EXTENDED ERRATUM TO "POROUS MEASURES ON $\mathbb{R}^n$: LOCAL STRUCTURE AND DIMENSIONAL PROPERTIES"

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The claim of [3, Proposition 2.7] is not correct as will be indicated in Example 3 below. Even though it fails for packing dimension, it holds, as all other results in [3], if one replaces packing dimension by Hausdorff dimension defined for a Radon measure μ on $[0,1]^n$ as follows:

$$\begin{aligned}\dim_{\mathrm{H}}(\mu) &= \sup\{s \geq 0 \mid \liminf_{i\to\infty} \frac{\log \mu(D_i(x))}{\log 2^{-i}} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\} \\ &= \inf\{\dim_{\mathrm{H}}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\}.\end{aligned}$$

Here $D_i(x)$ is the closed dyadic cube of side-length 2^{-i} containing x. When verifying these statements, the only corrections needed are in the proof of [3, Proposition 2.7] which should be stated as follows:

1. Proposition. *Let μ be a Radon probability measure on $[0,1]^n$ such that $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. Let $p \leq 2^{-kn}$ and $L \in I = \{1, \dots, 2^{kn}\}$. Assume that $\limsup_{l\to\infty} \frac{1}{l}\sum_{i=1}^{l} \mathrm{P}_i^{\mu}(\{j\}) \leq p$ for all $j = 1, \dots, L$. Then*

$$\dim_{\mathrm{H}} \mu \leq -\frac{1}{\log 2^k}(Lp\log p + (1-Lp)\log(\frac{1-Lp}{2^{kn}-L})) =: \alpha(p,L).$$

Proof. For fixed l, let ν be the self-similar measure on $[0,1]^n$ determined by $\alpha_{\mathbf{i}}$, $\mathbf{i} \in \mathbf{I}^l = \{\mathbf{i} = (i_1, \dots, i_l) \mid i_p \in I \text{ for all } p = 1, \dots, l\}$ with $\sum_{\mathbf{i}\in\mathbf{I}^l} \alpha_{\mathbf{i}} = 1$. Assuming that for all $j = 1, \dots, L$ we have

$$\frac{1}{l}\sum_{m=1}^{l} \sum_{\substack{\mathbf{i}\in\mathbf{I}^l \\ i_m = j}} \alpha_{\mathbf{i}} = \frac{1}{l}\sum_{m=1}^{l} \mathrm{P}_m^{\nu}(\{j\}) \leq p,$$

we will first prove that

$$\dim_{\mathrm{H}} \nu = -\frac{1}{\log 2^{kl}} \sum_{\mathbf{i}\in\mathbf{I}^l} \alpha_{\mathbf{i}} \log \alpha_{\mathbf{i}} \leq \alpha(p,L). \tag{2}$$

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The equality in (2) follows similarly as [1, (14.4) p. 139]. For the purpose of proving the inequality in (2) we will determine the maximum of the concave function

$$f(\alpha_{\mathbf{i}}) = -\sum_{\mathbf{i}\in\mathbf{I}^l} \alpha_{\mathbf{i}} \log \alpha_{\mathbf{i}}$$

on the unit cube of $\mathbb{R}^{2^{kln}}$ given the linear restrictions

$$h_1(\alpha_{\mathbf{i}}) = \sum_{\mathbf{i}\in\mathbf{I}^l} \alpha_{\mathbf{i}} = 1$$

and

$$0 \le h_{j+1}(\alpha_{\mathbf{i}}) = \frac{1}{l}\sum_{m=1}^{l} \sum_{\substack{\mathbf{i}\in\mathbf{I}^l \\ i_m = j}} \alpha_{\mathbf{i}} \le p$$

for all $j = 1, \dots, L$. Since $\bigtriangledown f = 0$ if and only if $\alpha_{\mathbf{i}} = e^{-1}$ for all $\mathbf{i} \in \mathbf{I}^l$ the maximum is obtained on the boundary. Using the method of Lagrange's multipliers we will end up solving for all $\mathbf{i} \in \mathbf{I}^l$ the equations

$$-\log \alpha_{\mathbf{i}} - 1 = \lambda_1 + \frac{1}{l}\sum_{j=1}^{L} \lambda_{j+1}\, \mathrm{n}_j(\mathbf{i})$$

with $h_1 = 1$ and $h_{j+1} = p$ for all $j = 1, \dots, L$. Here $\mathrm{n}_j(\mathbf{i})$ is the number of j's in $\mathbf{i}$. Since f is concave and the restrictions are linear there is only one solution for these equations. Choosing $e^{-\lambda_1 - 1} = (\frac{1-Lp}{2^{kn}-L})^l$ and $\lambda_{j+1} = -l\log(\frac{p(2^{kn}-L)}{1-Lp})$ for $j = 1, \dots, L$ it is easily checked that $\alpha_{\mathbf{i}} = p^{\sum_{j=1}^{L} \mathrm{n}_j(\mathbf{i})}\ (\frac{1-Lp}{2^{kn}-L})^{\sum_{j=L+1}^{2^{kn}} \mathrm{n}_j(\mathbf{i})}$ is the unique solution. Note that these $\alpha_{\mathbf{i}}$'s are the l^{th} iterates of the self-similar measure determined by attaching L times the weight p and $2^{kn} - L$ times the weight $\frac{1-Lp}{2^{kn}-L}$ to the dyadic cubes of side-length 2^{-k}. Since this measure has Hausdorff dimension $\alpha(p, L)$ the inequality in (2) is proved in this special case.

Now take an arbitrary Radon probability measure μ satisfying the assumptions of the proposition. Let $\mathrm{d}^-(x) = \liminf_{i\to\infty} \frac{\log \mu(D_i(x))}{\log 2^{-i}}$. Given $\varepsilon > 0$ there exists l_0 such that for all $l \ge l_0$ one has

$$\frac{1}{l}\sum_{i=1}^{l} P_i^{\mu}(\{j\}) \le p + \varepsilon = p'$$

for all $j = 1, \dots, L$ and

$$\mathrm{d}^-(x) - \varepsilon \le \frac{\log \mu(D_{lk}(x))}{\log 2^{-lk}}$$

for all $x \in B$ with $\mu(B) \ge 1 - \varepsilon$. We may assume that $\dim_{\mathrm{H}} \mu \le \mathrm{d}^-(x)$ for all $x \in B$. For $l \ge l_0$ let μ_s be the self-similar measure determined by $\alpha_{\mathbf{i}} = \mu(D_{\mathbf{i}})$ for $\mathbf{i} \in \mathbf{I}^l$. Here $D_{\mathbf{i}}$ is the closed subcube of $[0,1]^n$ of side-length 2^{-kl} consisting of those points whose expansions start with $\mathbf{i} \in \mathbf{I}^l$. Defining

$$G = \{\mathbf{i} \in \mathbf{I}^l \mid\ D_{\mathbf{i}} \cap B \ne \emptyset\}$$

and noting that $\sum_{i=1}^{l} P_i^{\mu_s}(\{j\}) = \sum_{i=1}^{l} P_i^{\mu}(\{j\})$ for all $j = 1, \ldots, L$, we obtain by applying (2) to μ_s

$$-\frac{1}{\log 2^k}(Lp' \log p' + (1 - Lp') \log(\frac{1 - Lp'}{2^{kn} - L})) \geq -\frac{1}{\log 2^{kl}} \sum_{\mathbf{i} \in \mathbf{I}^l} \alpha_{\mathbf{i}} \log \alpha_{\mathbf{i}}$$
$$\geq (\dim_{\mathrm{H}} \mu - \varepsilon) \sum_{\mathbf{i} \in G} \mu(D_{\mathbf{i}}) \geq (\dim_{\mathrm{H}} \mu - \varepsilon)\mu(B) \geq (\dim_{\mathrm{H}} \mu - \varepsilon)(1 - \varepsilon).$$

Letting ε tend to zero we obtain the claim. □

The following example shows that the assumptions in Proposition 1 do not necessarily guarantee that the upper bound is valid for the packing dimension as is erroneously stated in [3, Proposition 2.7]. In the construction we will denote by $\mathbf{I}^l$ the set of all l-term sequences of integers 0 and 1. For any $\mathbf{i} = (i_1, \ldots, i_l) \in \mathbf{I}^l$ let $D_{\mathbf{i}}$ be the closed dyadic subinterval of $[0, 1]$ of length 2^{-l} consisting of numbers whose binary expansions begin with $\mathbf{i}$.

3. Example. *Let* $0 \leq \alpha \leq \frac{1}{2}$. *We will construct a measure* μ *on* $[0, 1]$ *satisfying*

$$\limsup_{l \to \infty} \frac{1}{l} \sum_{k=1}^{l} \sum_{\substack{\mathbf{i} \in \mathbf{I}^k \\ i_k = 0}} \mu(D_{\mathbf{i}}) \leq \alpha \tag{4}$$

such that $\dim_{\mathrm{p}} \mu = 1$.

The main idea is the following. First divide the unit interval into subintervals such that all of them have measure less than α. Then organize the local dimension (at some scale) close to one inside the first interval. Since the total mass of this first interval is less than α one can distribute the measure inside other intervals such that (4) is valid. Next organize the local dimension close to one inside the second interval and the other intervals will guarantee that (4) is valid. Continue in this way until all the intervals have local dimension close to one at some scale. Then start again from the first interval and go on in this cyclic manner.

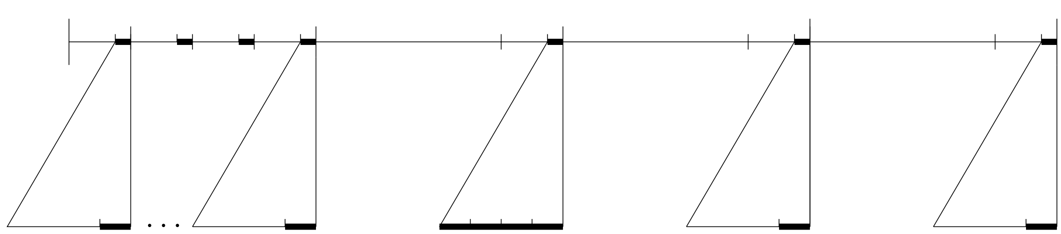

FIGURE. First four steps of the construction. Here $k_0 = k_1 = k_2 = k_3 = 2$. The mass is concentrated on the thick intervals.

We start with k_0 steps of the construction of the self-similar measure determined by α and $1 - \alpha$. Choosing k_0 so large that $(1 - \alpha)^{k_0} \leq \alpha$, the measure of each of the 2^{k_0} intervals $J_1, \ldots, J_{2^{k_0}}$ of length 2^{-k_0} is at most α. Divide each interval J_i into 2^{k_1} dyadic subintervals. Inside J_1 distribute the measure evenly and inside every other interval give all the measure to the rightmost subinterval. Choose k_1

large enough such that the local dimension at scale $2^{-k_0-k_1}$ is greater than $1-\frac{1}{3}$ for points in J_1 and less than $\frac{1}{3}$ for points in other intervals, that is,

$$\frac{\log\mu(D_{k_0+k_1}(x))}{\log 2^{-k_0-k_1}}\quad\begin{cases} = \frac{\log(2^{-k_1}\mu(D_{k_0}(x)))}{\log 2^{-k_0-k_1}} \geq 1-\frac{1}{3} & \text{for } x\in J_1 \\ \leq \frac{\log\mu(D_{k_0}(x))}{\log 2^{-k_0-k_1}} \leq \frac{1}{3} & \text{otherwise}\ . \end{cases}$$

Next divide each interval with positive measure into 2^{k_2} subintervals and give all the mass to the rightmost subinterval. Let k_2 be so large that the local dimension at scale $2^{-k_0-k_1-k_2}$ is smaller than $\frac{1}{4}$ at all points meaning that

$$\frac{\log\mu(D_{k_0+k_1+k_2}(x))}{\log 2^{-k_0-k_1-k_2}} \leq \frac{\log\mu(D_{k_0+k_1}(x))}{\log 2^{-k_0-k_1-k_2}} \leq \frac{1}{4} \quad \text{for all } x.$$

Continue by dividing every interval with positive measure into 2^{k_3} subintervals. Distribute the mass evenly inside J_2 and inside every other interval give all the mass to the rightmost subinterval. Again choosing k_3 large enough guarantees that inside J_2 the local dimension at scale $2^{-k_0-k_1-k_2-k_3}$ is at least $1-\frac{1}{5}$ and everywhere else it is at most $\frac{1}{5}$. Continue in this way until all the intervals $J_1,\ldots,J_{2^{k_0}}$ have been handled. Then start the process again from J_1 and organize the local dimension close to 1 at every non-empty subinterval of J_1 and so on. In this way the upper local dimension equals 1 at (almost) every point and thus the packing dimension equals 1. The condition (4) is satisfied because it is valid at every step of the construction. Note that $\dim_{\mathrm{H}}\mu=0$.

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References

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