{ is a Borel set with } \mu(\mathbb{R}^n \setminus A) = 0\} \\ &= \mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \underline{\dim}_{\mathrm{loc}}\, \mu(x) \quad \text{and} \\ \dim_{\mathrm{p}}^* \mu &= \inf\{\dim_{\mathrm{p}} A \mid A \text{ is a Borel set with } \mu(\mathbb{R}^n \setminus A) = 0\} \\ &= \mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\dim}_{\mathrm{loc}}\, \mu(x). \end{aligned}$$

If $\mu = 0$ we define $\dim_{\mathrm{H}}^* \mu = \dim_{\mathrm{p}}^* \mu = 0$. Recall that the (lower) Hausdorff and packing dimensions, denoted by $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{p}}$, are defined by replacing Borel sets with full measure by Borel sets with positive measure (and μ- ess sup by μ- ess inf) in the above definitions.

Now we are ready to state the result from [JM] that is our starting point. Note that

$$I_t(\mu) < \infty \implies \dim_{\mathrm{H}} \mu \ge t. \tag{2.18}$$

This follows immediately from (2.3).

2.10. Theorem (Järvenpää, Mattila). *Let μ be a Radon measure on $\mathbb{R}^n$ with compact support. Then for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ we have*

(1) $\mathcal{H}^m\text{-}\operatorname{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{H}}^* \mu_{V,a} = \dim_{\mathrm{H}}^* \mu - m$ *provided that* $\dim_{\mathrm{H}} \mu > m$ *and*

(2) $\mathcal{H}^m\text{-}\operatorname{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}^* \mu_{V,a} = \mu\text{-}\operatorname{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) - m$ *provided that* $I_{m+d}(\mu) < \infty$ *for some* $d > 0$.

Proof. See [JM, Theorem 4.2 and Theorem 6.4].

Based on an application of the above result we will prove that corresponding assumptions lead to equalities in Theorem 2.8.

2.11. Theorem. *Let μ be a finite Radon measure on $\mathbb{R}^n$. Then for μ-almost all $x \in \mathbb{R}^n$ and $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ we have*

(1) $\underline{\dim}_{\mathrm{loc}} \mu_{V,x}(x) = \underline{\dim}_{\mathrm{loc}} \mu(x) - m = \underline{\mathrm{d}}_\mu(x) - m$ *provided that* $\dim_H \mu > m$ *and*

(2) $\overline{\dim}_{\mathrm{loc}} \mu_{V,x}(x) = \overline{\mathrm{d}}_\mu(x) - m$ *provided that* $I_{m+d}(\mu) < \infty$ *for some* $d > 0$.

Proof. As in the proof of Theorem 2.8 we may assume that μ has a compact support. The lower bounds for local dimensions of sliced measures follow from Theorem 2.1.

To prove that the opposite inequalities hold, we use the notation $\dim_{\mathrm{loc}}$, d_μ, and $\dim^*$ for both the triplet $\underline{\dim}_{\mathrm{loc}}$, $\underline{\mathrm{d}}_\mu$, and $\dim_{\mathrm{H}}^*$, and for the triplet $\overline{\dim}_{\mathrm{loc}}$, $\overline{\mathrm{d}}_\mu$, and $\dim_{\mathrm{p}}^*$.

Noting that under the assumption $\dim_{\mathrm{H}} \mu > m$ (or $I_{m+d}(\mu) < \infty$) the measure $\mu_{V,x}$ is defined for $\mu \times \gamma_{n,n-m}$-almost all $(x, V) \in \mathbb{R}^n \times G_{n,n-m}$ (see Remark 2.7 (a) and Proposition 2.5 (3)), assume to the contrary that there are real numbers $t_1 > t_2$ such that

$$0 < \mu \times \gamma_{n,n-m}(\{(x, V) \in \mathbb{R}^n \times G_{n,n-m} \mid \dim_{\mathrm{loc}} \mu_{V,x}(x) > t_1 > t_2 > \mathrm{d}_\mu(x) - m\}).$$

Recalling that $(x, V) \mapsto \dim_{\mathrm{loc}} \mu_{V,x}(x)$ and $x \mapsto \mathrm{d}_\mu(x)$ are Borel functions (see [Mat2, Lemma 4.2]), Fubini's theorem implies the existence of a Borel set

$$B \subset \{x \in \mathbb{R}^n \mid \mathrm{d}_\mu(x) - m < t_2\}$$

with $\mu(B) > 0$ such that for all $x \in B$ we have $\gamma_{n,n-m}(G_x) > 0$ for the Borel set

$$G_x = \{V \in G_{n,n-m} \mid \dim_{\mathrm{loc}} \mu_{V,x}(x) > t_1\}.$$

Hence

$$\inf_{x \in B} \inf_{V \in G_x} \dim_{\mathrm{loc}} \mu_{V,x}(x) \geq t_1 > t_2 \geq \sup_{x \in B} \mathrm{d}_\mu(x) - m.$$

For the compactly supported Radon measure $\nu = \mu|_B$ we have $\dim_{\mathrm{H}} \nu > m$ if $\dim_{\mathrm{H}} \mu > m$ (and $I_{m+d}(\nu) < \infty$ if $I_{m+d}(\mu) < \infty$), and furthermore $(P_{V^\perp})_* \nu \ll \mathcal{H}^m$ for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ by [Mat3, Theorem 9.7] and [FM, Lemma 4.1].

The disintegration formula (2.10), Remark 2.3 (a), and Fubini's theorem imply the existence of $P \subset G_{n,n-m}$ with $\gamma_{n,n-m}(P) > 0$ such that for all $V \in P$

$$\begin{aligned}\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a\in V^\perp} \dim^* \nu_{V,a} &= \mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a\in V^\perp}\ \nu_{V,a}\text{-}\operatorname*{ess\,sup}_{x\in V_a} \dim_{\mathrm{loc}} \nu_{V,a}(x)\\ &\geq t_1 > t_2 \geq \nu\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \mathrm{d}_\nu(x) - m\end{aligned}$$

giving a contradiction with Theorem 2.10. Note that by (2.3) and (2.8)

$$\nu\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \underline{\mathrm{d}}_\nu(x) = \dim_{\mathrm{H}}^* \nu$$

under the assumption $\dim_{\mathrm{H}} \nu > m$. □

2.12. Remarks. *(a) It is not sufficient to assume that* $\dim_{\mathrm{H}} \mu \geq m$ *in Theorem 2.11 (1). Indeed, let* $F \subset \mathbb{R}^2$ *be a compact set such that* $0 < \mathcal{H}^1(F) < \infty$ *and* $\mathcal{H}^1(\mathrm{P}_{L^\perp}(F)) = 0$ *for all* $L \in G_{2,1}$. *(For the existence of* F*, see* [F1, Theorem 6.15]*.) Taking* $\mu = \mathcal{H}^1|_F$*, we have* $\dim_{\mathrm{H}} \mu = 1$*, and furthermore,* $\mu(E(\mu, L)) = 0$ *for all* $L \in G_{2,1}$*, since* $(P_{L^\perp})_*\mu$ *and* $\mathcal{H}^1$ *are always mutually singular, see Proposition 2.5.*

b) Theorem 2.11 (1) is valid under a weaker assumption $\int |x-y|^{-m}\, d\mu(y) < \infty$ *for* μ*-almost all* $x \in \mathbb{R}^n$*. Indeed, from the proof of* [JM, Lemma 4.1] *(which gives the upper bound in Theorem 2.10 (1)) one sees that the only assumption needed is the absolute continuity* $(\mathrm{P}_{V^\perp})_*\mu \ll \mathcal{H}^m$ *(see also (2.8) and Remark 2.7 (a)). It follows immediately from (2.4) that this condition is also necessary for the second equality in Theorem 2.11 (1). However, the first equality may be valid even though the local energy condition fails (see Remark 2.9 (c)).*

(c) We do not know whether one can replace the condition $I_{m+d}(\mu) < \infty$ *by* $I_m(\mu) < \infty$ *in Theorem 2.11 (2).*

3. Stability results for packing dimensions of sections of sets

In this section we consider packing dimensional properties of sections $A \cap V_a$ where $A \subset \mathbb{R}^n$, $V \in G_{n,n-m}$, and $a \in V^\perp$. It follows directly from [Mat3, Corollary 9.4] that we may restrict our attention to sets A with $\dim_{\mathrm{H}} A \geq m$. In fact, if A is a Borel set with $\dim_{\mathrm{H}} A < m$ then for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ we have $\dim_{\mathrm{H}} P_{V^\perp}(A) = \dim_{\mathrm{H}} A$ implying that $A \cap V_a = \emptyset$ for $\mathcal{H}^m$-almost all $a \in V^\perp$.

Our aim is to achieve a better understanding about the structure of sets for which the essential supremum of packing dimensions of plane sections is almost surely a constant which is independent of the plane. As noted in the Introduction the examples in [Cs] and [FJM] show that this is not true for all sets $A \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}}(A) \geq m$. However, the properties of the Hausdorff dimension imply that stability results of this kind are valid for a natural subset of the class

$$\mathcal{S}(\mathbb{R}^n) = \{A \subset \mathbb{R}^n \mid \dim_{\mathrm{p}} A = \dim_{\mathrm{H}} A \geq m\}.$$

Indeed, if $A \in \mathcal{S}(\mathbb{R}^n)$ is a Borel set such that $0 < \mathcal{H}^d(A) < \infty$ for $d = \dim_{\mathrm{p}} A = \dim_{\mathrm{H}} A$, it is a straightforward consequence of (1.7) and (1.6) that for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a\in V^\perp} \dim_{\mathrm{p}}(A \cap V_a) = d - m. \tag{3.1}$$

In Theorem 3.10 we will verify (3.1) for a wider class of sets. Toward this generalization, define for all $A \subset \mathbb{R}^n$

$$\mathcal{M}(A) = \{\mu \mid \mu \text{ is a Radon measure on } \mathbb{R}^n \text{ such that } \operatorname{spt}\mu \text{ is compact}, \\ \operatorname{spt}\mu \subset A, \text{ and } 0 < \mu(\mathbb{R}^n) < \infty\}.$$

Here $\operatorname{spt}\mu$ is the support of μ. It is well known that for any analytic set $A \neq \emptyset$ there is $\mu \in \mathcal{M}(A)$ such that both $\dim_{\mathrm{p}} \mu$ and $\dim_{\mathrm{p}}^* \mu$ and both $\dim_{\mathrm{H}} \mu$ and $\dim_{\mathrm{H}}^* \mu$ are arbitrarily close to $\dim_{\mathrm{p}} A$ and $\dim_{\mathrm{H}} A$, respectively, that is, [Cu, Theorem 1.5]

$$\dim_{\mathrm{p}} A = \sup\{\dim_{\mathrm{p}} \mu \mid \mu \in \mathcal{M}(A)\} = \sup\{\dim_{\mathrm{p}}^* \mu \mid \mu \in \mathcal{M}(A)\} \tag{3.2}$$

and

$$\dim_{\mathrm{H}} A = \sup\{\dim_{\mathrm{H}} \mu \mid \mu \in \mathcal{M}(A)\} = \sup\{\dim_{\mathrm{H}}^* \mu \mid \mu \in \mathcal{M}(A)\}. \tag{3.3}$$

To be precise, in [Cu, Theorem 1.5] it is proved that (3.2) and (3.3) hold when the requirement $\operatorname{spt}\mu \subset A$ is replaced by the condition $\mu(\mathbb{R}^n \setminus A) = 0$. However, [Cu, Theorem 1.5] leads immediately to (3.2) and (3.3). In the case of packing dimension it follows directly from [Cu, Lemma 2.5 and Lemma 3.4]) and for the Hausdorff dimension it is a straightforward consequence of [Cu, Lemma 2.5] and Frostman's lemma [Mat3, Theorem 8.8].

Generalizing (3.1) leads to the concept of thickness. For all $A \subset \mathbb{R}^n$ and $\varepsilon > 0$, let

$$\mathcal{M}_\varepsilon(A) = \{\mu \in \mathcal{M}(A) \mid I_\varepsilon(\mu) < \infty\}.$$

3.1. Definition. *Let $V \in G_{n,n-m}$ and $\varepsilon > 0$. A set $A \subset \mathbb{R}^n$ is (V,ε)-thick if*

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a) = \mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \sup\{\dim_{\mathrm{p}}^* \mu \mid \mu \in \mathcal{M}_\varepsilon(A \cap V_a)\}.$$

For $V \in G_{n,n-m}$, the set A is V-thick if it is (V,ε)-thick for some $\varepsilon > 0$. Finally, A is $(n-m)$-thick if it is V-thick for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$.

3.2. Remarks. *(a) By (3.2)*

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a) \geq \mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \sup\{\dim_{\mathrm{p}}^* \mu \mid \mu \in \mathcal{M}_\varepsilon(A \cap V_a)\}$$

for all analytic sets $A \subset \mathbb{R}^n$, $V \in G_{n,n-m}$, and $\varepsilon > 0$.

(b) The condition $I_\varepsilon(\mu) < \infty$ in Definition 3.1 guarantees that the packing dimensions of plane sections of thick sets can be typically estimated by upper packing dimensions of measures having positive Hausdorff dimensions.

(c) If A is (V,ε)-thick, then $\mathcal{M}_\varepsilon(A \cap V_a) \neq \emptyset$ for positively many $a \in V^\perp$. (Recall that $\sup E = -\infty$ if $E = \emptyset$.)

(d) Let $A \subset \mathbb{R}^n$ be a Borel set such that $\dim_{\mathrm{H}} A > m$ and let $0 < \varepsilon < \dim_{\mathrm{H}} A - m$. Then for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$

$$\mathcal{M}_\varepsilon(A \cap V_a) \neq \emptyset$$

for positively many $a \in V^\perp$.

To see this, let $\mu \in \mathcal{M}(A)$ such that $I_{m+\varepsilon}(\mu) < \infty$ [Mat3, Theorem 8.9]*. Combining Remark 2.7 (a), Proposition 2.5, and* [Mat3, Theorem 10.7] *gives that for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ we have*

$$\mu_{V,a} \in \mathcal{M}_\varepsilon(A \cap V_a)$$

for positively many $a \in V^\perp$.

3.3. Examples. *(a) If $A \in \mathcal{S}(\mathbb{R}^n)$ is a Borel set such that $0 < \mathcal{H}^d(A) < \infty$ for $d \equiv \dim_{\mathrm{H}} A > m$ (see Corollary 3.9), then A is $(n-m)$-thick.*

In fact, given such A and $0 < \varepsilon < d - m$, [Mat3, Theorem 8.9] *implies the existence of $\nu \in \mathcal{M}(A)$ with $I_{m+\varepsilon}(\nu) < \infty$. From Remark 3.2 (d) and* [JM, Theorem 3.3] *for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ we have $\nu_{V,a} \in \mathcal{M}_\varepsilon(A \cap V_a)$ and*

$$\dim_{\mathrm{p}}^* \nu_{V,a} \geq \dim_{\mathrm{H}} \nu_{V,a} \geq \dim_{\mathrm{H}} \nu - m \geq \varepsilon$$

for positively many $a \in V^\perp$. Thus

$$\varepsilon \leq \mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \sup\{\dim_{\mathrm{p}}^* \mu \mid \mu \in \mathcal{M}_\varepsilon(A \cap V_a)\}$$

for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$. The $(n-m)$-thickness of A follows by (3.1) and Remark 3.2 (a).

(b) The Hausdorff and packing dimensions of thick sets do not necessarily coincide, and therefore the class of thick sets is strictly larger than $\mathcal{S}(\mathbb{R}^n)$. The verification of this statement is based on [FM, Example 5.2 and Theorem 4.5].

Let $m < s < t < n$ and let F be the support of the Radon probability measure μ constructed in [FM, Example 5.2]. *Then F is compact, $\dim_{\mathrm{H}} F = s$, $\dim_{\mathrm{p}} F = t$, and for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$*

$$\dim_{\mathrm{p}}(F \cap V_a) \leq \frac{(n-m)t(s-m)}{ns - mt} \equiv C \tag{3.4}$$

for all $a \in V^\perp$ [FM, Example 5.2].

For the purpose of showing that F is $(n-m)$-thick, by (3.4) and Remark 3.2 (a) it suffices to prove that for some $\varepsilon > 0$

$$C \leq \mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \sup\{\dim_{\mathrm{p}}^* \mu \mid \mu \in \mathcal{M}_\varepsilon(F \cap V_a)\} \tag{3.5}$$

for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$.

Since $\dim_{\mathrm{H}} \mu > m$, [FM, Theorem 4.5], *Remark 2.7 (a), and the disintegration formula (2.10) imply that for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$*

$$\dim_{\mathrm{p}}^* \mu_{V,a} \geq C \text{ and } \mu_{V,a}(\mathbb{R}^n) > 0 \tag{3.6}$$

for positively many $a \in V^\perp$. Furthermore, letting $0 < \delta < s - m$ and using the fact that there is a constant c such that $\mu(B(x,r)) \leq cr^s$ for all $x \in F$ and $0 < r \leq 1$ (see [FM, Example 5.2]*), it follows easily that $I_{s-\delta}(\mu) < \infty$. By* [Mat3, Theorem 10.7] *this in turn gives that for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$*

$$I_{s-\delta-m}(\mu_{V,a}) < \infty$$

for $\mathcal{H}^m$-almost all $a \in V^\perp$. Combining this with (3.6) gives (3.5) for $\varepsilon = s - \delta - m$.

Next we prove two technical lemmas. As the first consequence of them we state in Corollary 3.9 a relation between thickness and dimension.

We use the notation $\mathrm{C}_0^+(\mathbb{R}^n)$ for the space of continuous and non-negative functions on $\mathbb{R}^n$ having compact support.

3.4. Lemma. *For $V \in G_{n,n-m}$ let $K \subset V^\perp$ be a bounded $\mathcal{H}^m$-measurable set such that $\mathcal{H}^m(K) > 0$. Let $A \subset \mathbb{R}^n$ be compact. Assume that there is $\varepsilon > 0$ such that for all $a \in K$ there is a probability measure $\nu_a \in \mathcal{M}_\varepsilon(A \cap V_a)$ such that the function $a \mapsto \int \phi \, d\nu_a$ is $\mathcal{H}^m$-measurable for all $\phi \in \mathrm{C}_0^+(\mathbb{R}^n)$. For $a \in V^\perp \setminus K$, set $\nu_a = 0$. Then the Radon measure μ defined for all $\phi \in \mathrm{C}_0^+(\mathbb{R}^n)$ as*

$$\int \phi \, d\mu = \int \int \phi \, d\nu_a \, d\mathcal{H}^m(a). \tag{3.7}$$

has the following properties:

(1) *For $\mathcal{H}^m$-almost all $a \in V^\perp$ we have $\mu_{V,a}(B) = \nu_a(B)$ for all Borel sets $B \subset \mathbb{R}^n$*
(2) $(\mathrm{P}_{V^\perp})_* \mu \ll \mathcal{H}^m$
(3) $\dim_{\mathrm{H}} \mu \geq m + \varepsilon$.

Before the proof we state two remarks:

3.5. Remarks. *(a) The existence of the Radon measure μ in (3.7) follows from Riesz representation theorem* [Mat3, Theorem 1.16].

(b) By the monotone convergence theorem the equation (3.7) is valid for all non-negative lower semicontinuous functions g.

Proof of Lemma 3.4. Given $V \in G_{n,n-m}$ and $\varphi \in \mathrm{C}_0^+(\mathbb{R}^n)$ we obtain from Remark 3.5 (b) and [Mat3, Corollay 2.14]

$$\begin{aligned}\lim_{\delta \to 0} (2\delta)^{-m} \int_{V_a(\delta)} \varphi(x) \, d\mu(x) &= \lim_{\delta \to 0} (2\delta)^{-m} \int_{\{b \in V^\perp \mid |a-b| \leq \delta\}} \int \varphi(x) \, d\nu_b(x) \, d\mathcal{H}^m(b) \\ &= \int \varphi(x) \, d\nu_a(x)\end{aligned}$$

for $\mathcal{H}^m$-almost all $a \in V^\perp$. Note that the separability of $\mathrm{C}_0^+(\mathbb{R}^n)$ implies that the exceptional set of points a may be chosen to be independent on the choice of $\varphi \in \mathrm{C}_0^+(\mathbb{R}^n)$. This gives (1) by (2.9) and Riesz representation theorem [Mat3, Theorem 1.16].

The second claim follows from [Mat3, Theorem 2.12] since by Remark 3.5 (b)

$$\underline{D}((\mathrm{P}_{V^\perp})_* \mu, \mathcal{H}^m, a) \leq \liminf_{\delta \to 0} (2\delta)^{-m} \int_{\{b \in V^\perp \mid |a-b| \leq \delta\}} \nu_b(\mathbb{R}^n) \, d\mathcal{H}^m(b) \leq 1$$

for all $a \in V^\perp$.

According to the proof of [JM, Lemma 3.1]

$$\mathcal{H}^m\text{-} \operatorname*{ess\,inf}_{a \in V^\perp} \{\dim_{\mathrm{H}} \mu_{V,a} \mid \mu_{V,a}(\mathbb{R}^n) > 0\} \leq \dim_{\mathrm{H}} \mu - m$$

provided $(\mathrm{P}_{V^\perp})_* \mu \ll \mathcal{H}^m$. Combining this with (2.18), (1), and (2), gives (3). □

We continue by introducing the notation needed in Lemma 3.7 which is an important tool in both Corollary 3.9 and Theorem 3.10.

For any complete separable metric space Y, let $\mathcal{K}(Y)$ be the space of all non-empty compact subsets of Y equipped with the Hausdorff metric. Denote by $\mathcal{P}(Y)$ the space of all Borel probability measures on Y with a metric comparable to the weak*-topology. Then both $\mathcal{K}(Y)$ and $\mathcal{P}(Y)$ are complete separable metric spaces. Note that the definition of packing dimension extends naturally to Y.

3.6. Remark. *Let $\mu_i \in \mathcal{P}(Y)$ and $K_i \in \mathcal{K}(Y)$ such that $(\mu_i, K_i) \to (\mu, K) \in \mathcal{P}(Y) \times \mathcal{K}(Y)$. Then for all $\varepsilon > 0$ we have $K_i \subset K(\varepsilon)$ for all large i, and therefore from the Portmanteau theorem* [Ke, Theorem 17.20] $\limsup_{i\to\infty} \mu_i(K_i)) \leq \limsup_{i\to\infty} \mu_i(K(\varepsilon)) \leq \mu(K(\varepsilon))$ *giving*

$$\limsup_{i\to\infty} \mu_i(K_i) \leq \mu(K).$$

Here $K(\varepsilon)$ is the closed ε-neighbourhood of K in Y.

3.7. Lemma. *Let X and Y be complete separable metric spaces. Assume that $A \subset X \times Y$ is compact. Then for all real numbers t and $\varepsilon > 0$ the set*

$$B = \{(x, \mu) \in X \times \mathcal{P}(Y) \mid \dim_{\mathrm{p}}^* \mu \geq t, \operatorname{spt} \mu \subset A_x, \text{ and } I_\varepsilon(\mu) < \infty\}$$

is analytic. Here $A_x = \{y \in Y \mid (x, y) \in A\}$ for all $x \in X$.

Proof. Approximating the kernel $(y_1, y_2) \mapsto d(y_1, y_2)^{-\varepsilon}$ by an increasing sequence of continuous bounded functions on $Y \times Y$, the monotone convergence theorem implies that the set $\{\mu \in \mathcal{P}(Y) \mid I_\varepsilon(\mu) > c\}$ is open for all $c \geq 0$. Therefore

$$E = \{(x, \mu, K) \in X \times \mathcal{P}(Y) \times \mathcal{K}(Y) \mid I_\varepsilon(\mu) < \infty\}$$

is a Borel set.

The set $N = \{(\mu, K) \in \mathcal{P}(Y) \times \mathcal{K}(Y) \mid \mu(K) > 0\}$ is Borel since the set $\{(\mu, K) \in \mathcal{P}(Y) \times \mathcal{K}(Y) \mid \mu(K) \geq c\}$ is closed for all $c > 0$ by Remark 3.6. For all $\mu \in \mathcal{P}(Y)$, $c > 0$ and $r > 0$ define

$$L(\mu, c, r) = \{y \in Y \mid \mu(B(y, r)) < r^c\}$$

and

$$L(\mu, c) = \{y \in Y \mid \overline{\dim}_{\mathrm{loc}} \mu(y) \geq c\}.$$

Let $\{r_i \mid i \in \mathbb{N}\}$ be an enumeration of the rational numbers in the open unit interval $(0, 1)$. By Remark 3.6 the set

$$D_{c,i,k} = \{(\mu, K) \in \mathcal{P}(Y) \times \mathcal{K}(Y) \mid K \subset \bigcup_{\substack{l=k \\ r_l \leq \frac{1}{k}}}^{\infty} L(\mu, c - \tfrac{1}{i}, r_l)\}$$

is open for all $c > 0$ and positive integers i and k. Noting that in the definition (2.2) r may be restricted to positive rationals, this gives that

$$D_c = \{(\mu, K) \in \mathcal{P}(Y) \times \mathcal{K}(Y) \mid K \subset L(\mu, c)\} = \bigcap_{i=1}^{\infty} \bigcap_{k=1}^{\infty} D_{c,i,k}$$

is a Borel set. Since by [Ke, Theorem 17.11] $\dim_{\mathrm{p}}^* \mu \geq c$ if and only if for all $\delta > 0$ there is a compact set K such that $\mu(K) > 0$ and $K \subset L(\mu, c - \delta)$, we obtain the analyticity of the set

$$\{\mu \in \mathcal{P}(Y) \mid \dim_{\mathrm{p}}^* \mu \geq c\} = \bigcap_{j=1}^{\infty} \pi_1(D_{c-\frac{1}{j}} \cap N).$$

Here π_1 is the projection from $\mathcal{P}(Y) \times \mathcal{K}(Y)$ onto $\mathcal{P}(Y)$. Hence

$$D = \{(x, \mu, K) \in X \times \mathcal{P}(Y) \times \mathcal{K}(Y) \mid \dim_{\mathrm{p}}^* \mu \geq t\}$$

is analytic.

By [Ke, Theorem 14.12] the set $\{(x, K) \in X \times \mathcal{K}(Y) \mid K = A_x\}$ is a Borel set as the graph of the Borel measurable function $x \mapsto A_x$ [Ku, p. 58], implying the Borel measurability of

$$J = \{(x, \mu, K) \in X \times \mathcal{P}(Y) \times \mathcal{K}(Y) \mid K = A_x\}.$$

Finally, using Remark 3.6, one easily verifies that

$$T = \{(x, \mu, K) \in X \times \mathcal{P}(Y) \times \mathcal{K}(Y) \mid \operatorname{spt} \mu \subset K\}$$

is closed. This completes the proof since

$$B = \pi_{12}(E \cap D \cap J \cap T)$$

where $\pi_{12} : X \times \mathcal{P}(Y) \times \mathcal{K}(Y) \to X \times \mathcal{P}(Y)$ is the projection. $\square$

3.8. Remark. *Proof of Lemma 3.7 shows that the mapping $\mu \mapsto \dim_{\mathrm{p}}^* \mu$ is measurable with respect to the σ-algebra generated by analytic sets, denoted by $\mathcal{B}(\mathcal{A})$.*

According to the next corollary, thick sets have large dimension.

3.9. Corollary. *Let $V \in G_{n,n-m}$ and $\varepsilon > 0$. If $A \subset \mathbb{R}^n$ is a compact set such that $\mathcal{M}_\varepsilon(A \cap V_a) \neq \emptyset$ for positively many $a \in V^\perp$, then $\dim_{\mathrm{H}} A \geq m + \varepsilon$. In particular, $\dim_{\mathrm{H}} A \geq m + \varepsilon$ for compact (V, ε)-thick sets $A \subset \mathbb{R}^n$.*

Proof. We may assume that $A \subset [0,1]^n$. The analyticity of the set

$$B = \{(a, \nu) \in [0,1]^m \times \mathcal{P}([0,1]^{n-m}) \mid I_\varepsilon(\nu) < \infty \text{ and } \operatorname{spt} \nu \subset A_a\}$$

follows from Lemma 3.7, and from the assumption one obtains $\mathcal{H}^m(\pi_1(B)) > 0$. According to Jankov-von Neuman theorem [Ke, Theorem 18.1] there exists a $\mathcal{B}(\mathcal{A})$-measurable mapping $f : \pi_1(B) \to \mathcal{P}([0,1]^{n-m})$ whose graph is a subset of B. In this way we find a analytic set $K \subset V^\perp$ with positive $\mathcal{H}^m$-measure such that for all $a \in K$ there is $\nu_a \in \mathcal{M}_\varepsilon(A \cap V_a)$ such that the function $a \mapsto \int \varphi \, d\nu_a$ is $\mathcal{B}(\mathcal{A})$-measurable for all $\varphi \in \mathcal{C}_0^+(\mathbb{R}^n)$. (For the relation between Radon and Borel measures in this context, see [Mat3, Corollary 1.11] and [Ke, Theorem 17.10].) Setting $\nu_a = 0$ for $a \notin K$ and defining the Radon measure μ on A by the formula

$$\int \varphi \, d\mu = \iint \varphi \, d\nu_a \, d\mathcal{H}^m(a)$$

for all $\varphi \in \mathcal{C}_0^+(\mathbb{R}^n)$, the first claim follows from Lemma 3.4 (3). The second claim is an immediate consequence of Remark 3.2 (c). $\square$

Now we are ready to prove the main theorem of this section. For all $A \subset \mathbb{R}^n$, let

$$\mathcal{M}_*(A) = \{\mu \in \mathcal{M}(A) \mid \dim_{\mathrm{H}} \mu > m\}. \tag{3.8}$$

3.10. Theorem. *Let $A \subset \mathbb{R}^n$ be a compact $(n-m)$-thick set. Then for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$*

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a) = \sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} - m. \tag{3.9}$$

3.11. Remarks. *(a) It follows directly from (2.6) and (3.2) that*

$$\sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} \leq \dim_{\mathrm{p}} A$$

for all $A \subset \mathbb{R}^n$.

(b) For all analytic $A \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}} A > m$ we have by (3.3), Remark 2.7 (a), and (2.8) that

$$\sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} \geq \dim_{\mathrm{H}} A.$$

In particular, by Corollary 3.9 the constant on the right hand side of (3.9) is positive for analytic $(n-m)$-thick sets.

(c) If $A \subset \mathbb{R}^n$ is analytic such that $d = \dim_{\mathrm{H}} A = \dim_{\mathrm{p}} A > m$, then

$$\sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} = d$$

by (a) and (b) (see (3.1)).

Proof of Theorem 3.10. We may assume that $A \subset [0,1]^n$. We will first verify that the following series of inequalities

$$\begin{aligned}
&\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \sup\{\dim_{\mathrm{p}}^* \nu \mid \nu \in \mathcal{M}(A \cap V_a)\} \\
&\qquad \geq \sup\{\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}^* \mu_{V,a} \mid \mu \in \mathcal{M}(A)\} && (3.10)\\
&\qquad \geq \sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\dim}_{\mathrm{loc}} \mu_{V,x}(x) \mid \mu \in \mathcal{M}(A)\} && (3.11)\\
&\qquad \geq \sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} - m && (3.12)
\end{aligned}$$

is valid for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ implying by (3.2) that

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a) \geq \sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} - m.$$

To see that (3.10) holds, note that for any $\mu \in \mathcal{M}(A)$ the slice $\mu_{V,a}$ is defined for $\mathcal{H}^m$-almost all $a \in V^\perp$ and $\mu_{V,a} \in \mathcal{M}(A \cap V_a)$ if $\mu_{V,a} \neq 0$.

For (3.11), let $V \in G_{n,n-m}$, $s \geq 0$, and $\mu \in \mathcal{M}(A)$ such that $\mu(E) > 0$ for the Borel set

$$E = \{x \in \mathbb{R}^n \mid \overline{\dim}_{\mathrm{loc}} \mu_{V,x}(x) > s\}.$$

(The Borel measurability of the function $x \mapsto \overline{\dim}_{\mathrm{loc}} \mu_{V,x}(x)$ can be verified by means of the methods introduced in [Mat2, Lemma 4.2].)

Letting the measures $\mu_{\text{sing}}^{\text{V}}$ and $\mu_{\text{abs}}^{\text{V}}$ be as in Lemma 2.4, we obtain $\mu_{\text{sing}}^{\text{V}}(E) = 0$ since $\mu_{\text{sing}}^{\text{V}}(\{x \in \mathbb{R}^n \mid \mu_{V,x} \text{ is defined}\}) = 0$ by Corollary 2.6, and therefore

$$0 < \mu_{\text{abs}}^{\text{V}}(E) = \int (\mu_{\text{abs}}^{\text{V}})_{V,a}(\{x \in \mathbb{R}^n \mid \overline{\dim}_{\text{loc}}(\mu_{\text{abs}}^{\text{V}})_{V,a}(x) > s\})\, d\mathcal{H}^m(a)$$

by (2.10) and Proposition 2.5 (a). Hence

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\text{p}}^* (\mu_{\text{abs}}^{\text{V}})_{V,a} > s$$

giving (3.11) since $\mu_{\text{abs}}^{\text{V}} \in \mathcal{M}(A)$.

Finally, let

$$L = \sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) \mid \mu \in \mathcal{M}_*(A)\} - m.$$

Taking a sequence $\mu^i \in \mathcal{M}_*(A)$ such that $\mu^i\text{-}\operatorname{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_{\mu^i}(x) - m \to L$ as $i \to \infty$ and using Theorem 2.1 (see Remarks 2.7), we find for all i a set $\mathrm{P}_i \subset G_{n,n-m}$ with $\gamma_{n,n-m}(P_i) = 0$ such that

$$\mu^i\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_{\mu^i}(x) - m \le \mu^i\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\dim}_{\text{loc}}(\mu^i)_{V,x}(x)$$

for all $V \in G_{n,n-m} \setminus P_i$. Defining

$$P = \bigcup_{i=1}^{\infty} P_i,$$

and taking $V \in G_{n,n-m} \setminus P$, we have

$$\sup\{\mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\dim}_{\text{loc}}\, \mu_{V,x}(x) \mid \mu \in \mathcal{M}(A)\} \ge \mu^i\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_{\mu^i}(x) - m$$

for all i. Letting i tend to infinity (3.12) follows.

When proving the remaining inequality in (3.9), we need the following result from [JM]: Let μ be a Radon measure on $\mathbb{R}^n$ with compact support and $V \in G_{n,n-m}$ such that $(\mathrm{P}_{V^\perp})_*\mu \ll \mathcal{H}^m$. Assume that there is $\varepsilon > 0$ such that $I_\varepsilon(\mu_{V,a}) < \infty$ for $\mathcal{H}^m$-almost all $a \in V^\perp$. Then

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\text{p}}^* \mu_{V,a} \le \mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) - m. \tag{3.13}$$

The verification of this statement may be read from the proof of [JM, Theorem 6.4].

For the purpose of completing the proof of (3.9), let $V \in G_{n,n-m}$ and $\varepsilon > 0$ be such that the equality in Definition 3.1 is valid. Taking

$$s < \mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \sup\{\dim_{\text{p}}^* \nu \mid \nu \in \mathcal{M}_\varepsilon(A \cap V_a)\} \tag{3.14}$$

and proceeding by means of Jankov-von Neuman theorem [Ke, Theorem 18.1] as in Corollary 3.9, we find a bounded analytic set $K \subset V^\perp$ with $\mathcal{H}^m(K) > 0$ such that for all $a \in K$ there is a probability measure $\nu_a \in \mathcal{M}_\varepsilon(A \cap V_a)$ with $\dim_{\text{p}}^* \nu_a \ge s$ such that the function $a \mapsto \nu_a$ is $\mathcal{B}(\mathcal{A})$-measurable. Defining the Radon measure μ as in Lemma 3.4, we have $(\mathrm{P}_{V^\perp})_*\mu \ll \mathcal{H}^m$, and $\mu_{V,a} = \nu_a$ for $\mathcal{H}^m$-almost all $a \in V^\perp$, and therefore (3.13) gives

$$s \le \mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \overline{\mathrm{d}}_\mu(x) - m.$$

This completes the proof since $\mu \in \mathcal{M}_*(A)$ by Lemma 3.4. $\square$

3.12. Remarks. *(a) Theorem 3.10 is clearly valid for σ-compact sets. The compactness of A is needed only for measurability arguments in the proof of Lemma 3.7 when using the Borel measurability of the mapping $x \mapsto A_x$.*

(b) One may read from the proof of Theorem 3.10 that the statement (3.9) is valid under the following weaker condition than thickness: For all $V \in G_{n,n-m}$, let $u_V = \mathcal{H}^m\text{-}\operatorname{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a)$. Suppose that $\gamma_{n,n-m}$-almost every $V \in G_{n,n-m}$ satisfies the property stated as follws: For all $t < u_V$ there are $\varepsilon > 0$ and a $\mathcal{H}^m$-measurable set $B_t \subset V^\perp$ with $\mathcal{H}^m(B_t) > 0$ such that for all $b \in B_t$ there exists a measure $\mu_t \in \mathcal{M}_\varepsilon(A \cap V_b)$ with $\dim_{\mathrm{p}}^* \mu \geq t$ such that the function $b \mapsto \int \varphi \, d\mu_b$ is $\mathcal{H}^m$-measurable for all $\varphi \in \mathcal{C}_0^+(\mathbb{R}^n)$. Here ε may tend to zero as t goes to u_V. Note that in Definition 3.1 ε depends only on V.

(c) The supremum in Theorem 3.10 can be taken over $\mathcal{M}(A)$ by the last statement in Theorem 2.8.

(d) In [FJ, Theorem 10] *it is proved that for all compact sets $A \subset \mathbb{R}^n$ and $V \in G(n, n-m)$*

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{p}}(A \cap V_a) = \dim_{\mathrm{p}}^{\mathrm{V}} A$$

where $\dim_{\mathrm{p}}^{\mathrm{V}}$ is a generalized direction dependent packing dimension introduced in [FJ]*. By Theorem 3.10 $\dim_{\mathrm{p}}^{\mathrm{V}} A$ is almost surely a constant provided that A is a compact $(n-m)$-thick set.*

(e) Combining (1.5) and (1.7) on obtains

$$\dim_{\mathrm{H}}(A \cap V_x) = s - m \tag{3.15}$$

for $\mathcal{H}^s \times \gamma_{n,n-m}$-almost all $(x, V) \in A \times G_{n,n-m}$ if $m < s \leq n$ and $A \subset \mathbb{R}^n$ such that $\mathcal{H}^s(A) < \infty$. It appears that a slight modification of Example 3.3 (b) gives a compact $(n-m)$-thick set $A \subset \mathbb{R}^n$ with $0 < \mathcal{P}^s(A) < \infty$ for $s = \dim_{\mathrm{p}} A > m$ such that $\dim_{\mathrm{p}}(A \cap V_x)$ is not constant for $\mathcal{P}^s \times \gamma_{n,n-m}$-almost all $(x, V) \in A \times G_{n,n-m}$. This means that the analogue to (3.15) is not valid for the packing dimension even in the case where A is thick.

To see this, fix $m < t_1 < t_2 < s < n$. Let μ_1 and μ_2 be compactly supported Radon probability measures constructed as in [FM, Example 5.2] *such that $\dim_{\mathrm{p}} F_1 = \dim_{\mathrm{p}} F_2 = s$, $\dim_{\mathrm{H}} F_1 = t_1$, and $\dim_{\mathrm{H}} F_2 = t_2$ for $F_1 = \operatorname{spt} \mu_1 \subset B_1$ and $F_2 = \operatorname{spt} \mu_2 \subset B_2$ where B_1 and B_2 are disjoint closed balls. Using standard methods and the scaling properties in* [FM, Example 5.2 (a) and (b)]*, one easily checks that for $i = 1, 2$ we have $0 < \mathcal{P}^s(F_i) < \infty$ and $\underline{D}(\mu_i, \mathcal{P}^s|_{F_i}, x) < \infty$ for all $x \in F_i$ giving $\mu_i \ll \mathcal{P}^s|_{F_i}$ by* [Mat3, Theorem 2.12 (3)]*. Combining this with* [FM, Example 5.2 (c)] *and* [FM, Theorem 4.5] *gives that for $i = 1, 2$ and for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$*

$$\dim_{\mathrm{p}}(F_i \cap V_x) = c_i$$

for $\mathcal{P}^s$-positively many $x \in F_i$. Here $c_1 \neq c_2$ are constants. Taking $A = F_1 \cup F_2$ completes the construction. Note that A is $(n-m)$-thick as the union of $(n-m)$-thick sets (see Example 3.3 (b)).

As illustrated by the examples in [FJM] and [Cs], Theorem 3.10 is not valid for all compact sets. Taking into consideration that in both [FJM] and [Cs] the constructions have Hausdorff dimension equal to m, it is natural to ask whether $\dim_{\mathrm{H}} A > m$ is a sufficient condition for the statement in Theorem 3.10. (Recall that by Corollary 3.9 any compact $(n-m)$-thick set has Hausdorff dimension strictly greater than m.) The answer is negative as seen by the following example: Let $E \subset \mathbb{R}^n$ be a compact set such that for positively many $V \in G_{n,n-m}$ we have $E \cap V_a = \emptyset$ for $\mathcal{H}^m$-almost all $a \in V^\perp$, and for positively many $V \in G_{n,n-m}$ we have $\dim_{\mathrm{p}}(E \cap V_a) = n - m$ for positively many $a \in V^\perp$. (For the construction of E, see [FJM, Theorem 4.1].) Taking $A = E \cup F$, where F is a Cantor set with $m < \dim_{\mathrm{H}} F = \dim_{\mathrm{p}} F < n$, gives $\dim_{\mathrm{H}} A > m$. However, for positively many $V \in G_{n,n-m}$ we have $\dim_{\mathrm{p}}(A \cap V_a) = \dim_{\mathrm{H}} F - m$ for positively many $a \in V^\perp$, and for positively many $V \in G_{n,n-m}$ we have $\dim_{\mathrm{p}}(E \cap V_a) = n - m$ for positively many $a \in V^\perp$.

Based on the above examples one might try to argue that it is sufficient to assume some kind of local dimension condition for the stability result (3.9). However, this is not possible at least in the following sense: We say that $A \subset \mathbb{R}^n$ satisfies the local dimension condition if for all $x \in \mathbb{R}^n$ and $r > 0$ with $\dim_{\mathrm{p}}(A \cap B(x,r)) > m$ we have $\dim_{\mathrm{H}}(A \cap B(x,r)) > m$. The local dimension condition does not imply stability result akin to (3.9). This is seen by putting a scaled copy of a Cantor set C with $m < \dim_{\mathrm{H}} C < n$ inside each construction parallelepiped of the set E in [FJM, Theorem 4.1] such that it is disjoint from the next generation parallelepipeds of E. Note that taking a suitable scaled copy of the set in [FM, Example 5.2] instead of the Cantor set C in the above example, shows that the condition

$$\dim_{\mathrm{p}}(A \cap B(x,r)) = \sup\{\dim_{\mathrm{p}}^* \mu \mid \mu \in \mathcal{M}_{m+\varepsilon}(A \cap B(x,r))\}$$

for all $x \in \mathbb{R}^n$ and $r > 0$ with $\dim_{\mathrm{p}}(A \cap B(x,r)) > m$ does not imply the $(n-m)$-thickness of A.

Our final remark is concerned with a global energy condition on a set A according to which for some $\varepsilon > 0$

$$I_{m+\varepsilon}(\mu) < \infty \tag{3.16}$$

provided that $\mu \in \mathcal{M}(A)$ with $\dim_{\mathrm{p}}^* \mu > m$. Clearly any set satisfying (3.16) is $(n-m)$-thick but, for example, the unit ball supports a Radon measure μ with $\dim_{\mathrm{p}} \mu = n$ and $I_s(\mu) = \infty$ for all $s > 0$.

Intuitively, all the above remarks seem to suggest that the instable behaviour of packing dimensions of sections of sets described in [FJM] and [Cs] is due to the fact that there are parts of the sets with packing dimension strictly greater than m and Hausdorff dimension equal to m. It is an interesting open problem to find out whether there exists a characterization of stable sets without using any information on sections.

4. Exceptional sets of planes for subsets of a given set

In this section we indicate another difference between Hausdorff and packing dimensional properties of sections of sets.

In [Mar] Marstrand proved that for projections the exceptional set of planes can be chosen to be independent of subsets of a given set when considering Hausdorff dimension. More precisely, let $A \subset \mathbb{R}^2$ be a Borel set with $0 < \mathcal{H}^s(A) < \infty$ for some $1 < s \leq 2$. Then there exists $D \subset G_{2,1}$ with $\gamma_{2,1}(G_{2,1} \setminus D) = 0$ such that for any Borel set $B \subset A$ with $\mathcal{H}^s(B) > 0$ we have

$$\mathcal{H}^1(\mathrm{P}_L(B)) > 0$$

for all $L \in D$ [Mar, Lemma 13]. This clearly extends to higher dimensions.

According to the the following proposition the analogue of the above result holds for sections.

4.1. Proposition. *Let $A \subset \mathbb{R}^n$ be a Borel set such that $0 < \mathcal{H}^s(A) < \infty$ for $m < s < n$. There exists $D \subset G_{n,n-m}$ with $\gamma_{n,n-m}(G_{n,n-m} \setminus D) = 0$ such that for all Borel sets $B \subset A$ with $\mathcal{H}^s(B) > 0$*

$$\dim_{\mathrm{H}}(B \cap V_x) = s - m$$

for all $V \in D$ and for $\mathcal{H}^s$-almost all $x \in B$.

Proof. We may assume that A is bounded. Applying [FM, Lemma 4.1] and [JM, Theorem 3.8] to the compactly supported Radon measure $\mu = \mathcal{H}^s|_A$, we find $D \subset G_{n,n-m}$ with $\gamma_{n,n-m}(G_{n,n-m} \setminus D) = 0$ such that for all $V \in D$

$$(P_{V^\perp})_*\mu \ll \mathcal{H}^m \text{ and} \tag{4.1}$$

$$\mathcal{H}^m\text{- ess inf}\{\dim_{\mathrm{H}} \mu_{V,a} \mid a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^n) > 0\} = s - m. \tag{4.2}$$

Consider a Borel set $B \subset A$ with $\mathcal{H}^s(B) > 0$. Let $V \in D$. From (4.1), (4.2), and Remark 2.3 (c)

$$\mathcal{H}^m\text{-}\operatorname*{ess\,inf}_{a \in E_V} \dim_{\mathrm{H}}(\mu|_B)_{V,a} \geq s - m$$

where

$$E_V = \{a \in V^\perp \mid (\mu|_B)_{V,a}(\mathbb{R}^n) > 0\}$$

is a Borel set (see the proof of [Mat2, Lemma 3.4] where it is verified that $a \mapsto \int g \, d\nu_{V,a}$ is a Borel function for all non-negative lower semicontinuous functions g on $\mathbb{R}^n$ with $\int g \, d\nu < \infty$). By Remark 2.3 (c) this gives $\mathcal{H}^m(F_V) = 0$ for

$$F_V = \{a \in E_V \mid \dim_{\mathrm{H}}(B \cap V_a) < s\},$$

implying the existence of a Borel set C_V with $F_V \subset C_V \subset E_V$ and $\mathcal{H}^m(C_V) = 0$ [Mat3, Corollary 4.5]. Having proved

$$\mu(B \setminus \mathrm{P}_{V^\perp}^{-1}(E_V \setminus F_V)) = 0 \tag{4.3}$$

we have $\dim_{\mathrm{H}}(B \cap V_x) \geq s - m$ for μ-almost all $x \in B$ since $\dim_{\mathrm{H}}(B \cap V_x) \geq s - m$ for all $x \in B \cap \mathrm{P}_{V^\perp}^{-1}(E_V \setminus F_V)$. For (4.3) note that by (4.1), the disintegration formula (2.10), and Remark 2.3 (c)

$$\mu(B \cap \mathrm{P}_{V^\perp}^{-1}(V^\perp \setminus E_V)) = \int_{V^\perp \setminus E_V} (\mu|_B)_{V,a}(\mathrm{P}_{V^\perp}^{-1}(V^\perp \setminus E_V)) \, d\mathcal{H}^m(a) = 0.$$

Similarly $\mu(\mathrm{P}_{V^\perp}^{-1}(C_V)) = 0$. Therefore the inclusion $B \setminus \mathrm{P}_{V^\perp}^{-1}(E_V \setminus F_V) \subset B \cap \mathrm{P}_{V^\perp}^{-1}(C_V \cup (V^\perp \setminus E_V))$ gives (4.3). Finally, using once again (4.1) completes the proof by (1.5). □

4.2. Remark. *The construction of Remark 3.12 (e) shows that the analogue to Proposition 4.1 is not valid for the packing dimension even in the case where A and B are thick.*

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