LOWER BOUNDS FOR THE TWO WELL PROBLEM WITH SURFACE ENERGY I: REDUCTION TO FINITE ELEMENTS

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^2$, let H be a 2×2 matrix with $\det(H)=1$. Let $\epsilon>0$ and consider the functional $I_\epsilon(u):=\int_\Omega \operatorname{dist}(Du(z), SO(2)\cup SO(2)H)+\epsilon\left|D^2u(z)\right| dL^2 z$ over the class $\mathcal{B}_F$ of Lipschitz functions from Ω satisfying affine boundary condition F. It can be shown by convex integration that there exists $F\notin SO(2)\cup SO(2)H$ and $u\in\mathcal{B}_F$ with $I_0(u)=0$. In this paper we begin the study of the asymptotics of $m_\epsilon:=\inf_{\mathcal{B}_F\cap W^{2,1}} I_\epsilon$ for such F. This is the simplest minimisation problem involving surface energy in which we can hope to see the effects of convex integration solutions. The only known lower bounds are $\liminf_{\epsilon\to 0}\frac{m_\epsilon}{\epsilon}=\infty$. In this paper we link the behavior of m_ϵ to the minimum of I_0 over a suitable class of piecewise affine functions. Let $\{\tau_i\}$ be a triangulation of Ω by triangles of diameter less than h and let A_F^h denote the class of continuous functions that are piecewise affine on a triangulation $\{\tau_i\}$. For function $u\in\mathcal{A}_F$ let $\tilde{u}\in A_F^h$ be the interpolant, i.e. the function we obtain by defining $\tilde{u}_{\lfloor\tau_i}$ to be the affine interpolation of u on the corners of τ_i. We show that if for some small $\beta>0$ there exists $u\in\mathcal{B}_F\cap C^2\cap\text{Bilip}$ with

$$\frac{I_\epsilon(u)}{\epsilon}\leq\epsilon^{-\beta}$$

then for $h\approx\epsilon^{\sqrt{\beta}}$ the interpolant $\tilde{u}\in A_F^h$ satisfies $I_0(\tilde{u})\leq h^{1-c\sqrt{\beta}}$.

Note that it is conjectured that $\inf_{v\in A_F^h} I_0(v)\approx h^{\frac{1}{3}}$ and it is trivial that $\inf_{v\in A_F^h} I_0(v)\geq c_0 h$ so we reduce the problem of non-trivial (scaling) lower bounds on $\inf_{\mathcal{B}_F\cap C^2\cap\text{Bilip}}\frac{I_\epsilon}{\epsilon}$ to the problem of non-trivial lower bounds on $\inf_{v\in A_F^h} I_0$. This latter point will be addressed in a forthcoming paper.

Contents

1. Introduction

In the 1980's from the work of Ball, James [1], [2] and Chipot, Kinderlehrer [5] a now well known model for solid-solid phase transformations arose. In the model, microstructures observed in phase mixtures where explained in terms of energy minimisation of deformations of the material.

Let $u : \Omega \to \mathbb{R}^3$ be a deformation of the material which occupies a reference configuration Ω, the total free energy of this deformation is given by

$$I(u) = \int_\Omega \phi\left(Du\left(x\right), \theta\right) dL^3 x \tag{1}$$

where $\phi\left(., \theta\right)$ is the free energy per unit volume in Ω at temperature θ. We fix θ and we normalize ϕ such that $\inf_F \phi\left(F, \theta\right) = 0$.

Formation of microstructure was shown to be closely related to the behavior of minimising sequences of I. Many features of minimising sequences can be understood from the set $\{F : \phi\left(F\right) = 0\}$. This set is known as the energy *wells* of the functional I.

Certain natural assumptions on the behavior of ϕ, in particular frame indifference, imply that K has to be of the form

$$K = \{SO\left(3\right) A_i : i = 1, 2, \dots n\} \tag{2}$$

where the A_i are symmetry related and depend on the action of the phase transition.

Given $F \in M^{n\times n}$ let $\mathcal{B}_F$ denote the set of functions $u : \Omega \to \mathbb{R}^n$ satisfying $u\left(x\right) = F\left(x\right)$ for all $x \in \partial\Omega$. The set of F for which $\inf_{u\in\mathcal{B}_F} I = 0$ turns out to agree with the quasiconvex hull K^{qc} (see [22] for the relevant notions). For any $F \in \mathrm{int}\left(K^{qc}\right)$ it is possible to lower the energy of functional I with a relatively simple function $u \in \mathcal{B}_F$ that is built up from a simple (finite) layering of regions on which Du is made to be affine, these functions are known as *laminates* .

Mathematically speaking, the first real surprise in this theory is the existence of exact minimisers of functional I. Formally; given $F \in K^{qc}$ there exists a function $u \in \mathcal{B}_F$ such that

$$Du\left(x\right) \in K \text{ for } a.e.\ x \in \Omega. \tag{3}$$

Even though functional I is not quasiconvex (by the very existence of such exact solutions) and therefor not lower semicontinuous with respect to weak convergence, absolute minimizers exist and can be constructed.

Following the work of Dacorogna and Marcellini [8], Müller and Šverák [18], and later by Sychev [24] and Kirchheim [10] there now exist a wide variety of methods to prove the existence of such solutions.

The approach of Müller, Šverák was to apply the theory of convex integration developed by Gromov [12]; convex integration is a far reaching generalisation of the methods developed by Nash and Kuiper in their work on isometric embeddings.

Dacorogna and Marcellini used Baire category methods that were introduced by Cellina [3] and developed by DeBlasi and Pianigiani in the context of Cauchy problems for ordinary differential inclusions.

The method of Müller, Šverák is in some sense more constructive in that the functions u satisfying (3) are the limit in the $W^{1,1}$ norm of a sequence of explicitly constructed functions. These functions are, roughly speaking, "laminate like" in nature. Surprisingly, strong convergence in $W^{1,1}$ norm is achieved by making the functions oscillate faster and faster. The limiting function is a wild object. In fact it has been proved that solutions to (3) must be such that $Du \notin BV(\Omega)$, see [7].

So exact minimisers of functional I are only possible due to the fact that I takes no account of the "cost" of oscillations. This is physically unrealistic. Since oscillations in minimisers occur when the derivative of the function jumps from one well, say $SO(3)A_i$, to another well $SO(3)A_j$, the "amount" of oscillation is related to the total "surface area" of the regions in which the derivative of the minimizer lies in specific wells, this is referred to as the *surface energy* . The *bulk energy* is the $\int_\Omega \phi(\cdot)\, dL^2x$ part of the functional.

Functional I was designed to model situations for which the *surface energy* is small. From the mathematical perspective the simplest adaption of the functional that takes account of surface energy is:

$$I_\epsilon(u) = \int_\Omega \phi(Du(x)) + \epsilon\left|D^2u(x)\right| dL^2x \tag{4}$$

This functional is minimised over functions $u \in W^{2,1}(\Omega) \cap \mathcal{B}_F$.

1.1. **The question: The effect of Surface Energy on Microstructure.** The question of interest is whether the unexpected existence of exact solutions to inclusion (3) having affine boundary condition has any effect on the scaling of $\inf_{W^{2,1}\cap\mathcal{B}_F} I_\epsilon$ as $\epsilon \to 0$. In some sense this could be expected, in words; as $\epsilon \to 0$ surface energy becomes arbitrarily cheap, we can concern ourselves less and less with oscillations and just concentrate on minimizing the bulk part of the functional. It may there for be reasonable to expect that minimisers for sufficiently small ϵ are something like slightly smoothed out solutions of (3).

This question is important because convex integration solutions are important. Recently, long standing questions as to the regularity of systems of elliptic and parabolic equations have received surprising counter examples via convex integration methods, [20], [21]. Specifically it has been proved there exist nowhere C^1 solutions to the Euler Lagrange equations of a strictly quasiconvex functional. This is in contrast to the well known result of Evans [9] that minimisers of strictly quasiconvex functions are $C^{1,\alpha}$ on a dense open, full measure subset of Ω. In [21] a parabolic system that starts from smooth initial data and evolves into a function that is nowhere C^1 is exhibited.

Let $K = SO(2) \cup SO(2)H$, $F \in \mathrm{int}K^{qc}$. The differential inclusion

$$Du \in K \;\; a.e. \tag{5}$$

for function $u \in \mathcal{B}_F$ is the simplest convex integration result. And the minimisation problem

$$\inf_{u\in\mathcal{B}_F\cap W^{2,1}} I_\epsilon(u) \tag{6}$$

is the simplest "physical" situation where we could hope to see the effects of convex integration. The question of asymptotics of $\inf_{\mathcal{B}_F} \frac{I_\epsilon}{\epsilon}$ is a simple case of the more fundamental question; how much do convex integration solutions oscillate?

The only known lower bounds on (6) are $\inf_{u\in\mathcal{B}_F} \frac{I_\epsilon(u)}{\epsilon} \to \infty$ which follows from the result of Dolzmann, Müller [7].

As a consequence of Šverák's characterization of the wells K, [23] (namely that the quasiconvex hull is in the second laminate convex hull) it is easy to see

$$\inf_{u\in\mathcal{B}_F\cap W^{2,1}} \frac{I_\epsilon(u)}{\epsilon} < c\epsilon^{-\frac{2}{3}}.$$

If convex integration type solutions start having an effect on our functional for sufficiently small ϵ then we can expect to be able to "beat" the scaling $c\epsilon^{-\frac{2}{3}}$. Conversely if it could be shown that $\inf_{\mathcal{A}_F\cap W^{2,1}} \frac{I_\epsilon}{\epsilon} \geq c'\epsilon^{-\frac{2}{3}}$ this would say that convex integration solutions do not affect functional I_ϵ.

Note that the only method by which non-trivial lower bounds on surface energy have previously been obtained for simplified (finite well) versions of functional I_ϵ is to use the smallness of the bulk energy of function u to show that u must lie close to the affine boundary condition. Since the affine boundary condition is a non-trivial laminate convex combination of matrices in the wells, the only way u can remain close the affine boundary is if the derivative of u, going up through region Ω, jumps continuously from one well to another. In this way lower bounds on surface energy can be easily harvested [17],[13].

In our case, by the very existence of convex integration solutions, functions even with zero bulk energy need not behave anything like the affine boundary. Hence it is necessary to somehow use the smallness of surface and bulk energies in combination to control the function.

The main contribution of this paper is to reduce the problem of (non-trivial) lower bounds for $\frac{I_\epsilon}{\epsilon}$ to the problem of (non-trivial) lower bounds for the finite element approximation of I. In the follow up to this paper [14] we will establish such bounds. Before stating our results we need to introduce the notation and background to explain the scaling of the finite element approximations of I.

2. Background and Notation

We let H denote a diagonal matrix of the form

$$H = \begin{pmatrix} \tilde{\sigma} & 0 \\ 0 & \tilde{\sigma}^{-1} \end{pmatrix}$$

for some $\tilde{\sigma} \in (0,1)$. Let $\sigma = \min\left\{\tilde{\sigma}, 10000^{-1}\right\}$ and we assume throughout that σ is radically smaller than any small constant that might appear in the proof.

Let $P(a, \phi_1, \phi_2, r)$ denote the parallelogram centered on a of side length equal to r with sides parallel to ϕ_1 and ϕ_2. We will refer to this as a *skewcube* .

We let $\mathfrak{F}(a, v_1, v_2)$ denote the parallelogram centered on a with one side parallel to v_1 of length $|v_1|$ and the other side parallel to v_2 of length $|v_2|$.

We will often be required to consider ODEs of the form

$$X(0) = x_0 \quad \text{and} \quad \frac{dX}{dt}(t_0) = D\Psi(X(t_0))$$

where Ψ is some C^2 scalar function. Informally when we use the expression *"run X forward in time until it hits .."* we consider the set $\{X(t) : t > 0\}$ and we find the smallest $t_1 > 0$ such that $X(t_1)$ reaches the boundary of some set. The expression *"run X backward in time until it hits .."* is defined similarly.

Let

$$\mathfrak{D}(\sigma) := \left\{ M \in M^{2\times 2} : \sup_{v\in S} |Mv| \leq \frac{1}{\sigma^2} \text{ and } \inf_{v\in S} |Mv| \geq \sigma^2 \right\}.$$

and we let

$$\mathcal{A}_F(\Omega) = \left\{ u \in C^2(\operatorname{int}(\Omega)) : u(x) = F(x) \text{ for } x \in \partial\Omega \text{ and } Du(x) \in \mathfrak{D}(\sigma) \text{ for any } x \in \operatorname{int}(\Omega) \right\}.$$

We will be considering minimisers of functional I_ϵ over this function class. As we will be dealing with the case $\det(H) = 1$ a solution to the differential inclusion (5) is given by method of Müller and Šverák ([19]). Their method yields easily the existence of a sequence $u_k \in \mathcal{A}_F(\Omega)$ such that $u_k \to u$ in $W^{1,1}(\Omega)$ for some Lipschitz function u that solves (5).

In this paper we will have to deal with many constants, all of them in one way or the other dependent on the eigenvalues of H. We adopt the following convention; constants that carry through the whole proof from lemma to lemma will be denoted $\mathfrak{c}_1, \mathfrak{c}_2, \ldots$. Inside each lemma the "local" constant will be denoted $c_1, c_2, \ldots$. In each lemma "we start the clock back" and begin by numbering our local constants from c_1. We also make the convention that $c_1 \leq c_2 \leq c_3 \ldots$.

2.1. Finite Element Approximations.

As is standard in finite element approximations, we will say a *triangulation* (denoted $\triangle_\epsilon$) of Ω of size ϵ is a collection of pairwise disjoint triangles $\{\tau_i\}$ all of diameter ϵ such that

$$\Omega \subset \bigcup_{\tau_i \in \triangle_\epsilon} \tau_i.$$

Given a function u, we can approximate u uniformly by a function $\tilde{u}$ that is piecewise affine on the triangles of $\triangle_\epsilon$ by letting $\tilde{u}_{\lfloor \tau_i}$ be the affine map we obtain from interpolating u on the corners of τ_i. We will call $\tilde{u}$ the *F.E. approximation* of u. When we replace the function class of a minimisation problem with respect to functional J, with a the class of F.E. approximations to the functions in the function class, this is known as the finite element approximation to J.

Finite element approximations of functionals such as I have received much interest, for example see [15],[17], [4]. In this paper our interest in these approximations comes mainly from the fact that they provide a convenient intermediary step for the study of surface energy problems: Given a triangulation for which the edges of the triangles are not parallel to the rank-1 connections of the wells K, every time the F.E. approximation to a function jumps from one well to another, there must be at least one triangle which is nowhere near the wells. More informally; if for example we have an F.E. approximation to a laminate, every triangle that cuts through an interface between the regions where the derivative of the laminate takes different wells will be such that the affine map we get from interpolating the laminate on the corners of the triangle will have its linear part some distance from the wells.

In this way, F.E. approximations reflect a competition between "surface energy" as given by the error contributed from jumps in the derivative, and bulk energy which in the case of "laminate like" F.E. approximations is basically the width of the interpolation layer.

F.E. approximations of a three well functional $\tilde{I}$ of the form I, over a function class having affine boundary condition in the second laminate convex hull of the wells have been studied by Chipot [4] and the author [13]. If $\triangle_h$ denotes a triangulation of size h and A_F^h denotes the set of functions that are piecewise affine on $\triangle_h$ satisfying the affine boundary condition F. Chipot showed $\inf_{u\in A_F^h}\tilde{I}(u)\leq Ch^{\frac{1}{3}}$ and the lower bound of $\inf_{u\in A_F^h}\tilde{I}(u)\geq ch^{\frac{1}{3}}$ was provided by the author. From Šverák's characterization [23] we even know the exact arrangement of rank-1 connections between the wells $SO(2)\cup SO(2)H$ and a matrix in the interior of the quasiconvex hull. The finite well functional studied in [13] precisely mimics these rank-1 connections. We feel confident in conjecturing:

Conjecture 1. *Let $K:=SO(2)\cup SO(2)H$, $H=\begin{pmatrix}\sigma & 0\\ 0 & \sigma^{-1}\end{pmatrix}$. Let $\triangle_h$ be a triangulation of Ω of size h with the edges of the triangles not in the set of rank-1 direction of K.*

Let A_F^h denote the set of function with affine boundary condition F that are piecewise affine on the triangulation $\triangle_h$. Let $d(\cdot,B)$ denote the Euclidean distance away from set B. Let $I(u):=\int_\Omega d(Du(x),K)\,dL^2x$ then we have

$$\inf_{u\in A_F^h} I(u)\geq ch^{\frac{1}{3}}$$

The contribution of this paper that is most recognizably relevant to the asymptotics of $\inf_{u\in\mathcal{A}_F}\frac{I_\epsilon(u)}{\epsilon}$ is to reduce the proof of lower bounds of the form $\inf_{u\in\mathcal{A}_F}\frac{I_\epsilon(u)}{\epsilon}\geq\epsilon^{-\beta}$ (for sufficiently small β) to lower bounds of the form $\inf_{v\in A_F^h} I(v)\geq h^{1-c\sqrt{\beta}}$ for $h\approx\epsilon^{\sqrt{\beta}}$.

3. Statement of Results

Theorem 1. *Given region Ω and triangulation $\triangle_\epsilon$ with triangulation size ϵ. Let $m_1\geq 2048$. Let $v\in\mathcal{A}_F(\Omega)$. If we have for some small ϵ*

$$\int_\Omega d(Dv(x),SO(2)\cup SO(2)H)\,dL^2x\leq\epsilon^{m_1}$$

and

$$\int_\Omega\left|D^2v(x)\right|dL^2x\leq\epsilon^{-\frac{4096}{m_1}}$$

then the F.E. approximation $\tilde{v}_\epsilon$ of v on $\triangle_\epsilon$ satisfies

$$I(\tilde{v}_\epsilon)\leq\mathfrak{c}\epsilon^{1-\frac{8192}{m_1}}.$$

This Theorem is basically a consequence of the following Theorem.

Theorem 2. *Given region Ω and triangulation $\triangle_\epsilon$ with triangulation size ϵ. Let $m_0\geq 2048$. Let $v\in\mathcal{A}_F(\Omega)$ and let skewcube $S:=P(a,\phi_1,\phi_2,\mathfrak{c}\epsilon)$ be such that $N_{\frac{\epsilon}{\sigma^2}}(S)\subset\Omega$. If for some small $\kappa>0$ we have*

$$\int_S d(Dv(x),SO(2)\cup SO(2)H)\,dL^2x\leq\kappa^{\frac{7m_0}{2}+8}\epsilon^2$$

then either

- $$\int_S \left|D^2 v\left(x\right)\right| dL^2 x \geq \mathfrak{c}_0 \kappa \epsilon$$
- *or given triangle $\tau_i \in \triangle_\epsilon$ containing a, if L_i denotes the linear part of the affine map obtain from interpolating v on the corners of τ_i then*
$$d\left(L_i, SO\left(2\right) \cup SO\left(2\right) H\right) < \kappa^{\frac{m_0}{1024}}.$$

Theorem 1 has as an easy corollary

Corrolary 1. *Given region Ω and triangulation $\triangle_h$ with triangulation size h. Let $m_1 \geq 2048$. If $v \in \mathcal{A}_F\left(\Omega\right)$ is such that*
$$\frac{I_\epsilon\left(v\right)}{\epsilon} \leq \epsilon^{-\frac{2048}{m_1^2}}$$
then for $h := \epsilon^{\frac{m_1^2 - 2048}{m_1^3}}$, the F.E. approximation $\tilde{v}_h$ of v on $\triangle_h$ satisfies
$$I\left(\tilde{v}_h\right) \leq h^{1-\frac{8192}{m_1}}.$$

The bulk of this paper will be devoted to proving Theorem 2.

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4. Plan of Proof

The proof is long, but essentially made up of only four basic ideas. We explain them in chronological order (which is not the order they appear in the proof) because this still seems the best way to explain how the proof builds up.

4.1. **The pull back idea.** To begin with note that there are two linearly independent vectors ϕ_1 and ϕ_2 such that $|H\phi_i| = 1$ for $i = 1, 2$. See for example Fig. 14. Every unit vector between ϕ_1 and ϕ_2 (denoted by Ξ_2) will be mapped by H to a vector of size strictly less than 1. Similarly every vector between $-\phi_1$ and $-\phi_2$ (denoted by Ξ_1) will by shrunk by H. For this reason $\Xi_1 \cup \Xi_2$ will be called the *shrink directions*.

Now the most basic example of a function satisfying the affine boundary condition that minimizes bulk energy is a *laminate* . In the reference configuration this can be seen as a function defined on a collection of strips running parallel to either ϕ_1 or ϕ_2 for which the derivative of the *laminate* alternates from one strip to the next from being in $SO(2)$ to being in $SO(2)\,H$. For simplicity, let us suppose for the time being the strips are parallel to ϕ_1 and let us denote the *laminate* by u. Now if all our strips are of width w, by Fubini and the fact that $\det(H) = 1$ and $|H\phi_1| = 1$ we know that the images of our strips under the action of u will be to send them to strips of width w, as shown Fig. 1.

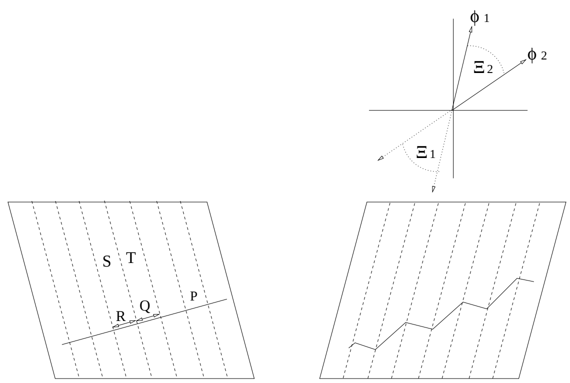

Figure 1

Now if we want a path that travels through the strips in the image in the quickest way possible (i.e. a path that goes from one strip to the next with the minimum length possible) then clearly such a path would traverse at right angles to the strips in the image, as shown in Fig. 1. Let P denote this path. Now we consider the pull back of P in the image. Let $t(z)$ denote the tangent to the path $u^{-1}(P)$ at point z, so we have the following formula: $H^1(P) = \int_{u^{-1}(P)} |Du(x)\,t(x)|\,dH^1 x$. Let Q denote a subsegment of the path that is given by the path intersected with one strip in the image (denoted S, see Fig. 1) for which $Du_{\lfloor u^{-1}(S)} \in SO(2)$. Now $H^1(Q) = w$ and as $u^{-1}(Q)$ connects the edges of $u^{-1}(S)$, so $H^1\left(u^{-1}(Q)\right) \geq w$. Since $Du_{\lfloor u^{-1}(S)} \in SO(2)$ we have

$$w = H^1(Q) = \int_{u^{-1}(Q)} |Du(x)\,t(x)|\,dH^1 x = H^1\left(u^{-1}(Q)\right).$$

Thus $u^{-1}(Q)$ must be a straight line going through S perpendicular to ϕ_1. Now let R denote the subsegment of the path given by the path intersected with a strip in the image (denoted by T) for which the $Du_{\lfloor u^{-1}(T)} \in SO(2)\,H$. It should be obvious that $u^{-1}(R)$ is not going to be a line perpendicular to ϕ_1, since if our laminate pulled back two linearly independent straight lines to straight lines it would be affine. On the other hand, by the same argument as for Q, if $u^{-1}(R)$ isn't perpendicular to ϕ_1 and so $H^1\left(u^{-1}(R)\right) > w$, then (by the fact that u is a laminate and so $u^{-1}(R)$ is a straight line) all the points $x \in u^{-1}(R)$ must be such that $t(x) \in \Xi_1$, i.e. $t(x)$ must be in the shrink directions. We examine the situation more closely, see Fig. 2.

Now $u^{-1}(Q)$ needs to connect the edges of $u^{-1}(T)$ (which are of course distance w apart) whilst keeping the integral $\int_{u^{-1}(Q)} |Du(x)\,t(x)|\,dH^1 x$ small. As can be seen from Fig. 14, the vector that shrinks most under the action of H is right in the middle of $-\phi_1$, $-\phi_2$. We denote this vector by ψ_0

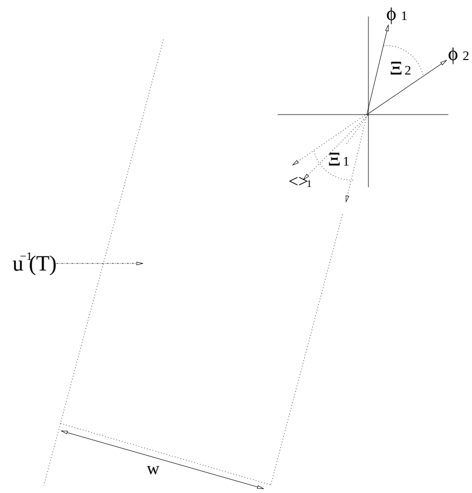

Figure 2

Let W be the straight line connecting the edges of $u^{-1}(T)$, which minimizes $\int_W |Ht(x)|\, dH^1 x$. In deciding the angle of W there is a compromise between having W close to parallel with ψ_0 and trying to keep the length of W not too much bigger than w. Its an exercise to calculate the optimal direction , we will denote it $\Diamond_1$.

Now $u^{-1}(Q)$ connects the edges of $u^{-1}(T)$ and $w = H^1(Q) = \int_{u^{-1}(Q)} |Ht(x)|\, dH^1 x$. So since w minimizes we must have that $\int_W |Ht(x)|\, dH^1 x \leq w$. If the inequality was strictly less, then as $u(W)$ connects the edges of T, so we would have

$$w > \int_W |Ht(x)|\, dH^1 x = H^1(u(W)) \geq w,$$

contradiction. So $u^{-1}(Q)$ is the minimiser and hence $u^{-1}(Q)$ must be parallel to $\Diamond_1$.

So we know exactly what the pull back of p looks like; in strips in the reference with derivative in $SO(2)$ it is forms a line perpendicular to ϕ_1. And in strips in the reference with derivative in $SO(2)H$ it is forms a line parallel to $\Diamond_1$. As shown on Fig. 1.

Now we wish to apply what we have learned to a general function $v : \Omega \to \mathbb{R}^2$ with small bulk energy (i.e. $\int_\Omega d(Dv(x), SO(2) \cup SO(2)H)\, dL^2 x < \epsilon L^2(\Omega)$). We take lines through the reference going in direction ϕ_1 going through Ω and consider their image under v, as shown in Fig. 3.

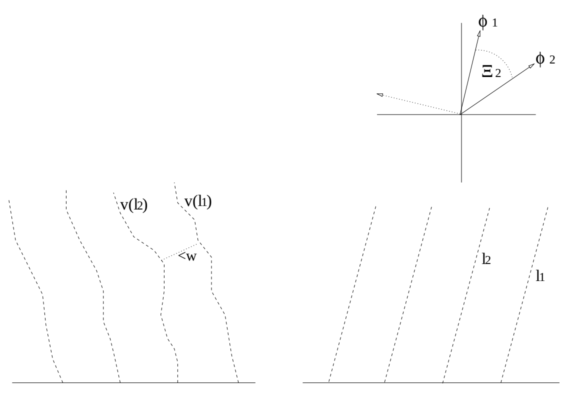

Figure 3

Suppose for two lines l_1, l_2 we have that $v(l_1)$, $v(l_2)$ are distance less than w apart at some point, as shown. Let l be the line of length less than w joining $v(l_1)$ to $v(l_2)$. We consider the pull back $v^{-1}(l)$. Let n_1 denote the anticlockwise normal to ϕ_1. Now as we have an L^1 bound we know that most of the points $x \in v^{-1}(l)$ are close to the wells, if $x \in v^{-1}(l)$ such that $Du(x) \in N_{\sqrt{\epsilon}}(SO(2))$ it doesn't matter which direction the tangent $t(x)$ is in. On the other hand for those $x \in v^{-1}(l)$ such that $Dv(x) \in N_{\sqrt{\epsilon}}(SO(2)H)$ the worst thing that can happen is that $t(x) = \Diamond_1$. But even in this case (as we know from the example we studied) l will "fill up" according to how far in direction n_1 path $v^{-1}(l)$ travels. So we have

$$H^1(l) = \int_{v^{-1}(l)} |Dv(x)\, t(x)|\, dH^1 x \geq (1-\sqrt{\epsilon})L^1\left(P_{\phi_1^{\perp}}\left(v^{-1}(l)\right)\right) = \left(1-\sqrt{\epsilon}\right) w.$$

So this implies the images of lines l_1 and l_2 must be (by at least $(1-\sqrt{\epsilon})\, w$) "pushed over" from one another. This is our first restriction on the geometry of the function we want to study, just coming from smallness of bulk energy.

4.2. **ODE method.** We consider the same picture as before but from a different perspective. So $l_1, l_2, \dots$ are lines in direction ϕ_1 going through Ω and we consider the images $v(l_1), v(l_2), \dots$. Now supposing we were on a point $x \in v(l_1)$ and we wanted to get to $v(l_2)$ via a path of the shortest length.

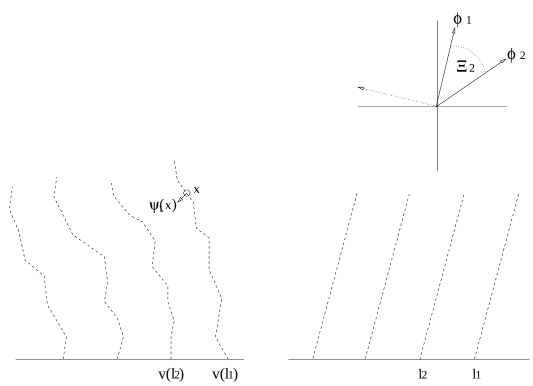

FIGURE 4

The most natural way to do it would be to consider the vector field given by the derivative of the function $\Psi_1 : v(\Omega) \to \mathbb{R}^2$ defined by $\Psi_1(x) := v^{-1}(x) \cdot n_1$. If we "follow" the vector field from point x it will indeed take us along the optimal path to $v(l_2)$. But "following" a vector field is exactly finding an integral curve for a vector field, which means solving the following ODE

$$X(0) = x \quad \frac{dX}{dt}(t_1) = D\Psi_1(X(t_1)).$$

Now if point $y \in \{X(t) : t > 0\}$ is such that $Dv\left(v^{-1}(y)\right) \in N_{\sqrt{\epsilon}}(SO(2) \cup SO(2)H)$ we calculate that $D\Psi_1(y) = Dv^{-T}(y) \cdot n_1$. Letting $R\left(v^{-1}(y)\right) S\left(v^{-1}(y)\right) := Dv\left(v^{-1}(y)\right)$ be the polar decomposition of $Dv\left(v^{-1}(y)\right)$ (i.e. $R\left(v^{-1}(y)\right) \in SO(2)$ and $S\left(v^{-1}(y)\right) \in M^{\text{sym}}$) we have $Dv^{-T}(y)\, n_1 = R\left(v^{-1}(y)\right) S^{-1}\left(v^{-1}(y)\right) n_1$ and as $S\left(v^{-1}(y)\right) \in N_{\sqrt{\epsilon}}(\{Id, H\})$ so either $S\left(v^{-1}(y)\right) \in N_{\sqrt{\epsilon}}(Id)$ and so $\left|S\left(v^{-1}(y)\right) n_1\right| \approx 1$ or $S\left(v^{-1}(y)\right) \in N_{\sqrt{\epsilon}}(H)$ and so $\left|S\left(v^{-1}(y)\right) n_1\right| \approx \left|H^{-1} n_1\right| = 1$ (see Fig. 14). So assuming the path of the vector field is such that Dv stays close to the wells, if Λ is a connected subset of the set $\{X(t) : t > 0\}$ with end points $e \in v(l_2)$, $s \in v(l_1)$ and with the property $\int_{\Lambda} d\left(Dv\left(v^{-1}(x)\right), SO(2) \cup SO(2)H\right) dH^1 x < \epsilon H^1(\Lambda)$ then

$$|\Psi_1(e) - \Psi_1(s)| = \left|\left(v^{-1}(e) - v^{-1}(s)\right) \cdot n_1\right| \approx H^1(\Lambda).$$

The precise statement of this is given by Lemma 3. So on Fig. 4, if $v^{-1}(s) \in l_1$ and $v^{-1}(e) \in l_2$ then $H^1(\Lambda) \approx w$, however if Λ is a wavy line, then $|e - s| < H^1(\Lambda)$ and so we have $e \in v(l_2)$ is distance less than w away from $s \in v(l_1)$. This contradicts the "pull back" idea. And so Λ must form a nearly straight line.

4.3. **The Coarea Alternative.** From the "pull back idea" and the "ODE method" it seems we are able to gain quite a lot of control of our function v just by using bulk energy. The catch is that whilst it is not hard (by Fubini and the Area Formula) to find lines in the image for which $Dv\left(v^{-1}(\cdot)\right)$ stays close to the wells, its much harder to find integral curves to the vector field $D\Psi_1$ with this property. We require a kind of curvilinear version of Fubini and this of course is nothing other than the Coarea formula.

To invoke the Coarea formula we need to define a function $\Theta_1 : v(\Omega) \to \mathbb{R}$ such that the level sets $\Theta_1^{-1}(t)$ form integral curves of $D\Psi_1$. By smoothness of u and hence of $D\Psi_1$ and so by uniqueness of ODE solutions, its easy to see that such a function exists. Let a be the center of Ω, in its crudest possible manifestation we define Θ_1 in the following way: For any $x \in v(\Omega)$ we run the ODE that forms the integral curve of $D\Psi_1$ containing x until we reach $v((a + \langle \phi_1 \rangle) \cap \Omega)$ (for easy reasons the integral curve must intersect this 1-set at only one point) we define $p(x)$ to be the point of $v((a + \langle \phi_1 \rangle) \cap \Omega)$ that we reach, then we define $\Theta_1(x) := v^{-1}(p(x)) \cdot \phi_1$. Let $J(x) := d\left(Dv\left(v^{-1}(x)\right), SO(2) \cup SO(2)H\right)$, the coarea formula tells us

$$\int_{\Omega} J(x)\,|D\Theta_1(x)|\,dL^2 x = \int_{\mathbb{R}} \int_{\Theta_1^{-1}(t)} J(x)\,dH^1 x dL^1 t.$$

So assuming the expression on the left hand side is small we are guaranteed the existence of many integral curves with small bulk energy. However by the existence of an abundance of functions with arbitrarily small bulk energy that are nothing like close to being affine, smallness of $\int_{\Omega} J(x)\,|D\Theta_1(x)|\,dL^2 x$ is obvious a non-trivial issue. It is here that we finally have to use the information we have about the surface energy of v.

If $\int_{v(\Omega)} J(x)\,|D\Theta_1(x)|\,dL^2 x$ is large, then we must (by an application of the coarea formula with respect to the lines $v((z + \langle \phi_1 \rangle) \cap \Omega)$ using $|D\Psi_1|$ as the Jacobian) be able to find a point $z \in \Omega$ such that $\int_{v((z+\langle \phi_1 \rangle)\cap\Omega)} J(x)\,dH^1 x$ is very small but

$$\int_{v((z+\langle \phi_1 \rangle)\cap\Omega)} J(x)\,|D\Theta_1(x)|\,dL^2 x \text{ is big.} \tag{7}$$

So there must exist a set $B \subset v((z + \langle \phi_1 \rangle) \cap \Omega)$ of quite small H^1 measure such that $\int_B |D\Theta_1(x)|\,dH^1 x$ is big. Now considering the pull back of the integral curves into the reference configuration gives us Figure 5.

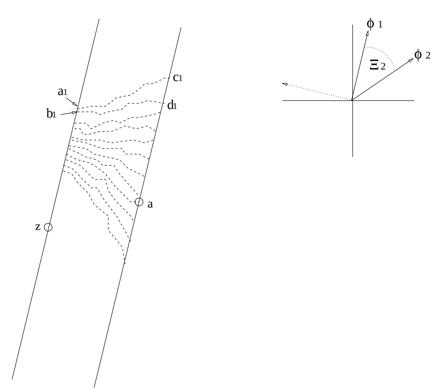

FIGURE 5

We arrive at this diagram in the following way: Firstly its an exercise to see that lines of the form $v(\langle \phi_1 \rangle + z)$ form integral curves to $D\Theta_1$. So we can find a collection of intervals $[a_k, b_k] \subset (\langle \phi_1 \rangle + z)$ such that

$$\int_{\bigcup_k v([a_k,b_k])} |D\Theta_1(x)|\,dH^1 x = \sum_{k\in\mathbb{N}} \Theta_1(b_k) - \Theta_1(a_k) \text{ is big.}$$

On the other hand by definition of Θ_1 this means the integral curves running from the endpoints of each interval $[a_k, b_k]$ must be splayed out as is shown. We let c_k denote the point in $a + \langle \phi_1 \rangle$ reached from a_k by the pullback of the integral curve that goes through $v(a_k)$ and let d_k be the point in $a + \langle \phi_1 \rangle$ reached from b_k by the pullback of the integral curve that goes through $v(b_k)$, as shown on Fig. 5.

Recall by polar decomposition of the derivative; $Dv(x) =: R(x)S(x)$ we have $R(x) \in SO(2)$, $S(x) \in M^{\mathrm{sym}}$. Its a calculation to see that if $t(x)$ denotes the tangent to pull back of the integral curve at x, then

$$t(x) := S^{-1}(x) S^{-1}(x) n_1. \tag{8}$$

Now we know from the fundamental theorem of calculus that the integral of the difference of the tangents to the pull backs of the integral curves starting at a_k, b_k must be bigger than $\frac{|c_k - d_k|}{2}$. From (8) the difference in tangents from one curve to the next is a lower bound for the difference in Dv between these two curves, so we have an inequality of the following form.

$$\int_\Omega \left|D^2 v(x)\right| dL^2 x \geq \sum_k \frac{|c_k - d_k|}{2} \approx \frac{1}{2}\int_B |D\Theta_1(x)|\, dH^1 x \approx c_0 \int_{v(z+\langle\phi_1\rangle)} J(x)\,|D\Theta_1(x)|\, dH^1 x.$$

So we have the "Coarea alternative" either

$$\int_\Omega \left|D^2 v(x)\right| dL^2 x \text{ is not small}$$

or

$$\int_{v(\Omega)} J(x)\,|D\Theta_1(x)|\, dL^2 x \text{ is small}$$

and we can find many integral curves with small bulk energy. The most basic form of the coarea alternative is given by Lemma 10.

4.4. **Finite element reduction.** If we have region Π such that $\int_{v(\Pi)} J(x)\,|D\Theta_i(x)|\, dL^2 x$ is small for $i = 1, 2$, by the pull back idea and the ODE method we know that we can find many integral curves of the form $\Theta_1^{-1}(t)$ and $\Theta_2^{-1}(t)$ (where Θ_i denotes the level set function for Ψ_i) which form approximate straight lines. We know from our analysis of the laminate example in section 4.1, exactly what the pull back of straight lines look like. Since Dv stays close to the wells along our integral curves $\Theta_i^{-1}(t)$, by the same arguments we end up being able to show that the pull back of $\Theta_1^{-1}(t)$ and $\Theta_2^{-1}(t)$ have very much the same form. Informally; our control of the integral curves $\left\{\Theta_1^{-1}(t) : t \in R\right\}$ says that our function v has to be something like a laminate with strips parallel to ϕ_1. And our control of integral curves $\left\{\Theta_2^{-1}(t) : t \in R\right\}$ says that v has to be something like a laminate with strips parallel to ϕ_2. The only way v can be both these things is if $Dv_{\lfloor\Pi} \approx R_1 H$ for some $R_1 \in SO(2)$ or $Dv_{\lfloor\Pi} \approx R_2$ for some $R_2 \in SO(2)$.

So the natural idea is to cut Ω into triangular subregions (i.e. take a triangulation of Ω) of roughly fixed size. Denote these regions as $\{\tau_i\}$. For each τ_i, by the "coarea alternative" and what we have shown, either

$$\int_{\tau_i} \left|D^2 v(x)\right| dL^2 x \text{ is not small}$$

or

$$Dv_{\lfloor\tau_i} \approx R\mathfrak{S}$$

for some $R \in SO(2)$ and some $\mathfrak{S} \in \{Id, H\}$. So if we let $\tilde{v}$ be the function we obtain by interpolating v on the corners of each τ_i (i.e. for each τ_i the affine function we obtain from interpolating v on the corners of τ_i is given by $\tilde{v}_{\lfloor\tau_i}$), then we can expect $I(\tilde{v})$ to be quite small, assuming the bulk and surfaces energies of v are small enough.

This is how we reduce the problem to the problem of lower bounds for the finite element approximation of I.

In truth, the "coarea alternative" we apply to each τ_i is considerably more subtle than the argument described here, but the basic ideas are the same. When needed we will preface the proof of the more intricate lemmas with a preproof to indicate how the argument goes.

5. Preliminary Lemmas

5.1. The vector field $D\Psi_i$.

5.1.1. *Traveling in Cones.*
Here we set up one of the basic lemmas about integral curves to the vector field $D\Psi_i$. Quite simply this lemma says that when we pull back with v^{-1} one such integral curve, the resulting curve will be a Lipschitz graph over the line $\phi_i^{\perp}$. More informally, the pull back of the integral curve will always travel in cones. The proof is just a calculation.

Lemma 1. *Let $v \in \mathcal{A}_F(\Omega)$. Let $\phi_1, \phi_2 \in S^1$ be the rank-1 directions of H, let $n_i \in S^1$ be the counterclockwise normal to ϕ_i for $i = 1, 2$. Let*

$$X^+(x, v, \alpha) := \left\{z \in \mathbb{R}^2 : \left|(z-x)\cdot v^{\perp}\right| \leq \alpha (z-x)\cdot v, \ \ (z-x)\cdot v > 0\right\}.$$

be the standard one sided cone. For $i \in \{1, 2\}$ given function $v \in \mathcal{A}_F(\Omega)$ we can define a function $\Psi_i : v(\Omega) \to \mathbb{R}$ in the following way:

$$\Psi_i(x) = v^{-1}(x)\cdot n_i.$$

By smoothness of v the vector field

$$D\Psi_i : v(\Omega) \to \mathbb{R}^2$$

is smooth. For any $x_0 \in \mathbb{R}^2$ if we solve the ODE

$$\frac{dX}{dt}(t) = D\Psi(X(t))$$
$$X(0) = x_0,$$

then the path X has the property

$$v^{-1}(X(t)) \in X^+\left(v^{-1}(x_0), n_i, \frac{1}{\sigma^6}\right) \ \ \forall\, t \in \mathbb{R}_+ \cap \{t : X(t) \in v(\Omega)\}$$

and

$$v^{-1}(X(t)) \in X^+\left(v^{-1}(x_0), -n_i, \frac{1}{\sigma^6}\right) \ \ \forall\, t \in \mathbb{R}_- \cap \{t : X(t) \in v(\Omega)\}$$

Proof. Let $i \in \{1, 2\}$ and let

$$\Psi_i(x) = v^{-1}(x)\cdot n_i.$$

Given $x_0 \in \mathbb{R}^2$ and let $X : \mathbb{R}_+ \to \mathbb{R}^2$ be a solution of the ode; $X(0) = x_0$,

$$\frac{dX}{dt}(t_0) = D\Psi_i(X(t_0)).$$

Let $z_0 = X(t_0)$ for some $t_0 > 0$, so

$$\frac{dX}{dt}(t_0) = D\Psi_i(z_0)$$

and note

$$D\Psi_i(z_0) = D\left(v^{-1}(z_0)\cdot n_i\right) = Dv^{-T}\left(v^{-1}(z_0)\right) n_i. \tag{9}$$

Since $det(Dv(x)) > \sigma^2$ for every $x \in \mathbb{R}^2$, we have the following decomposition;

$$Dv(x) = R(x)\, S(x) \tag{10}$$

for some $R(x) \in SO(2)$ and some positive definite symmetric matrix $S(x)$. So

$$Dv^{-T}\left(v^{-1}(z_0)\right) = R\left(v^{-1}(z_0)\right) S^{-1}\left(v^{-1}(z_0)\right). \tag{11}$$

Hence

$$\frac{dX}{dt}(t_0) = R\left(v^{-1}(X(t_0))\right) S^{-1}\left(v^{-1}(X(t_0))\right) n_i. \tag{12}$$

Let $Y : \mathbb{R}_+ \to \mathbb{R}^2$ be the path defined by

$$Y(t) := v^{-1}(X(t)),$$

so for any $t_0 > 0$ let v_{t_0} be the (non-normalised) tangent to path Y at point $Y(t_0)$, so

$$v_{t_0} = \frac{dY}{dt}(t_0) = Dv^{-1}(X(t_0))\frac{dX}{dt}(t_0), \tag{13}$$

and as $Dv^{-1}(X(t_0)) = \left(Dv\left(v^{-1}(X(t_0))\right)\right)^{-1}$ putting (10), (12) and (13) together

$$\begin{aligned} [Dv(Y(t_0))]\, v_{t_0} &\overset{(10)}{=} [R(Y(t_0))\, S(Y(t_0))]\, v_{t_0} \\ &\overset{(13)}{=} \frac{dX}{dt}(t_0) \\ &\overset{(12)}{=} \left[R(Y(t_0))\, S^{-1}(Y(t_0))\right] n_i. \end{aligned}$$

This gives;

$$v_{t_0} = \left[S^{-1}(Y(t_0))\, S^{-1}(Y(t_0))\right] n_i. \tag{14}$$

Now its easy to see that $S^{-1}(\cdot)\, S^{-1}(\cdot)$ is symmetric. If we let $\lambda_2 > 0$ and $\lambda_2 > 0$ denote the eigenvalues of S^{-1} then the eigenvalues $S^{-1}(\cdot)\, S^{-1}(\cdot)$ will be $\lambda_2^2 > 0$ and $\lambda_2^2 > 0$ and in particular there exits a unitary matrix U such that $U^T S^{-1}(\cdot)\, S^{-1}(\cdot)\, U = D$ where D is a diagonal matrix with entries λ_2^2 and λ_2^2. So for any $m = \binom{m_1}{m_2} \in S^1$, letting $\tilde{m} = U^T m$ we have

$$m \cdot S^{-1}(\cdot)\, S^{-1}(\cdot)\, m = \tilde{m} \cdot D\tilde{m} = \lambda_1^2 m_1^2 + \lambda_2^2 m_2^2 > \sigma^2$$

and so we have

$$\frac{dY}{dt}(t_0) \cdot n_i \geq \sigma^2. \tag{15}$$

Now from (13) and (14) we have

$$\left|\frac{dY}{dt}(t_0)\right|^2 = \left|\left(S^{-1}(Y(t_0)) \cdot S^{-1}(Y(t_0))\right) \cdot n_i\right|^2 \leq \frac{1}{\sigma^4}. \tag{16}$$

Note from (15), (16) we have

$$Y(t) \in X^+\left(v^{-1}(x_0), n_i, \frac{1}{\sigma^6}\right) \tag{17}$$

for all $t > 0$.

In exactly the same way, if we solve the ode $X(0) = x_0$ and

$$\frac{dX}{dt}(t_0) = -D\Psi_i(X(t_0))$$

we can show that if $Y(t_0) = v^{-1}(X(t_0))$ then

$$Y(t) \in X^+\left(v^{-1}(x_0), -n_i, \frac{1}{\sigma^6}\right), \tag{18}$$

and this completes the proof. □

5.1.2. *The level set function.*
As noted in the introduction (section 4.3), we will need to integrate up our integral curves with with the coarea formula. To do this we require a function whose level sets form the integral curves. Here such a function is defined and its basic properties are proved.

Lemma 2. *Let $v \in \mathcal{A}_F(\Omega)$ and let $S = P(a, \phi_1, \phi_2, \epsilon)$ be such that $N_{\frac{\epsilon}{\sigma^9}}(S) \subset \Omega$. Let $i \in \{1, 2\}$ and let $\Psi_i(x) := v^{-1}(x) \cdot n_i$.*

Take $q \in S$. We define $\Upsilon_q := (q + \langle\phi_1\rangle) \cap \Omega$. We can define function

$$\Theta_q^i : v(S) \to \mathbb{R}$$

such that for any $x_0 \in v(S)$, the path defined by the ODE

$$\begin{aligned} &X(0) = x_0 \\ &\frac{dX}{dt}(t) = D\Psi_i(X(t)) \end{aligned} \tag{19}$$

is such that there exists a unique $t_0 \in \mathbb{R}$ *for which*

$$X(t_0) \in v\left(\Upsilon_q^i\right),$$
$$\{X(t) : t \in \mathbb{R}\} = \Theta_q^{i\,-1}\left(\Theta_q^i\left(X(t_0)\right)\right).$$

and

$$\sup_{x \in v(S)} \left|\Theta_q^i(x)\right| < \frac{2\epsilon}{\sigma^2}. \tag{20}$$

Proof. To start, we note that for any $x_0 \in v(S)$ by smoothness of v on Ω^0 we can uniquely solve the ode

$$\begin{aligned} &X(0) = x_0 \\ &\frac{dX}{dt}(t) = D\Psi_i(X(t)). \end{aligned} \tag{21}$$

Let $s(x_0)$ be the first point of $\partial v(\Omega)$ to be hit by the path X going backwards in time from x_0. And let $e(x_0)$ be the first point of $\partial v(\Omega)$ to be hit by the path X going forwards in time from x_0. Let $\pi_1 \in \mathbb{R}$ be the unique number such that $X(\pi_1) = s(x_0)$ and let $\pi_2 \in \mathbb{R}$ be the unique number such that $X(\pi_2) = e(x_0)$. By definition; for any $t \in [\pi_1, \pi_2]$ we have $v(t) \in v(\Omega)$. Let $\mathcal{I}(t_0) := \{X(t) : t \in [\pi_1, \pi_2]\}$. Let $T(x_0) := [\pi_1, \pi_2]$.

Now $\mathcal{I}(x_0)$ is the continuous image of a connected interval, so is connected.

By Lemma 1 we know

$$\begin{aligned} &\left\{v^{-1}(X(t)) : t \in \mathbb{R}_+ \cap T(x_0)\right\} \subset X^+\left(v^{-1}(x_0), n_i, \frac{1}{\sigma^6}\right) \\ &\left\{v^{-1}(X(t)) : t \in \mathbb{R}_- \cap T(x_0)\right\} \subset X^+\left(v^{-1}(x_0), -n_i, \frac{1}{\sigma^6}\right). \end{aligned} \tag{22}$$

So from (22) we have that for any $x \in P_{\langle n_i^{\perp}\rangle}\left(v^{-1}(\mathcal{I}(x_0))\right)$ we have $P^{-1}_{\langle n_i^{\perp}\rangle}(x) \cap v^{-1}(\mathcal{I}(x_0))$ consists of one point.

So since $P_{\langle n_1^{\perp}\rangle}\left(v^{-1}(\mathcal{I}(x_0))\right)$ is a connected set, if $v^{-1}(\mathcal{I}(x_0)) \cap \Upsilon_q^i = \emptyset$ it can only be because path X has run out of region $v(\Omega)$ before crossing $v\left(\Upsilon_a^i\right)$. However since any point $x_0 \in S$ is at least distance $\frac{\epsilon}{\sigma^7}$ away from $\partial\Omega$ by (22) this can not happen. So the exists a unique point $t_0 \in \mathbb{R}$ such that $X(t_0) \in v\left(\Upsilon_q^i\right)$.

Now we define $\Theta_q^i : v(S) \to \mathbb{R}$ as follows; For any $x_0 \in v(S)$ let X be the solution of

$$\begin{aligned} &X(0) = v(x_0) \\ &\frac{dX}{dt}(t) = D\Psi_i(X(t)). \end{aligned}$$

Let $t(x_0) \in \mathbb{R}$ be the unique real number such that $v^{-1}(X(t(x_0))) \cap \Upsilon_q \neq \emptyset$. We define

$$\Theta_q^i(x_0) := v^{-1}(X(t(x_0))) \cdot \phi_i,$$

by uniqueness of $t(x_0)$, $\Theta_q^i(x_0)$ is well defined and its clear that $\Theta_q^{i\,-1}\left(\Theta_q^i(x_0)\right) = \{X(t) : t \in \mathbb{R}\}$. $\square$

5.1.3. *Integrating along integral curves.*
As mentioned in the introduction, section 4.2, one of the main observations in this proof is that for integral curves for which the derivative $Dv\left(v^{-1}(\cdot)\right)$ *stays close the wells, we have that* $|D\Psi_i(\cdot)| \approx 1$*. So for a subsegment* U *of such an integral curve of* $D\Psi_i$ *with end points* s *and* e*, the* $H^1(U)$ *is approximately* $|\Psi_i(e) - \Psi_i(s)|$*. This is the contents of the statement of Lemma 3, the proof is just a calculation.*

Lemma 3. *Let* $v \in \mathcal{A}_F(\Omega)$*. Let* $S := P(a, \phi_1, \phi_2, \epsilon) \subset \Omega$ *and let* $i \in \{1,2\}$*. Suppose we have for some* $t_0 \in \mathbb{R}$ *we have a connected subset* $U \subset \Theta_a^{i\,-1}(t_0) \cap v(S)$ *such that*

$$\int_U J(x)\, dH^1 x \leq \alpha H^1(U). \tag{23}$$

If we let s, e *be the endpoints of* U *then we have*

$$\left(1 - \frac{\mathfrak{c}_1 \alpha}{\sigma^4}\right) |\Psi_i(e) - \Psi_i(s)| \leq H^1(U) \leq \left(1 + \frac{\mathfrak{c}_1 \alpha}{\sigma^4}\right) |\Psi_i(e) - \Psi_i(s)|.$$

Proof. For each $x \in U$ since K is compact we can find $P(x) \in K$ such that

$$d\left(Dv\left(v^{-1}(x)\right), K\right) = \left|Dv\left(v^{-1}(x)\right) - P(x)\right|.$$

Let $E(x) = Dv\left(v^{-1}(x)\right) - P(x)$.

Let $B = \left\{x \in U : d\left(Dv\left(v^{-1}(x)\right), SO(2)\right) = d\left(Dv\left(v^{-1}(x)\right), SO(2)H\right)\right\}$ this is a closed set and since we have (23) we know

$$H^1(B) < c_1 \alpha H^1(U). \tag{24}$$

Let

$$R = \left\{x \in U : d\left(Dv\left(v^{-1}(x)\right), SO(2)\right) < d\left(Dv\left(v^{-1}(x)\right), SO(2)H\right)\right\}.$$

And let

$$S = \left\{x \in U : d\left(Dv\left(v^{-1}(x)\right), SO(2)H\right) < d\left(Dv\left(v^{-1}(x)\right), SO(2)\right)\right\}.$$

Now S and R are open in $\Theta_a^{-1}(q)$ and so

$$R = \cup_n K_n \quad \text{and} \quad S = \cup_n I_n$$

where I_n and K_n are open connected sets in U.

Let s_n be the starting point of segment I_n (the point coming from the right) and let e_n be the endpoint. For each $x \in I_n$ let $t_x \in S^1$ denote the tangent to the curve I_n at point x. We will show

$$\int_{I_n} D\Psi_i(x)\, t_x - \frac{3\,|E(x)|}{\sigma^4} dH^1 x < H^1(I_n) < \int_{I_n} D\Psi_i(x)\, t_x + \frac{3\,|E(x)|}{\sigma^4} dH^1 x. \tag{25}$$

To begin with note

$$\int_{I_n} D\Psi_i(x)\, t_x\; dH^1 x = \Psi_i(e_n) - \Psi_i(s_n).$$

Recall,

$$R(\cdot)\, S(\cdot) = Dv(\cdot) \tag{26}$$

is the polar decomposition of of $Dv(\cdot)$.

We have already calculated that $D\Psi_i(x) = R\left(v^{-1}(x)\right) S^{-1}\left(v^{-1}(x)\right) n_i$. We let $X : \mathbb{R} \to \mathbb{R}^2$ be a solution of

$$\begin{aligned} X(0) &= s_n \\ \frac{dX}{dt}(t) &= D\Psi_i(X(t)). \end{aligned} \tag{27}$$

Let $Y(s) = v^{-1}(X(s))$. We also have calculated that

$$\frac{dY}{dt}(s) = S^{-1}(Y(s))\, S^{-1}(Y(s))\, n_i, \tag{28}$$

for $s > 0$. Let $x_0 \in I_n$ and let $s_0 > 0$ be such that $X(s_0) = x_0$. So as $v(Y(s_0)) = X(s_0)$ we have

$$\frac{dX}{dt}(s_0) = Dv(Y(s_0))\, \frac{dY}{dt}(s_0), \tag{29}$$

and as $Dv(Y(s_0)) = R(Y(s_0))\,S(Y(s_0))$ so by (26), (28) and (29) we have

$$\begin{aligned} D\Psi_i(x_0) &= \frac{dX}{dt}(s_0) \\ &\overset{(28),(29)}{=} R(Y(s_0))\,S(Y(s_0))\,S^{-1}(Y(s_0))\,S^{-1}(Y(s_0))\,n_i \\ &= R(Y(s_0))\,S^{-1}(Y(s_0))\,n_i. \end{aligned} \tag{30}$$

So to estimate the length of $D\Psi_i(x_0)$ we need only know $S^{-1}(Y(s_0))\,n_i$ however as we know $Dv(Y(s_0)) =: P(Y(s_0)) + E(Y(s_0))$ for some $P(Y(s_0)) \in SO(2)\,H$ we have

$$\begin{aligned} S(Y(s_0))^2 &= Dv(Y(s_0))^T\,Dv(Y(s_0)) \\ &= (P(Y(s_0)) + E(Y(s_0)))^T\,(P(Y(s_0)) + E(Y(s_0))) \\ &= P(Y(s_0))\,P(Y(s_0))^T + E(Y(s_0))^T\,P(Y(s_0)) + E(Y(s_0))\,P(Y(s_0))^T \\ &\quad + E(Y(s_0))\,E(Y(s_0))^T, \end{aligned} \tag{31}$$

and recall $P(Y(s_0)) = \tilde{R}H$ for some $\tilde{R} \in SO(2)$ and so $P(Y(s_0))^T\,P(Y(s_0)) = H^T R^T R H = H^T H$. So $S(Y(s_0))^2 \in N_{\frac{3|E(x)|}{\sigma^2}}\left(H^2\right)$ and so

$$S^{-1}(Y(s_0)) \in N_{\frac{3|E(x)|}{\sigma^4}}\left(H^{-1}\right). \tag{32}$$

And we claim for $i = 1, 2$;

$$\left|H^{-1} n_i\right| = 1. \tag{33}$$

First we have to calculate ϕ_1, ϕ_2 and subsequently n_1, n_2. So we require

$$\left|\begin{pmatrix} \tilde{\sigma} & 0 \\ 0 & \frac{1}{\tilde{\sigma}} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right| = (\tilde{\sigma}a)^2 + \left(\frac{b}{\tilde{\sigma}}\right)^2 = 1,$$

for vector

$$\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = 1.$$

To simplify expression we let $\lambda = \frac{1}{\tilde{\sigma}}$.

So $a^2 = 1 - b^2$ and inserting this into the first equation we have

$$\tilde{\sigma}^2\left(1 - b^2\right) + \lambda^2 b^2 = \tilde{\sigma}^2 + \left(\lambda^2 - \tilde{\sigma}^2\right) b^2 = 1.$$

Hence

$$b^2 = \frac{1 - \tilde{\sigma}^2}{\lambda^2 - \tilde{\sigma}^2},$$

$$a^2 = 1 - b^2 = \frac{\left(\lambda^2 - \tilde{\sigma}^2\right) - \left(1 - \tilde{\sigma}^2\right)}{\lambda^2 - \tilde{\sigma}^2} = \frac{\lambda^2 - 1}{\lambda^2 - \tilde{\sigma}^2}.$$

Thus

$$\phi_1 = \begin{pmatrix} \sqrt{\frac{\lambda^2 - 1}{\lambda^2 - \tilde{\sigma}^2}} \\ \sqrt{\frac{1 - \tilde{\sigma}^2}{\lambda^2 - \tilde{\sigma}^2}} \end{pmatrix} \quad \phi_2 = \begin{pmatrix} \sqrt{\frac{\lambda^2 - 1}{\lambda^2 - \tilde{\sigma}^2}} \\ -\sqrt{\frac{1 - \tilde{\sigma}^2}{\lambda^2 - \tilde{\sigma}^2}} \end{pmatrix}. \tag{34}$$

And hence

$$n_1 = \begin{pmatrix} -\sqrt{\frac{1 - \tilde{\sigma}^2}{\lambda^2 - \tilde{\sigma}^2}} \\ \sqrt{\frac{\lambda^2 - 1}{\lambda^2 - \tilde{\sigma}^2}} \end{pmatrix} \quad n_2 = \begin{pmatrix} \sqrt{\frac{1 - \tilde{\sigma}^2}{\lambda^2 - \tilde{\sigma}^2}} \\ \sqrt{\frac{\lambda^2 - 1}{\lambda^2 - \tilde{\sigma}^2}} \end{pmatrix}. \tag{35}$$

Thus

$$\begin{aligned} \left|H^{-1}n_i\right| &= \left|\begin{pmatrix} \lambda & 0 \\ 0 & \tilde{\sigma} \end{pmatrix}\begin{pmatrix} \pm\sqrt{\frac{1-\tilde{\sigma}^2}{\lambda^2-\tilde{\sigma}^2}} \\ \sqrt{\frac{\lambda^2-1}{\lambda^2-\tilde{\sigma}^2}} \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} \pm\lambda\sqrt{\frac{1-\tilde{\sigma}^2}{\lambda^2-\tilde{\sigma}^2}} \\ \tilde{\sigma}\sqrt{\frac{\lambda^2-1}{\lambda^2-\tilde{\sigma}^2}} \end{pmatrix}\right| \\ &= \lambda^2\frac{\left(1-\tilde{\sigma}^2\right)}{\left(\lambda^2-\tilde{\sigma}^2\right)}+\tilde{\sigma}^2\frac{\left(\lambda^2-1\right)}{\left(\lambda^2-\tilde{\sigma}^2\right)} \\ &= \frac{\lambda^2-\tilde{\sigma}^2}{\lambda^2-\tilde{\sigma}^2} \\ &= 1. \end{aligned} \tag{36}$$

So we have proved (33).

Thus from (32), (30) and (33) we have that

$$\left|D\Psi_i\left(x_0\right)\right| \in \left(1-\frac{3\left|E\left(x\right)\right|}{\sigma^4}, 1+\frac{3\left|E\left(x\right)\right|}{\sigma^4}\right). \tag{37}$$

Let

$$\beta\left(x_0\right) := \left|D\Psi_i\left(x_0\right)\right| = \left|\frac{dX}{dt}\left(s_0\right)\right|. \tag{38}$$

Now as X is a solution of (27) we have

$$t_{x_0} = \frac{\frac{dX}{dt}\left(s_0\right)}{\left|\frac{dX}{dt}\left(s_0\right)\right|} = \frac{D\Psi_i\left(x_0\right)}{\beta\left(x_0\right)}. \tag{39}$$

and so from (39)

$$\begin{aligned} D\Psi_i\left(x_0\right)\cdot t_{x_0} &= \beta\left(x_0\right)t_{x_0}\cdot t_{x_0} \\ &= \beta\left(x_0\right). \end{aligned}$$

And so from (37) and (38)

$$\int_{I_n} D\Psi_i\left(x\right)t_x\, dH^1x \in \left(H^1\left(I_n\right)-\int_{I_n}\frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x, H^1\left(I_n\right)+\int_{I_n}\frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x\right)$$

which implies (25).

Now we argue a similar inequality for the $\{K_n\}$. So let e_n, s_n be endpoints of segment K_n. From (31) we see that $S\left(v^{-1}\left(x\right)\right) \in N_{\frac{3|E(x)|}{\sigma^4}}\left(I\right)$ for any $x \in K_n$ so again we have

$$\int_{K_n} D\Psi_i\left(x\right)t_x - \frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x \leq H^1\left(K_n\right) \leq \int_{K_n} D\Psi_i\left(x\right)t_x + \frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x. \tag{40}$$

Now from (25) we have

$$\sum_n\int_{I_n} D\Psi_i\left(x\right)t_x - \frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x \leq \sum_n H^1\left(I_n\right) \leq \sum_n\int_{I_n} D\Psi_i\left(x\right)t_x + \frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x \tag{41}$$

and from (40) we have

$$\sum_n\int_{K_n} D\Psi_i\left(x\right)t_x - \frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x \leq \sum_n H^1\left(K_n\right) \leq \sum_n\int_{K_n} D\Psi_i\left(x\right)t_x + \frac{3\left|E\left(x\right)\right|}{\sigma^4}dH^1x. \tag{42}$$

Since for any subsegment $I\subset U$ where I has endpoint a and b we have

$$\int_I D\Psi_i(x)\,t_x dH^1x=\Psi_i(b)-\Psi_i(a)=P_{\langle\phi_i^{\perp}\rangle}\left(v^{-1}(I)\right).$$

Let $\varpi=\int_{\Theta_a^{-1}(t)\cap v(S)}\frac{3|E(x)|}{\sigma^4}dH^1x$, note that by assumption we know $\varpi<\frac{3\alpha}{\sigma^4}H^1(U)$. Putting together (41) and (42) we have that

$$\begin{aligned}\sum_n L^1\left(P_{\langle\phi_1^{\perp}\rangle}\left(v^{-1}(K_n)\cup v^{-1}(I_n)\right)\right)-\varpi\quad&\leq\quad\sum_n H^1(K_n)+H^1(I_n)\\&\leq\sum_n L^1\left(P_{\langle\phi_1^{\perp}\rangle}\left(v^{-1}(K_n)\cup v^{-1}(I_n)\right)\right)+\varpi.\end{aligned}\tag{43}$$

Since we know from (24)

$$H^1\left(U\backslash\left(\bigcup_n(K_n\cup I_n)\right)\right)\leq c_1\alpha H^1(U)$$

so

$$H^1\left(v^{-1}\left(U\backslash\left(\bigcup_n(K_n\cup I_n)\right)\right)\right)\leq\frac{c_1\alpha H^1(U)}{\sigma^2}\tag{44}$$

and thus we have from (43) and (44)

$$\begin{aligned}H^1(U)\quad&\leq\quad\left(\sum_n H^1(K_n)+H^1(I_n)\right)+c_1\alpha H^1(U)\\&\leq\quad\sum_{n\in\mathbb{N}}L^1\left(P_{\langle\phi_1^{\perp}\rangle}\left(v^{-1}(K_n)\cup v^{-1}(I_n)\right)\right)+\varpi+c_1\alpha H^1(U)\\&\leq\quad L^1\left(P_{\langle\phi_1^{\perp}\rangle}\left(v^{-1}(U)\right)\right)+\left(\frac{3}{\sigma^4}+\frac{2c_1}{\sigma^2}\right)\alpha H^1(U).\end{aligned}\tag{45}$$

and similarly

$$\begin{aligned}L^1\left(P_{\langle\phi_1^{\perp}\rangle}\left(v^{-1}(U)\right)\right)-\left(\frac{3}{\sigma^4}+\frac{2c_1}{\sigma^2}\right)\alpha H^1(U)\quad&\leq\quad\sum_{n\in\mathbb{N}}H^1(K_n)+H^1(I_n)\\&\leq\quad H^1(U)\end{aligned}$$

and this establishes our claim.

□

5.2. **The pullback idea.**

This next lemma is a formalization of what has been described in section 4.1 of introduction as the pull back idea. Essentially what it means is that for a function v of small bulk energy, lines of the form $v(\langle\phi_1\rangle+z_1)$, $v(\langle\phi_1\rangle+z_2)$ are pushed over from one another. A better explanation can be obtained from section 4.1 of the introduction.

Lemma 4. *Given function $v\in\mathcal{A}_F(\Omega)$. Let $S=P(a,\phi_1,\phi_2,\epsilon)$ be such that $S\subset\Omega$. Let $i\in\{1,2\}$. For any $b,e\in v(S)$, let $\eta:=[b,e]$ if we have*

$$\int_\eta J(z)\,dH^1z<\alpha\,|b-e|$$

then

$$|e-b|>\left(1-2\sigma^{-2}\sqrt{\alpha}\right)|\Psi_i(e)-\Psi_i(b)|.$$

Proof. Now letting $t_x \in S^1$ denote the tangent to the curve $v^{-1}(\eta)$ at point x.

$$\begin{aligned}\int_\eta J(z)\, dH^1 z &= \int_{v^{-1}(\eta)} |Dv(x)\, t_x|\, J(v(x))\, dH^1 x \\ &\geq \sigma^2 \int_{v^{-1}(\eta)} J(v(x))\, dH^1 x,\end{aligned}$$

so we have

$$\sigma^{-2}\alpha\, |b-e| \geq \int_{v^{-1}(\eta)} J(v(x))\, dH^1 x.$$

Now $v^{-1}(\eta)$ connects $v^{-1}(b)$ to $v^{-1}(e)$. Let

$$T = \left\{x \in v^{-1}(\eta) : J(v(x)) < \sqrt{\alpha}\right\},$$

so we know that

$$H^1\left(v^{-1}(\eta)\setminus T\right) \leq \sigma^{-2}\sqrt{\alpha}\, |b-e|. \tag{46}$$

Now T is the countable union of connected segments $I_n \subset v^{-1}(\eta)$. Now let a_n, b_n be the end points of the segment I_n. For $i \in \{1,2\}$ we will show

$$|v(a_n) - v(b_n)| \geq L^1\left(P_{\langle \phi_i^\perp\rangle}(I_n)\right) - \sqrt{\alpha} H^1(I_n) \tag{47}$$

Now recall we calculated n_i at (35) (where $\lambda := \tilde{\sigma}^{-1}$)

$$n_1 = \begin{pmatrix} -\sqrt{\frac{1-\tilde{\sigma}^2}{\frac{1}{\tilde{\sigma}^2}-\tilde{\sigma}^2}} \\ \sqrt{\frac{\frac{1}{\tilde{\sigma}^2}-1}{\frac{1}{\tilde{\sigma}^2}-\tilde{\sigma}^2}} \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{\tilde{\sigma}^2(1-\tilde{\sigma}^2)}{(1-\tilde{\sigma}^2)(1+\tilde{\sigma}^2)}} \\ \sqrt{\frac{1-\tilde{\sigma}^2}{(1-\tilde{\sigma}^2)(1+\tilde{\sigma}^2)}} \end{pmatrix} = \begin{pmatrix} -\frac{\tilde{\sigma}}{\sqrt{(1+\tilde{\sigma}^2)}} \\ \frac{1}{\sqrt{(1+\tilde{\sigma}^2)}} \end{pmatrix}, \quad n_2 = \begin{pmatrix} \frac{\tilde{\sigma}}{\sqrt{(1+\tilde{\sigma}^2)}} \\ \frac{1}{\sqrt{(1+\tilde{\sigma}^2)}} \end{pmatrix}$$

now as a first step to proving equation (47) we will prove the following:
We firstly we define the *shrink directions*. Let Ξ_2 be the subset of vectors of S^1 between ϕ_1 and ϕ_2 and let Ξ_1 be the subset of vectors of S^1 between $-\phi_1$ and $-\phi_2$. Its easy to see that for any $v \in S^1$, $|Hv| \leq 1 \Leftrightarrow v \in \Xi_1 \cup \Xi_2$ hence the name *shrink directions*.

Claim Let $i \in \{1,2\}$. We will show that there exists vector $\Diamond_i \in \Xi_i$ such that for any $\psi \in \Xi_i$

$$|H\psi| \geq \psi \cdot n_i + \frac{\mathfrak{c}_2 \left(\operatorname{ang}(\psi, \Diamond_i)\right)^2}{4}. \tag{48}$$

First we consider the inequality

$$|H\psi| \geq \psi \cdot n_2 + \frac{\mathfrak{c}_2 \left(\operatorname{ang}(\psi, \Diamond_2)\right)^2}{4}. \tag{49}$$

Let

$$\psi = \begin{pmatrix} \cos a \\ \sin a \end{pmatrix}, \tag{50}$$

so equation (49) is equivalent to

$$\sqrt{\tilde{\sigma}^2 \cos^2 a + \frac{\sin^2 a}{\tilde{\sigma}^2}} \geq \frac{\tilde{\sigma}\cos a}{\sqrt{\tilde{\sigma}^2+1}} + \frac{\sin a}{\sqrt{\tilde{\sigma}^2+1}} + \frac{\mathfrak{c}_2 \left(\operatorname{ang}(\psi, \Diamond_2)\right)^2}{4}. \tag{51}$$

and we will prove (51) in due course. Firstly we will show why inequality (48) for $i=1$ follows from inequality (49). Give $\psi \in S^1$ of the form (50) we define

$$\bar{\psi} = \begin{pmatrix} -\cos a \\ \sin a \end{pmatrix}. \tag{52}$$

When we calculate $\Diamond_1$ and $\Diamond_2$ it will turn out that $\bar{\Diamond}_1 = \Diamond_2$, see (247) in the Appendix.

Hence (see fig 6) if we have (49) then

$$\begin{aligned}\left|\widetilde{H}\phi\right| &= \left|\widetilde{H}\bar{\phi}\right| \\ &\geq \bar{\phi}\cdot n_2 + \frac{\mathfrak{c}_2\left(\operatorname{ang}\left(\bar{\phi},\lozenge_2\right)\right)^2}{4} \\ &= \phi\cdot n_1 + \frac{\mathfrak{c}_2\left(\operatorname{ang}\left(\phi,\lozenge_1\right)\right)^2}{4},\end{aligned}$$

so all that remains is to establish (49) which as we noted is equivalent to (51).

The proof of inequality (51) is quite involved, partly due to the fact its sharp. Let

$$f(a) := \tilde{\sigma}^2\cos^2 a + \frac{\sin^2 a}{\tilde{\sigma}^2} - \frac{\tilde{\sigma}^2\cos^2 a}{\tilde{\sigma}^2+1} - \frac{\sin^2 a}{\tilde{\sigma}^2+1} - \frac{2\tilde{\sigma}\cos a\sin a}{\tilde{\sigma}^2+1},$$

so

$$\begin{aligned}f(a) &= \tilde{\sigma}^2\cos^2 a\left(1-\frac{1}{\tilde{\sigma}^2+1}\right) + \sin^2 a\left(\frac{1}{\tilde{\sigma}^2}-\frac{1}{\tilde{\sigma}^2+1}\right) - \frac{2\tilde{\sigma}\cos a\sin a}{\tilde{\sigma}^2+1} \\ \left(\tilde{\sigma}^2+1\right)f(a) &= \tilde{\sigma}^4\cos^2 a + \sin^2 a\left(\frac{\tilde{\sigma}^2+1}{\tilde{\sigma}^2}-1\right) - 2\tilde{\sigma}\cos a\sin a \\ \tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)f(a) &= \tilde{\sigma}^6\cos^2 a + \sin^2 a - 2\tilde{\sigma}^3\cos a\sin a,\end{aligned}$$

now using standard trigonometric identities we have

$$\begin{aligned}\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)f(a) &= \tilde{\sigma}^6\left(\frac{1+\cos 2a}{2}\right) + \frac{1-\cos 2a}{2} - \tilde{\sigma}^3\sin 2a \\ 2\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)f(a) &= \left(\tilde{\sigma}^6-1\right)\cos 2a - 2\tilde{\sigma}^3\sin 2a + \tilde{\sigma}^6 + 1.\end{aligned}$$

Now

$$2\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)f'(a) = -2\left(\tilde{\sigma}^6-1\right)\sin 2a - 4\tilde{\sigma}^3\cos 2a$$

So

$$\begin{aligned}f'(\tilde{a}) = 0 \quad &\Leftrightarrow \quad -2\sin 2\tilde{a}\left(\tilde{\sigma}^6-1\right) - 4\tilde{\sigma}^3\cos 2\tilde{a} = 0 \\ &\Leftrightarrow \quad -2\sin 2\tilde{a}\left(\tilde{\sigma}^6-1\right) = 4\tilde{\sigma}^3\cos 2\tilde{a} \\ &\Leftrightarrow \quad \tan 2\tilde{a} = \frac{2\tilde{\sigma}^3}{\left(1-\tilde{\sigma}^6\right)}.\end{aligned} \tag{53}$$

Now as $\tilde{a}$ is chosen from an interval of length less than π, there is only one $\tilde{a}$ for which (53) is true. Let

$$\lozenge_2 := \left(\begin{smallmatrix}\cos\tilde{a}\\ \sin\tilde{a}\end{smallmatrix}\right). \tag{54}$$

Let $p = \sqrt{\left(4\tilde{\sigma}^6 + \left(1-\tilde{\sigma}^6\right)^2\right)} = \left(1+\tilde{\sigma}^6\right)$, so $\sin 2\tilde{a} = \frac{2\tilde{\sigma}^3}{p}$ and $\cos 2\tilde{a} = \frac{\left(1-\tilde{\sigma}^6\right)}{p}$. Now

$$\begin{aligned}2\tilde{\sigma}\left(\tilde{\sigma}^2+1\right)f(\tilde{a}) &= \left(\tilde{\sigma}^6-1\right)\cos 2\tilde{a} - 2\tilde{\sigma}^3\sin 2\tilde{a} + \tilde{\sigma}^6 + 1 \\ &= -\frac{\left(\tilde{\sigma}^6-1\right)^2}{p} - \frac{4\tilde{\sigma}^6}{p} + \tilde{\sigma}^6 + 1 \\ &= -\frac{\left(4\tilde{\sigma}^6 + \left(\tilde{\sigma}^6-1\right)^2\right)}{p} + \tilde{\sigma}^6 + 1 \\ &= -\left(\tilde{\sigma}^6+1\right) + \tilde{\sigma}^6 + 1 \\ &= 0.\end{aligned}$$

Now note

$$2\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)f''(a) = -4\left(\tilde{\sigma}^6-1\right)\cos 2a + 8\tilde{\sigma}^2\sin 2a. \tag{55}$$

Before continuing we need to estimate the lenght of Ξ_i. Observe fig 6.

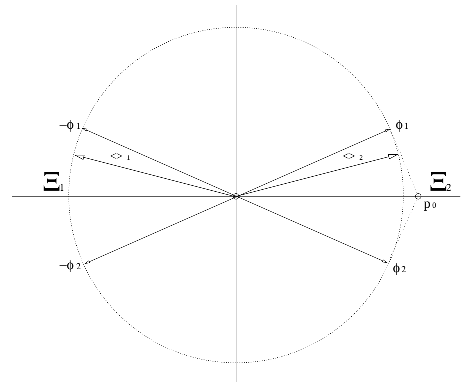

FIGURE 6

We let $p_0 = \left(\phi_1 + \phi_1^{\perp}\right) \cap \{x : x \cdot e_2 = 0\}$. Its clear from fig 6 that $H^1(\Xi_2) \leq 2|\phi_1 - p_0|$. So we have to find point p_0. From (34) we see that p_0 is given by the following formula

$$p_0 := \phi_1 + \lambda_0 \begin{pmatrix} \sqrt{\left(\frac{1-\tilde{\sigma}^2}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} \\ -\sqrt{\left(\frac{\tilde{\sigma}^{-2}-1}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} \end{pmatrix}$$

where $\lambda_0 > 0$ is some number such that

$$e_2 \cdot \left(\phi_1 + \lambda_0 \begin{pmatrix} \sqrt{\left(\frac{1-\tilde{\sigma}^2}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} \\ -\sqrt{\left(\frac{\tilde{\sigma}^{-2}-1}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} \end{pmatrix} \right) = 0. \tag{56}$$

Now by (34), (56) is equivilant to

$$\sqrt{\left(\frac{1-\tilde{\sigma}^2}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} = \lambda_0 \sqrt{\left(\frac{\tilde{\sigma}^{-2}-1}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} \Leftrightarrow \lambda_0 = \frac{\sqrt{\left(\frac{1-\tilde{\sigma}^2}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)}}{\sqrt{\left(\frac{\tilde{\sigma}^{-2}-1}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)}} = \sqrt{\left(\frac{1-\tilde{\sigma}^2}{\tilde{\sigma}^{-2}-1}\right)} = \tilde{\sigma}.$$

So

$$|\phi_1 - p_0| = \left| \lambda_0 \begin{pmatrix} \sqrt{\left(\frac{1-\tilde{\sigma}^2}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} \\ -\sqrt{\left(\frac{\tilde{\sigma}^{-2}-1}{\tilde{\sigma}^{-2}-\tilde{\sigma}^2}\right)} \end{pmatrix} \right| = \tilde{\sigma}.$$

Thus we have the estimate we want;

$$H^1(\Xi_2) = H^1(\Xi_1) \leq 2\tilde{\sigma}.$$

Now we will use this together with (55) to get a lower bound on f'' for those $a \geq$ such that $\left(\begin{smallmatrix} \cos a \\ \sin a \end{smallmatrix}\right) \in \Xi_2$:

Since $a \in [0, \tilde{\sigma}] \subset [0, 1)$ so from (55)

$$\begin{aligned} 2\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right) f''(a) &\geq \min\left\{4\left(1-\tilde{\sigma}^6\right), 8\tilde{\sigma}^2\right\}(\cos 2a + \sin 2a) \\ &\geq \min\left\{4\left(1-\tilde{\sigma}^6\right), 8\tilde{\sigma}^2\right\}(\cos 2 + \sin 2) \\ &> \frac{\min\left\{4\left(1-\tilde{\sigma}^6\right), 8\tilde{\sigma}^2\right\}}{4}. \end{aligned}$$

And so

$$f''(a) \geq \frac{\min\left\{4\left(1-\tilde{\sigma}^6\right), 8\tilde{\sigma}^2\right\}}{8\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)}. \tag{57}$$

Now note that for $a<0$ such that $\binom{\cos a}{\sin a}\in\Xi_2$, since $|a|<\tilde\sigma<1$ we know that $\cos a>0$ and $\sin a<0$ so we have that

$$\begin{aligned}\sqrt{\left(\tilde\sigma^2\cos^2 a+\frac{\sin^2 a}{\tilde\sigma^2}\right)}-\frac{\tilde\sigma\cos a}{\sqrt{\tilde\sigma^2+1}}-\frac{\sin a}{\sqrt{\tilde\sigma^2+1}} &> \tilde\sigma\cos a-\frac{\tilde\sigma\cos a}{\sqrt{\tilde\sigma^2+1}}-\frac{\sin a}{\sqrt{\tilde\sigma^2+1}}\\ &= \tilde\sigma\left(1-\frac{1}{\sqrt{\tilde\sigma^2+1}}\right)\cos a+\frac{|\sin a|}{\sqrt{\tilde\sigma^2+1}}\\ &> \min\left\{\tilde\sigma\left(1-\frac{1}{\sqrt{\tilde\sigma^2+1}}\right),\frac{1}{\sqrt{\tilde\sigma^2+1}}\right\}(|\cos a|+|\sin a|)\\ &> \frac14\min\left\{\tilde\sigma\left(1-\frac{1}{\sqrt{\tilde\sigma^2+1}}\right),\frac{1}{\sqrt{\tilde\sigma^2+1}}\right\}\end{aligned}\tag{58}$$

From (57) and (58) we let

$$\mathfrak{c}_2:=\min\left\{\frac{\left(1-\tilde\sigma^6\right)}{32\tilde\sigma^2\left(\tilde\sigma^2+1\right)},\frac{1}{16\left(\tilde\sigma^2+1\right)},\frac{\tilde\sigma}{64}\left(1-\frac{1}{\sqrt{1+\tilde\sigma^2}}\right),\frac{1}{64\sqrt{1+\tilde\sigma^2}}\right\}$$

and we define $t(x):=8\mathfrak{c}_2(x-\tilde a)$. We have

- $f'(\tilde a)=t(\tilde a)=0$.
- By (57), for all $a\in[0,\tilde\sigma]$ we have $f''(a)-t'(a)\geq 0$ and hence for any $a\in[\max\{\tilde a,\tilde\sigma\},0]$ we have $f'(a)-t(a)\geq 0$ and for any $a\in[0,\max\{\tilde a,0\}]$ we have $f'(a)-t(a)\leq 0$.

So for any $a\in[0,\tilde\sigma]$ we have

$$\int_{\tilde a}^{a} f'(x)-t(x)\,dL^1x=f(a)-8\mathfrak{c}_2(a-\tilde a)^2\geq 0.$$

which is equivalent to

$$\tilde\sigma^2\cos^2 a+\frac{\sin^2 a}{\tilde\sigma^2}\geq\frac{\tilde\sigma^2\cos^2 a+\sin^2 a+2\tilde\sigma\cos a\sin a}{\tilde\sigma^2+1}+8\mathfrak{c}_2(a-\tilde a)^2\tag{59}$$

.

Now in order to understand (59) note that

$$\left(\frac{\tilde\sigma\cos a+\sin a}{\sqrt{(\tilde\sigma^2+1)}}\right)^2+8\mathfrak{c}_2(a-\tilde a)^2\quad\leq\quad\frac{1}{(\tilde\sigma^2+1)}+\tilde\sigma^2<2$$

if we let $g(x):=\sqrt{x}$, we note that g' is greater than $\frac{1}{\sqrt2}$ on the interval $[0,2]$ and so by considering the integral of g' between $\left(\frac{\tilde\sigma\cos a+\sin a}{\sqrt{(\tilde\sigma^2+1)}}\right)^2$ and $\left(\frac{\tilde\sigma\cos a+\sin a}{\sqrt{(\tilde\sigma^2+1)}}\right)^2+8\mathfrak{c}_2(a-\tilde a)^2$.

We get

$$\sqrt{\left(\left(\frac{\tilde\sigma\cos a+\sin a}{\sqrt{\tilde\sigma^2+1}}\right)^2+8\mathfrak{c}_2(a-\tilde a)^2\right)}-\left(\frac{\tilde\sigma\cos a+\sin a}{\sqrt{\tilde\sigma^2+1}}\right)\geq\frac{8\mathfrak{c}_2}{\sqrt2}(a-\tilde a)^2.$$

So putting this together with (59)

$$\sqrt{\left(\tilde\sigma^2\cos^2 a+\frac{\sin^2 a}{\tilde\sigma^2}\right)}\geq\frac{\tilde\sigma\cos a+\sin a}{\sqrt{(\tilde\sigma^2+1)}}+\frac{8\mathfrak{c}_2}{\sqrt2}(a-\tilde a)^2$$

and this establishes the claim for $a\in[0,\tilde\sigma]$.

Now we need to deal with the case $a\in[-\tilde\sigma,0]$. Since $\tilde\sigma\in(0,1)$ from (58) we have

$$\sqrt{\left(\tilde\sigma^2\cos^2 a+\frac{\sin^2 a}{\tilde\sigma^2}\right)}-\frac{\tilde\sigma\cos a}{\sqrt{\tilde\sigma^2+1}}-\frac{\sin a}{\sqrt{\tilde\sigma^2+1}}\geq\frac{8\mathfrak{c}_2}{\sqrt2}(a-\tilde a)^2,$$

as $(\mathrm{ang}(\psi,\Diamond_2))^2\leq 10(a-\tilde a)^2$ this completes the proof of (51).

So for each $x \in I_n$ we know that for some $G(x) \in SO(2) \cup SO(2)H$ and some $E(x) \in M^{2\times 2}$ with $\|E(x)\| < \sqrt{\alpha}$ we have

$$Dv(x) = G(x) + E(x).$$

Now from (48)

$$\begin{aligned} |[Dv(x)]\psi| &> |(G(x)+E(x))\psi| \\ &\geq |G(x)\psi| - |E(x)\psi| \\ &> \psi \cdot n_i - \sqrt{\alpha}. \end{aligned}$$

So

$$\begin{aligned} H^1(v(I_n)) &= \int_{I_n} |Dv(x)\cdot t_x|\, dH^1 x \\ &\geq \int_{I_n} t_x \cdot n_i - \sqrt{\alpha} dH^1 x \\ &= (a_n - b_n)\cdot n_i - \sqrt{\alpha} H^1(I_n). \end{aligned}$$

Now $(a_n - b_n)\cdot n_i = L^1\left(P_{\langle \phi_i^\perp\rangle}(I_n)\right)$ and as $v(I_n)$ is a straight line, so $H^1(v(I_n)) = |v(a_n) - v(b_n)|$ and thus we have established inequality (47). Now by the fact that the line segments connecting $v(a_n)$ and $v(b_n)$ are subsets of η and by using (47) we have,

$$\begin{aligned} |e-b| &= H^1(\eta) \\ &\geq \sum_{k=1}^{\infty} |v(a_n) - v(b_n)| \\ &\geq \left(\sum_{k=1}^{\infty} L^1\left(P_{\langle\phi_i^\perp\rangle}(I_n)\right)\right) - \sqrt{\alpha}\left(\sum_{k=1}^{\infty} H^1(I_n)\right). \end{aligned} \tag{60}$$

Now from (46)

$$L^1\left(P_{\langle\phi_i^\perp\rangle}\left(v^{-1}(\eta)\setminus(\cup_n I_n)\right)\right) < \tilde{\sigma}^2\sqrt{\alpha}\,|b-e|,$$

and thus

$$L^1\left(P_{\langle\phi_i^\perp\rangle}(\cup_n I_n)\right) > L^1\left(P_{\langle\phi_i^\perp\rangle}\left(v^{-1}(\eta)\right)\right) - \tilde{\sigma}^2\sqrt{\alpha}\,|b-e|.$$

So inserting this into equation (60) we have

$$\begin{aligned} |e-b| &= L^1\left(P_{\langle\phi_i^\perp\rangle}\left(v^{-1}(\eta)\right)\right) - \tilde{\sigma}^2\sqrt{\alpha}\,|b-e| - \sqrt{\alpha}H^1\left(v^{-1}(\eta)\right) \\ &\geq |\Psi_i(e) - \Psi_i(s)| - 2\tilde{\sigma}^2\sqrt{\alpha}\,|b-e|. \end{aligned}$$

$\square$

5.2.1. *Forcing integral curves into straight lines.*
This coming lemma is elementary. If we have the conditions to invoke both Lemma 3 and Lemma 4 then (as indicated in section 4.2 of the introduction) we get sufficiently strict bounds from above on the length of the curves and bounds from below on the distance between the end points that we are able to force the integral curves to run in straight lines.

Lemma 5. *Let* $v \in \mathcal{A}_F(\Omega)$.
Given skew rectangular region $R = \mathfrak{F}(a, w\phi_2, r\phi_1)$ *where* $\frac{w}{r} < \sigma^2$. *We assume* $N_{\frac{w}{\sigma^2}}(R) \subset \Omega$.
Define $\Theta_i^a : R \to \mathbb{R}$ *as in Lemma 2.*
Suppose for some $t \in (a + \langle\phi_i\rangle) \cap R$ *we have*

$$\int_{\Theta_i^{a\,-1}(t)} J(x)\, dH^1 x \leq \alpha w \tag{61}$$

then let s be the first point of $\Theta_i^{a\ -1}(t)$ (going backwards in time) to hit ∂R and e be the first point (going forwards) to hit ∂R. Let U denote the connected component of $\Theta_i^{a\ -1}(t)$ between s and e, then firstly by Lemma 3

$$\left(1-\frac{\mathfrak{c}_1\alpha}{\sigma^4}\right)|\Psi_i(e)-\Psi_i(s)| \leq H^1(U) \leq \left(1+\frac{\mathfrak{c}_1\alpha}{\sigma^4}\right)|\Psi_i(e)-\Psi_i(s)|.$$

If in addition we have

$$\int_{N_{\frac{w}{\sigma^2}}(R)} T(x)\,dL^2x \leq \alpha^3 w^2 \tag{62}$$

then

$$\Theta_i^{a\ -1}(t)\cap R \in N_{\mathfrak{c}_3\sqrt{\alpha}w}(l_t)$$

for $l_t := t+l$, $l \in G(1,2)$.

Proof. So U is the connected component of $\Theta_a^{i\ -1}(t_0)\cap v(S)$ between s and e. We will show U has to lie very close to $[s,e]$.

Let $w_0 := \sup\left\{\left|P_{(e-s)^\perp}(e-z)\right| : z \in U\right\}$ and let $\varpi \in U$ be the point such that

$$\left|P_{(e-s)^\perp}(e-\varpi)\right| = w_0.$$

From figure 7, it should be clear that $H^1(U) \geq \sqrt{\left(|e-s|^2+w_0^2\right)}$.

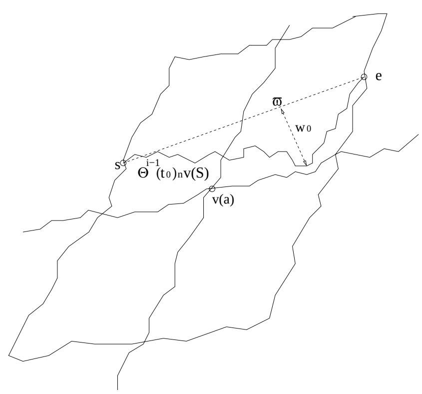

Figure 7

Now from (61), by Lemma 3 we have

$$H^1(U) \leq |\Psi_i(e)-\Psi_i(s)|\left(1+\frac{\mathfrak{c}_1\alpha}{\sigma^4}\right). \tag{63}$$

By Fubini we know that for some $p \in \left(s+\langle(e-s)^\perp\rangle\right)\cap B_{\alpha w}(s)$ we have

$$\int_{(p+\langle s-e\rangle)\cap v(S)} J(x)\,dH^1x \leq \sigma^{-2}\alpha^2 w.$$

So let $q \in (p+\langle s-e\rangle)\cap B_{\alpha w}(e)$. By Lemma 4 we have

$$|p-q| \geq |\Psi_i(p)-\Psi_i(q)|\left(1-2\sigma^{-6}\alpha\right).$$

and by bilipshitness and the fact that $|e-s| > c_1 w$ this gives

$$|s-e| \geq |\Psi_i(s)-\Psi_i(e)|\left(1-4\sigma^{-6}\alpha\right). \tag{64}$$

So by (63) and (64) we have

$$
\begin{aligned}
& w_0^2 + \left(\left|\Psi_i(s) - \Psi_i(e)\right| \left(1 - 4\sigma^{-6}\alpha\right)\right)^2 \\
& \quad \leq |e - s|^2 + w_0^2 \\
& \quad \leq \left(H^1(U)\right)^2 \\
& \quad \leq \left(\left|\Psi_i(s) - \Psi_i(e)\right| \left(1 + \frac{\mathfrak{c}_1 \alpha}{\sigma^4}\right)\right)^2 \\
& \quad \leq \left|\Psi_i(s) - \Psi_i(e)\right|^2 + 2\frac{\mathfrak{c}_1 \alpha}{\sigma^4}\epsilon \left|\Psi_i(s) - \Psi_i(e)\right|^2 + \left(\frac{\mathfrak{c}_1 \alpha}{\sigma^4} \left|\Psi_i(s) - \Psi_i(e)\right|\right)^2.
\end{aligned}
$$

So

$$\frac{8\mathfrak{c}_1 \alpha}{\sigma^{12}} \left|\Psi_i(e) - \Psi_i(s)\right|^2 \geq w_0^2.$$

Thus

$$w_0 \leq \mathfrak{c}_3 \sqrt{\alpha} w. \tag{65}$$

So let $\Gamma_0 := t_0 + \langle s - e \rangle$, (65) implies

$$U \subset N_{\mathfrak{c}_3 \sqrt{\alpha} w}(\Gamma_0) \tag{66}$$

and this concludes the proof of the Lemma.

□

6. Fundamental Lemmas

6.1. **Precise control on the pullbacks of integral curves that form straight lines.**

This coming Lemma is fundamental. We know from Lemma 5 that when we have the conditions to invoke Lemma 3 and Lemma 4 the integral curves are forced into something like straight lines. In this lemma we obtain very precise information about the pull back of such integral curves, we show that they are in effect, very much like the pullback of straight lines in the image of laminates. The proof is heuristically quite similar to the way we analyzed the the pullback of the straightline in the laminate in section 4.1 of the introduction.

Lemma 6. *Let $v \in \mathcal{A}_F(\Omega)$. Let $i \in \{1,2\}$.*

Given skew rectangular region $R := \mathfrak{F}(a, w\phi_2, r\phi_1)$ where $\frac{w}{r} < \sigma^2$. We assume $N_{\frac{w}{\sigma^2}(R)} \subset \Omega$. Let a be the central point of R and define $\Theta_i^a : R \to \mathbb{R}$ as in Lemma 2. Let $\alpha > 0$ be a sufficiently small number.

Suppose for some $t \in (a + \langle \phi_i \rangle) \cap B_{\frac{w}{\sigma^4}}(a)$ we have

$$\int_{\Theta_i^{a\ -1}(t) \cap R} J(x)\, dH^1 x \leq \alpha w. \tag{67}$$

And we have the bulk energy estimate

$$\int_{P\left(a,\phi_1,\phi_2,\frac{w}{\sigma^2}\right)} d\left(Dv(x), SO(2) \cup SO(2)H\right) dL^2 x \leq \alpha^3 w^2 \tag{68}$$

then let s be the first point of $\Theta_i^{a\ -1}(t)$ (going backwards in time) to hit ∂R and e be the first point (going forwards) to hit ∂R. Let U denote the connected component of $\Theta_i^{a\ -1}(t)$ between s and e, then the following statement holds true.

Firstly recall by Lemma 5 and Lemma 3 we have

- $$\left(1 - \frac{\mathfrak{c}_1 \alpha}{\sigma^4}\right) |\Psi_i(s) - \Psi_i(e)| \leq H^1(U) \leq \left(1 + \frac{\mathfrak{c}_1 \alpha}{\sigma^4}\right) |\Psi_i(s) - \Psi_i(e)|. \tag{69}$$
- *For some $l_t := l + v(t)$, $l \in G(1,2)$ we have*
$$U \in N_{\mathfrak{c}_3 \sqrt{\alpha} w}(l_t). \tag{70}$$

We define the clockwise normal w_t, to l_t as follows. If we let $\vartheta \in l_t \cap B_{\sqrt{\alpha} w}(e)$ and $\xi \in l_t \cap B_{\sqrt{\alpha} w}(s)$ then we define w_t to be the clockwise normal to vector $\frac{\vartheta - \xi}{|\vartheta - \xi|}$.

We will prove that.

Given $\mathbb{A}_t := N_{\sqrt{\alpha} w}(l_t) \cap R$. There exists a set $\mathbb{B}_t \subset \mathbb{A}_t$ with the following properties; $L^2(\mathbb{A}_t \backslash \mathbb{B}_t) \leq \tilde{\mathfrak{c}}_4 \alpha^{\frac{5}{8}} w^2$ and for any $x \in \mathbb{B}_t$ we have

$$\left| Dv\left(v^{-1}(x)\right) \phi_1 - w_t \right| < \mathfrak{c}_4 \sqrt{\alpha} \tag{71}$$

Proof. So to begin with, as noted in the statement, the first part is just from Lemma 5 and Lemma 3. Now by Fubini, the area formula and assumption (68) we can find the existence of a set

$$C \subset \left(l^{\perp} + v(t)\right) \cap B_{\sqrt{\alpha} w}(t) \text{ with } L^1(C) > \left(1 - \sigma^{-2}\sqrt{\alpha}\right) 2\sqrt{\alpha} w \tag{72}$$

and the property that for any $\mathfrak{s} \in C$ we have

$$\int_{(l+\mathfrak{s}) \cap v(R)} J(x)\, dH^1 x < \alpha w. \tag{73}$$

Now as $\Theta_i^{a\ -1}(\mathfrak{s}) \cap v(R)$ is connected (by Lemma 1) if we let η denote the connected component of $(l + \mathfrak{s}) \cap v(R)$ containing $\mathfrak{s}$, we will show that the endpoints of η must be within $\mathfrak{c}_3 \sigma^{-4} \sqrt{\alpha} w$ of the endpoints of $\Theta_i^{a\ -1}(s) \cap R$. Formally; let $\tilde{s}$ denote the endpoint of η closest to s, and let $\tilde{e}$ denote the endpoint closest to e. We will show

$$\tilde{e} \in N_{\mathfrak{c}_3 \sigma^{-4} \sqrt{\alpha} w}(e) \text{ and } \tilde{s} \in N_{\mathfrak{c}_3 \sigma^{-4} \sqrt{\alpha} w}(s). \tag{74}$$

To see this firstly note by bilipschitzness from (70) and by the fact that $\tilde{e}, \tilde{s} \in v(\partial R)$ we have

$$v^{-1}(\tilde{s}) \in N_{\frac{\mathfrak{c}_3 \sqrt{\alpha}}{\sigma^2}}\left(v^{-1}\left(\Theta_i^{a\ -1}(\mathfrak{s})\right)\right) \cap \partial R \text{ and } v^{-1}(\tilde{e}) \in N_{\frac{\mathfrak{c}_3 \sqrt{\alpha}}{\sigma^2}}\left(v^{-1}\left(\Theta_i^{a\ -1}(\mathfrak{s})\right)\right) \cap \partial R. \tag{75}$$

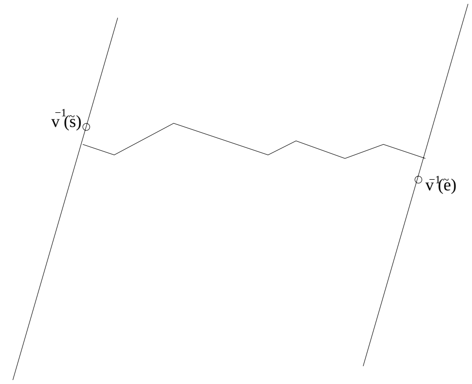

FIGURE 8

By Lemma 1 we know $v^{-1}\left(\Theta_i^{a\ -1}(\mathfrak{s})\right)$ only passes through each side of R once, so as is shown on fig 8, the end points of $v^{-1}\left(\Theta_i^{a\ -1}(\mathfrak{s})\right)\cap R$ are given by $v^{-1}(e)$, $v^{-1}(s)$. Formally $v^{-1}\left(\Theta_i^{a\ -1}(\mathfrak{s})\right)\cap\partial R=\left\{v^{-1}(e),v^{-1}(s)\right\}$ and so by (75) and the fact that s, $\tilde{s}$ are the (say) rightmost endpoints of $\Theta_i^{a\ -1}(\mathfrak{s})$, η respectively, e, $\tilde{e}$ are the (say) leftmost endpoints; by bilipschitzness we have (74).

Note that from (74) and (69) we have that

$$\begin{aligned} |\tilde{e}-\tilde{s}| &\leq H^1(U)+4\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}w \\ &\leq \left(1+8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right)|\Psi_i(\tilde{e})-\Psi_i(\tilde{s})|. \end{aligned} \tag{76}$$

And in the same way

$$|\tilde{e}-\tilde{s}|\geq\left(1-8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right)|\Psi_i(\tilde{e})-\Psi_i(\tilde{s})|. \tag{77}$$

It is also not hard to see that

$$L^1\left((l_{\mathfrak{s}}+v(\mathfrak{s}))\cap v(R)\setminus\eta\right)\leq c_1\sqrt{\alpha}w. \tag{78}$$

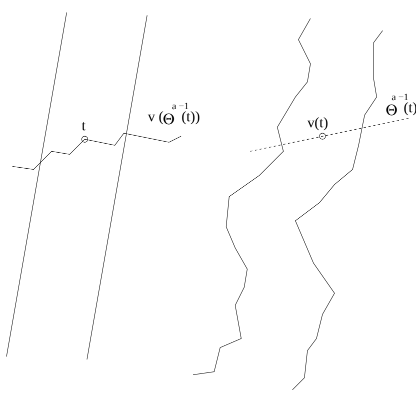

FIGURE 9

This essentially follows from fig 9. The reader who is already convinced is invited to skip the next paragraph.

By Lemma 1 we know that $v^{-1}\left(\Theta_i^{a\ -1}(\mathfrak{s})\right)$ must go through $v(R)$ at a definite angle. And so the line $l+v(\mathfrak{s})$ must cut through the boundary of $v(R)$ quite cleanly. Formally, if we let L denote the right hand side boundary of R then $\operatorname{diam}(v(L)\cap(l+v(t)))\leq c_1\sqrt{\alpha}w$ since otherwise by (70) we would be able to find points $y_1,y_2\in\Theta_i^{a\ -1}(\mathfrak{s})$ with $|y_1-y_2|>\frac{c_1}{2}\sqrt{\alpha}w$ and $d(y_1,v(L))<\sqrt{\alpha}w$, $d(y_2,v(L))<\sqrt{\alpha}w$ and by bilipschitzness, assuming constant c_1 is chosen big enough this contradicts Lemma 1.

Let

$$\mathbb{K}:=\left\{x\in\eta:d\left(Dv\left(v^{-1}(x)\right),SO(2)\right)<d\left(Dv\left(v^{-1}(x)\right),SO(2)H\right)\right\}$$

let

$$\mathbb{L} := \left\{x \in \eta : d\left(Dv\left(v^{-1}(x)\right), SO(2)H\right) < d\left(Dv\left(v^{-1}(x)\right), SO(2)\right)\right\}$$

and let

$$\mathbb{E} := \left\{x \in \eta : d\left(Dv\left(v^{-1}(x)\right), SO(2)H\right) = d\left(Dv\left(v^{-1}(x)\right), SO(2)\right)\right\}.$$

Now for some constant $c_2 := c(\sigma)$ we have

$$\begin{aligned} c_2 L^1(\mathbb{E}) &\leq \int_\eta d\left(Dv(x), SO(2) \cup SO(2)H\right) dL^1 x \\ &\leq \sqrt{\alpha} w. \end{aligned} \tag{79}$$

Which implies $L^1(\mathbb{E}) \leq c_2^{-1}\sqrt{\alpha}w$.

Now $\mathbb{K}$ is the countable union of connected subintervals $\{K_n\}$, ie. $\mathbb{K} = \bigcup_{n\in\mathbb{N}} K_n$ and similarly $\mathbb{L} = \bigcup_{n\in\mathbb{N}} L_n$.

Let

$$A_0 = \left\{n \in \mathbb{N} : \int_{K_n} d\left(Dv\left(v^{-1}(x)\right), SO(2)\right) dL^1 x \leq \sqrt{\alpha} H^1(K_n)\right\}$$

and let

$$B_0 = \left\{n \in \mathbb{N} : \int_{L_n} d\left(Dv\left(v^{-1}(x)\right), SO(2)H\right) dL^1 x \leq \sqrt{\alpha} H^1(L_n)\right\}.$$

As v is C^2 on the compact set Ω we know A_0 and B_0 are finite. We also know

$$\begin{aligned} \sum_{n\in A_0^c} \sqrt{\alpha} L^1(K_n) + \sum_{n\in B_0^c} \sqrt{\alpha} L^1(L_n) &\leq \sum_{n\in A_0^c} \int_{K_n} d\left(Dv\left(v^{-1}(x)\right), SO(2)\right) dL^1 x \\ &\quad + \sum_{n\in B_0^c} \int_{L_n} d\left(Dv\left(v^{-1}(x)\right), SO(2)H\right) dL^1 x \\ &\leq \int_{(l_{\mathfrak{s}}+\mathfrak{s})\cap v(R)} J(x)\, dH^1 x \\ &\leq \alpha w. \end{aligned}$$

So

$$\sum_{n\in A_0^c} L^1(K_n) + \sum_{n\in B_0^c} L^1(L_n) \leq \sqrt{\alpha} w. \tag{80}$$

We point out that from Lemma 3 we have

$$\left(1 - \sigma^{-4}\mathfrak{c}_1\sqrt{\alpha}\right) L^1\left(P_{\phi_i^\perp}(I_k)\right) \leq L^1(v(I_k)) \leq \left(1 + \sigma^{-4}\mathfrak{c}_1\sqrt{\alpha}\right) L^1\left(P_{\phi_i^\perp}(I_k)\right). \tag{81}$$

Let $n_1 = \mathrm{Card}(A_0)$ and $m_1 = \mathrm{Card}(B_0)$. Let $\{I_k : k = 1, \ldots n_1 + m_1\}$ be a reordering of the set $\{K_j : j \in A_0\} \cup \{L_k : k \in B_0\}$ so that I_1 is the rightmost interval. I_2 the second rightmost interval, ectra.

Note from from the fact that $\eta = \mathbb{K} \cup \mathbb{L} \cup \mathbb{E}$, (80), (79) and bilipschitzness we have

$$H^1\left(v^{-1}(\eta) \setminus \bigcup_{k=1}^{n_1+m_1} I_k\right) \leq 2c_2^{-1}\sigma^{-2}\sqrt{\alpha} w. \tag{82}$$

Let $m_2 := m_1 + n_1$. Now we consider point $v^{-1}(\tilde{s})$ to be the start of the path $v^{-1}(\eta)$ and point $v^{-1}(\tilde{e})$ to be the end. So going from right to left in this way; we denote the first point of segment I_k by α_k and the last point β_k. Let $d_k = \beta_k - \alpha_k$ for $k = 1, 2 \ldots m_2$. See figure 10.

Let $e_0 = \alpha_1 - v^{-1}(\tilde{s})$ and $e_{m_2} = v^{-1}(\tilde{e}) - \beta_{m_2}$. Also let $e_k = \alpha_{k+1} - \beta_k$ for $k \in \{1, 2, \ldots m_2 - 1\}$. So we have

$$v^{-1}(\tilde{e}) - v^{-1}(\tilde{s}) = \sum_{k=1}^{m_2} d_k + \sum_{j=0}^{m_2} e_j. \tag{83}$$

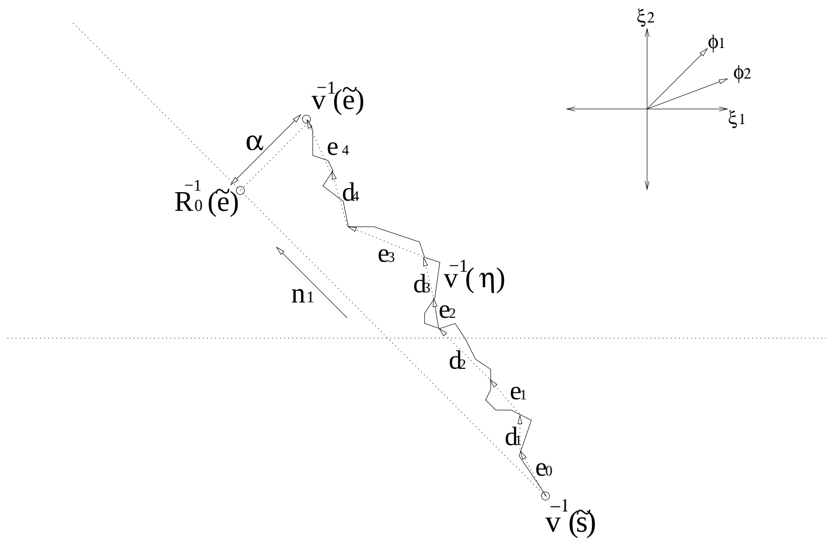

FIGURE 10

Let E_k be the subsegment of $v^{-1}(\eta)$ between β_k and α_{k+1} for $k = 1, 2 \ldots m_2 - 1$. Let E_0 be the subsegment between $v^{-1}(\tilde{s})$ and α_1 and let E_{m_2} be the subsegment between β_{m_2} and $v^{-1}(\tilde{e})$. So we know

$$\bigcup_{j=1}^{m_2} I_j \cup \bigcup_{j=0}^{m_2} E_j = v^{-1}(\eta) \tag{84}$$

and thus from (82) and (84) we have

$$\begin{aligned} \sum_{j=0}^{m_2} |e_j| &\leq \sum_{j=0}^{m_2} H^1(E_j) \\ &\leq 2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w. \end{aligned} \tag{85}$$

Now let $\mathcal{K}_1 = \{k \in \{1, \ldots m_2\} : d(Dv(x), SO(2)H) < d(Dv(x), SO(2)) \;\; \forall x \in I_k\}$ and let $\mathcal{K}_2 = \{1, \ldots m_2\} \backslash \mathcal{K}_1$. For subsegment I_k for $k \in \mathcal{K}_1$ we have that

$$\begin{aligned} \int_{I_k} |Dv(z) \cdot t_z| \, dH^1 z &\geq \int_{I_k} (1 - 2d(Dv(x), SO(2)H)) |Ht_z| \, dH^1 z \\ &\geq |Hd_k| - 2\sigma^{-2}\sqrt{\alpha} H^1(I_k) \\ &\geq \left(1 - 2\sigma^{-4}\sqrt{\alpha}\right) |Hd_k|. \end{aligned} \tag{86}$$

And for subsegment I_k with $k \in \mathcal{K}_2$ we have

$$\int_{I_k} |Dv(z) \cdot t_z| \, dH^1 z \geq \left(1 - 2\sigma^{-4}\sqrt{\alpha}\right) |d_k|. \tag{87}$$

So let $\tilde{d} = \sum_{k \in \mathcal{K}_2} d_k$ and $\tilde{u} = \sum_{k \in \mathcal{K}_1} d_k$, so since $\{1, 2, \ldots m_2\} = \mathcal{K}_1 \cup \mathcal{K}_2$ and from (85) and (83) we have

$$\left| \left(v^{-1}(\tilde{e}) - v^{-1}(\tilde{s})\right) - \left(\tilde{d} + \tilde{u}\right) \right| \leq 2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w. \tag{88}$$

So from (86) and (87) we also have

$$\begin{aligned}|\tilde{e}-\tilde{s}| &= \int_{v^{-1}(\eta)} |Dv(z)\,t_z|\,dH^1 z \\ &\geq \sum_{k=1}^{m_2}\int_{I_k} |Dv(z)\,t_z|\,dH^1 z \\ &\geq \sum_{k\in\mathcal{K}_2}\left(1-2\sigma^{-4}\sqrt{\alpha}\right)|d_k| + \sum_{k\in\mathcal{K}_1}\left(1-2\sigma^{-4}\sqrt{\alpha}\right)|Hd_k| \\ &\geq \left(1-2\sigma^{-4}\sqrt{\alpha}\right)\left(\left|\tilde{d}\right| + |H\tilde{u}|\right). \end{aligned} \tag{89}$$

Letting $\tilde{e}=\sum_{j=0}^{m_2} e_j$ from (83) we have $v^{-1}(\tilde{e})-v^{-1}(\tilde{s})=\tilde{d}+\tilde{u}+\tilde{e}$ and so by (88) we know $|\tilde{e}|\leq 2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w$.

Step 1

The first thing we will show is that vector $\tilde{u}$ points in direction $\Diamond_1$ and vector $\tilde{d}$ points in direction n_1. Strictly speaking this is not necessary for the proof however as it will be of great physiological comfort to know that where $\tilde{u}$ and $\tilde{d}$ point and as it will serve as an introduction to the ideas we will use repeatedly we give the details.

Formally; we will show

$$\operatorname{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right) < c_4\alpha^{\frac{1}{8}} \tag{90}$$

and

$$\operatorname{ang}\left(\frac{\tilde{d}}{\left|\tilde{d}\right|},n_i\right) < c_4\alpha^{\frac{1}{8}}. \tag{91}$$

Now from (49) Lemma 4 we have that

$$|H\psi| \geq \psi\cdot n_i + \frac{c_2}{4}\left(\operatorname{ang}(\psi,\Diamond_i)\right)^2$$

for all $\psi\in\Xi_i$.

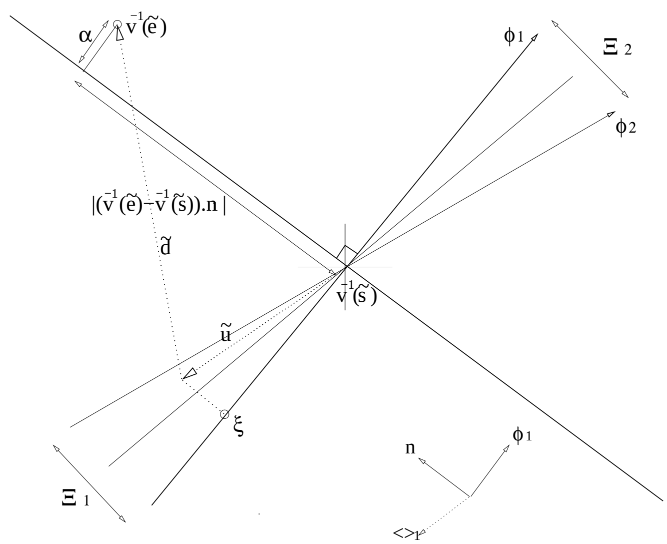

FIGURE 11

Its easy to see from fig 3 that $\frac{\tilde{u}}{|\tilde{u}|}\in\Xi_1$ since otherwise either $\tilde{u}$ is very small or

$$\left|\tilde{d}\right| + |H\tilde{u}| > \left|\tilde{d}\right| + |\tilde{u}| >> \left|v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right|$$

however by (89) this implies $|\tilde{e}-\tilde{s}| >> \left|v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right|$ and this contradicts (76).

So

$$|H\tilde{u}| = |\tilde{u}|\left|H\frac{\tilde{u}}{|\tilde{u}|}\right| > |\tilde{u}|\left(\frac{\tilde{u}\cdot n_i}{|\tilde{u}|}+\frac{\mathfrak{c}_2}{4}\left(\text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)\right)^2\right).$$

As can be easily seen from figure 11 we have

$$\left|\tilde{d}\right| \geq \left(v^{-1}(\tilde{e})-\left(v^{-1}(\tilde{s})+\tilde{u}\right)\right)\cdot n_i - 2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w. \tag{92}$$

So

$$\begin{aligned}
|H\tilde{u}|+\left|\tilde{d}\right| &\geq \tilde{u}\cdot n_i+\frac{\mathfrak{c}_2|\tilde{u}|}{4}\left(\text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)\right)^2+\left(v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right)\cdot n_i-\tilde{u}\cdot n_i-2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w \\
&= \left(v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right)\cdot n_i+\frac{\mathfrak{c}_2|\tilde{u}|}{4}\left(\text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)\right)^2-2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w. \qquad (93)
\end{aligned}$$

Hence by (76), (89), (93) we have

$$\begin{aligned}
8\sigma^{-4}\mathfrak{c}_3\sqrt{\alpha}w+\left(v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right)\cdot n_i &\geq H^1(\eta) \\
&\geq \left(1-2\sigma^{-4}\sqrt{\alpha}\right)\left(\left|\tilde{d}\right|+|H\tilde{u}|\right) \\
&\geq \left|\left(v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right)\cdot n_i\right|+\frac{\mathfrak{c}_2|\tilde{u}|}{4}\left(\text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)\right)^2-4c_2^{-1}\sigma^{-2}\sqrt{\alpha}w.
\end{aligned}$$

Thus

$$\frac{|\tilde{u}|}{4}\left(\text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)\right)^2 \leq c_3\sqrt{\alpha}w.$$

So either $|\tilde{u}|<\alpha^{\frac{1}{4}}w$ or we have

$$\begin{aligned}
c_3\sqrt{\alpha}w \geq \frac{\alpha^{\frac{1}{4}}w}{4}\left(\text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)\right)^2 \quad &\Longleftrightarrow \quad 4c_3\alpha^{\frac{1}{4}} \geq \left(\text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)\right)^2 \\
&\Longleftrightarrow \quad 2\sqrt{c_3}\alpha^{\frac{1}{8}} \geq \text{ang}\left(\frac{\tilde{u}}{|\tilde{u}|},\Diamond_i\right)
\end{aligned}$$

and so we have established (90).

Now we establish (91). To start, we know

$$|H\tilde{u}| \geq |\tilde{u}\cdot n_i| \tag{94}$$

from (48), Lemma 4. Now by (88)

$$\tilde{d}\cdot n_i+\tilde{u}\cdot n_i \geq \left(v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right)\cdot n_i-2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w \tag{95}$$

and $\left|\tilde{d}\right| = \sqrt{\left|\tilde{d}\cdot n_i\right|^2+\left|\tilde{d}\cdot\phi_i\right|^2}$ as we have seen before, since $\left|\tilde{d}\right|<1$ we have

$$\begin{aligned}
\sqrt{\left|\tilde{d}\cdot n_i\right|^2+\left|\tilde{d}\cdot\phi_i\right|^2}-\sqrt{\left|\tilde{d}\cdot n_i\right|^2} &= \int_{\left|\tilde{d}\cdot n_i\right|^2}^{\left|\tilde{d}\cdot n_i\right|^2+\left|\tilde{d}\cdot\phi_i\right|^2}\frac{x^{-\frac{1}{2}}}{2}dL^1x \\
&\geq \frac{\left|\tilde{d}\cdot\phi_i\right|^2}{2}. \qquad (96)
\end{aligned}$$

so putting (94), (95), (96) we get

$$\begin{aligned}
\left|\tilde{d}\right|+|H\tilde{u}| &\overset{(94),(96)}{\geq} \left|\tilde{d}\cdot n_i\right|+|\tilde{u}\cdot n_i|+\frac{\left|\tilde{d}\cdot\phi_i\right|^2}{2} \\
&\overset{(95)}{\geq} \left(v^{-1}(\tilde{e})-v^{-1}(\tilde{s})\right)\cdot n_i+\frac{\left|\tilde{d}\cdot\phi_i\right|^2}{2}-2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w. \qquad (97)
\end{aligned}$$

So putting (97) together with (76) (recall $\tilde{s}$,$\tilde{e}$ are the endpoints of η), (89) and (88) we get

$$\begin{aligned}\left(1+8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right)\left|\left(\left(v^{-1}\left(\tilde{e}\right)-v^{-1}\left(\tilde{s}\right)\right)\cdot n_i\right|\right. &\geq H^1\left(\eta\right)\\ &\geq \left(1-2\sigma^{-4}\sqrt{\alpha}\right)\left(\left|\tilde{d}\right|+\left|H\tilde{u}\right|\right)\\ &\geq \left(1-2\sigma^{-4}\sqrt{\alpha}\right)\left(\left(v^{-1}\left(\tilde{e}\right)-v^{-1}\left(\tilde{s}\right)\right)\cdot n_i-2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w\right)+\frac{\left|\tilde{d}\cdot\phi_i\right|^2}{4}.\end{aligned}$$

So

$$2c_2^{-1}\sigma^{-2}\sqrt{\alpha}w+10\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\left|\left(v^{-1}\left(\tilde{e}\right)-v^{-1}\left(\tilde{s}\right)\right)\cdot n_i\right|\geq\frac{\left|\tilde{d}\cdot\phi_i\right|^2}{4},$$

thus

$$\tilde{c}_4\sqrt{\alpha}w\geq\frac{\left|\tilde{d}\cdot\phi_i\right|^2}{4}=\frac{\left|\tilde{d}\right|^2}{4}\left|\frac{\tilde{d}}{\left|\tilde{d}\right|}\cdot\phi_i\right|^2.$$

So

$$\sqrt{\tilde{c}_4}\alpha^{\frac{1}{4}}\sqrt{w}\geq\frac{\left|\tilde{d}\right|}{2}\left|\frac{\tilde{d}}{\left|\tilde{d}\right|}\cdot\phi_i\right|$$

assuming $\left|\tilde{d}\right|\geq\alpha^{\frac{1}{8}}\sqrt{w}$ we have

$$\sqrt{\tilde{c}_4}\sigma^{-2}\alpha^{\frac{1}{8}}\geq\left|\frac{\tilde{d}}{\left|\tilde{d}\right|}\cdot\phi_i\right|$$

which establishes (91).

Step 2

Now we use similar arguments to establish that most of the subsegments $\{I_k : k\in\mathcal{K}_2\}$ lie roughly parallel to n_i and most of the subsegments $\{I_k : k\in\mathcal{K}_1\}$ lie roughly parallel to $\diamondsuit_i$.

Let $P_2 := \left\{k\in\mathcal{K}_2 : \text{ang}\left(\frac{d_k}{|d_k|},n_i\right)>\alpha^{\frac{1}{16}}\right\}$ and $P_1 := \left\{k\in\mathcal{K}_1 : \text{ang}\left(\frac{d_k}{|d_k|},\diamondsuit_i\right)>\alpha^{\frac{1}{16}}\right\}$. Firstly we will show that for any $k\in P_1\cup P_2$ we have

$$L^1\left(v\left(I_k\right)\right)\geq\left(1+\frac{\mathfrak{c}_2\alpha^{\frac{1}{8}}}{16}\right)L^1\left(P_{\langle\phi_i^{\perp}\rangle}\left(I_k\right)\right). \tag{98}$$

Look at case $k\in P_2$, see fig 12. Firstly, we know from Lemma 1 that $\text{ang}\left(\frac{d_k}{|d_k|},n_i\right)\leq\pi$. So

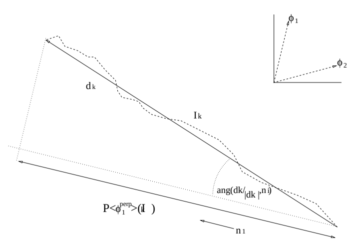

Figure 12

$$\begin{aligned}\left|d_k\right|^2 &\geq \left(L^1\left(P_{\phi_i^{\perp}}\left(I_k\right)\right)\right)^2+\left(\sin\text{ang}\left(\frac{d_k}{|d_k|},n_i\right)L^1\left(P_{\phi_i^{\perp}}\left(I_k\right)\right)\right)^2\\ &\geq \left(L^1\left(P_{\phi_i^{\perp}}\left(I_k\right)\right)\right)^2\left(1+\left(\sin\alpha^{\frac{1}{16}}\right)^2\right).\end{aligned} \tag{99}$$

Now as we have seen before, by considering the integral $\int_1^{1+\left(\sin\alpha^{\frac{1}{16}}\right)^2} \frac{x-\frac{1}{2}}{2} dL^1 x$

$$L^1\left(P_{\phi_i^{\perp}}(I_k)\right)\sqrt{\left(1+\left(\sin\alpha^{\frac{1}{16}}\right)^2\right)} \quad \geq \quad L^1\left(P_{\phi_i^{\perp}}(I_k)\right)\left(1+\frac{\left(\sin\alpha^{\frac{1}{16}}\right)^2}{4}\right) \tag{100}$$

And since $|\sin x - x| \leq \left(\sum_{n=3}^{\infty}\frac{1}{n!}\right)x^3$, assuming $\alpha > 0$ is small enough (in the applications of this lemma, α will be some power of κ or ϵ and so will indeed be small enough) we have $\left(\sin\alpha^{\frac{1}{16}}\right)^2 \geq \frac{\alpha^{\frac{1}{8}}}{2}$.

So putting this together with (99) and (100) we have

$$|d_k| \geq L^1\left(P_{\phi_i^{\perp}}(I_k)\right)\left(1+\frac{\alpha^{\frac{1}{8}}}{8}\right). \tag{101}$$

Now as

$$\int_{v(I_k)} d\left(Dv\left(v^{-1}(x)\right), SO(2)\right) dL^1 x \leq \sqrt{\alpha}L^1\left(v(I_k)\right),$$

we have that

$$\begin{aligned}\int_{I_k} d\left(Dv(z), SO(2)\right) dH^1 z &\leq \sigma^{-2}\sqrt{\alpha}L^1\left(v(I_k)\right)\\ &\leq \sigma^{-4}\sqrt{\alpha}H^1(I_k).\end{aligned} \tag{102}$$

For each $z \in I_k$ let $R(z) \in SO(2)$ be such that $|Dv(x) - R(z)| = d(Dv(z), SO(2))$. So by (101) and (102) we have

$$\begin{aligned}L^1\left(v(I_k)\right) &= \int_{I_k}|Dv(z)\,t(z)|\,dH^1 z\\ &\geq \int_{I_k}|R(z)\,t(z)|\,dH^1 z - \int_{I_k} d\left(Dv(z), SO(2)\right) dH^1 z\\ &\geq H^1(I_k)\left(1-\sigma^{-4}\sqrt{\alpha}\right)\\ &\geq |d_k|\left(1-\sigma^{-4}\sqrt{\alpha}\right)\\ &\geq L^1\left(P_{\phi_1^{\perp}}(I_k)\right)\left(1+\frac{\alpha^{\frac{1}{8}}}{8}\right)\left(1-\sigma^{-4}\sqrt{\alpha}\right)\\ &\geq L^1\left(P_{\phi_1^{\perp}}(I_k)\right)\left(1+\frac{\alpha^{\frac{1}{8}}}{16}\right)\end{aligned}$$

for α small enough. This establishes the claim in the case $k \in P_1$.

Let $k \in P_2$. Observe the fig 13.

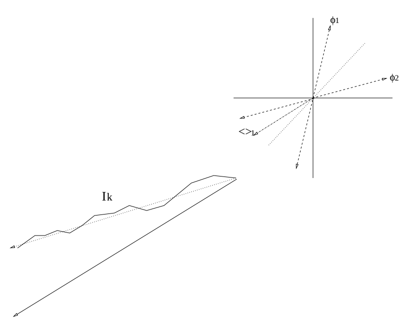

FIGURE 13

As before letting $T(z) \in SO(2)H$ be such that $|Dv(z) - T(z)| = d(Dv(z), SO(2)H)$ we have from (102)

$$\begin{aligned} L^1(v(I_k)) &= \int_{I_k} |Dv(z)t(z)|\, dH^1 z \\ &\geq \int_{I_k} |T(z)t(z)|\, dH^1 z - \int_{I_k} d(Dv(z), SO(2)H)\, dH^1 z \\ &\geq \int_{I_k} |T(z)t(z)|\, dH^1 z - \sigma^{-4}\sqrt{\alpha} H^1(I_k). \end{aligned} \tag{103}$$

And we know

$$\begin{aligned} \int_{I_k} |T(z)t(z)|\, dH^1 z &= \int_{I_k} |Ht(z)|\, dH^1 z \\ &\geq |d_k| \left| H\frac{d_k}{|d_k|} \right|. \end{aligned} \tag{104}$$

Now by (48) Lemma 4 we know

$$\begin{aligned} \left| H\frac{d_k}{|d_k|} \right| &\geq \frac{d_k}{|d_k|} \cdot n_i + \frac{\mathfrak{c}_2}{4} \left(\operatorname{ang}\left(\frac{d_k}{|d_k|}, \Diamond_i \right) \right)^2 \\ &\geq \frac{d_k}{|d_k|} \cdot n_i + \frac{\mathfrak{c}_2 \alpha^{\frac{1}{8}}}{4}. \end{aligned} \tag{105}$$

Putting together (103),(104) and (105) gives

$$L^1(v(I_k)) \geq d_k \cdot n_i + \frac{\mathfrak{c}_2 |d_k|}{4} \alpha^{\frac{1}{8}} - \sigma^{-4}\sqrt{\alpha} H^1(I_k)$$

Now $d_k \cdot n_i = L^1\left(P_{\phi_i^\perp}(I_k)\right)$ and so $|d_k| \geq L^1\left(P_{\phi_i^\perp}(I_k)\right)$ and by Lemma 1 we have $H^1(I_k) \leq \sigma^{-4} L^1\left(P_{\phi_i^\perp}(I_k)\right)$.
So

$$\begin{aligned} L^1(v(I_k)) &\geq L^1\left(P_{\phi_i^\perp}(I_k)\right)\left(1 + \frac{\mathfrak{c}_2\alpha^{\frac{1}{8}}}{4}\right) - \sigma^{-4}\sqrt{\alpha} H^1(I_k) \\ &\geq L^1\left(P_{\phi_i^\perp}(I_k)\right)\left(1 + \frac{\mathfrak{c}_2\alpha^{\frac{1}{8}}}{8}\right) \end{aligned}$$

for small enough α, hence we have established (98).
Step 3 In this step we bound the cardinality of P_1 and P_2.
Recall by (76) and (77)

$$|\tilde{s} - \tilde{e}| \in \left(L^1\left(P_{\phi_i^\perp}\left(v^{-1}([\tilde{s}, \tilde{e}]) \right) \right)\left(1 - 8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right), L^1\left(P_{\phi_i^\perp}\left(v^{-1}([\tilde{s}, \tilde{e}]) \right) \right)\left(1 + 8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right) \right) \tag{106}$$

and from (82)

$$L^1\left([\tilde{s}, \tilde{e}] \setminus \bigcup_{k \in \mathcal{K}_1 \cup \mathcal{K}_2} v(I_k) \right) \leq 2c_2^{-1}\sigma^{-4}\sqrt{\alpha}\, |\tilde{s} - \tilde{e}|. \tag{107}$$

Now using (106), (81) and (98) we have

$$\begin{aligned}\left(1+8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right)L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right) &\geq |\tilde{s}-\tilde{e}| \\ &\geq \sum_{k\in\mathcal{K}_1\cup\mathcal{K}_2} L^1\left(v\left(I_k\right)\right) \\ &\geq \sum_{k\in P_1\cup P_2} L^1\left(v\left(I_k\right)\right)+\sum_{k\in(\mathcal{K}_1\cup\mathcal{K}_2)\backslash(P_1\cup P_2)} L^1\left(v\left(I_k\right)\right) \\ &\geq \left(1+\frac{\mathfrak{c}_2\alpha^{\frac{1}{8}}}{16}\right)\sum_{k\in P_1\cup P_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\ &\quad +\left(1-\sigma^{-4}\mathfrak{c}_1\sqrt{\alpha}\right)\sum_{k\in(\mathcal{K}_1\cup\mathcal{K}_2)\backslash(P_1\cup P_2)} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\ &\geq \sum_{k\in\mathcal{K}_1\cup\mathcal{K}_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right)+\sum_{k\in P_1\cup P_2}\frac{\mathfrak{c}_2\alpha^{\frac{1}{8}}}{16}L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\ &\quad -\sigma^{-4}\mathfrak{c}_1\sqrt{\alpha}\left|\tilde{s}-\tilde{e}\right|. \end{aligned} \tag{108}$$

Now from (82) we have

$$L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\backslash\bigcup_{k\in\mathcal{K}_1\cup\mathcal{K}_2} I_k\right)\right)\leq 4c_2^{-1}\sigma^{-2}\sqrt{\alpha}\left|\tilde{s}-\tilde{e}\right|. \tag{109}$$

Putting this together with (108) we get

$$\begin{aligned}\left(1+8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right)L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right) &\geq L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right)-4c_2^{-1}\sigma^{-2}\sqrt{\alpha}\left|\tilde{s}-\tilde{e}\right| \\ &\quad +\sum_{k\in P_1\cup P_2}\frac{\mathfrak{c}_2\alpha^{\frac{1}{8}}}{8}L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right)-\sigma^{-4}\sqrt{\alpha}\left|\tilde{s}-\tilde{e}\right|.\end{aligned}$$

Which implies

$$\sum_{k\in P_1\cup P_2}\alpha^{\frac{1}{8}}L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right)\leq c_5\sqrt{\alpha}\left|\tilde{s}-\tilde{e}\right|.$$

So by bilipschitzness and Lemma 1 we have

$$\begin{aligned}\sigma^{-2}c_5\alpha^{\frac{3}{8}}L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right) &\geq c_5\alpha^{\frac{3}{8}}\left|\tilde{s}-\tilde{e}\right| \\ &\geq \sum_{k\in P_1\cup P_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\ &\geq \sigma^2\sum_{k\in P_1\cup P_2} H^1\left(I_k\right).\end{aligned} \tag{110}$$

By Lemma 1 this gives us a bound on the cardinality of $P_1\cup P_2$ of the following form

$$\sigma^{-6}c_5\alpha^{\frac{3}{8}}H^1\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\geq\sum_{k\in P_1\cup P_2} H^1\left(I_k\right). \tag{111}$$

As a consequence by (82) we have

$$\begin{aligned}H^1\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\backslash\bigcup_{k\in\mathcal{K}_1\cup\mathcal{K}_2\backslash(P_1\cup P_2)} I_k\right) &\leq H^1\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\backslash\bigcup_{k\in\mathcal{K}_1\cup\mathcal{K}_2} I_k\right)+H^1\left(\bigcup_{k\in P_1\cup P_2} I_k\right) \\ &\leq 2c_5\sigma^{-6}\alpha^{\frac{3}{8}}H^1\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right).\end{aligned} \tag{112}$$

<u>Step 4</u>

Now let

$$O_2 := \left\{ \begin{array}{ll} k \in \mathcal{K}_2 \backslash P_2 : & \exists\, J_k \subset I_k \text{ with } H^1(J_k) \geq \left(1 - \alpha^{\frac{1}{8}}\right) H^1(I_k) \\ & \text{and for each } z \in J_k, \operatorname{ang}(t(z), n_i) > \alpha^{\frac{1}{32}} \end{array} \right\} \tag{113}$$

and let

$$O_1 := \left\{ \begin{array}{ll} k \in \mathcal{K}_1 \backslash P_1 : & \exists\, J_k \subset I_k \text{ with } H^1(J_k) \geq \left(1 - \alpha^{\frac{1}{8}}\right) H^1(I_k) \\ & \text{and for each } z \in J_k, \operatorname{ang}(t(z), \lozenge_i) > \alpha^{\frac{1}{32}} \end{array} \right\} \tag{114}$$

We will show that for $k \in O_1 \cup O_2$

$$L^1(v(I_k)) \geq \left(1 + \frac{\mathfrak{c}_2 \alpha^{\frac{1}{16}}}{8}\right) L^1\left(P_{\phi_i^\perp}(I_k)\right). \tag{115}$$

First we consider the case $k \in O_1$ because its more intricate.

As before letting $t(z)$ be the tangent to I_k at z and letting $U(z) \in SO(2)H$ be such that $d(Dv(z), SO(2)H) = |Dv(z) - U(z)|$ we have by definition of the set $\{I_k : k = 1, \dots m_0\}$ that

$$\int_{I_k} d(Dv(z), SO(2)H)\, dH^1 z \leq \sqrt{\alpha} H^1(I_k).$$

Now using again (48) from Lemma 4 and for the final inequality using Lemma 1 we have

$$\begin{aligned} L^1(v(I_k)) &= \int_{I_k} |Dv(z)\, t(z)|\, dH^1 z \\ &\geq \int_{I_k} |Ht(z)|\, dH^1 z - \sqrt{\alpha} H^1(I_k) \\ &\geq \int_{J_k} |Ht(z)|\, dH^1 z + \int_{I_k \backslash J_k} |Ht(z)|\, dH^1 z - \sqrt{\alpha} H^1(I_k) \\ &\geq L^1\left(P_{\phi_i^\perp}(J_k)\right)\left(1 + \frac{\mathfrak{c}_2 \alpha^{\frac{1}{16}}}{4}\right) - \sqrt{\alpha} H^1(I_k) \\ &\geq L^1\left(P_{\phi_i^\perp}(I_k)\right)\left(1 + \frac{\mathfrak{c}_2 \alpha^{\frac{1}{16}}}{4}\right) - 4\alpha^{\frac{1}{8}} H^1(I_k) \\ &\geq L^1\left(P_{\phi_i^\perp}(I_k)\right)\left(1 + \frac{\mathfrak{c}_2 \alpha^{\frac{1}{16}}}{8}\right). \end{aligned} \tag{116}$$

The case where $k \in O_1$ can be argued with a simple Pythagoras type argument. We do not go into the details.

<u>Step 5</u>

Now in the same way as we showed the cardinality of P_1 and P_2 are bounded, we will show the cardinality of O_1 and O_2 are bounded. The reader who is already convinced is invited to skip to Step 6.

Using (81), (112), (106) and (115) we have

$$\begin{aligned}
\left(1+8\mathfrak{c}_3\sigma^{-4}\sqrt{\alpha}\right)L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right) &\overset{(106)}{\geq} |\tilde{s}-\tilde{e}| \\
&\geq \sum_{k\in(\mathcal{K}_1\cup\mathcal{K}_2\backslash(P_1\cup P_2\cup O_1\cup O_2))} L^1\left(v\left(I_k\right)\right)+\sum_{k\in O_1\cup O_2} L^1\left(v\left(I_k\right)\right) \\
&\overset{(81),(115)}{\geq} \left(1-\sigma^{-4}\mathfrak{c}_1\sqrt{\alpha}\right)\sum_{k\in(\mathcal{K}_1\cup\mathcal{K}_2\backslash(P_1\cup P_2\cup O_1\cup O_2))} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\
&\quad +\left(1+\frac{\mathfrak{c}_2\alpha^{\frac{1}{16}}}{8}\right)\sum_{k\in O_1\cup O_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\
&= \sum_{k\in(\mathcal{K}_1\cup\mathcal{K}_2)\backslash(P_1\cup P_2)} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right)+\frac{\mathfrak{c}_2\alpha^{\frac{1}{16}}}{8}\sum_{k\in O_1\cup O_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\
&\quad -\sigma^{-4}\mathfrak{c}_1\sqrt{\alpha}L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right) \\
&\overset{(112)}{\geq} L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right)+\frac{\mathfrak{c}_2\alpha^{\frac{1}{16}}}{8}\sum_{k\in O_1\cup O_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\
&\quad -\left(\mathfrak{c}_1\sqrt{\alpha}\sigma^{-4}-2c_5\sigma^{-6}\alpha^{\frac{3}{8}}\right)L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right).
\end{aligned}$$

Putting things together we have

$$\frac{\mathfrak{c}_2\alpha^{\frac{1}{16}}}{8}\sum_{k\in O_1\cup O_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right)\leq c_6\alpha^{\frac{3}{8}}L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right)$$

so by Lemma 1

$$\begin{aligned}
\sum_{k\in O_1\cup O_2} H^1\left(I_k\right) &\leq \sigma^{-2}\sum_{k\in O_1\cup O_2} L^1\left(P_{\phi_i^\perp}\left(I_k\right)\right) \\
&\leq 8\sigma^{-2}\mathfrak{c}_2^{-1}c_6\alpha^{\frac{1}{4}}L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right). \qquad (117)
\end{aligned}$$

<u>Step 6</u>

Claim 1. We will show the existence of a set $Y\subset[\tilde{s},\tilde{e}]$ with $L^1\left([\tilde{s},\tilde{e}]\backslash Y\right)\leq\left(1-c_7\alpha^{\frac{1}{8}}\right)|\tilde{s}-\tilde{e}|$ for which if we let R^* be the clockwise rotation by $\frac{\pi}{2}$ and let $w_t:=R^*\left(\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}\right)$ for any $z\in Y$ we have

$$Dv\left(v^{-1}\left(x\right)\right)\phi_i\in N_{c_8\sqrt{\alpha}}\left(w_t\right). \qquad (118)$$

Proof of Claim:

Let $D:=\left(\mathcal{K}_1\backslash\left(P_1\cup O_1\right)\right)\cup\left(\mathcal{K}_2\backslash\left(P_2\cup O_2\right)\right)$. For any $k\in D\cap\mathcal{K}_1$ (by definition (114)) we have

$$H^1\left(\left\{x\in I_k:\operatorname{ang}\left(t\left(z\right),\Diamond_i\right)>\alpha^{\frac{1}{32}}\right\}\right)\leq\alpha^{\frac{1}{8}}H^1\left(I_k\right)$$

so $W_k:=\left\{x\in I_k:\operatorname{ang}\left(t\left(z\right),\Diamond_i\right)\leq\alpha^{\frac{1}{32}}\right\}$ is such that $H^1\left(W_k\right)\geq\left(1-\alpha^{\frac{1}{8}}\right)H^1\left(I_k\right)$.

Similarly, for any $k\in D\cap\mathcal{K}_2$ (by definition (113)), $W_k:=\left\{x\in I_k:\operatorname{ang}\left(t\left(z\right),n_i\right)\leq\alpha^{\frac{1}{32}}\right\}$ is such that

$$H^1\left(W_k\right)\geq\left(1-\alpha^{\frac{1}{8}}\right)H^1\left(I_k\right).$$

So from (111) and (117) we have

$$\sum_{k\in P_1\cup P_2\cup O_1\cup O_2} H^1\left(I_k\right)\leq 9\sigma^{-6}\mathfrak{c}_2^{-1}c_6\alpha^{\frac{1}{4}}L^1\left(P_{\phi_i^\perp}\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right)\right)$$

so from the definition of D and by (82) and Lemma 1 we have

$$H^1\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\setminus\bigcup_{k\in D}I_k\right)\leq 11\sigma^{-6}\mathfrak{c}_2^{-1}c_2^{-1}c_6\alpha^{\frac{1}{4}}H^1\left(v^{-1}\left([\tilde{s},\tilde{e}]\right)\right). \tag{119}$$

Note from (73) we have

$$\int_{v^{-1}(\eta)}d\left(Dv\left(z\right),SO\left(2\right)\cup SO\left(2\right)H\right)dH^1x\leq\sigma^{-2}\alpha^2w.$$

Take $z\in\left(\bigcup_{k\in D}W_k\right)\cap\left\{z\in v^{-1}\left(\eta\right):d\left(Dv\left(z\right),SO\left(2\right)\cup SO\left(2\right)H\right)<\alpha^{\frac{1}{2}}\right\}$. We have that either $z\in\bigcup_{k\in D\cap\mathcal{K}_1}W_k$ or $z\in\bigcup_{k\in D\cap\mathcal{K}_2}W_k$. Supposing the later, then

$$\text{ang}\left(t\left(z\right),n_i\right)\leq\alpha^{\frac{1}{32}}. \tag{120}$$

Now as we have already calculated (Lemma 1, (14)) $t\left(z\right):=\left[S^{-1}\left(z\right)S^{-1}\left(z\right)\right]n_i$ where $Dv\left(z\right):=R\left(z\right)S\left(z\right)$ is the polar decomposition of the matrix $Dv\left(z\right)$. So $S\left(z\right)\in N_{\sqrt{\alpha}}\left(\{H,Id\}\right)$.

Its a lengthy calculation to see that

$$H^{-1}H^{-1}n_i=\lozenge_i \tag{121}$$

for $i=1,2$. The proof is relegated to the Appendix 1.

So we know that $Dv\left(z\right)\in N_{\sqrt{\alpha}}\left(SO\left(2\right)\right)$ or $Dv\left(z\right)\in N_{\sqrt{\alpha}}\left(SO\left(2\right)H\right)$ and we can not have the latter case because that would imply $t\left(z\right):=\left[S^{-1}\left(z\right)S^{-1}\left(z\right)\right]n_i\in N_{c_9\sqrt{\alpha}}\left(\lozenge_i\right)$ which contradicts (120) and so we must have $S\left(z\right)\in B_{\sqrt{\alpha}}\left(Id\right)$.

Let $R_1\in SO\left(2\right)$ be the rotation such that

$$R_1n_i=\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}. \tag{122}$$

Let $t\left(z\right)$ denote the (non-normalised) tangent to the curve $v^{-1}\left(\eta\right)$; formally $t\left(z\right):=\left[Dv\left(z\right)\right]^{-1}\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}$, as already noted, since $Dv\left(z\right)\in N_{\sqrt{\alpha}}\left(SO\left(2\right)\right)$ we know

$$S\left(x\right)\in B_{\sqrt{\alpha}}\left(Id\right) \tag{123}$$

as this (by the fact that $t\left(z\right):=\left[S^{-1}\left(z\right)S^{-1}\left(z\right)\right]n_i$) implies $|t\left(z\right)-n_i|<c_7\sqrt{\alpha}$ and using (122) we have

$$\begin{aligned}|Dv\left(z\right)n_i-R_1n_i|&\leq|Dv\left(z\right)t\left(z\right)-Dv\left(z\right)n_i|+|Dv\left(z\right)t\left(z\right)-R_1n_i|\\&\leq\sigma^{-2}\left|t\left(z\right)-n_i\right|\\&\leq\sigma^{-2}c_7\sqrt{\alpha}.\end{aligned}$$

So letting R^* be the clockwise rotation by $\frac{\pi}{2}$ we have by (123) and (122)

$$\begin{aligned}\sigma^{-2}c_7\sqrt{\alpha}&\geq|Dv\left(z\right)n_i-R_1n_i|\\&=\left|R\left(z\right)R^{*-1}\phi_i-\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}\right|-\sqrt{\alpha}\\&=\left|Dv\left(z\right)\phi_i-R^*\left(\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}\right)\right|-2\sqrt{\alpha}.\end{aligned} \tag{124}$$

Now in the case $z\in\bigcup_{k\in D\cap\mathcal{K}_1}W_k$, from (121) we can see $S\left(z\right)\in B_{\sqrt{\alpha}}\left(H\right)$. Let $R_2\in SO\left(2\right)$ be the rotation such that $R_2H^{-1}n_i=\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}$.

Observe figure 14.

As can be seen from figure 14 letting R^* again denote the clockwise rotation by $\frac{\pi}{2}$, $R^{*\,-1}\phi_i=n_i$ and $R^{*\,-1}H\phi_i=H^{-1}n_i$. Now

$$Dv\left(z\right)t\left(z\right)=R\left(z\right)S\left(z\right)\left[S^{-1}\left(z\right)S^{-1}\left(z\right)\right]n_i=R\left(z\right)S^{-1}\left(z\right)n_i,$$

so

$$\left|Dv\left(z\right)t\left(z\right)-R\left(z\right)H^{-1}n_i\right|\leq\left|S^{-1}\left(z\right)-H^{-1}\right|\leq c_8\sqrt{\alpha}.$$

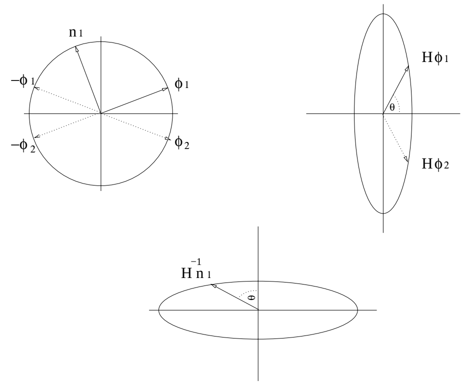

FIGURE 14

Again as $Dv(z)\,t(z) = \frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}$ so (as $R^*H^{-1}n_i = H\phi_i$) by multiplying through on the right by R^* this gives

$$\left|R^*\left(\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}\right) - R(z)\,H\phi_i\right| \le c_8\sqrt{\alpha}$$

so as $S(z) \in B_{\sqrt{\alpha}}(H)$

$$\left|R^*\left(\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|}\right) - Dv(z)\,\phi_i\right| \le 2c_8\sqrt{\alpha}. \tag{125}$$

And as $\frac{\tilde{e}-\tilde{s}}{|\tilde{e}-\tilde{s}|} = \frac{\vartheta-\xi}{|\vartheta-\xi|}$ this gives

$$\left|R^*\left(\frac{\vartheta-\xi}{|\vartheta-\xi|}\right) - Dv(z)\,\phi_i\right| < 2c_8\sqrt{\alpha}.$$

So let $Y = v\left(\bigcup_{k\in D} W_k\right) \cap \{z \in I_k : d(Dv(z), SO(2) \cup SO(2)\,H) < \sqrt{\alpha}\}$. Now as $H^1\left(\bigcup_{k\in D} I_k \backslash W_k\right) \le \alpha^{\frac{1}{8}} \sum_{k\in D} H^1(I_k) \le \alpha^{\frac{1}{8}}\,|\tilde{s}-\tilde{e}|$ so by bilipschitzness $H^1\left(\bigcup_{k\in D} v(I_k) \backslash v(W_k)\right) \le \sigma^{-2}\alpha^{\frac{1}{8}}\,|\tilde{s}-\tilde{e}|$. So by (119) and (73) we have

$$H^1([\tilde{s},\tilde{e}] \backslash Y) \le c_9\alpha^{\frac{1}{8}}\,|\tilde{s}-\tilde{e}| \tag{126}$$

and for any $x \in Y$, by (124), (125), letting $w_t := R^*\left(\frac{\vartheta-\xi}{|\vartheta-\xi|}\right)$ we have

$$\left|Dv\left(v^{-1}(x)\right)\phi_i - w_t\right| < 2c_8\sqrt{\alpha}. \tag{127}$$

and this establishes claim 1. $\diamondsuit$

Now recall we chose η as a subset of $(l_{\mathfrak{s}} + v(\mathfrak{s})) \cap v(R)$ when $\mathfrak{s}$ was an arbitrary point in the set C (see 72). So η and hence Y depend implicitly on $\mathfrak{s}$, now it will be convenient to make the dependence explicit. So let $\eta_{\mathfrak{s}} := \eta$ and $Y_{\mathfrak{s}} := Y$.

Let $\mathbb{B}_t := \bigcup_{\mathfrak{s}\in C} Y_{\mathfrak{s}}$. And recall $\mathbb{A}_t := N_{\sqrt{\alpha}w}(l_t) \cap v(R)$. Now from (78) and (126) we have for every $\mathfrak{s} \in C$

$$L^1(l_{\mathfrak{s}} \cap v(R) \backslash Y_{\mathfrak{s}}) \le 2c_9\alpha^{\frac{1}{8}}\,|\tilde{s}-\tilde{e}|$$

and from (72) we have

$$\begin{aligned} L^2(\mathbb{A}_t \backslash \mathbb{B}_t) &\le \int_{(l^\perp+t)\cap B_{\sqrt{\alpha}w}(t)} L^1(l_{\mathfrak{s}} \cap v(R) \backslash Y_{\mathfrak{s}})\,dL^1\mathfrak{s} \\ &\le 2c_9\alpha^{\frac{1}{8}}\,|e-s|\,L^1(C) + |e-s|\,L^1\left(\left(l^\perp+t\right) \cap B_{\sqrt{\alpha}w}(t) \backslash C\right) \\ &\le 4c_9\alpha^{\frac{5}{8}}w^2. \end{aligned}$$

And by (127) and point $x \in B_t$ satisfies (71) and the proof is complete.

$\square$

7. The Coarea alternative: Part I

In section 4.3 we described the "coarea alternative", roughly speaking, this was that for a function v on a region S (diameter ϵ say) with small bulk energy and small "coarea integral" with respect to the level set functions Θ^1 and Θ^2, function v on S must behave very much like an affine map whose linear part is in $SO(2) \cup SO(2)H$. And for a function v with small bulk energy and not small "coarea integral", v must oscillate by a not small amount in S. If we argue in the simplest way, for a function with small bulk energy (say $\int_S d(Dv(x), SO(2) \cup SO(2)H)\, dL^2x \leq \kappa^{m_0}\epsilon$, for some large integer m_0) then $\int_S |D^2v(x)|^2 dL^2x \leq \kappa\epsilon$ implies $\int_{v(S)} J(x) |D\Theta^i(x)| dL^2x \leq \kappa\epsilon^2$ for $i \in \{1,2\}$ and from this we can show that the linear part (denoted by L) of the affine map obtained form interpolating v on the corners of a triangle inside S is such that.

$$d(L, SO(2) \cup SO(2)H) < \kappa^{\frac{1}{8}} \tag{128}$$

So we take a triangulation $\{\tau_i\}$ of Ω (with triangulation size ϵ), by the "alternative", for all triangles τ_i with small bulk energy, either: (1); the linear part L_i of the interpolation of v on τ_i is such that L_i is less than $\kappa^{\frac{1}{8}}$ away from the wells. Or, (2); $\int_{\tau_i} |D^2v(x)| dL^2x > \kappa\epsilon$. If we want to apply this to finite element approximations we end up having to argue as follows: We can for simplicity assume all triangles τ_i have very low bulk energy. Let $\kappa = \epsilon^\alpha$ for some $\alpha > 0$ we decide on later. Let $B_1 := \{\tau_i : d(L_i, SO(2) \cup SO(2)H) > \epsilon^{\frac{\alpha}{8}}\}$. If $\int_\Omega |D^2v(x)| dL^2x \leq \epsilon^{-\beta}$ then $\mathrm{Card}(B_1) \leq \epsilon^{-\beta-\alpha-1}$. So if $\tilde{v}$ denotes the function obtained by taking the affine interpolation of v on the triangulation $\{\tau_i\}$ then we have.

$$\begin{aligned} \int_\Omega d(D\tilde{v}(x), SO(2) \cup SO(2)H)\, dL^2x &\leq \sum_{\tau_i \in B_1} \sigma^{-2}\epsilon^2 + \epsilon^{\frac{\alpha}{8}} \\ &\leq \sigma^{-2}\epsilon^{1-\beta-\alpha} + \epsilon^{\frac{\alpha}{8}}. \end{aligned} \tag{129}$$

Now matter how small β is or how we chose α as we expect $\int_\Omega d(D\tilde{v}(x), SO(2) \cup SO(2)H)\, dL^2x \approx \epsilon^{\frac{1}{3}}$ we do not get any kind of contradiction from this!

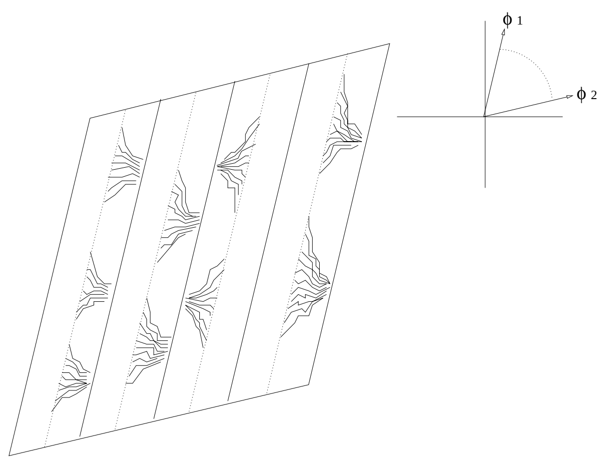

Figure 15

Hence we need a much more subtle invocation of the coarea alternative. What we failed to do, is to exploit the extremely good control we have on bulk energy. Recall, we bounded the coarea integral $\int_{v(S)} J(x) |D\Theta^i(x)| dL^2x$ for $i = 1,2$ from above by the surface energy. We did this by considering the pullbacks of the integral curves and we obtained a picture like fig 5. The point being somehow that if the bulk energy is sufficiently small we can chose the line $(z + \langle\phi_1\rangle)$ for which $\int_{v(z+\langle\phi_1\rangle)} J(x)\, dH^1x$ is small but $\int_{v(z+\langle\phi_1\rangle)} J(x) |D\Theta^1(x)| dH^1x$ is not small, to be very close to the line $(a + \langle\phi_1\rangle)$. So all the oscillation we pick up from the argument is concentrated in a thin strip around $(a + \langle\phi_1\rangle)$. This leads us naturally to the idea of considering the coarea integral in a thin strips parallel to ϕ_i running though S. As shown in fig 15.

Let $\left\{C_k^i\right\}$ denote the set of strips, width $\kappa^{m_0}\epsilon$ going through S in direction ϕ_i and let Θ_k^i denote the level set function defined in each strip, then we have $\sum_k(\kappa\epsilon)^{-1}\int_{v\left(C_k^i\right)}J(x)\left|D\Theta_k^i(x)\right|dL^2x\leq\kappa\epsilon$. So "on average" we can except to have $(\kappa\epsilon)^{-1}\int_{v\left(C_k^i\right)}J(x)\left|D\Theta_k^i(x)\right|dL^2x\leq\kappa^{m_0}\kappa\epsilon$. Thus we can find many integral curves in $v\left(C_k\right)$ with.

$$\int_{\Theta_k^{i-1}(t)\cap v\left(C_k\right)}J(x)\,dH^1x\leq\kappa^{\frac{m_0}{2}}\kappa^{m_0}\epsilon. \tag{130}$$

This gives us very good control over the short length of our little integral curves. Note we get better and better control by taking more and columns and the only thing we need to take more columns is lower bulk energy. This is how we exploit our control on bulk energy. So for any subskewcube $S_{k,l}\subset C_l^1\cap C_l^2$ we have many integral curves with respect to both Θ_k^1 and Θ_k^2 for which (130) is true.

Now we argue as indicated in the introduction. Lemma 6 says that the pullback of the integral curves with respect to Θ_k^i must look very much like laminates whose interfaces (in the reference configuration) are parallel to ϕ_i for $i=1,2$. So $v_{\lfloor S}$ must look very much like a laminate with respect to both ϕ_1 and ϕ_2 and the only way this can happen is if $v_{\lfloor S}\approx R_1H$ for some $R_1\in SO(2)$ or $v_{\lfloor S}\approx R_2$ for some $R_2\in SO(2)$. So we have very good control on a large number of subskewcubes inside our subcube S, by using this in combination with smallness of bulk and surface energy, we will in Lemma 9 show that for a triangle τ contained in S, the linear part of the affine interpolation of v on τ (denoted by L) will be such that $d(L,SO(2)\cup SO(2)H)<\kappa^{\frac{m_0}{1024}}\epsilon$ where m_0 is some large integer depending on bulk energy. By inserting this "strengthened" "coarea alternative" into the calculation (129) we see that we obtain unrealistic upper bounds on the scaling of the finite element approximation. This is how the coarea alternative works.

Proposition 1. *Given skewcube $S:=P\left(a,\phi_1,\phi_2,\tilde{\mathfrak{c}}\epsilon\right)\subset\Omega$.*

Assume we have;

- $$\int_S d\left(Dv(x),K\right)dL^2x\leq\kappa^{\frac{7m_0}{2}+8}\epsilon^2. \tag{131}$$
- *Let $\left\{C_k^{(p)}:k\in\{1,\dots[\kappa^{m_0}]+1\}\right\}$ denote the set of columns width $\kappa^{m_0}\epsilon$ going through S, parallel to ϕ_p.*

  Let $a_k^{(p)}$ denote the center point in $C_k^{(p)}$. Let $\Theta_k^{(p)}$ denote the level set function defined with respect to the line $\left\{a_k^{(p)}+\langle\phi_k\rangle\right\}$. Let $E_k^{(p)}=N_{\mathfrak{c}_5\kappa^{m_0}\epsilon}\left(C_k^{(p)}\right)\cap S$ for $k=1,2,\dots,[\kappa^{-m_0}]+1$.

  From (131), for each $p\in\{1,2\}$ we can find a distinct set of numbers $\left\{k_1^p,\dots k_{Q_0^p}^p\right\}\subset\{1,\dots[\kappa^{-m_0}]\}$ with $Q_0^p\geq\left(1-\kappa^{\frac{m_0}{2}}\right)[\kappa^{-m_0}]$ and

  $$\int_{v\left(E_{k_j^p}^{(p)}\right)}J(x)\,dL^2x\leq\kappa^{3m_0+7}\epsilon^2$$

  for each $j\in\{1,\dots Q_0^p\}$.

  We assume we have the following inequalities

  $$\sum_{j=1}^{Q_0}\int_{v\left(E_{k_j^p}^{(p)}\right)}J(x)\left|D\Theta_{k_j^p}^{(p)}\right|dL^2x\leq\kappa^{m_0+1}\epsilon^2 \tag{132}$$

  for $p=1$ and $p=2$.
- $$\int_S\left|D^2v(x)\right|dL^2x<\epsilon\kappa.$$

then we following statement holds true:

Let

$$\left\{\mathfrak{L}_{i,j}:=P\left(a_{i,j},\phi_1,\phi_2,\kappa^{m_0}\epsilon\right):i,j\in\left\{1,2,\dots\left[\kappa^{-m_0}\right]+1\right\}\right\}$$

be a set of pairwise disjoint skewcubes such that $S \subset \cup_{i,i\in\{1,\dots[\kappa^{-m_0}]+1\}} \mathfrak{L}_{i,j}$. *Let*

$$S_{i,j} := P\left(a_{i,j}, \phi_1, \phi_2, \mathfrak{c}_8 \kappa^{m_0}\epsilon\right)$$

for some constant $\mathfrak{c}_8 > 1$ *we will decide on later.*

There exists a set $G_0 \subset \{S_{i,j} : i,j \in \{1,\dots[\kappa^{-m_0}]+1\}\}$ *such that*

- $$L^2\left(\left\{x \in S_{i,j} : d\left(Dv\left(x\right), R_{i,j}T_{i,j}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) < 20\sigma^{-8}\tilde{\mathfrak{c}}_4\mathfrak{c}_7^2\kappa^{\frac{m_0}{64}}\left(\epsilon\kappa^{m_0}\right)^2$$

  for some $R_{i,j} \in SO(2)$, $T_{i,j} \in \{Id, H\}$
- $$\mathrm{Card}\left(G_0\right) \geq \frac{1-16\sigma^{-2}\kappa^{\frac{m_0}{8}}}{\kappa^{2m_0}}$$

Before proving this we need to prove a number of elementary Lemmas first.

7.1. Integral curves in a controlled subskewcube must run parallel.

This lemma is the essential step in showing that a controlled subskewcube (ie. a subskewcube $S_{k_1,k_2} \subset C^1_{k_1} \cap C^2_{k_2}$ *for columns* $C^1_{k_1}$, $C^2_{k_2}$ *such that* $\int_{v\left(C^i_{k_i}\right)} J\left(x\right)\left|D\Theta^i_k\left(x\right)\right| dL^2x$ *is small for* $i = 1,2$*) is such that* $v_{\lfloor S_{k_1,k_2}}$ *is very much like a laminate with respect to both rank-1 directions. Essentially what we show is that integral curves given by level sets of the form* $\Theta^{i\ -1}_{k_i}\left(t_1\right)$ *and* $\Theta^{i\ -1}_{k_i}\left(t_2\right)$ *are roughly parallel. The proof is more or less a calculation. We know that if we consider the pullback of an integral curve then (by Lemma 6) for most points* $x \in v^{-1}\left(\Theta^i_{k_i}\left(t\right)\right)$, $Dv\left(x\right)\phi_i$ *points in the clockwise normal direction to* $\Theta^i_{k_i}\left(t\right)$ *(recall* $\Theta^i_{k_i}\left(t\right)$ *is very much close to being a line). Let* a *denote the center of the skewcube. We chose* $t_1, t_2 \in \{a + \langle\phi_1\rangle\}$ *and* $t_3 \in \{a + \langle\phi_2\rangle\}$ *such that* $\Theta^2_{k_2}\left(t_3\right)$ *crosses* $\Theta^1_{k_1}\left(t_1\right)$, $\Theta^1_{k_1}\left(t_2\right)$. *If it happens that the intersection points* $v^{-1}\left(\Theta^2_{k_2}\left(t_3\right) \cap \Theta^1_{k_1}\left(t_1\right)\right)$ *and* $v^{-1}\left(\Theta^2_{k_2}\left(t_3\right) \cap \Theta^1_{k_1}\left(t_2\right)\right)$ *are such that* $Dv\left(\cdot\right)$ *at these points lie close to different components of the wells (ie.* $Dv\left(v^{-1}\left(\Theta^2_{k_2}\left(t_3\right) \cap \Theta^1_{k_1}\left(t_1\right)\right)\right) \approx SO(2)$ *and* $Dv\left(v^{-1}\left(\Theta^2_{k_2}\left(t_3\right) \cap \Theta^1_{k_1}\left(t_2\right)\right)\right) \approx SO(2)H$ *or vice versa) by the fact that the angle between the normals to the lines* $\Theta^2_{k_2}\left(t_3\right)$ *and* $\Theta^1_{k_1}\left(t_1\right)$ *is roughly the same as the angle between* $Dv\left(\cdot\right)\phi_1$ *and* $Dv\left(\cdot\right)\phi_2$ *at this intersection point, and this is in turn prescribed by which component of the wells* $Dv\left(\cdot\right)$ *is in, so the intersection points belonging to different components means that the angle of the lines* $\Theta^2_{k_2}\left(t_3\right)$ *and* $\Theta^1_{k_1}\left(t_3\right)$ *at their intersection point will be radically different from the angle of the lines* $\Theta^2_{k_2}\left(t_3\right)$ *and* $\Theta^1_{k_1}\left(t_2\right)$ *at their intersection point. Hence the lines* $\Theta^1_{k_1}\left(t_1\right)$ *and* $\Theta^1_{k_2}\left(t_2\right)$ *will be so radically non-parallel that (assuming we chose* t_1, t_2 *close enough to* a*)* $\Theta^1_{k_1}\left(t_1\right) \cap \Theta^1_{k_2}\left(t_2\right) \neq \emptyset$. *This is a contradiction and so the intersection points must be near the same component of the wells. Given this fact, almost exactly the same argument implies that the angle between the lines* $\Theta^2_{k_2}\left(t_3\right)$, $\Theta^1_{k_1}\left(t_1\right)$ *is very close to the angle between the lines* $\Theta^2_{k_2}\left(t_3\right)$ *and* $\Theta^1_{k_1}\left(t_2\right)$ *and this means* $\Theta^1_{k_2}\left(t_1\right)$ *is almost parallel to* $\Theta^1_{k_1}\left(t_2\right)$.

Lemma 7. *Let* $v \in \mathcal{A}_F\left(\Omega\right)$. *Given skew cube* $S := P\left(a, \phi_1, \phi_2, \mathfrak{c}_6 w\right)$ *with the following properties:*

- S *is contain in a skew rectangles* $R_1 := \mathfrak{F}\left(a, \mathfrak{c}_7 w\phi_0, r_1\phi_2\right) \subset \Omega$ *and* $R_2 := \mathfrak{F}\left(a, r_2\phi_0, \mathfrak{c}_7 w\phi_2\right) \subset \Omega$ *with* $\frac{w}{r_i} < \sigma^2$ *for* $i = 1,2$ *and* $\mathfrak{c}_7$ *is some constant bigger than* $\mathfrak{c}_6$.
- *We have a level set function* $\Theta^a_i : R_i \to \mathbb{R}$ *for* $i = 1,2$ *such that if we let*

  $$G_i := \left\{t \in \left(a + \langle\phi_i\rangle\right) \cap B_{\frac{w}{\sigma^2}}\left(a\right) : \int_{\Theta^{a\ -1}_i\left(t\right)\cap R_i} J\left(x\right) dH^1x \leq \alpha w\right\}$$

  we have

  $$L^1\left(\left(a + \langle\phi_i\rangle\right) \cap B_{\frac{w}{\sigma^4}}\left(a\right)\backslash G_i\right) \leq \alpha w \text{ for } i \in \{1,2\}. \tag{133}$$
- $$\int_{N_{\frac{w}{\sigma^2}}\left(S\right)} d\left(Dv\left(x\right), SO\left(2\right) \cup SO\left(2\right)H\right) dL^2x \leq \alpha^3 w^2. \tag{134}$$

The following holds true:

- *Firstly recall by Lemma 5 we have that for each $t \in G_i$, let e_t be the first point (going forward in time) to hit ∂R and s_t be the first point (going backward in time) to hit ∂R. Let U_t be the connected subset subset of $\Theta_i^{a\;-1}(t)$ between e_t and s_t then we have for some $l_t \in G(1,2)$*

$$U_t \subset N_{\mathfrak{c}_3\sqrt{\alpha}w}\left(l_t + v(t)\right).$$

- *Let $\mathbb{D}_t := N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(l_t + v(t)\right) \cap R$. As before, we define the clockwise normal to l_t by $w_t := R^* \frac{\tilde{e_t} - \tilde{s_t}}{|\tilde{e_t} - \tilde{s_t}|}$ where $\tilde{e_t} \in (l_t + v(t)) \cap B_{\sqrt{\alpha}w}(e_t)$, $\tilde{s_t} \in (l_t + v(t)) \cap B_{\sqrt{\alpha}w}(s_t)$ and R^* is a clockwise rotation by $\frac{\pi}{2}$.*
  There exists a set $\mathbb{C}_t \subset \mathbb{D}_t$ such that

$$L^2\left(\mathbb{D}_t \backslash \mathbb{C}_t\right) \leq \sigma^{-6}\tilde{\mathfrak{c}}_4 \alpha^{\frac{3}{16}} w^2$$

  and for any $x \in \mathbb{C}_t$ we have

$$\left|Dv\left(v^{-1}(x)\right)\phi_i - w_t\right| \leq 128\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{16}}.$$

Secondly for any two points $t_1, t_2 \in G_1 \cap B\left(a, \frac{w}{\sigma}\right)$ and $t_3 \in G_2 \cap B\left(a, \frac{w}{\sigma}\right)$ if we let $\theta_1 \in \Theta_1^{a\;-1}(t_1) \cap \Theta_2^{a\;-1}(t_3)$ and $\theta_2 \in \Theta_1^{a\;-1}(t_2) \cap \Theta_2^{a\;-1}(t_3)$ then we can find $\mathfrak{S} \in \{H, I\}$ such that

- $$L^2\left(\left\{x \in B_{\frac{\alpha^{\frac{1}{16}}w}{4}}(\theta_i) : Dv(z) \notin N_{\sqrt{\alpha}}\left(SO(2)\mathfrak{S}\right)\right\}\right) \leq 4\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{3}{16}}w^2.$$

  for $i = 1, 2$.
- *For any two points $t_1, t_2 \in G_i \cap B\left(a, \frac{w}{\sigma}\right)$ we have*

$$\left|w_{t_1} - w_{t_2}\right| \leq 2048\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}.$$

Proof. To begin with we assume $\mathfrak{c}_6 > \sigma^{-7}$ so we have $B_{\sigma^{-5}w}(v(a)) \subset v(S)$. This gives us some room to work in.

Step 1.
For $k \in \{1, 2\}$, let

$$\mathbb{C}_k := \left\{y \in N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(v(t_k) + l_{t_k}\right) \cap B_{\frac{w}{4\sigma^5}}(v(a)) : \left|Dv\left(v^{-1}(y)\right)\phi_i - w_{t_k}\right| < 128\sigma^{-5}\mathfrak{c}_4\alpha^{\frac{1}{16}}\right\},$$

we will show that

$$L^2\left(N_{\frac{\sigma^2\alpha^{\frac{1}{16}}w}{4}}\left(v(t_k) + l_{t_k}\right) \backslash \mathbb{C}_k\right) \leq \sigma^{-6}\mathfrak{c}_4\alpha^{\frac{3}{16}}w^2.$$

First by (133) we can pick a chain of points

$$\left\{z_n^k : n = 1, 2, \ldots N_0\right\} \subset G_1 \cap B_{\frac{\alpha^{\frac{1}{16}}w}{\sigma^2}}(a)$$

such that

$$B_{\frac{\alpha^{\frac{1}{16}}w}{\sigma^2}}(a) \cap \{a + \langle\phi_1\rangle\} \subset \bigcup_{n=1}^{N_0} B_{\sigma^4\sqrt{\alpha}w}\left(z_n^k\right) \cap \{a + \langle\phi_1\rangle\}.$$

Note $N_0 \leq \frac{\alpha^{\frac{1}{16}}w}{\sigma^6\alpha^{\frac{1}{2}}w} = \sigma^{-6}\alpha^{-\frac{7}{16}}$. By Lipschitzness for v, in particular the fact that $\mathrm{Lip}(v) \leq \sigma^{-2}$ we have

$$\begin{aligned}
B_{\alpha^{\frac{1}{16}}w}(v(a)) \cap v(\{a + \langle\phi_1\rangle\}) &\subset v\left(B_{\frac{\alpha^{\frac{1}{16}}w}{\sigma^2}}(a) \cap \{a + \langle\phi_1\rangle\}\right) \\
&\subset \bigcup_{n=1}^{N_0} v\left(B_{\sqrt{\alpha}\sigma^4 w}\left(z_n^k\right) \cap \{a + \langle\phi_1\rangle\}\right) \\
&\subset \bigcup_{n=1}^{N_0} B_{\sqrt{\alpha}\sigma^2 w}\left(v\left(z_n^k\right)\right) \cap v(\{a + \langle\phi_1\rangle\}).
\end{aligned}$$

Now as $B_{\frac{w}{\sigma^5}}(v(a)) \subset v\left(P\left(a, \phi_1, \phi_2, \frac{w}{\sigma^7}\right)\right) \subset v(S)$ we consider the integral curves given by

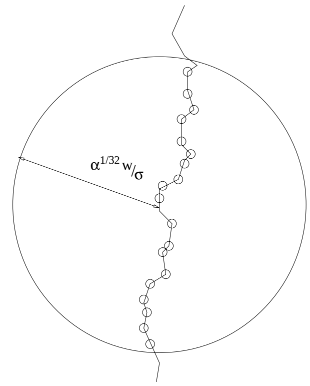

FIGURE 16

$$\left\{ \Theta_1^{a\ -1}\left(z_n^k\right) \cap B_{\frac{w}{\sigma^5}}\left(v\left(a\right)\right) : n = 1, 2, \dots N_0 \right\}.$$

We know that this set is disjoint by uniqueness of solutions of ODE and of course by Lemma 5 we have

$$\Theta_1^{a\ -1}\left(z_n^k\right) \cap B_{\frac{w}{\sigma^5}}\left(v\left(a\right)\right) \subset N_{\mathfrak{c}_3\sqrt{\alpha}w}\left(v\left(z_n^k\right) + l_{z_n^k}\right). \tag{135}$$

Now as can be seen from fig 17.

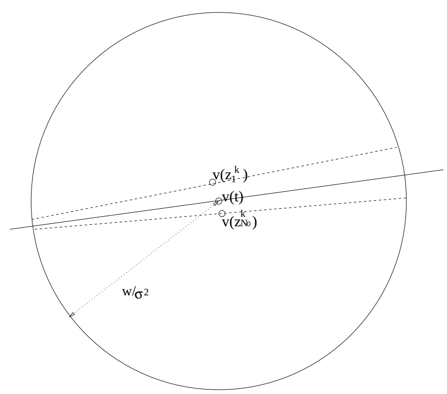

FIGURE 17

$$\left(\left\{v\left(z_{N_0}^k\right) + l_{z_{N_0}^k}\right\} \cap B_{\frac{w}{2\sigma^5}}\left(v\left(a\right)\right)\right) \cap \left(\left\{v\left(z_1^k\right) + l_{z_1^k}\right\} \cap B_{\frac{w}{2\sigma^5}}\left(v\left(a\right)\right)\right) = \emptyset, \tag{136}$$

since otherwise by (135) and the fact that $\left|v\left(z_1^k\right) - v\left(z_{N_0}^k\right)\right| > \sigma^2 \alpha^{\frac{1}{16}} w$ we have

$$\left(\Theta_1^{k\ -1}\left(z_1^k\right) \cap B_{\frac{w}{\sigma^5}}\left(v\left(a\right)\right)\right) \cap \left(\Theta_1^{k\ -1}\left(z_{N_0}^k\right) \cap B_{\frac{w}{\sigma^5}}\left(v\left(a\right)\right)\right) = \emptyset \tag{137}$$

and this is a contradiction.

Now since we have (136), as can be seen from fig 17, for any point $y \in \left\{v\left(z_1^k\right)+l_{z_1^k}\right\} \cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)$ we have

$$\begin{aligned}\frac{\alpha^{\frac{1}{16}}w}{2} &< \frac{\left|v\left(z_1^k\right)-v\left(z_{N_0}^k\right)\right|}{2}\\ &< d\left(y,\left\{v\left(z_{N_0}^k\right)+l_{z_{N_0}^k}\right\}\cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)\right)\\ &< 2\left|v\left(z_1^k\right)-v\left(z_{N_0}^k\right)\right|\\ &< \frac{4\alpha^{\frac{1}{16}}w}{\sigma^4}. \qquad (138)\end{aligned}$$

Now in the same way as (136), since $\left|v\left(z_m^k\right)-v\left(z_{m+1}^k\right)\right| \leq \sqrt{\alpha}\sigma^2 w$, for $m \in \{1,2,\dots N_0-1\}$ and for any $y \in \left\{v\left(z_{m+1}^k\right)+l_{z_{m+1}^k}\right\} \cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)$ we have

$$\begin{aligned}d\left(y,\left\{v\left(z_m^k\right)+l_{z_m^k}\right\}\cap B_{\frac{w}{2\sigma^5}}\left(v\left(a\right)\right)\right) &< \sigma^{-2}\left|v\left(z_m^k\right)-v\left(z_{m+1}^k\right)\right|\\ &< \sqrt{\alpha}w, \qquad (139)\end{aligned}$$

since otherwise we will have that

$$\left(\Theta_1^{k\ -1}\left(z_m^k\right)\cap B_{\frac{w}{2\sigma^5}}\left(v\left(a\right)\right)\right)\cap\left(\Theta_1^{k\ -1}\left(z_{m+1}^k\right)\cap B_{\frac{w}{2\sigma^5}}\left(v\left(a\right)\right)\right)\neq\emptyset.$$

And again in the same way as (136) for any $y \in \left\{v\left(z_m^k\right)+l_{z_m^k}\right\} \cup \left\{v\left(z_{N_0}^k\right)+l_{z_{N_0}^k}\right\} \cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)$ we have

$$\frac{\alpha^{\frac{1}{16}}w}{4} < \frac{1}{2}\min\left\{\left|v\left(z_1^k\right)-v\left(a\right)\right|,\left|v\left(z_{N_1}^k\right)-v\left(a\right)\right|\right\}\ | < d\left(y,\left\{v\left(t\right)+l_t\right\}\cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)\right).$$

Thus the set $N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(v\left(t\right)+l_t\right)\cap B_{\frac{w}{4\sigma^5}}\left(a\right)$ is contained in the region of $B_{\frac{w}{4\sigma^5}}\left(a\right)$ between the two lines $\left(v\left(z_1^k\right)+l_{z_1^k}\right)\cap B_{\frac{w}{4\sigma^5}}\left(a\right)$ and $\left(v\left(z_{N_0}^k\right)+l_{z_{N_0}^k}\right)\cap B_{\frac{w}{4\sigma^5}}\left(a\right)$.

Now (139) implies that for any $m \in \{1,2,\dots N_0-1\}$ we have

$$\left(v\left(z_{m+1}^k\right)+l_{z_{m+1}^k}\right)\cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)\subset N_{\sqrt{\alpha}w}\left(v\left(z_m^k\right)+l_{z_m^k}\right)\cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right). \qquad (140)$$

As any point $z \in N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(v\left(t\right)+l_t\right)\cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)$ is either on line $\left(v\left(z_m^k\right)+l_{z_m^k}\right)\cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)$ for $k = 1,\dots N_0$ or lies between two such lines. So from (140) we have

$$N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(v\left(t\right)+l_t\right)\cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right)\subset\bigcup_{m=1}^{N_0}N_{\sqrt{\alpha}w}\left(v\left(z_m^k\right)+l_{z_m^k}\right)\cap B\left(a,\frac{w}{4\sigma^5}\right). \qquad (141)$$

For each $m \in \{1,2,\dots N_0\}$ let $w_m := w\left(z_m^k\right)$ be the clockwise normal to $l_{z_m^k}$. Let $\mathbb{A}_m := N_{\sqrt{\alpha}w}\left(v\left(z_m^k\right)+l_{z_m^k}\right)\cap R$. By Lemma 6 we know that there exists a set $\mathbb{B}_m \subset \mathbb{A}_m$ such that $L^2\left(\mathbb{A}_m\backslash\mathbb{B}_m\right) \leq \tilde{\mathfrak{c}}_4\alpha^{\frac{5}{8}}w^2$ and for any $x \in \mathbb{B}_m$ we have $\left|Dv\left(v^{-1}\left(x\right)\right)\phi_i - w_m\right| \leq \mathfrak{c}_4\sqrt{\alpha}$

Now we can see from the fig 18.

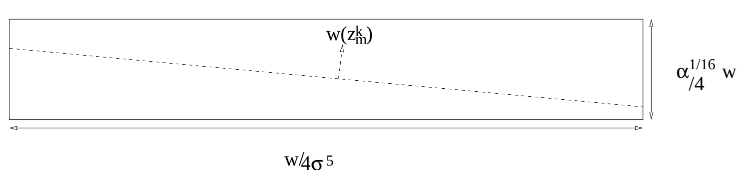

FIGURE 18

The difference in angle between w_{m_1} and w_{m_2} for any $m_1, m_2 \in \{1,2,\dots N_0\}$ is less than $2\tan^{-1}\left(16\sigma^{-5}\alpha^{\frac{1}{16}}\right)$ so we must have $|w_{m_1}-w_{m_2}| \leq 64\sigma^{-5}\alpha^{\frac{1}{16}}$.

So by (141) if we let $\mathbb{C}_k := \left\{ y \in N_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(v\left(t_k\right) + l_{t_k}\right) \cap B_{\frac{w}{4\sigma^5}}\left(v\left(a\right)\right) : \left|Dv\left(v^{-1}\left(y\right)\right)\phi_i - w_k\right| \leq 128\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{16}} \right\}$ we see

$$N_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(v\left(t_k\right) + l_{t_k}\right) \backslash \mathbb{C}_k \subset \bigcup_{m=1}^{N_0} \mathbb{A}_m \backslash \mathbb{B}_m .$$

Thus

$$\begin{aligned} L^2\left(N_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(v\left(t_k\right) + l_{t_k}\right) \backslash \mathbb{C}_k\right) &\leq N_0 \tilde{\mathfrak{c}}_4 \alpha^{\frac{5}{8}} w^2 \\ &\leq \sigma^{-6} \tilde{\mathfrak{c}}_4 \alpha^{\frac{3}{16}} w^2. \end{aligned} \tag{142}$$

Step 2.

Firstly we note that from the fact we established in Lemma 6, namely that path $v^{-1}\left(\Theta_i^{a\ -1}\left(t\right)\right) \cap R$ has its tangents mostly in directions n_i and $\diamondsuit_i$. We can see from the figure 19, for $t_1, t_2 \in G_1 \cap B_{\frac{w}{\sigma^2}}\left(a\right)$ and point $t_3 \in G_2 \cap B_{\frac{w}{\sigma^2}}\left(a\right)$ the points given by

$$\theta_1 \in \Theta_1^{a\ -1}\left(t_1\right) \cap \Theta_2^{a\ -1}\left(t_3\right) \text{ and } \theta_2 \in \Theta_1^{a\ -1}\left(t_2\right) \cap \Theta_2^{a\ -1}\left(t_3\right)$$

are such that $v^{-1}\left(\theta_1\right), v^{-1}\left(\theta_1\right) \in B_{\frac{w}{\sigma^3}}\left(a\right)$.

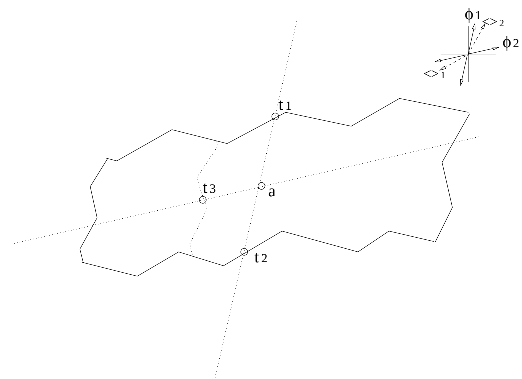

Figure 19

So by Lipschitzness we have

$$\theta_1, \theta_2 \in B_{\frac{w}{\sigma^5}}\left(v\left(a\right)\right). \tag{143}$$

Observe the figure 20.

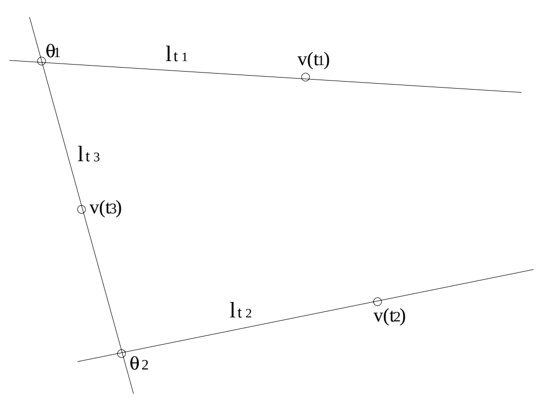

Figure 20

Now as $\theta_1 \in N_{\sqrt{\alpha}w}\left(l_{t_1} + v\left(t_1\right)\right) \cap N_{\sqrt{\alpha}w}\left(l_{t_3} + v\left(t_3\right)\right)$ and as the angle between lines l_{t_1} and l_{t_3} can not be too sharp (by 143) if we let $\tilde{\theta_1} := \left(v\left(t_1\right) + l_{t_1}\right) \cap \left(v\left(t_3\right) + l_{t_3}\right)$ then we have $\left|\theta_1 - \tilde{\theta_1}\right| < c_1\sqrt{\alpha}w$.

Similarly if we let $\tilde{\theta_2} := \left(v\left(t_3\right) + l_{t_3}\right) \cap \left(v\left(t_2\right) + l_{t_2}\right)$ we have $\left|\theta_2 - \tilde{\theta_2}\right| < c_1\sqrt{\alpha}w$. So

$$B_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(\tilde{\theta_1}\right) \subset \left(N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(l_{t_1} + v\left(t_1\right)\right) \cap B_{\frac{w}{4\sigma^5}}\left(a\right)\right) \cap \left(N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(l_{t_3} + v\left(t_3\right)\right) \cap B_{\frac{w}{4\sigma^5}}\left(a\right)\right)$$

and

$$B_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(\tilde{\theta_2}\right) \subset \left(N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(l_{t_2} + v\left(t_2\right)\right) \cap B_{\frac{w}{4\sigma^5}}\left(a\right)\right) \cap \left(N_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(l_{t_3} + v\left(t_3\right)\right) \cap B_{\frac{w}{4\sigma^5}}\left(a\right)\right).$$

So by (142) we have

$$L^2\left(B_{\frac{\alpha^{\frac{1}{4}}w}{16}}\left(\tilde{\theta_1}\right) \setminus \left(\mathbb{C}_1 \cup \mathbb{C}_3\right)\right) \leq 2\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{3}{16}}w^2$$

and

$$L^2\left(B_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(\tilde{\theta_2}\right) \setminus \left(\mathbb{C}_2 \cup \mathbb{C}_3\right)\right) \leq 2\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{3}{16}}w^2.$$

Now by assumption (134) we can find sets; $F_1 \subset B_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(\theta_1\right) \cap \mathbb{C}_1 \cap \mathbb{C}_3$ such that

$$L^2\left(F_1\right) \geq \frac{\pi}{2}\left(\frac{\alpha^{\frac{1}{16}}w}{4}\right)^2\left(1 - 64\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{1}{16}}\right)$$

and for any $z \in F_1$ we have $Dv\left(v^{-1}\left(z\right)\right) \in N_{\sqrt{\alpha}}\left(SO\left(2\right) \cup SO\left(2\right)H\right)$. In the same way we can find $F_2 \subset B_{\frac{\alpha^{\frac{1}{16}}w}{4}}\left(\theta_1\right) \cap \mathbb{C}_2 \cap \mathbb{C}_3$ such that $L^2\left(F_2\right) \geq \frac{\pi}{2}\left(\frac{\alpha^{\frac{1}{16}}w}{4}\right)^2\left(1 - 64\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{1}{16}}\right)$ and for any $z \in F_2$ we have $Dv\left(v^{-1}\left(z\right)\right) \in N_{\sqrt{\alpha}}\left(SO\left(2\right) \cup SO\left(2\right)H\right)$.

We will show if we have a point $z_1 \in F_1$ such that $Dv\left(v^{-1}\left(z_1\right)\right) \in N_{\sqrt{\alpha}}\left(SO\left(2\right)\right)$ and $z_2 \in F_2$ such that $Dv\left(v^{-1}\left(z_2\right)\right) \in N_{\sqrt{\alpha}}\left(SO\left(2\right)H\right)$ we get a contradiction from the fact that w_{t_1} and w_{t_2} (the orientations of l_{t_1} and l_{t_2} respectively) must point in the same direction.

Formally; we will show that for some $\mathfrak{S} \in \{H, Id\}$ we have

$$\left\{Dv\left(v^{-1}\left(z\right)\right) : z \in F_1 \cup F_2\right\} \subset N_{\sqrt{\alpha}}\left(SO\left(2\right)\mathfrak{S}\right). \tag{144}$$

Before we do so we will have to establish some things about w_{t_1} (the clockwise normal of line l_{t_1}) and w_{t_2} (the clockwise normal to line l_{t_2}).

Step 3 We will show $w_{t_1} \cdot w_{t_2} \in \left[\frac{4}{5}, 1\right]$.

Firstly we note that since we can run the ODE all the way to the boundary of S, as $\left|t_1 - t_2\right| < \frac{w}{\sigma}$ and

$$\left(\Theta_1^{a\ -1}\left(t_1\right) \cap v\left(S\right)\right) \cap \left(\Theta_1^{a\ -1}\left(t_2\right) \cap v\left(S\right)\right) = \emptyset$$

assuming the constant $\mathfrak{c}_6$ has been chosen big enough, the lines l_{t_1} and l_{t_2} must be roughly parallel, see fig 21.

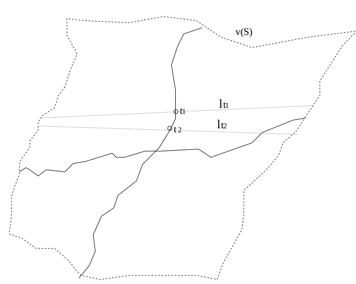

FIGURE 21

Now we consider the pullback of lines l_{t_1}, l_{t_2} in the reference. See figure 22. By bilipschitzness we know

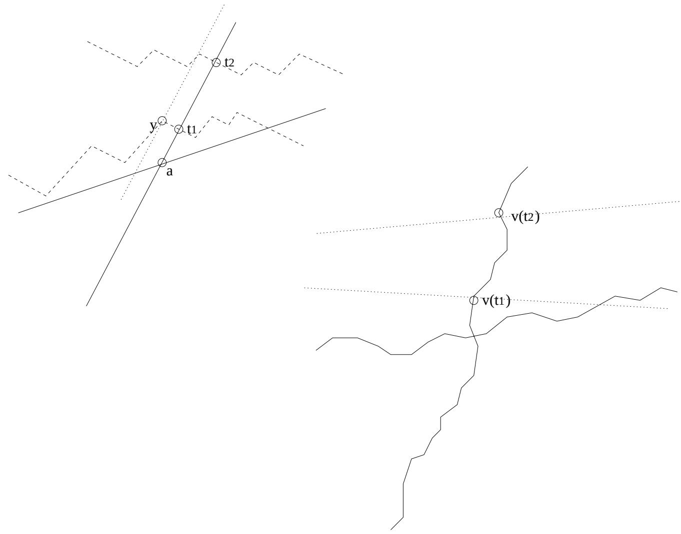

FIGURE 22

$$N_{\frac{\sigma^2\alpha^{\frac{1}{16}}w}{4}}\left(v^{-1}\left(v\left(t_k\right)+l_{t_k}\right)\right)\cap B_{\frac{w}{\sigma^3}}\left(a\right)\subset v^{-1}\left(\mathbb{C}_k\right)\cap B_{\frac{w}{\sigma^3}}\left(a\right)\text{ for }k=1,2.$$

And by Step 1 or (142)

$$L^2\left(\left\{z\in N_{\frac{\sigma^2\alpha^{\frac{1}{16}}w}{4}}\left(v^{-1}\left(v\left(t_k\right)+l_{t_k}\right)\right)\cap B_{\frac{w}{\sigma^3}}\left(a\right):\left|Dv\left(y\right)\phi_1-w_{t_k}\right|>128\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{16}}\right\}\right)\leq\sigma^{-8}\tilde{\mathfrak{c}}_4\alpha^{\frac{3}{16}}w^2.$$

So we can find a point $y_0\in\left\{a+\phi_1^{\perp}\right\}\cap B_{\alpha^{\frac{1}{16}}w}\left(a\right)$ such that

$$L^1\left(\left\{z\in N_{\frac{\sigma^2\alpha^{\frac{1}{16}}w}{4}}\left(v^{-1}\left(v\left(t_k\right)+l_{t_k}\right)\cap B_{\frac{w}{\sigma^3}}\left(a\right)\right)\cap\left(y_0+\langle\phi_1\rangle\right):\left|Dv\left(y\right)\phi_1-w_{t_k}\right|>128\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{16}}\right\}\right)\leq\sigma^{-8}\tilde{\mathfrak{c}}_4\alpha^{\frac{1}{8}}w$$

for $k=1,2$.

Let

$$\Upsilon_k:=N_{\frac{\sigma^2\alpha^{\frac{1}{16}}}{4}}\left(v^{-1}\left(l_{t_k}\right)\cap B_{\frac{w}{\sigma^3}}\left(a\right)\right)\cap\left(y_0+\langle\phi_1\rangle\right)$$

for $k=1,2$ and let

$$\widetilde{\Upsilon}_k:=\left\{y\in\Upsilon_k:\left|Dv\left(y\right)\phi_i-w_{t_k}\right|<128\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{16}}\right\}$$

for $k=1,2$. In words; for any $z_1\in\widetilde{\Upsilon}_1$ we know that w_{t_1} is (approximately) equal to $Dv\left(z_1\right)\phi_1$ and for any $z_2\in\Upsilon_2$ we have that w_{t_2} is (approximately) equal to $Dv\left(z_2\right)\phi_1$. And $L^1\left(\Upsilon_k\backslash\widetilde{\Upsilon}_k\right)\leq\sigma^{-8}\alpha^{\frac{1}{8}}w$.

So as l_{t_1} and l_{t_2} are roughly parallel we have (for big enough constant c_1) we have $w_{t_1}\cdot w_{t_2}\in\left[\frac{4}{5},1\right]$ or $w_{t_1}\cdot w_{t_2}\in\left[-1,-\frac{4}{5}\right]$. Recall we want to show $w_{t_1}\cdot w_{t_2}\in\left[\frac{4}{5},1\right]$. Suppose not; so $w_{t_1}\cdot w_{t_2}\in\left[-1,-\frac{4}{5}\right]$.

As $Dv\left(z\right)\phi_1$ is the tangent to the line $v\left(\left\{y_0+\langle\phi_1\rangle\right\}\cap S\right)$ at point $v\left(z\right)$ we have that $v\left(\Upsilon_k\right)$ (which is a connected segment of $v\left(\left\{y_0+\langle\phi_1\rangle\right\}\cap S\right)$) has for most of its points, a tangent pointing roughly in direction w_{t_2}.

Formally, we can prove (just by considering the integral $v\left(z\right)-v\left(y_0\right)=\int_{y_0}^{z}Dv\left(x\right)\phi_i dL^1x$) that if we let $\xi_k:=v^{-1}\left(l_{t_k}\right)\cap\left\{y_0+\langle\phi_1\rangle\right\}$ then

$$H\left(v\left(\Upsilon_k\right),\left[v\left(\xi_k\right)-\frac{\sigma^2\alpha^{\frac{1}{16}}w}{4}w_{t_2},v\left(\xi_k\right)+\frac{\sigma^2\alpha^{\frac{1}{16}}w}{4}w_{t_2}\right]\right)\leq c_2\alpha^{\frac{1}{16}}w. \tag{145}$$

More important however is that for $z_1 \in \widetilde{\Upsilon}_1$ and $z_2 \in \widetilde{\Upsilon}_2$ we have that $Dv(z_1)\phi_1$ and $Dv(z_2)\phi_1$ point in roughly opposite directions. The only way this can happen is if $v((a+\langle\phi_i\rangle)\cap S)$ "turns around" by π as shown on fig 23.

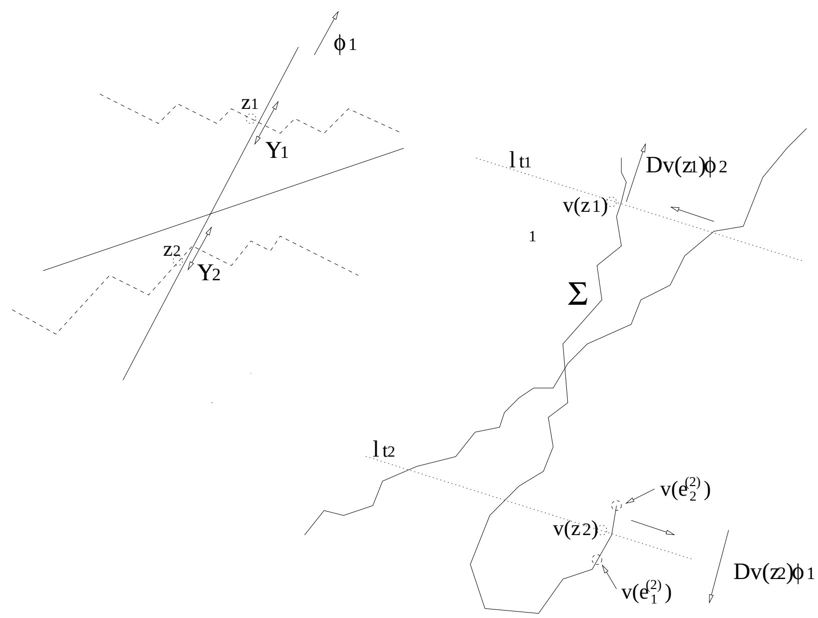

FIGURE 23

Now let $e_1^{(2)}$ and $e_2^{(2)}$ be the endpoints of Υ_2 as shown, from (145) we have that $N_{\sqrt{\alpha}w}(l_{t_2})$ passes in between $v\left(e_1^{(2)}\right)$ and $v\left(e_2^{(2)}\right)$. Let $\Sigma := v\left(\left[e_1^{(2)}, t_1\right]\right)$. Its easy to see that

$$P_{l_{t_2}^{\perp}}\left(\left[v\left(e_1^{(2)}\right), v\left(e_2^{(2)}\right)\right]\right) \subset P_{l_{t_2}^{\perp}}(\Sigma)$$

and so its clear that $N_{\sqrt{\alpha}w}(l_{t_2})$ must pass through both $v(\Upsilon_k)$ and Σ which implies

$$\Theta^{a\ -1}(t_2) \cap B\left(a, \frac{w}{\sigma^5}\right) \cap v(\Upsilon_k) \neq \emptyset$$

and

$$\Theta^{a\ -1}(t_2) \cap B\left(a, \frac{w}{\sigma^5}\right) \cap \Sigma \neq \emptyset.$$

By Lemma 1 $\Theta^{a\ -1}(t_2)$ can only pass through $v(\{y_0 + \langle\phi_1\rangle\})$ once so this is a contradiction. Thus we have finally established that $w_{t_1} \cdot w_{t_2} \in \left[\frac{4}{5}, 1\right]$.

Before continuing on our mission to establish (144) we note the following. If θ_1 denotes the angle between ϕ_1 and ϕ_2 then the angle between $H\phi_1$ and $H\phi_2$ is given by $\pi - \theta_1$. This can be seen either by direction calculation or by noting that since $detH = 1$ and $|H\phi_1| = |H\phi_2| = 1$ a parallelogram with sides parallel to ϕ_1, ϕ_2 must be sent to parallelogram with the same volume and same side length and hence the same pair of internal angles.

Now we can proceed with the proof of (144). See fig 24.

We start with point z_1. Now we know $Dv\left(v^{-1}(z_1)\right) \approx R_0$ for some $R_0 \in SO(2)$. Since $z_1 \in \mathbb{C}_1$ we know $Dv\left(v^{-1}(z)\right)\phi_1 \approx R_0\phi_1$ points (roughly) in the direction to w_{t_1} as shown. Similarly since $z \in \mathbb{C}_3$, $Dv\left(v^{-1}(z)\right)\phi_2 \approx R_0\phi_2$ points (roughly) in the direction w_{t_3}. So the angle between lines l_{t_1} and l_{t_3} at the intersection point is approximately the angle between ϕ_1 and ϕ_2.

Now we go to point z_2. We have $Dv\left(v^{-1}(z_2)\right) \approx R_1H$ for some $R_1 \in SO(2)$. As $z_2 \in \mathbb{C}_3$ we have $Dv\left(v^{-1}(z_2)\right)\phi_2 \approx R_2H\phi_2$ points in the direction w_{t_3} as shown. And as $z_2 \in \mathbb{C}_2$ we should have that

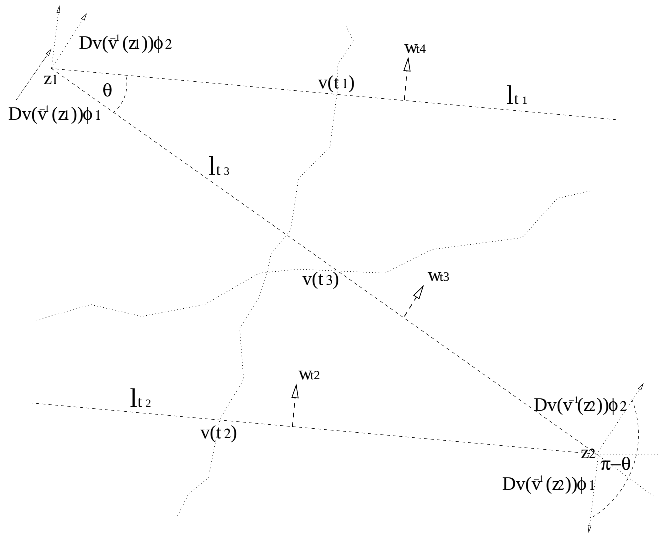

FIGURE 24

$Dv\left(v^{-1}\left(z_2\right)\right)\phi_1 \approx H\phi_1$ should point in the direction w_{t_2} however as the action of H on ϕ_1 and ϕ_2 is to stretch them apart so that $H\phi_1$ and $H\phi_2$ meet at angle of $\pi-\theta_1$, as we can see from the diagram $H\phi_1$ points in the opposite direction to w_{t_2} so this is a contradiction.

In an identical way we get a contradiction from the possibility that $z_1 \in F_1$ with $Dv\left(v^{-1}\left(z_1\right)\right) \in N_{\sqrt{\alpha}}\left(SO\left(2\right)H\right)$ and $z_2 \in F_2$ with $Dv\left(v^{-1}\left(z_2\right)\right) \in N_{\sqrt{\alpha}}\left(SO\left(2\right)\right)$. So there exists $\mathfrak{S} \in \{Id, H\}$ such that

$$Dv\left(v^{-1}\left(z\right)\right) \in SO\left(2\right)\mathfrak{S} \text{ for each } z \in F_1 \cup F_2.$$

This completes the proof of the first part of the lemma.

Now we establish the second part of the lemma. See fig 25.

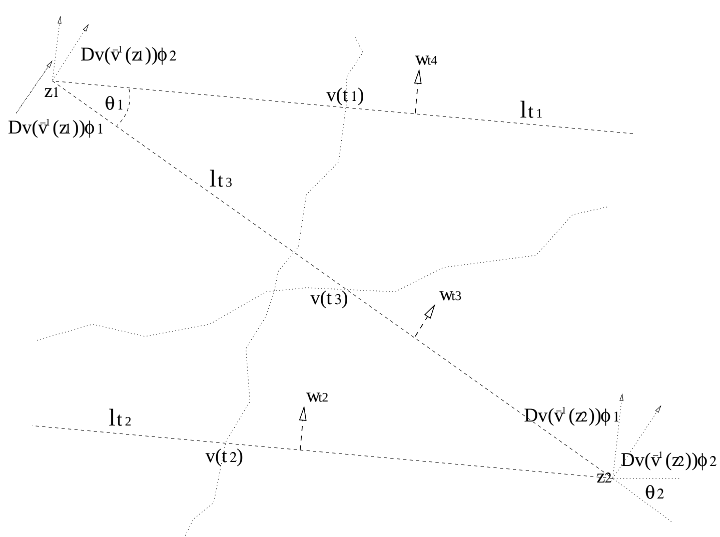

FIGURE 25

By the first part of the lemma $Dv\left(v^{-1}\left(z_1\right)\right) \in B_{\sqrt{\alpha}}\left(R_0\right)$ for some $R_0 \in SO\left(2\right)$ and $Dv\left(v^{-1}\left(z_2\right)\right) \in B_{\sqrt{\alpha}}\left(R_1\right)$ for some $R_1 \in SO\left(2\right)$.

Similarly to as before. As $z_1 \in \mathbb{C}_1 \cap \mathbb{C}_3$ we have

$$\left|Dv\left(v^{-1}\left(z_1\right)\right)\phi_1 - w_{t_1}\right| \leq 128\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{128}}$$

and

$$\left|Dv\left(v^{-1}\left(z_1\right)\right)\phi_2 - w_{t_3}\right| \leq 128\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{128}}.$$

And so the angle between w_{t_1} and w_{t_3} is within $512\mathfrak{c}_4\sigma^{-5}\alpha^{\frac{1}{16}}$ of the angle between ϕ_1 and ϕ_2.

Formally; let θ_1 be the angle between w_{t_1} and w_{t_3} and let θ_2 be the angle between w_{t_2} and w_{t_3}. So we have

$$\begin{aligned}
\left|w_{t_1}\cdot w_{t_3} - \phi_1\cdot\phi_2\right| &\leq \left|w_{t_1}\cdot\left(w_{t_3} - R_0\phi_2\right)\right| + \left|R_0\phi_2\cdot\left(w_{t_1} - R_0\phi_1\right)\right| \\
&\leq \left|w_{t_3} - Dv\left(v^{-1}\left(z_1\right)\right)\phi_2\right| + \left|Dv\left(v^{-1}\left(z_1\right)\right)\phi_2 - R_0\phi_2\right| + \left|w_{t_1} - Dv\left(v^{-1}\left(z_1\right)\right)\phi_1\right| \\
&\quad + \left|Dv\left(v^{-1}\left(z_1\right)\right)\phi_1 - R_0\phi_1\right| \\
&\leq 512\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}. \qquad (146)
\end{aligned}$$

In exactly the same way we can see

$$\left|w_{t_2}\cdot w_{t_3} - \phi_1\cdot\phi_2\right| \leq 512\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}$$

and so putting things together we get

$$\left|\left(w_{t_1} - w_{t_2}\right)\cdot w_{t_3}\right| \leq 1024\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}. \qquad (147)$$

Let R^* be a rotation by $\frac{\pi}{2}$ in the clockwise direction, we have

$$\begin{aligned}
\left|w_{t_1}\cdot R^*w_{t_3} - \phi_1\cdot R^*\phi_2\right| &\leq \left|w_{t_1}\cdot\left(R^*w_{t_3} - R^*R_0\phi_2\right)\right| + \left|R^*R_0\phi_2\cdot\left(w_{t_1} - R_0\phi_1\right)\right| \\
&\leq \left|w_{t_3} - Dv\left(v^{-1}\left(z_1\right)\right)\phi_2\right| + \left|Dv\left(v^{-1}\left(z_1\right)\right)\phi_2 - R_0\phi_2\right| + \left|w_{t_1} - Dv\left(v^{-1}\left(z_1\right)\right)\phi_1\right| \\
&\quad + \left|Dv\left(v^{-1}\left(z_1\right)\right)\phi_1 - R_0\phi_1\right| \\
&\leq 512\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}. \qquad (148)
\end{aligned}$$

And in the same way we can see $\left|w_{t_2}\cdot R^*w_{t_3} - \phi_1\cdot R^*\phi_2\right| \leq 512\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}$ so $\left|\left(w_{t_1} - w_{t_2}\right)\cdot R^*w_{t_3}\right| \leq 1024\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}$. Together with (147) this implies $\left|w_{t_1} - w_{t_2}\right| \leq 2048\mathfrak{c}_4\sigma^{-6}\alpha^{\frac{1}{16}}$.

$\square$

7.2. **Controlled subskewcubes have derivative mostly in one well.**
This next lemma essentially follows from Lemma 6, as most of the integral curves given by $\left\{\Theta_{k_1}^{i\,-1}(t) : t \in \mathbb{R}\right\}$ are straight parallel lines for $i=1,2$. Given that from Lemma 6 we know that $Dv(\cdot)\phi_i$ is clockwise the normal to these classes of "lines" for $i=1,2$, we only have to show that significant subskewcube of our skewcube is contained within a neighborhood of these lines to conclude that for most points x within this subskewcube, the directions of $Dv(x)\phi_1$ and $Dv(x)\phi_2$ are roughly fixed. So (as a weak conclusion) we have that most points in our subskewcube are such that $Dv(\cdot)$ remain close to one component of the wells.

Lemma 8. *Given skew cube $S := P(a,\phi_1,\phi_2,\mathfrak{c}_6 w)$ with the following properties:*

- *S is contained in skew rectangles $R_1 := \mathfrak{F}(a,\mathfrak{c}_7 w\phi_2, r_1\phi_1)$, $R_2 := \mathfrak{F}(a, r_2\phi_2, \mathfrak{c}_7 w\phi_1)$ with $\frac{w}{r_i} < \sigma^2$ for $i=1,2$ where $\mathfrak{c}_7$ is some constant bigger than $\mathfrak{c}_6$.*
- *We have level set functions $\Theta_i^a : R_i \to \mathbb{R}$ such that if we let*

$$G_i := \left\{ t \in (a + \langle \phi_i \rangle) \cap B_{\frac{w}{\sigma^6}}(a) : \int_{\Theta_i^{a\;-1}(t)\cap R_i} J(x)\, dH^1 x \le \alpha w \right\}$$

we have

$$L^1\left((a+\langle\phi_i\rangle)\cap B_{\frac{w}{\sigma^6}}(a)\backslash G_i\right) \le \alpha w \tag{149}$$

for $i=1,2$.

- $$\int_{N_{\frac{w}{\sigma^2}}(S)} d\left(Dv(x), SO(2)\cup SO(2)H\right) dL^2 x < \alpha^3 w^2$$

then if we take skewcube $\widetilde{S} := P(a,\phi_1,\phi_2,\mathfrak{c}_8 w)$ (for some constant $\mathfrak{c}_8$ strictly less than $\mathfrak{c}_7$) there exists rotation $R_0 \in SO(2)$ and $\mathfrak{S} \in \{Id, H\}$ such that

$$L^2\left(\left\{x \in \widetilde{S} : |Dv(x) - R_0\mathfrak{S}| > \sqrt{\alpha}\right\}\right) < 20\sigma^{-8}\tilde{\mathfrak{c}}_4\alpha^{\frac{1}{8}} w^2.$$

Proof. This lemma follows from Lemma 6 in a relatively straightforward way. Let

$$t_1 := \partial P\left(a,\phi_1,\phi_2,\frac{2w}{\sigma^2}\right)\cap\{a+\lambda\phi_1 : \lambda > 0\},\; t_2 := \partial P\left(a,\phi_1,\phi_2,\frac{2w}{\sigma^2}\right)\cap\{a+\lambda\phi_1 : \lambda \le 0\}.$$

$$t_3 := \partial P\left(a,\phi_1,\phi_2,\frac{2w}{\sigma^2}\right)\cap\{a+\lambda\phi_2 : \lambda > 0\},\; t_4 := \partial P\left(a,\phi_1,\phi_2,\frac{2w}{\sigma^2}\right)\cap\{a+\lambda\phi_2 : \lambda \le 0\}.$$

Observe fig 26.

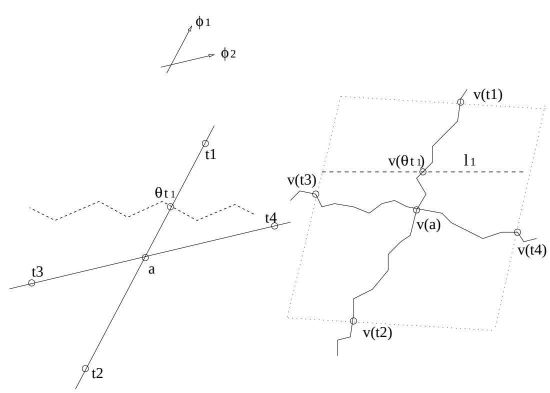

FIGURE 26

Take point $\vartheta_{t_1} \in G_1 \cap [t_1,t_2]$. By Lemma 1 and bilipschitzness we have that line $[v(t_1), v(t_2)]$ is some distinct angle away from line $l_1 := l_{\vartheta_{t_1}}$. Specifically there exists a constant $c_1 := c(\sigma) \in (0,1)$ such that

$$L^1\left(P_{l_1^\perp}([v(t_1), v(t_2)])\right) \ge c_1 |v(t_1) - v(t_2)|.$$

Similarly if we chose point $\vartheta_{t_2} \in G_2 \cap [t_3, t_4]$ let $l_2 := l_{\vartheta_{t_2}}$ we have

$$L^1\left(P_{l_1^\perp}\left(\left[v\left(t_3\right), v\left(t_4\right)\right]\right)\right) \geq c_1\left|v\left(t_3\right) - v\left(t_4\right)\right|.$$

So for some $c_2 := c(\sigma)$ we have

$$B_{c_2 w}\left(v\left(a\right)\right) \subset P_{l_1^\perp}^{-1}\left(P_{l_1^\perp}\left(\left[v\left(t_1\right), v\left(t_2\right)\right]\right)\right) \tag{150}$$

and

$$B_{c_2 w}\left(v\left(a\right)\right) \subset P_{l_2^\perp}^{-1}\left(P_{l_2^\perp}\left(\left[v\left(t_3\right), v\left(t_4\right)\right]\right)\right).$$

Let $Q_1 := P_{\langle l_1^\perp\rangle}\left(\left[v\left(t_1\right), v\left(t_2\right)\right]\right)$ and $Q_2 := P_{\langle l_2^\perp\rangle}\left(\left[v\left(t_3\right), v\left(t_4\right)\right]\right)$. We cut Q_1 into intervals of width $\frac{\alpha^{\frac{1}{16}} w}{2}$. Formally, we can find a set $\left\{B_{\frac{\alpha^{\frac{1}{16}} w}{8}}\left(\zeta_k^1\right) : k = 1, 2, \ldots M_0\right\}$ that are pairwise disjoint, with $\left\{\zeta_k^1 : k = 1, 2, \ldots M_0\right\} \subset Q_2$,

$$Q_2 \subset \bigcup_{k=1}^{M_0} B_{\frac{\alpha^{\frac{1}{16}} w}{4}}\left(\zeta_k^1\right)$$

and $M_0 := \left[\frac{4L^1(Q_1)}{\alpha^{\frac{1}{16}} w}\right] + 1 \leq 5\left[\alpha^{\frac{1}{16}}\right]$.

Similarly we cut Q_1 into intervals of width $\frac{\alpha^{\frac{1}{16}} w}{2}$. So we find $\left\{B_{\frac{\alpha^{\frac{1}{16}} w}{8}}\left(\zeta_k^2\right) : k = 1, 2, \ldots M_1\right\}$ that are pairwise disjoint, with $\left\{\zeta_k^2 : k = 1, 2, \ldots M_1\right\} \subset Q_2$,

$$Q_2 \subset \bigcup_{k=1}^{M_1} B_{\frac{\alpha^{\frac{1}{16}} w}{4}}\left(\zeta_k^1\right)$$

and $M_1 := \left[\frac{4L^1(Q_2)}{\alpha^{\frac{1}{16}} w}\right] + 1 \leq 5\left[\alpha^{-\frac{1}{16}}\right]$.

Now as $v\left(\left[t_1, t_2\right]\right)$ is connected, for each $k \in \{1, 2, \ldots M_0\}$ we have $P_{l_1^\perp}^{-1}\left(\zeta_k^1\right) \cap v\left(\left[t_1, t_2\right]\right) \neq \emptyset$. So we can pick $q_k^1 \in P_{l_1^\perp}^{-1}\left(\zeta_k^1\right) \cap v\left(\left[t_1, t_2\right]\right)$ for $k \in \{1, 2, \ldots M_0\}$ and $q_k^2 \in P_{l_2^\perp}^{-1}\left(\zeta_k^2\right) \cap v\left(\left[t_3, t_4\right]\right)$ for $k \in \{1, 2, \ldots M_1\}$.

Now observe fig 27 below.

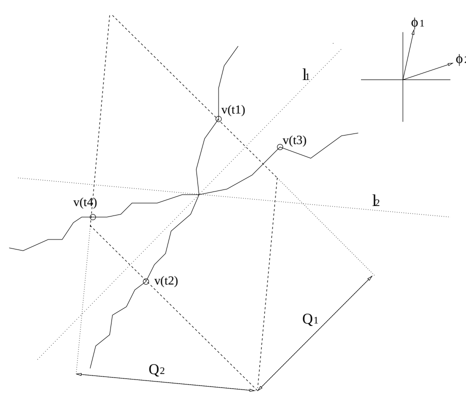

FIGURE 27

By (149) for each $k \in \{1, 2, \ldots M_0\}$ we must be able to chose $\tau_k^1 \in N_{\alpha w}\left(v^{-1}\left(q_k^1\right)\right) \cap G_1$ and for each $k \in \{1, 2, \ldots M_1\}$ we can chose $\tau_k^2 \in N_{\alpha w}\left(v^{-1}\left(q_k^1\right)\right) \cap G_2$ so we have

$$Q_1 \subset \bigcup_{k=1}^{M_0} B_{\frac{\alpha^{\frac{1}{16}} w}{4}}\left(P_{l_1^\perp}\left(v\left(\tau_k^1\right)\right)\right) \tag{151}$$

and

$$Q_2 \subset \bigcup_{k=1}^{M_1} B_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(P_{l_1^{\perp}} \left(v\left(\tau_k^2\right)\right)\right). \tag{152}$$

Now by Lemma 6 we know $w_{\tau_k^1}$ (the clockwise normal to $l_{\tau_k^1}$) and w_1 is the clockwise normal to l_1 are such that

$$\left| w_{\tau_k^1} - w_1 \right| < 2048\sigma^{-6}\alpha^{\frac{1}{16}} \text{ for any } k \in \{1, 2, \dots N_0\} \tag{153}$$

Observe fig 28.

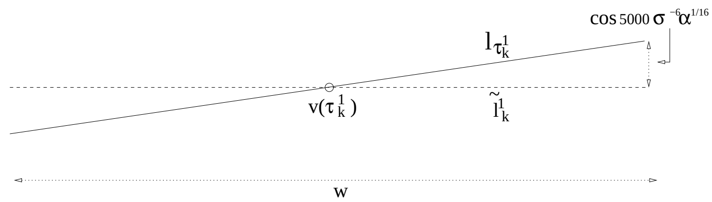

FIGURE 28

As the angle between $w_{\tau_k^1}$ and w_1 is less than $5000\sigma^{-6}\alpha^{\frac{1}{16}}$, if we let $\tilde{l_k^1}$ be the line centered on $v\left(\tau_k^1\right)$ parallel to l_1 as can be seen from fig 28

$$N_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(\tilde{l_k^1}\right) \subset N_{5000\sigma^{-6}\alpha^{\frac{1}{16}} w} \left(l_{\tau_k^1}\right). \tag{154}$$

So by Lemma 7, (153) and (154)

$$\begin{aligned}
&L^2\left(\left\{x \in N_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(\tilde{l_k^1}\right) \cap R_1 : \left|Dv\left(v^{-1}(x)\right)\phi_1 - w_1\right| > 6000\sigma^{-6}\alpha^{\frac{1}{16}}\right\}\right) \\
&\quad \leq L^2\left(\left\{x \in N_{\frac{\alpha^{\frac{1}{16}} w}{\sigma^7}} \left(\tilde{l}_{\tau_k^1}\right) \cap R_1 : \left|Dv\left(v^{-1}(x)\right)\phi_1 - w_{\tau_k^1}\right| > 3000\sigma^{-6}\alpha^{\frac{1}{16}}\right\}\right) \\
&\quad \leq \sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{3}{16}} w^2.
\end{aligned}$$

However from (150) and (151) we have

$$B_{c_2 w}(v(a)) \subset \bigcup_{k=1}^{N_0} N_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(\tilde{l_k^1}\right) \cap R_1.$$

If we let $\tilde{l_k^2}$ be the line centered on $v\left(\tau_k^2\right)$ parallel to l_2 for each $k \in \{1, 2, \dots N_1\}$. In exactly the same we have

$$B_{c_2 w}(v(a)) \subset \bigcup_{k=1}^{N_1} N_{\frac{\alpha^{\frac{1}{16}} w}{4}} \left(\tilde{l_k^2}\right) \cap R_2.$$

So

$$\begin{aligned}
&L^2\left(\left\{x \in B_{c_2 w}(v(a)) : \left|Dv\left(v^{-1}(x)\right)\phi_1 - w_1\right| > 6000\sigma^{-6}\alpha^{\frac{1}{16}}\right\}\right) \\
&\quad \leq \sum_{k=1}^{N_0} L^2\left(\left\{x \in N_{\alpha^{\frac{1}{16}} w} \left(\tilde{l_k^1}\right) \cap R_i : \left|Dv\left(v^{-1}(x)\right)\phi_1 - w_1\right| > 3000\sigma^{-6}\alpha^{\frac{1}{16}}\right\}\right) \\
&\quad \leq \sigma^{-6} M_0 \tilde{\mathfrak{c}}_4 \alpha^{\frac{3}{16}} w^2 \\
&\quad \leq 5\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{7}{8}} w^2
\end{aligned} \tag{155}$$

In exactly the same way we have

$$L^2\left(\left\{x \in B_{c_2 w}(v(a)) : \left|Dv\left(v^{-1}(x)\right)\phi_2 - w_2\right| > 6000\sigma^{-6}\alpha^{\frac{1}{16}}\right\}\right) \leq 5\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{1}{8}} w^2. \tag{156}$$

Let $J := \{x \in S : d(Dv(x), SO(2) \cup SO(2)H) < \sqrt{\alpha}\}$ and

$$Z := \left\{x \in B_{c_2 w}(v(a)) : \left|Dv\left(v^{-1}(x)\right)\phi_2 - w_2\right| < 6000\sigma^{-6}\alpha^{\frac{1}{16}} \text{ and } \left|Dv\left(v^{-1}(x)\right)\phi_1 - w_1\right| < 6000\sigma^{-6}\alpha^{\frac{1}{16}}\right\}$$

so by assumption (134) and (155), (156) we have

$$L^2\left(B_{c_2 w}(v(a)) \setminus (Z \cup v(J))\right) \leq 20\sigma^{-6}\tilde{\mathfrak{c}}_4\alpha^{\frac{1}{8}}w^2. \tag{157}$$

Now for any $z \in B_{c_2 w}(v(a)) \cap (Z \cup v(J))$ since for $z \in Z$ we have $\left|Dv\left(v^{-1}(z)\right) - R\mathfrak{S}\right| < \sqrt{\alpha}$ for some $R \in SO(2)$ and some $\mathfrak{S} \in \{Id, H\}$ and as the angle between $R\mathfrak{S}\phi_1$ and $R\mathfrak{S}\phi_2$ changes radically depending on whether $\mathfrak{S} = Id$ or $\mathfrak{S} = H$ whilst w_1 and w_2 are independent of x, so we either have

$$\left\{Dv\left(v^{-1}(z)\right) : z \in B_{c_2 w}(v(x)) \cap (Z \cup v(J))\right\} \subset N_{\sqrt{\alpha}}(SO(2)) \tag{158}$$

$$\left\{Dv\left(v^{-1}(z)\right) : z \in B_{c_2 w}(v(x)) \cap (Z \cup v(J))\right\} \subset N_{\sqrt{\alpha}}(SO(2)H). \tag{159}$$

Now assuming c_2 is big enough we have that

$$\widetilde{S} := P(a, \phi_1, \phi_2, \mathfrak{c}_8 w) \subset v^{-1}\left(B_{c_2 w}(v(a))\right)$$

and hence from (157)

$$L^2\left(\left\{x \in \widetilde{S} : d(Dv(x), SO(2)A) \geq \sqrt{\alpha}\right\}\right) \leq 20\sigma^{-8}\tilde{\mathfrak{c}}_4\alpha^{\frac{1}{8}}w^2$$

and hence we have established the lemma.

□

7.3. **Proof of Proposition 1 continued.**
Lemma 6 and Lemma 8 have essentially done most of the proof of Proposition 1 for us. The only "work" to be done in the coming proof is to set up all the conditions to apply Lemmas 6, 8 and simply to count up the consequences in terms our exponents κ and ϵ.

Proof. So recall, we have

$$\int_S d\left(Dv\left(x\right), SO\left(2\right)\cup SO\left(2\right)H\right)dL^2x \leq \kappa^{\frac{7m_0}{2}+8}\epsilon^2 \tag{160}$$

and for the set $\left\{C_k^{(p)}: k\in\{1,\dots[\kappa^{-m_0}]\}\right\}$ of pairwise disjoint columns width $\kappa^{m_0}\epsilon$ going through S parallel to ϕ_p centered on $a_k^{(p)}$. We let $E_k^{(p)} := N_{(\mathfrak{c}_7-1)\kappa^{m_0}\epsilon}\left(C_k^{(p)}\right)$ for $k\in\{1,\dots[\kappa^{-m_0}]+1\}$. From (160) we can find a set $\left\{k_1^p,\dots k_{Q_0^p}^p\right\}$ of distinct numbers satisfying the following inequalities

$$\int_{v\left(E_{k_j^p}^{(p)}\right)} J\left(x\right)dL^2x \leq \kappa^{3m_0+7}\epsilon^2$$

for each $j\in\{1,\dots Q_0^p\}$ and

$$Q_0^p \geq \left(1-\kappa^{\frac{m_0}{2}}\right)\left[\kappa^{-m_0}\right] \text{ for } p=1,2. \tag{161}$$

And by assumption (132) we have

$$\sum_{j=1}^{Q_0^p}\int_{v\left(E_{k_j^p}^{(p)}\right)} J\left(x\right)\left|D\Theta_{k_j^p}^{(p)}\right|dL^2x \leq \kappa^{m_0+1}\epsilon^2$$

for $p=1,2$.

We also have $\int_S\left|D^2v\left(x\right)\right|dL^2x < \epsilon\kappa$. Let

$$B_0^p = \left\{k_j^p : \int_{v\left(E_{k_j^p}^{(p)}\right)} J\left(x\right)\left|D\Theta_{k_j^p}^p\left(x\right)\right|dL^2x \geq \kappa^{\frac{m_0}{2}}\kappa^{m_0+1}\epsilon^2,\;\; j\in\{1,\dots Q_0^p\}\right\}.$$

So $\mathrm{Card}\left(B_0^p\right)\kappa^{\frac{m_0}{2}}\kappa^{m_0+1}\epsilon^2 \leq \kappa^{m_0+1}\epsilon^2$ implies $\mathrm{Card}\left(B_0^p\right)\leq\kappa^{\frac{-m_0}{2}}$. Let

$$G_0^p := \left\{k_j^p : j\in\{1,\dots Q_0^p\}\right\}\backslash B_0^p$$

thus

$$\mathrm{Card}\left(G_0^p\right) \geq \left[\kappa^{-m_0}\left(1-2\kappa^{\frac{m_0}{2}}\right)\right]. \tag{162}$$

Let $a_{i,j}$ be the centerpoint of $E_i^{(1)}\cap E_j^{(2)}$. Note that $S_{i,j} := P\left(a_{i,j},\phi_1,\phi_2,\mathfrak{c}_6\kappa^{m_0}\epsilon\right)\subset E_i^{(1)}\cap E_j^{(2)}$. Let $l_k^{(p)} := \left(a_k^{(p)}+\langle\phi_p\rangle\right)\cap S$. Let

$$U_k^p := \left\{t\in l_k^{(p)} : \int_{\Theta_k^{p\;-1}(t)\cap v\left(E_k^{(p)}\right)} J\left(x\right)dH^1x \geq \kappa^{\frac{m_0}{4}}\kappa^{m_0}\epsilon\right\}. \tag{163}$$

Let $k\in G_0^p$, by the coarea formula, for any $k\in G_0^p$

$$\begin{aligned}
\int_{t\in U_k^p}\int_{\Theta_k^{p\;-1}(t)\cap v\left(E_k^{(p)}\right)} J\left(x\right)dH^1x\,dL^1t &\leq \int_{t\in\mathbb{R}}\int_{\Theta_k^{p\;-1}(t)\cap v\left(E_k^{(p)}\right)} J\left(x\right)dH^1x\,dL^1t \\
&= \int_{v\left(E_k^{(p)}\right)} J\left(x\right)\left|D\Theta_k^{(p)}\right|dL^2x \\
&\leq \kappa^{\frac{m_0}{2}}\kappa^{m_0+1}\epsilon^2.
\end{aligned}$$

So $L^1(U_k^p)\kappa^{\frac{m_0}{4}}\kappa^{m_0}\epsilon \leq \kappa^{\frac{m_0}{2}}\kappa^{m_0+1}\epsilon^2$ which implies $L^1(U_k^p) \leq \kappa^{\frac{m_0}{4}}\kappa\epsilon$. Now let us temporarily fix $p=1$. Let $\Lambda_{k,r}^1 := l_k^{(1)} \cap N_{\frac{\kappa^{m_0}\epsilon}{\sigma^6}}(S_{k,r})$ for $k \in \{1,2,\dots[\kappa^{-m_0}]+1\}$, $r \in \{1,2,\dots[\kappa^{-m_0}]+1\}$. For $k \in G_0^1$ let

$$D_k^1 := \left\{r \in \left\{1,2,\dots\left[\kappa^{-m_0}\right]\right\} : L^1\left(\Lambda_{k,r}^1 \cap U_k^1\right) > \kappa^{1+\frac{m_0}{8}}\kappa^{m_0}\epsilon\right\}. \tag{164}$$

So since $\left\{\Lambda_{k,r}^1 : r \in \{1,2,\dots[\kappa^{-m_0}]+1\}\right\}$ do not overlap by more than $\left[\sigma^{-6}\right]+1$ we have

$$\begin{aligned}\frac{\sigma^6}{2}\mathrm{Card}\left(D_k^1\right)\kappa^{1+\frac{m_0}{8}}\kappa^{m_0}\epsilon &\leq L^1\left(U_k^1\right)\\ &\leq \kappa^{\frac{m_0}{4}}\kappa\epsilon\end{aligned}$$

and thus

$$\mathrm{Card}\left(D_k^1\right) \leq 2\sigma^{-6}\kappa^{\frac{m_0}{8}}\kappa^{-m_0}.$$

For $p=2$ we can define $\Lambda_{k,r}^2 := l_r^{(2)} \cap N_{\frac{\kappa^{m_0}\epsilon}{\sigma^6}}(S_{k,r})$ for $k,r \in \{1,\dots[\kappa^{-m_0}]+1\}$. And for $r \in G_0^2$ we let

$$D_r^2 := \left\{k \in \left\{1,2,\dots\left[\kappa^{-m_0}\right]\right\} : L^1\left(\Lambda_{k,r}^2 \cap U_k^2\right) > \kappa^{1+\frac{m_0}{8}}\kappa^{m_0}\epsilon\right\}.$$

We can see in exactly the some way that $\mathrm{Card}\left(D_r^2\right) \leq 2\sigma^{-6}\kappa^{\frac{m_0}{8}}\kappa^{-m_0}$.

Now we define the "bad" skew cubes $\widetilde{B_0}$ of the set $\left\{S_{i,j} : i \in G_0^1,\ j \in G_0^2\right\}$ as follows

$$S_{i,j} \in \widetilde{B_0} \text{ iff } N_{\frac{\kappa^{m_0}\epsilon}{\sigma^6}}(S_{i,j}) \cap l_i^{(1)} \in \left\{\Lambda_{i,r}^1 : r \in D_i^1\right\} \text{ or } N_{\frac{\kappa^{m_0}\epsilon}{\sigma^6}}(S_{i,j}) \cap l_j^{(2)} \in \left\{\Lambda_{k,j}^2 : k \in D_j^2\right\}. \tag{165}$$

So

$$\begin{aligned}\mathrm{Card}\left(\widetilde{B_0}\right) &\leq \sum_{k=1}^{\left[\kappa^{-m_0}\right]+1} D_k^1 + \sum_{r=1}^{\left[\kappa^{-m_0}\right]+1} D_r^2\\ &\leq 8\sigma^{-6}\left[\kappa^{-m_0}\right]\kappa^{\frac{m_0}{8}}\kappa^{-m_0}\\ &\leq \frac{8\sigma^{-6}\kappa^{\frac{m_0}{8}}}{\kappa^{2m_0}}.\end{aligned} \tag{166}$$

Let $\widetilde{G_0} := \left\{S_{i,j} : i \in G_0^1, j \in G_0^2\right\} \backslash \widetilde{B_0}$. So from (162) and (166) we have

$$\begin{aligned}\mathrm{Card}\left(\widetilde{G_0}\right) &= \frac{1}{\kappa^{2m_0}} - \mathrm{Card}\left(\left\{1,\dots\left[\kappa^{-m_0}\right]+1\right\}\backslash G_0^1\right)\kappa^{-m_0}\\ &\quad -\mathrm{Card}\left(\left\{1,\dots\left[\kappa^{-m_0}\right]+1\right\}\backslash G_0^2\right)\kappa^{-m_0} - \mathrm{Card}\left(\widetilde{B_0}\right)\\ &\geq \frac{1}{\kappa^{2m_0}}\left(1 - 4\kappa^{\frac{m_0}{2}} - 8\sigma^{-6}\kappa^{\frac{m_0}{8}}\right)\\ &\geq \frac{\left(1-16\sigma^{-6}\kappa^{\frac{m_0}{8}}\right)}{\kappa^{2m_0}}.\end{aligned} \tag{167}$$

Obviously we want to apply Lemma 8 to the skewcubes $\left\{S_{i,j} \in \widetilde{G_0}\right\}$. First we have to make a further selection. Let $w := \mathfrak{c}_7\kappa^{m_0}\epsilon$ and let $\alpha := \kappa^{\frac{m_0}{8}}$.

Now let

$$G_0 := \left\{S_{i,j} \in \widetilde{G_0} : \int_{N_{\frac{\kappa^{m_0}\epsilon}{\sigma^2}}(S_{i,j})} d\left(Dv\left(x\right), SO\left(2\right) \cup SO\left(2\right)H\right) dL^2x < \mathfrak{c}_7^2\kappa^{m_0\left(2+\frac{1}{4}\right)}\epsilon^2\right\}$$

assuming (160) we have that $\mathrm{Card}\left(\widetilde{G_0}\backslash G_0\right)\mathfrak{c}_7^2\kappa^{m_0\left(2+\frac{1}{2}\right)}\epsilon^2 \leq \kappa^{3m_0}\epsilon^2$ so

$$\widetilde{G_0}\backslash G_0 = \emptyset. \tag{168}$$

Given $S_{i,j} \in G_0$ let $R_1 := \mathfrak{F}(a_{i,j}, w\phi_2, \epsilon\phi_1)$ and $R_2 := \mathfrak{F}(a_{i,j}, \epsilon\phi_2, w\phi_1)$.

As $\alpha > \kappa^{\frac{m_0}{4}}$ and $w > \kappa^{m_0}\epsilon$ we know from definitions (163), (164) and (165) and the fact that $S_{i,j} \in G_0$ that

$$G_p := \left\{ t \in (a_{i,j} + \langle \phi_p \rangle) \cap B_{\frac{w}{\sigma^6}} (a_{i,j}) : \int_{\Theta_i^{a\ -1}(t) \cap R_p} J(x)\, dH^1 x \leq \alpha w \right\}$$

is such that

$$L^1 \left((a_{i,j} + \langle \phi_p \rangle) \cap B_{\frac{w}{\sigma^6}} (a_{i,j}) \backslash G_p \right) \leq \alpha \kappa^{m_0} \epsilon \leq \alpha w$$

for $p = 1, 2$. So we apply Lemma 8 to $S_{i,j}$ to conclude that for $\widetilde{S_{i,j}} := P(a_{i,j}, \phi_1, \phi_2, \mathfrak{c}_8 w)$ there exists rotation $R_{i,j} \in SO(2)$ and $\mathfrak{S}_{i,j} \in \{Id, H\}$ such that

$$L^2 \left(\left\{ x \in \widetilde{S_{i,j}} : |Dv(x) - R_{i,j} \mathfrak{S}_{i,j}| > \kappa^{\frac{m_0}{16}} \right\} \right) < 20 \sigma^{-8} \tilde{\mathfrak{c}}_4 \mathfrak{c}_7^2 \kappa^{\frac{m_0}{64}} (\kappa^{m_0} \epsilon)^2$$

so by (167) and (168) we have established the proof of Proposition 1.

$\square$

8. The Coarea alternative: Part II

In the first part of the "coarea alternative" we established that for a skewcube S of diameter ϵ with very small bulk energy and surface energy bounded by $\epsilon\kappa$, there must exist very many "controlled" subskewcubes of size roughly $\kappa^{m_0}\epsilon$; by this we mean that $Dv(\cdot)$ is these subskewcubes remains (mostly) close to one component of the wells. Recall that the goal of the "coarea alternative" was to show that for a function v and skewcube S under our hypotheses and with little surface energy over S, given triangle $\tau_i \subset S$, if we let L denote the linear part of the affine interpolation of v on the corners of L, then L will be very close to these wells. In the coming Lemma we will establish this for skewcube S under our hypotheses and additionally having the property of having many "controlled" subskewcubes.

The proof is based on two observations. Firstly we know from a calculation in the proof of Lemma 1 (specifically, equation (14)) that the tangent v_{x_0} at point x_0 of the pull back of an integral curve given by a level set of Θ_i is of the form $v_{x_0} := \left[S^{-1}(Y(t_0))\, S^{-1}(Y(t_0))\right] n_i$ where $Dv(\cdot) = R(\cdot)\, S(\cdot)$ is the polar decomposition of $Dv(\cdot)$. So the tangent doesn't depend of the rotational part of the derivative. Secondly, from our constraint on surface energy, $\int_S \left|D^2 v(x)\right| dL^2 x \leq \kappa\epsilon$, if we take any choice of direction $\psi_0 \in S^1$ then considering lines in direction ψ_0 going through S spaced out from one another by $\kappa^{m_0}\epsilon$, all but $\left[\frac{\kappa}{\kappa^{m_0}}\right]$ of these lines must be such that they only pass through controlled subskewcubes that have derivative close the same component of the wells. So suppose at least half our controlled subskewcubes are such that $Dv(\cdot)$ is close to $SO(2)H$, then we must be able to find many lines parallel to $H^{-2}n_1$ and $H^{-2}n_2$ running through S only touching controlled subskewcubes with derivative close to $SO(2)H$.

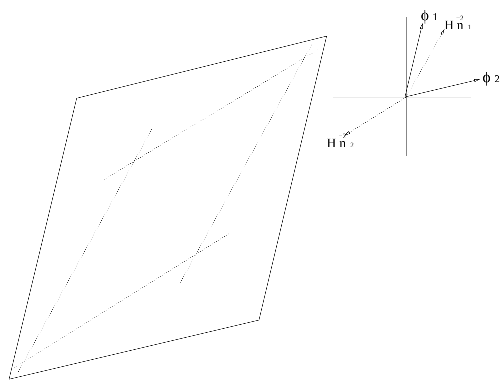

Figure 29

Recall from the introduction, in trying to control our function v on skewcube S our first step was to show that the integral curves run in straight lines, we did this by using the "pull back idea" (which only required information about the bulk energy) and the "ODE method" in which we needed somehow to find integral curves along which $Dv(\cdot)$ stays close the wells. In our situation now, we have that for many pulled back integral curves through S, $Dv(\cdot)$ along the pulled back curve does indeed stay close to the wells (in fact stays close to $SO(2)H$) and as such by using bulk energy as we have done before, its easy to show that these integral curves form something very much like straight lines.

Observe fig 29. Its not hard to see we can find lines parallel to $H^{-2}n_1$ and $H^{-2}n_2$ which enclose a large subset of S and have the property that they intersect only controlled skewcubes for which $Dv(\cdot)$ is close to $SO(2)H$. Let $\mathfrak{D}$ denote the region enclosed. Let B_1, B_2, B_3, B_4 denote the sides of $\mathfrak{D}$, ie. $\partial\mathfrak{D} = \bigcup_{i=1}^{4} B_i$. By using the "pull back idea" and the "ODE method" carefully its not hard to show that $v_{\lfloor B_i} \approx (R_i H + \zeta_i)_{\lfloor B_i}$ for some $R_i \in SO(2)$ and some $\zeta_i \in \mathbb{R}^2$. And then by using Lemma 6 and the fact that $\det(Dv(\cdot)) \approx 1$ for most points, we can (with some care) show that $v_{\lfloor \partial\mathfrak{D}} \approx \left(\widetilde{R}H + \zeta\right)_{\lfloor \partial\mathfrak{D}}$. From this, by considering the action of v on rank -1 lines parallel to ϕ_1 and ϕ_2 running through $\mathfrak{D}$, its easy to see that any triangle $\tau_i \subset \mathfrak{D}$ is such that if L denotes the linear part of the affine interpolation of v on the corners of τ_i, then L is very close to a matrix in $SO(2)H$.

This does the case where at least half the controlled subskewcubes are such that $Dv(\cdot)$ are close to $SO(2)H$. In the case where half the controlled subskewcubes are close to $SO(2)$ we can argue in an entirely analogous way. In this case we will consider lines going through S in directions n_1 and n_2, these form pullbacks of integral curves for which $Dv(\cdot)$ stays close to $SO(2)$. We can then argue the same way to get control of a region contained by four such lines, specifically we can conclude that any triangle τ_i contained in this region is such that the linear part of the affine interpolation of v on τ_i is close to a matrix in $SO(2)$. This is how the proof works.

Lemma 9. *Let $v \in \mathcal{A}_F(\Omega)$ and let $S := P(a, \phi_1, \phi_2, \mathfrak{c}_9\epsilon) \subset \Omega$ where $\mathfrak{c}_9$ is some large constant we will decide on later.*

Let

$$\left\{\mathfrak{L}_{i,j} := P\left(a_{i,j}, \phi_1, \phi_2, \kappa^{m_0}\epsilon\right) : i, j \in \left\{1, 2, \ldots \left[\kappa^{-m_0}\right] + 1\right\}\right\}$$

be a set of pairwise disjoint skewcubes such that $S \subset \cup_{i,j\in\{1,\ldots[\kappa^{-m_0}]+1\}}\mathfrak{L}_{i,j}$. Let

$$S_{i,j} := P\left(a_{i,j}, \phi_1, \phi_2, \mathfrak{c}_8\kappa^{m_0}\epsilon\right).$$

Suppose we have a set $G_0 \subset \{S_{i,j} : i, j \in \{1, \ldots [\kappa^{-m_0}] + 1\}\}$ such that for any $S_{i,j} \in G_0$

- $$L^2\left(\left\{x \in S_{i,j} : d\left(Dv(x), R_{i,j}\mathfrak{S}_{i,j}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) < 20\mathfrak{c}_7^2\tilde{\mathfrak{c}}_4\sigma^{-8}\kappa^{\frac{m_0}{64}}\left(\epsilon\kappa^{m_0}\right)^2$$

  for some $R_{i,j} \in SO(2)$, $\mathfrak{S}_{i,j} \in \{Id, H\}$
- $$\mathrm{Card}\left(G_0\right) \geq \frac{1 - 16\sigma^{-2}\kappa^{\frac{m_0}{8}}}{\kappa^{2m_0}}$$

And we suppose also that

$$\int_S \left|D^2 v(x)\right| dL^2 x \leq \kappa\epsilon \tag{169}$$

and

$$\int_S d(Dv, K)\, dL^2 x \leq \kappa^{\frac{m_0}{2}}\epsilon^2 \tag{170}$$

then if τ_i is the triangle in $\triangle_\epsilon$ that contains a, let L_i be the linear part of the affine map we get from the interpolation of v on the corners of τ_i, we have the following inequality;

$$d\left(L_i, SO(2) \cup SO(2)H\right) < \kappa^{\frac{m_0}{1024}}.$$

Proof. Let $B_0 := \{S_{i,j} : S_{i,j} \cap S \neq \emptyset$ and $S_{i,j} \notin G_0\}$. So $\mathrm{Card}(B_0) \leq \frac{16\sigma^{-2}\kappa^{\frac{m_0}{8}}}{\kappa^{2m_0}}$.

We start by having to consider two cases,

Let

$$G_0^{(1)} := \left\{S_{i,j} \in G_0 : Dv_{\lfloor S_{i,j}} \approx R_{i,j} \text{ for some } R_{i,j} \in SO(2)\right\}$$

and let

$$G_0^{(2)} := \left\{S_{i,j} \in G_0 : Dv_{\lfloor S_{i,j}} \approx R_{i,j}T_{i,j} \text{ for some } R_{i,j} \in SO(2) \text{ and } T_{i,j} \in \{Id, H\}\right\}.$$

<u>Case 1:</u> $\mathrm{Card}\left(G_0^{(2)}\right) \geq \mathrm{Card}\left(G_0^{(1)}\right)$.

<u>Case 2:</u> $\mathrm{Card}\left(G_0^{(1)}\right) \geq \mathrm{Card}\left(G_0^{(2)}\right)$.

We will have to argue the two cases in analogues, but different ways. As Case 1 is a little more intricate we chose to argue it in detail.

<u>Step 1</u> Take vector $\Diamond_1$. Recall definition (54).

We can find a chain of points $\{\tilde{x}_k : k = 1, 2, \ldots M_0\} \subset P_{\Diamond_1^\perp}(S)$, $M_0 = \left[\frac{L^1\left(P_{\Diamond_1^\perp}(S)\right)}{\kappa^{m_0}\epsilon}\right]$ such that $|\tilde{x}_{k-1} - \tilde{x}_k| = |\tilde{x}_k - \tilde{x}_{k+1}| = \epsilon\kappa^{m_0}$ for $k = 2, 3, \ldots M_0 - 1$. Now for $k \in \{2, 3, \ldots M_0\}$ let

$$\widetilde{Z_k} := \left\{S_{i,j} \in S : \mathfrak{L}_{i,j} \cap P^{-1}_{\Diamond_1^\perp}\left(\tilde{x}_k\right) \neq \emptyset\right\}.$$

Note $\{S_{i,j} \in S\} \subset \bigcup_{k=1}^{M_0} \widetilde{Z_k}$. Let

$$E_0 := \left\{ \begin{array}{ll} k \in \{1,2,\dots M_0\} : & \text{There exists } S_{i,j}^{(1)} \in \widetilde{Z_k} \cap G_0 \text{ with } Dv_{\lfloor S_{i,j}^{(1)}} \approx SO(2) \text{ and} \\ & \qquad S_{i,j}^{(2)} \in \widetilde{Z_k} \cap G_0 \text{ with } Dv_{\lfloor S_{i,j}^{(2)}} \approx SO(2)\,H \end{array} \right\}$$

By $5r$ Covering Theorem ([16] Theorem 2.1) can take some subset $E_1 \subset E_0$ which has the following properties

- For any $k_1, k_2 \in E_1$ that are not equal, $\widetilde{Z_{k_1}} \cap \widetilde{Z_{k_2}} = \emptyset$.
-
$$\bigcup_{k \in E_0} B_{2\kappa^{m_0}\epsilon}(\tilde{x}_k) \subset \bigcup_{k \in E_1} B_{10\kappa^{m_0}\epsilon}(\tilde{x}_k). \tag{171}$$

Now we clearly have that for each $k \in E_1$ we have

$$\int_{P^{-1}_{\Diamond_i^\perp}\left(B_{10\kappa^{m_0}\epsilon}(\tilde{x}_k)\right)} \left|D^2 v(x)\right| dL^2 x \geq \frac{\kappa^{m_0}\epsilon}{2}.$$

So from (170) $\mathrm{Card}(E_1)\frac{\kappa^{m_0}}{2}\epsilon \leq \frac{\kappa\epsilon}{2}$ which implies that

$$\mathrm{Card}(E_1) \leq \kappa^{1-m_0} \tag{172}$$

and from (171) we have

$$\begin{aligned} \mathrm{Card}(E_0)\,2\kappa^{m_0}\epsilon &= L^1\left(P_{\Diamond_1^\perp}\left(\bigcup_{k \in E_0} B_{2\kappa^{m_0}\epsilon}(\tilde{x}_k)\right)\right) \\ &\leq L^1\left(P_{\Diamond_1^\perp}\left(\bigcup_{k \in E_1} B_{10\kappa^{m_0}\epsilon}(\tilde{x}_k)\right)\right) \\ &\leq 10\kappa^{m_0}\epsilon\,\mathrm{Card}(E_1). \end{aligned}$$

So $\mathrm{Card}(E_1) \geq \frac{\mathrm{Card}(E_0)}{5}$ and thus from (172) we have

$$\mathrm{Card}(E_0) \leq 5\kappa^{1-m_0}. \tag{173}$$

Let $Q_0 := \{1,2,\dots M_0\} \backslash E_0$. So for any $k \in Q_0$, $\widetilde{Z_k} \cap G_0$ consists only of skewcubes $S_{i,j}$ for which $Dv_{\lfloor S_{i,j}} \approx SO(2)$ or consists only of skewcubes $S_{i,j}$ for which $Dv_{\lfloor S_{i,j}} \approx SO(2)\,H$. Let

$$Q_1 := \left\{ k \in Q_0 : \mathrm{Card}\left(\widetilde{Z_k} \cap B_0\right) \leq \kappa^{\frac{m_0}{16}}\kappa^{-m_0} \right\}$$

Now by a similar application of the $5r$ covering theorem

$$\mathrm{Card}(Q_0 \backslash Q_1)\,\kappa^{\frac{m_0}{16}}\kappa^{-m_0} \leq \frac{20\kappa^{\frac{m_0}{8}}}{\kappa^{2m_0}}$$

so

$$\mathrm{Card}(Q_0 \backslash Q_1) \leq \frac{20\kappa^{\frac{m_0}{16}}}{\kappa^{m_0}}.$$

Thus

$$\mathrm{Card}(Q_1) = \left(1 - 5\kappa - 20\kappa^{\frac{m_0}{16}}\right)\kappa^{-m_0}. \tag{174}$$

Finally for some constant we let

$$\widetilde{Q_2} = \left\{ k \in \{1,2,\dots M_0\} : L^1\left(P^{-1}_{\Diamond_1^\perp}(\tilde{x}_k) \cap S\right) \geq 10\kappa\epsilon \right\}$$

so its easy to see from the diagram $Q_1 \backslash \widetilde{Q_2} \approx 10c_1\kappa^{-m_0+1}$ for some $c_1 := c(\sigma) > 0$. Let $Q_2 := Q_1 \backslash \widetilde{Q_2}$. So from (174) we have

$$\mathrm{Card}(\{1,2,\dots M_0\} \backslash Q_2) \leq 20c_1\kappa^{1-m_0}. \tag{175}$$

<u>Step 2.</u>

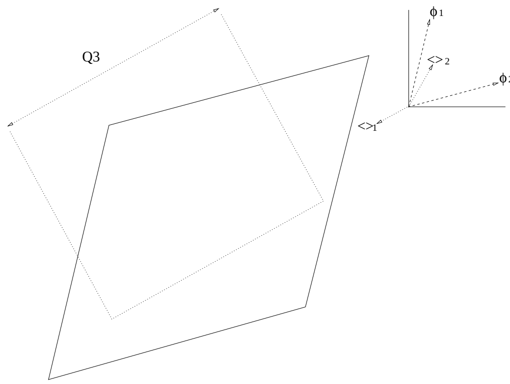

FIGURE 30

Now we will show that either

- For any $S_{i,j} \in \bigcup_{k \in Q_2} \widetilde{Z_k}$, $Dv_{\lfloor S_{i,j}} \approx A_{i,j}$ for $A_{i,j} \in SO(2)$
- For any $S_{i,j} \in \bigcup_{k \in Q_2} \widetilde{Z_k}$, $Dv_{\lfloor S_{i,j}} \approx B_{i,j}$ for $B_{i,j} \in SO(2)\,H$.

Suppose this was not true, so we can find $k_1, k_2 \in Q_2$ such that:

- For all $S_{i,j} \in \widetilde{Z_{k_1}}$
$$Dv_{\lfloor S_{i,j}} \approx A_{i,j} \text{ with } A_{i,j} \in SO(2)$$
- For all $S_{i,j} \in \widetilde{Z_{k_2}}$,
$$Dv_{\lfloor S_{i,j}} \approx B_{i,j} \text{ with } B_{i,j} \in SO(2)\,H.$$

Now by considering the change in derivative of Dv in direction $\Diamond_1$ from the set $\bigcup_{S_{i,j} \in \widetilde{Z_1}} S_{i,j}$ to the set $\bigcup_{S_{i,j} \in \widetilde{Z_2}} S_{i,j}$ we see that

$$\int_S \left|D^2 v(x)\right| dL^2 x \geq 10\kappa\epsilon$$

which contradicts assumption (169). So we have established the claim.

Now if we start the argument again from the beginning and instead of taking a chain of points in $P_{\Diamond_1^{\perp}}(S)$ we take a chain of points in $P_{\Diamond_2^{\perp}}(S)$, we can then run through the whole argument again to establish the following.

Let $\{\tilde{z_k} : k = 1, 2 \ldots M_1\} \subset P_{\Diamond_2^{\perp}}(S)$, ($M_1 = \left[\frac{L^1\left(P_{\Diamond_2^{\perp}}(S)\right)}{\kappa^{m_0}\epsilon}\right]$) be a chain of points such that

$$|\tilde{z}_k - \tilde{z}_{k+1}| = \epsilon\kappa^{m_0}$$

for $k = 1, 2, \ldots M_1 - 1$. Let $P_1 := \left\{k \in \{1, \ldots M_1\} : L^1\left(P^{-1}_{\Diamond_2^{\perp}}(\tilde{z_k}) \cap S\right) > 10\kappa\epsilon\right\}$. For $k \in \{1, 2, \ldots M_1\}$ we let $\widetilde{Y_k} := \left\{S_{i,j} : \mathfrak{L}_{i,j} \cap P^{-1}_{\Diamond_2^{\perp}}(\tilde{z_k}) \neq \emptyset\right\}$. We can find a subset $P_2 \subset P_1$ with the following properties.

There exists $\mathfrak{G} \in \{Id, H\}$ such that

- For any $S_{i,j} \in \bigcup_{k \in P_2} Y_k$ we have
$$Dv_{\lfloor S_{i,j}}(\cdot) \approx A_{i,j} \tag{176}$$
with $A_{i,j} \in SO(2)\,\mathfrak{G}$.
-
$$\mathrm{Card}\left(\{1, 2, \ldots M_1\} \setminus P_2\right) \leq 20 c_1 \kappa^{1-m_0} \tag{177}$$

Step 3. Now we refine the set up. Now for any $k_1 \in Q_2$ we have situation shown on fig 31.

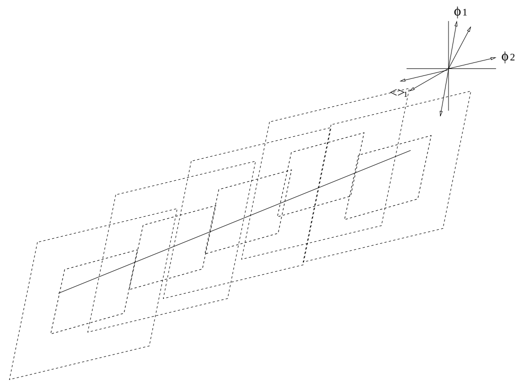

FIGURE 31

Since every point in $P^{-1}_{\Diamond_1^{\perp}}(x_{k_1}) \cap S$ is covered by skewcubes $\mathfrak{L}_{i,j}$ which are in turn covered by $S_{i,j}$. We have

$$N_{\sigma^2 c_2 \epsilon \kappa^{m_0}}\left(P^{-1}_{\Diamond_1^{\perp}}(x_{k_1})\right) \cap S \subset \bigcup_{S_{i,j} \in Z_k} S_{i,j}.$$

Now by definition of Q_2, for some $\mathfrak{S} \in \{Id, H\}$ we have that for any $S_{i,j} \in \bigcup_{k \in Q_2} \widetilde{Z_k}$,

$$L^2\left(\left\{x \in S_{i,j} : d\left(Dv(x), SO(2)\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) < 20\mathfrak{c}_7^2 \tilde{\mathfrak{c}}_4 \sigma^{-8} \kappa^{\frac{m_0}{64}} \left(\epsilon \kappa^{m_0}\right)^2.$$

So

$$\begin{aligned} & L^2\left(\left\{x \in N_{\sigma^2 c_2 \epsilon \kappa^{m_0}}\left(P^{-1}_{\Diamond_1^{\perp}}(x_{k_1})\right) \cap S : d\left(Dv(x), SO(2)\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) \\ & \leq \sum_{S_{i,j} \in \widetilde{Z_k}} L^2\left(\left\{x \in S_{i,j} : d\left(Dv(x), SO(2)\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) \\ & \leq 20\mathfrak{c}_7^2 \tilde{\mathfrak{c}}_4 \sigma^{-8} \mathrm{Card}\left(\widetilde{Z_k}\right) \kappa^{\frac{m_0}{64}} \left(\epsilon \kappa^{m_0}\right)^2. \end{aligned}$$

Now as $\{S_{i,j} : i, j \in \{1, 2, \ldots [\kappa^{-m_0}] + 1\}\}$ do not overlap by more than $\mathfrak{c}_8$ times, formally

$$\|\sum_{i,j \in \{1, \ldots [\kappa^{-m_0}]+1\}} X_{S_{i,j}}\|_\infty \leq \mathfrak{c}_8.$$

We have

$$\begin{aligned} \mathrm{Card}\left(\widetilde{Z_k}\right) & \leq \mathfrak{c}_8 \mathrm{Card}\left(\mathfrak{L}_{i,j} : \mathfrak{L}_{i,j} \cap P^{-1}_{\Diamond_1^{\perp}}(\tilde{x}_k) \neq \emptyset\right) \\ & \leq 2\mathfrak{c}_8 \kappa^{-m_0}. \end{aligned}$$

So

$$L^2\left(\left\{x \in N_{\mathfrak{c}_8 \sigma^2 \epsilon \kappa^{m_0}}\left(P^{-1}_{\Diamond_1^{\perp}}(\tilde{x}_{k_1})\right) \cap S : d\left(Dv(x), SO(2)\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) \leq c_2 \epsilon^2 \kappa^{m_0\left(1+\frac{1}{64}\right)}$$

and so by Fubini we must be able to find some $x_{k_1} \in B_{\kappa^{m_0}\left(1+\frac{1}{128}\right)\epsilon}(\tilde{x}_{k_1})$ so

$$L^1\left(\left\{x \in P^{-1}_{\Diamond_1^{\perp}}(x_{k_1}) \cap S : d\left(Dv(x), SO(2)\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) \leq c_2 \epsilon \kappa^{\frac{m_0}{128}}. \tag{178}$$

So by doing this for every $k \in Q_2$, we can find a set $\{x_k : k \in Q_2\}$ such that

- For every $k \in Q_2$
$$L^1\left(\left\{x \in P^{-1}_{\Diamond_1^{\perp}}(x_k) \cap S : d\left(Dv(x), SO(2)\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) \leq c_2 \epsilon \kappa^{\frac{m_0}{128}} \tag{179}$$
- $|x_k - \tilde{x}_k| \leq \kappa^{m_0\left(1+\frac{1}{128}\right)}\epsilon$ for each $k \in Q_2$.

Let $Z_k := \left\{S_{i,j} : \mathfrak{L}_{i,j} \cap P^{-1}_{\Diamond_1^\perp}(x_k) \neq \emptyset\right\}$.

Now in exactly the same way we can find a set $\{z_k : k \in P_2\}$ such that for some $\mathfrak{S} \in \{Id, H\}$

- For every $k \in P_2$
$$L^1\left(\left\{x \in P^{-1}_{\Diamond_2^\perp}(z_k) \cap S : d\left(Dv(x), SO(2)\,\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) \leq c_6 \epsilon \kappa^{\frac{m_0}{128}}$$
- $|z_k - \tilde{z_k}| \leq \kappa^{m_0\left(1+\frac{1}{128}\right)}\epsilon$ for each $k \in P_2$.

Let $Y_k := \left\{S_{i,j} : \mathfrak{L}_{i,j} \cap P^{-1}_{\Diamond_2^\perp}(z_k) \neq \emptyset\right\}$.

Step 4

Observe fig 32.

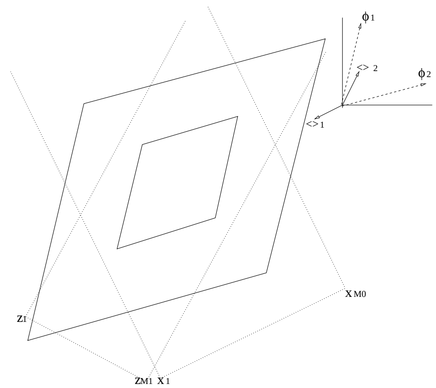

Figure 32

Let
$$\Delta_1 := \left\{S_{i,j} \in S : \mathfrak{L}_{i,j} \cap P^{-1}_{\Diamond_1^\perp}(x_k) \neq \emptyset \text{ for } k \in Q_2\right\}$$
and let
$$\Delta_2 := \left\{S_{i,j} \in S : \mathfrak{L}_{i,j} \cap P^{-1}_{\Diamond_2^\perp}(z_k) \neq \emptyset \text{ for } k \in P_2\right\}.$$
Now as can be seen from the diagram, assuming we chose $\mathfrak{c}_8$ big enough, then for some $c_3 > \sigma^{-1}$ we will decide on later, $I := P(a, \phi_1, \phi_2, \sigma c_3 \epsilon) \subset \bar{S}$ is such that
$$I \subset P^{-1}_{\Diamond_2^\perp}([z_1, z_{M_1}]) \cap P^{-1}_{\Diamond_1^\perp}([x_1, x_{M_0}]).$$

Now let $\Pi := \{S_{i,j} : S_{i,j} \cap I \neq \emptyset\}$. So if we have that $\Pi \cap \Delta_1 \cap \Delta_2 \neq \emptyset$ then we have that there exists $\mathfrak{S} \in \{Id, H\}$ such that for any $S_{i,j} \in \Pi \cap (\Delta_1 \cup \Delta_2)$ we have:
$$\text{There exists } R_{i,j} \in SO(2) \text{ such that } L^2\left(\left\{x \in S_{i,j} : d\left(Dv(x), R_{i,j}\mathfrak{S}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) < 20\mathfrak{c}_7^2 \tilde{\mathfrak{c}}_4 \sigma^{-8} \kappa^{\frac{m_0}{64}} \left(\epsilon \kappa^{m_0}\right)^2. \tag{180}$$
We will estimate $\mathrm{Card}\left(\Pi \backslash (\Delta_1 \cup \Delta_2)\right)$. Since for any $k_1 \in \{1, 2, \ldots M_1\}$, $k_2 \in \{1, 2, \ldots M_2\}$
$$\mathrm{Card}\left(Z_{k_1} \cap Y_{k_2}\right) \leq c_4$$
for some constant $c_4 = c(\sigma)$. So for any $k_1 \in \{1, 2, \ldots M_1\}$ we have
$$\begin{aligned} \mathrm{Card}\left(Z_{k_1} \cap \left(\bigcup_{k_2 \in \{1, \ldots M_1\} \backslash P_2} Y_{k_2}\right)\right) &= \mathrm{Card}\left(\bigcup_{k_2 \in \{1, \ldots M_1\} \backslash P_2} Z_{k_1} \cap Y_{k_2}\right) \\ &\leq \mathrm{Card}\left(\{1, \ldots M_1\} \backslash P_2\right) c_4. \end{aligned} \tag{181}$$

So by (177), (175)

$$\begin{aligned}\operatorname{Card}\left(\left(\bigcup_{k_1\in\{1,\ldots M_0\}\backslash Q_2} Z_{k_1}\right)\cap\left(\bigcup_{k_2\in\{1,\ldots M_1\}\backslash P_2} Y_{k_2}\right)\right) &\leq \sum_{k_1\in\{1,\ldots M_0\}\backslash Q_2}\operatorname{Card}\left(Z_{k_1}\cap\left(\bigcup_{k_2\in\{1,\ldots M_1\}\backslash P_2} Y_{k_2}\right)\right)\\ &\leq \operatorname{Card}\left(\{1,\ldots M_0\}\backslash Q_2\right)\operatorname{Card}\left(\{1,\ldots M_1\}\backslash P_2\right)c_4\\ &\leq 100c_1^2\frac{c_4\kappa^2}{\kappa^{2m_0}}.\end{aligned}$$

Now

$$\begin{aligned}\Pi\backslash\left(\Delta_1\cup\Delta_2\right) &= \left(\Pi\backslash\Delta_1\right)\cap\left(\Pi\backslash\Delta_2\right)\\ &\subset \left(\bigcup_{k_1\in\{1,\ldots M_0\}\backslash Q_2} Z_{k_1}\right)\cap\left(\bigcup_{k_2\in\{1,\ldots M_1\}\backslash P_2} Y_{k_2}\right)\end{aligned}$$

So

$$\operatorname{Card}\left(\Pi\backslash\left(\Delta_1\cup\Delta_2\right)\right)\leq 400c_1^2\frac{c_4\kappa^2}{\kappa^{2m_0}}$$

and thus $\Pi\cap\Delta_2\cap\Delta_2\neq\emptyset$ and so (180) is true.

Now the arguments are completely analogous no matter what $\mathfrak{S}$ is, however as the details are slightly more intricate when $\mathfrak{S}=H$ we will deal with this case.

See fig 33.

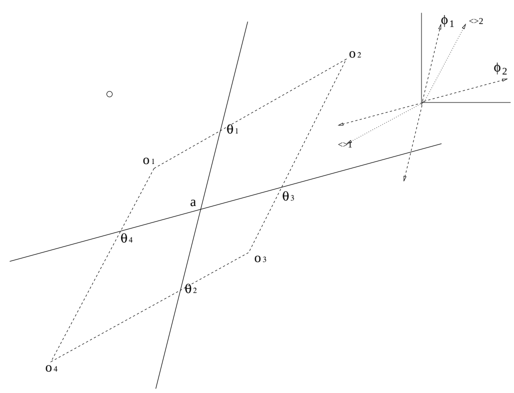

FIGURE 33

Let $\gamma_1\ldots\gamma_4>1$ be some numbers we will decide on later. Let $\theta_1:=a+\gamma_1\phi_1$, $\theta_2:=a-\gamma_2\phi_1$, $\theta_3:=a+\gamma_3\phi_2$ and $\theta_4:=a-\gamma_4\phi_2$.

We assume $\gamma_1\ldots\gamma_4$ have been chosen so that $\theta_1\in P^{-1}_{\Diamond_1^\perp}\left(x_{k_1}\right)$ for some $k_1\in Q_2$ and $\theta_2\in P^{-1}_{\Diamond_1^\perp}\left(x_{k_2}\right)$ for some $k_2\in Q_2$. And $\theta_3\in P^{-1}_{\Diamond_2^\perp}\left(z_{k_3}\right)$ for some $k_3\in P_2$ and $\theta_4\in P^{-1}_{\Diamond_1^\perp}\left(z_{k_4}\right)$ for some $k_4\in P_2$. We also assume (using (175),(177) that)

$$\left|\gamma_i-\gamma_j\right|<20c_1\kappa \text{ for any } i,j\in\{1,\ldots 4\}. \tag{182}$$

As shown on fig 33. Let $\mathfrak{D}$ denote the region enclosed by the lines $\{\theta_1+\langle\Diamond_1\rangle\}$, $\{\theta_2+\langle\Diamond_1\rangle\}$, $\{\theta_3+\langle\Diamond_2\rangle\}$ and $\{\theta_4+\langle\Diamond_2\rangle\}$ as shown.

Now we assume c_3 has been chosen such that $\{\gamma_i\}$ can be chosen so that $I_2:=P\left(a,\phi_1,\phi_2,c_{11}\kappa^{m_0}\epsilon\right)\subset\mathfrak{D}$ (for some $c_{11}>0$ we will decide on later) and $\mathfrak{D}\subset I$.

We assume $c_{11}>0$ is sufficiently smaller than γ_1 so that we can find $k_1\in Q_2$ such that $x_{k_1}\in P_{\langle\Diamond_1^\perp\rangle}\left(\mathfrak{D}\backslash I_2\right)$ with $P^{-1}_{\langle\Diamond_1^\perp\rangle}\left(x_{k_1}\right)\cap\{a+\alpha\phi_1:\alpha>0\}\neq\emptyset$. In the same way we can find $k_3\in Q_2$ such that $x_{k_3}\in P_{\langle\Diamond_1^\perp\rangle}\left(\mathfrak{D}\backslash I_2\right)$ and $P^{-1}_{\Diamond_1^\perp}\left(x_{k_3}\right)\cap\{a-\alpha\phi_1:\alpha>0\}$. We also can find k_2,k_4 with $z_{k_2},z_{k_4}\in P_{\langle\Diamond_2^\perp\rangle}\left(\mathfrak{D}\backslash I_2\right)$ and $P^{-1}_{\Diamond_2^\perp}\left(z_{k_2}\right)\cap\{a+\alpha\phi_2:\alpha>0\}\neq\emptyset$, $P^{-1}_{\Diamond_2^\perp}\left(z_{k_4}\right)\cap\{a+\alpha\phi_1:\alpha\leq 0\}\neq\emptyset$.

Step 5 We will show for $j \in \{1,3\}$

- For any $y_1, y_2 \in \mathfrak{D} \cap P^{-1}_{\diamondsuit_1^\perp}(x_{k_j})$ where $k_1 \in Q_2$ and $|y_1 - y_2| > \sqrt{\kappa}\epsilon$ we have

$$|v(y_1) - v(y_2)| \in \left[\left|P_{\phi_1^\perp}(y_1 - y_2)\right|\left(1 - c_5\kappa^{\frac{m_0}{256}}\right), \left|P_{\phi_1^\perp}(y_1 - y_2)\right|\left(1 + c_5\kappa^{\frac{m_0}{256}}\right)\right]. \tag{183}$$

  And specifically we have

$$H^1([y_1, y_2]) \leq \left(1 + c_5\kappa^{\frac{m_0}{256}}\right)\left|P_{\phi_1^\perp}(y_1 - y_2)\right| \tag{184}$$

- For $j \in \{2,4\}$ and for any $y_3, y_4 \in \mathfrak{D} \cap P^{-1}_{\diamondsuit_2^\perp}(z_{k_j})$ where $k_2 \in P_2$ and $|y_3 - y_4| > \sqrt{\kappa}\epsilon$ we have

$$|v(y_3) - v(y_4)| \in \left[\left|P_{\phi_2^\perp}(y_3 - y_4)\right|\left(1 - c_5\kappa^{\frac{m_0}{256}}\right), \left|P_{\phi_2^\perp}(y_3 - y_4)\right|\left(1 + c_5\kappa^{\frac{m_0}{256}}\right)\right]. \tag{185}$$

  And specifically we have

$$H^1([y_3, y_4]) \leq \left(1 + c_5\kappa^{\frac{m_0}{256}}\right)\left|P_{\phi_1^\perp}(y_3 - y_4)\right| \tag{186}$$

We will argue only the case where $y_1, y_2 \in \mathfrak{D} \cap P^{-1}_{\diamondsuit_1^\perp}(x_{k_j})$, $k_j \in Q_2$. The case where $y_3, y_4 \in \mathfrak{D} \cap P^{-1}_{\diamondsuit_2^\perp}(z_{k_j})$ for $k_j \in P_2$ follows in exactly the same way.

We start by establishing (184). Let $L = \left\{x \in [y_1, y_2] : d(Dv(x), SO(2)H) \leq \kappa^{\frac{m_0}{16}}\right\}$. For each $x \in [y_1, y_2]$ let $W(x) \in SO(2)H$ be such that

$$d(Dv(x), SO(2) \cup SO(2)H) = |Dv(x) - W(x)|.$$

So by (178) we have $L^1([y_1, y_2] \backslash L) \leq c_2\epsilon\kappa^{\frac{m_0}{128}}$ and so

$$\begin{aligned}
\left|H^1(v([y_1, y_2])) - |y_1 - y_2|\,|H\diamondsuit_1|\right| &= \left|\int_{[y_1, y_2]} (|Dv(x)\diamondsuit_1| - |H\diamondsuit_1|)\, dH^1x\right| \\
&\leq \int_L |Dv(x) - W(x)|\, dL^1x + \int_{[y_1, y_2]\backslash L} |Dv(x)| + |H\diamondsuit_1|\, dL^1x \\
&\leq \kappa^{\frac{m_0}{16}}|y_1 - y_2| + 2c_2\sigma^{-2}\epsilon\kappa^{\frac{m_0}{128}} \\
&\leq 4\sigma^{-2}c_2|y_1 - y_2|\kappa^{\frac{m_0}{128} - \frac{1}{2}}.
\end{aligned}$$

Now as we know from (48), $|y_1 - y_2|\,|H\diamondsuit_1| = \left|P_{\phi_1^\perp}(y_1 - y_2)\right|$ and thus

$$\left|H^1(v([y_1, y_2])) - \left|P_{\phi_1^\perp}(y_1 - y_2)\right|\right| \leq 4\sigma^{-2}c_2|y_1 - y_2|\kappa^{\frac{m_0}{128} - \frac{1}{2}}. \tag{187}$$

So this establishes (184).

Now we will show

$$|v(y_1) - v(y_2)| \geq \left|P_{\phi_1^\perp}(y_1 - y_2)\right|\left(1 - \kappa^{\frac{m_0}{256}}\right).$$

We appeal to Lemma 4 where it was shown that for any two points $\iota_1, \iota_2 \in v(S)$ such that $[\iota_1, \iota_2] \subset v(S)$ and

$$\int_{[\iota_1, \iota_2]} d\left(Dv\left(v^{-1}(z)\right), K\right) dL^1x < \kappa^{\frac{m_0}{128}}|\iota_1 - \iota_2|$$

we have

$$|\iota_1 - \iota_2| \geq \left(1 - 2\sigma^{-2}\kappa^{\frac{m_0}{256}}\right)|\Psi_1(\iota_1) - \Psi_1(\iota_2)|.$$

By the area formula we have $\int_{v(S)} J(x)\, dL^2x \leq \sigma^{-2}\kappa^{\frac{m_0}{2}}\epsilon^2$ so by Fubini we must be able to find interval $[\iota_1, \iota_2]$ parallel to $[v(y_1), v(y_2)]$ with

$$\int_{[\iota_1, \iota_2]} d\left(Dv\left(v^{-1}(x)\right), K\right) dL^1x \leq \kappa^{\frac{m_0}{64}}\epsilon \leq \kappa^{\frac{m_0}{128}}|l_1 - l_2|.$$

And $\iota_1 \in N_{\kappa^{\frac{m_0}{128}}\epsilon}(v(y_1))$ and $\iota_2 \in N_{\kappa^{\frac{m_0}{128}}\epsilon}(v(y_2))$. So by Lemma 4 we have

$$|\iota_1 - \iota_2| \geq \left(1 - 2\sigma^{-2}\kappa^{\frac{m_0}{128}}\right)|\Psi_1(\iota_1) - \Psi_1(\iota_2)|.$$

Now by Bilipschitzness we have

$$\begin{aligned} |v(y_1) - v(y_2)| &\geq \left(1 - 5\sigma^{-2}\kappa^{\frac{m_0}{128}}\right)|\Psi_1(v(y_1)) - \Psi_1(v(y_2))| \\ &= \left(1 - 5\sigma^{-2}\kappa^{\frac{m_0}{128}}\right)L^1\left(P_{\phi_1^\perp}([y_1, y_2])\right). \end{aligned}$$

So this together with (187) establishes Step 5

Step 6

Take $q \in \{2, 4\}$. Consider the line $(\theta_q + \langle\Diamond_2\rangle) \cap \mathfrak{D}$. We will show that there exists $R_q \in SO(2)$ such that for any $z \in (\theta_q + \langle\Diamond_2\rangle) \cap \mathfrak{D}$ with $|z - \theta_q| > \sqrt{\kappa}\epsilon$ we have

$$v(z) \in B_{c_6\kappa^{\frac{m_0}{1024}}\epsilon}(v(\theta_q) + R_q H(z - \theta_q)). \tag{188}$$

And similarly for $q \in \{1, 3\}$. For the line $(\theta_q + \langle\Diamond_1\rangle) \cap \mathfrak{D}$ we will show that there exists $R_q \in SO(2)$ such that for any $z \in (\theta_q + \langle\Diamond_1\rangle) \cap \mathfrak{D}$ with $|z - \theta_q| > \sqrt{\kappa}\epsilon$ we have

$$v(z) \in B_{c_6\kappa^{\frac{m_0}{1024}}\epsilon}(v(\theta_q) + R_q H(z - \theta_q)). \tag{189}$$

We argue only the case $q = 1$. All other cases follow in exactly the same way. Observe fig 34 below.

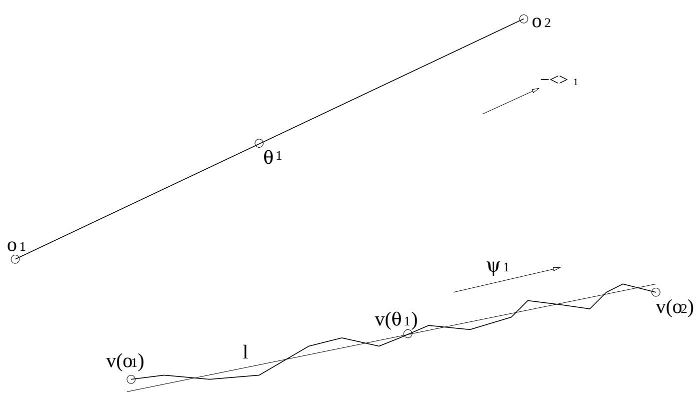

FIGURE 34

Firstly by the Pythagoras type arguments coming from Lemma 5 (see in particular fig 7) (183) implies that

$$v([o_1, o_2]) \subset N_{\kappa^{\frac{m_0}{512}}\epsilon}(v(\theta_1) + l) \tag{190}$$

for some $l \in G(1, 2)$.

Let $\psi_1 \in S^1$ be such that $\langle\psi_1\rangle = l$ and $\psi_1 \cdot \left(\frac{v(o_2) - v(o_1)}{|v(o_2) - v(o_1)|}\right) > 0$. Let $R_1 \in SO(2)$ be the rotation such that $R_1\left(\frac{o_2 - o_1}{|o_2 - o_1|}\right) = \psi_1$. Now for any $z \in (\theta_1 + \langle\Diamond_1\rangle) \cap \mathfrak{D}$ with $|z - \theta_1| > \sqrt{\kappa}\epsilon$ and $z \cdot (-\Diamond_1) \geq \theta_1 \cdot (-\Diamond_1)$. Let $\widetilde{R}$ be a rotation such that $\widetilde{R}\left(\frac{z - \theta_1}{|z - \theta_1|}\right) = \frac{v(z) - v(\theta_1)}{|v(z) - v(\theta_1)|} \approx \psi_1$.

So

$$v(z) = v(\theta_1) + |v(z) - v(\theta_1)|\,\widetilde{R}\left(\frac{z - \theta_1}{|z - \theta_1|}\right). \tag{191}$$

Now (190) we know $\left|(v(z)-v(\theta_1))\cdot\psi_1^{\perp}\right|\leq\kappa^{\frac{m_0}{512}}\epsilon$ and so

$$\begin{aligned}|(v(z)-v(\theta_1))\cdot\psi_1| &\geq |v(z)-v(\theta_1)|-\left|(v(z)-v(\theta_1))\cdot\psi_1^{\perp}\right|\\ &\geq |v(z)-v(\theta_1)|-\kappa^{\frac{m_0}{512}}\epsilon. \end{aligned}\tag{192}$$

So as we know by Bilipschitzness that

$$\begin{aligned}|v(z)-v(\theta_1)| &\geq \sigma^{-2}|z-\theta_1|\\ &\geq \sigma^{-2}\sqrt{\kappa}\epsilon. \end{aligned}\tag{193}$$

And as $\frac{o_2-o_1}{|o_2-o_1|}=\frac{z-\theta_1}{|z-\theta_1|}=-\Diamond_1$ we have

$$\begin{aligned}\left|\left(R_1-\widetilde{R}\right)\cdot\Diamond_1\right| &= \left|\psi_1-\frac{v(z)-v(\theta_1)}{|v(z)-v(\theta_1)|}\right|\\ &\leq \left|1-\frac{(v(z)-v(\theta_1))\cdot\psi_1}{|v(z)-v(\theta_1)|}\right|+\left|\frac{(v(z)-v(\theta_1))\cdot\psi_1^{\perp}}{|v(z)-v(\theta_1)|}\right|\\ &\leq |v(z)-v(\theta_1)|^{-1}\left(||v(z)-v(\theta_1)|-(v(z)-v(\theta_1))\cdot\psi_1|+\left|(v(z)-v(\theta_1))\cdot\psi_1^{\perp}\right|\right)\\ &\leq \frac{2\kappa^{\frac{m_0}{512}}\epsilon}{\sigma^2\sqrt{\kappa}\epsilon}\\ &\leq c_5\kappa^{\frac{m_0}{1024}}\end{aligned}$$

which implies $\left|R_1-\widetilde{R}\right|\leq c_5\kappa^{\frac{m_0}{1024}}$.

Now putting this together with (191) we have from (190), (192) and (193)

$$v(z)\in B_{c_5\epsilon\kappa^{\frac{m_0}{1024}}}\left(v(\theta_1)+|v(z)-v(\theta_1)|\,R_1\left(\frac{z-\theta_1}{|z-\theta_1|}\right)\right)$$

which completes the proof of (189) for the case $z\cdot(-\Diamond_1)\geq\theta_1\cdot(-\Diamond_1)$.

Now if $z\cdot(-\Diamond_1)<\theta_1\cdot(-\Diamond_1)$ we let $\bar{R}$ be the rotation such that

$$\bar{R}\left(\frac{z-\theta_1}{|z-\theta_1|}\right)=\frac{v(z)-v(\theta_1)}{|v(z)-v(\theta_1)|}\approx-\psi_1\tag{194}$$

but since $\frac{z-\theta_1}{|z-\theta_1|}=\Diamond_1$ we can see as before (since $R_1(\Diamond_1)=-\psi_1$) that $\left|\bar{R}-R_1\right|\leq c_5\kappa^{\frac{m_0}{1024}}$ so again by using this with an unscrambling of (194) we have complete the proof of (189).

Step 7. We will show $v(\mathfrak{D})$ is (roughly) mapped onto a parallelogram congruent to $H(\mathfrak{D})$.

First we find the corners of the shape $\mathfrak{D}$. Let $o_1:=(\theta_1+\langle\Diamond_1\rangle)\cap(\theta_4+\langle\Diamond_2\rangle)$, $o_2:=(\theta_1+\langle\Diamond_1\rangle)\cap(\theta_3+\langle\Diamond_2\rangle)$, $o_3:=(\theta_2+\langle\Diamond_1\rangle)\cap(\theta_3+\langle\Diamond_2\rangle)$, $o_4:=(\theta_2+\langle\Diamond_1\rangle)\cap(\theta_4+\langle\Diamond_2\rangle)$.

Let $\mathfrak{X}$ denote the shape contained by the lines

$$\{[v(o_1),v(o_2)],[v(o_1),v(o_4)],[v(o_3),v(o_4)],[v(o_3),v(o_2)]\}.$$

Note $\left|(o_1-o_4)\cdot\phi_1^{\perp}\right|=\left|(o_2-o_3)\cdot\phi_1^{\perp}\right|=:\alpha$ and $\left|(o_1-o_2)\cdot\phi_2^{\perp}\right|=\left|(o_3-o_4)\cdot\phi_2^{\perp}\right|=:\beta$ by Step 5 (see fig 33) we have

$$|v(o_4)-v(o_1)|\in\left[\left(1-c_5\kappa^{\frac{m_0}{256}}\right)\alpha,\left(1+c_5\kappa^{\frac{m_0}{256}}\right)\alpha\right]$$

and

$$|v(o_2)-v(o_3)|\in\left[\left(1-c_5\kappa^{\frac{m_0}{256}}\right)\alpha,\left(1+c_5\kappa^{\frac{m_0}{256}}\right)\alpha\right].$$

And also by Step 5 we have

$$|v(o_1)-v(o_2)|\in\left[\left(1-c_5\kappa^{\frac{m_0}{256}}\right)\beta,\left(1+c_5\kappa^{\frac{m_0}{256}}\right)\beta\right]$$

and

$$|v(o_4)-v(o_3)|\in\left[\left(1-c_5\kappa^{\frac{m_0}{256}}\right)\beta,\left(1+c_5\kappa^{\frac{m_0}{256}}\right)\beta\right].$$

So $\mathfrak{X}$ must be roughly a parallelogram. Formally, there exists a parallelogram $\widetilde{\mathfrak{X}}$ such that

$$H\left(\mathfrak{X},\widetilde{\mathfrak{X}}\right) \leq c_7\kappa^{\frac{m_0}{256}}\epsilon.$$

Now by Step 6 we have that $H\left(v\left(\mathfrak{D}\right),\mathfrak{X}\right) \leq c_6\kappa^{\frac{m_0}{1024}}\epsilon$ and so

$$H\left(\widetilde{\mathfrak{X}},v\left(\mathfrak{D}\right)\right) \leq 4c_7\kappa^{\frac{m_0}{1024}}\epsilon. \tag{195}$$

By the fact that $\int_{\mathfrak{D}} J\left(x\right)dL^2x \leq \sigma^{-2}\kappa^{\frac{m_0}{2}}L^2\left(\mathfrak{D}\right)$ we have

$$L^2\left(v\left(\mathfrak{D}\right)\right) \in \left(\left(1-\kappa^{\frac{m_0}{4}}\right)L^2\left(\mathfrak{D}\right),\left(1+\kappa^{\frac{m_0}{4}}\right)L^2\left(\mathfrak{D}\right)\right)$$

and so

$$L^2\left(\widetilde{\mathfrak{X}}\right) \in \left(\left(1-4c_7\kappa^{\frac{m_0}{1024}}\right)L^2\left(\mathfrak{D}\right),\left(1+4c_7\kappa^{\frac{m_0}{1024}}\right)L^2\left(\mathfrak{D}\right)\right).$$

Now given that the size lengths of $\widetilde{\mathfrak{X}}$ are prescribed and are (upto error $2c_7\kappa^{\frac{m_0}{512}}\epsilon$) the same side lengths as $H\left(\mathfrak{D}\right)$, there can only be two possible parallelogram with volume equal to $L^2\left(\mathfrak{D}\right)$. So (see fig 35) $v\left(\mathfrak{D}\right)$ is mapped (roughly) onto one of two possible parallelograms, as shown.

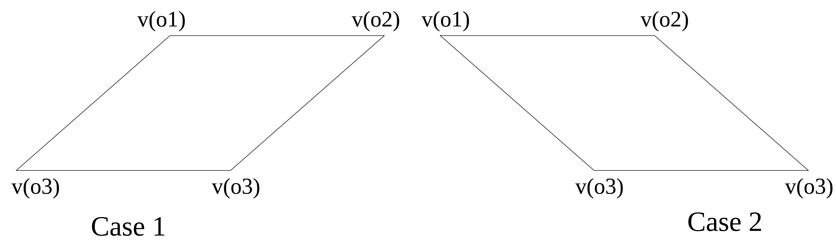

Figure 35

We know of course that H maps $\mathfrak{D}$ onto a parallelogram and $L^2\left(H\left(\mathfrak{D}\right)\right) = L^2\left(\mathfrak{D}\right)$. So $H\left(\mathfrak{D}\right)$ is one of the parallelograms shown in fig 35, in fact its parallelogram show is Case 2.

We will show that $v\left(\mathfrak{D}\right)$ is mapped onto the parallelogram given by $H\left(\mathfrak{D}\right)$. The fig 36 below shows $H\left(\mathfrak{D}\right)$.

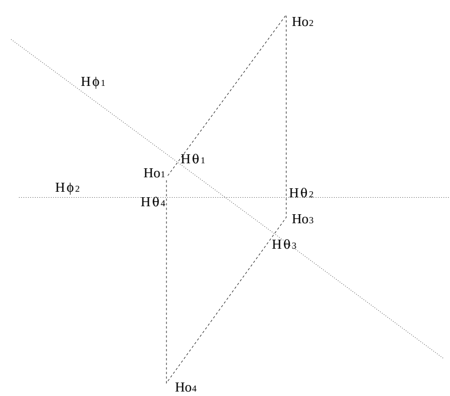

Figure 36

Now by Step 5 and (49) we see

$$\left|v\left(\theta_4\right)-v\left(o_1\right)\right| \leq \left(1+c_5\kappa^{\frac{m_0}{256}}\right)\left|H\left(\theta_4-o_1\right)\right|$$

and

$$\left|v\left(\theta_1\right)-v\left(o_1\right)\right| \leq \left(1+c_5\kappa^{\frac{m_0}{256}}\right)\left|H\left(\theta_1-o_1\right)\right|.$$

So if we had Case 2 and $v(o_1)$ was mapped to one of the sharp corners of the parallelograms, then $v(\theta_1)$ would be within $\left(1+c_5\kappa^{\frac{m_0}{256}}\right)|H(\theta_1-o_1)|$ of the sharp corner and $v(\theta_4)$ would also be within $\left(1+c_5\kappa^{\frac{m_0}{256}}\right)|H(\theta_4-o_1)|$ of the sharp corner.

In the same way $v(\theta_3)$ and $v(\theta_1)$ would be (respectively) within distance $\left(1+c_5\kappa^{\frac{m_0}{256}}\right)|H(\theta_1-o_3)|$, $\left(1+c_5\kappa^{\frac{m_0}{256}}\right)|H(\theta_3-o_3)|$ of the opposite sharp corner. Now as can be seen from the (36), in this case $|v(\theta_1)-v(\theta_3)|$ would have to be $\geq(1+c_8)|H(\theta_1-\theta_3)|$ for some not so small constant $c_8=c(\sigma)>0$. However as $\theta_1,\theta_3\in(a+\langle\phi_1\rangle)$ we have that $|v(\theta_1)-v(\theta_3)|\leq\left(1+\kappa^2\right)|\theta_1-\theta_3|=\left(1+\kappa^2\right)|H(\theta_1-\theta_3)|$ and this gives as a contradiction.

So we must have Case 1, which is to say $v(\mathfrak{D})$ is (roughly) mapped onto a parallelogram congruent $H(\mathfrak{D})$. Formally, (195) implies that for some $\zeta\in\mathbb{R}^2$ we have

$$H(v(\mathfrak{D}),H(\mathfrak{D})+\zeta)\leq 42c_7\kappa^{\frac{m_0}{1024}}\epsilon. \tag{196}$$

Step 8. We will use this to prove that for some $R\in SO(2)$ such that for any $x\in\partial\mathfrak{D}\backslash\bigcup_{i=1}^4 B_{\sqrt{\kappa}w}(\theta_i)$ we have

$$v(x)\in B_{42c_7\kappa^{\frac{m_0}{1024}}\epsilon}(v(\theta_1)+RH(x-\theta_1)). \tag{197}$$

We start by noting that since $v(\mathfrak{D})$ is an almost parallelogram (see (196)) we know that R_1 and R_3 from (189) are such that

$$|R_1-R_3|<42c_7\kappa^{\frac{m_0}{1024}} \tag{198}$$

and R_2, R_4 from (188) are such that

$$|R_2-R_4|<2c_6\kappa^{\frac{m_0}{1024}}. \tag{199}$$

So it suffices to show

$$|R_3-R_4|<8c_7\kappa^{\frac{m_0}{1024}}. \tag{200}$$

Now by (188) and (189) we have for any $z\in(\theta_4+\langle\Diamond_2\rangle)\cap\mathfrak{D}\backslash B_{\sqrt{\kappa}\epsilon}(o_4)$ that

$$v(z)\in B_{c_6\kappa^{\frac{m_0}{1024}}\epsilon}(v(o_4)+R_4H(z-o_4)) \tag{201}$$

and for any $z\in(\theta_3+\langle\Diamond_1\rangle)\cap\mathfrak{D}\backslash B_{\sqrt{\kappa}\epsilon}(o_4)$ we have

$$v(z)\in B_{c_6\kappa^{\frac{m_0}{1024}}\epsilon}(v(o_4)+R_3H(z-o_4)). \tag{202}$$

Now from (201), (202) we have

$$(v(o_1)-v(o_4))\cdot(v(o_3)-v(o_4))\in B_{2c_6\kappa^{\frac{m_0}{1024}}\epsilon}(R_4H(o_1-o_4)\cdot R_3H(o_3-o_4))$$

However by Step 7 we know $v(\mathfrak{D})$ is mapped onto a rectangle congruent to $H(\mathfrak{D})$ and so the arrangement of corners is as shown on Case 2, fig 36. And so, as can be seen from these diagrams, (196) implies that

$$(v(o_1)-v(o_4))\cdot(v(o_3)-v(o_4))\in B_{4c_7\kappa^{\frac{m_0}{1024}}\epsilon}(H(o_1-o_4)\cdot H(o_3-o_4))$$

So

$$R_4H(o_1-o_4)\cdot R_3H(o_3-o_4)\in B_{4c_7\kappa^{\frac{m_0}{1024}}}(H(o_1-o_4)\cdot H(o_3-o_4))$$

and thus

$$R_3^{-1}R_4H(o_1-o_4)\cdot H(o_3-o_4)\in B_{8c_7\kappa^{\frac{m_0}{1024}}}(H(o_1-o_4)\cdot H(o_3-o_4)).$$

So there are two possibilities. Either $R_3^{-1}R_4$ is close to the identity or $R_3^{-1}R_4$ flips $H(o_1-o_4)$ onto the other side of $H(o_3-o_4)$ as shown on fig 37. We will gain a contradiction from the possibility that $R_3^{-1}R_4$ is not close to the identity in the following way. Observe figure 38.

Now we can assume we have chosen $\{\theta_1,\theta_2,\theta_3,\theta_4\}$ such that

$$\int_{[o_4,o_1]}d(Dv(x),SO(2)\cup SO(2)H)\,dL^1x<\kappa|o_4-o_1|$$

and

$$\int_{[o_4,o_3]}d(Dv(x),SO(2)\cup SO(2)H)\,dL^1x<\kappa|o_4-o_3|$$

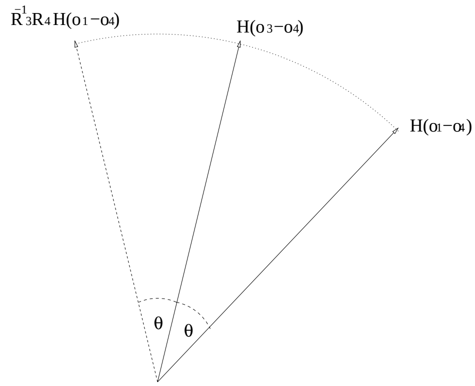

FIGURE 37

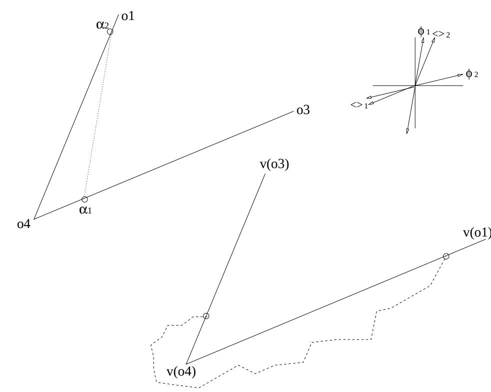

FIGURE 38

and so (as shown) we must be able to find $\alpha_1 \in [o_4, o_3]$ and $\alpha_2 \in [o_4, o_1]$ such that $\alpha_2 - \alpha_1$ is parallel to ϕ_1,

$$d\left(Dv\left(\alpha_1\right), SO\left(2\right) \cup SO\left(2\right) H\right) < \kappa$$

and

$$\int_{[\alpha_1,\alpha_2]} d\left(Dv\left(z\right), SO\left(2\right) \cup SO\left(2\right) H\right) dL^1 z \leq \kappa \left|\alpha_1 - \alpha_2\right|. \tag{203}$$

Now since $v\left(\left(\alpha_1, \alpha_2\right)\right)$ can not pass through $v\left(\left[o_4, o_3\right]\right)$ or $v\left(\left[o_4, o_1\right]\right)$ by Bilipschitzness. And more importantly since at point α_1 (as the $Dv\left(\alpha_1\right)$ is orientation preserving) the "triple junction" formed by the lines $[o_4, o_3]$ and $[\alpha_1, \alpha_2]$ must be mapped to a "similar" "triple junction" of $v\left(\left[o_4, o_3\right]\right)$ and $v\left(\left[\alpha_1, \alpha_2\right]\right)$. See figure 39.

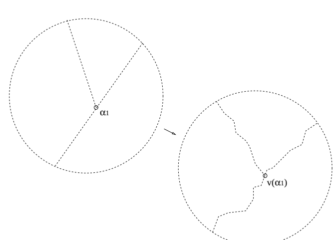

FIGURE 39

So $v([\alpha_1,\alpha_2])$ is forced to take a journey around the outside of $v([o_3,o_4])\cup v([o_1,o_4])$ as shown in fig 38. But this means

$$H^1(v([\alpha_1,\alpha_2]))>(1+c_{10})|\alpha_1-\alpha_2| \tag{204}$$

for some not small constant $c_{10}=c(\sigma)>0$.

However (203) since ϕ_1 is a rank-1 direction for H we have

$$H^1(v([\alpha_1,\alpha_2]))=\int_{[\alpha_1,\alpha_2]}|Dv(z)\phi_1|\,dL^1z\le(1+\kappa)|\alpha_1-\alpha_2|$$

which contradicts (204). So finally we have a contradiction from the assumption that $R_3^{-1}R_4$ flips $H(o_1-o_4)$ onto the other side of $H(o_3-o_4)$. So we must have the only other possibility which is

$$R_3^{-1}R_4\in N_{8c_7\kappa\frac{m_0}{1024}}(Id), \tag{205}$$

and this establishes (200).

Now we will use this prove (197). We will argue for $z\in[o_4,o_3]\backslash B_{\sqrt{\kappa}\epsilon}(\theta_2)$. See fig 33, by (189) and (198),(199), (205) we have

$$v(z)-v(o_4)\in B_{20c_7\kappa\frac{m_0}{1024}\epsilon}(R_1H(z-o_4)) \tag{206}$$

and by (188), (198), (199), (205)

$$v(o_4)-v(o_1)\in B_{20c_7\kappa\frac{m_0}{1024}\epsilon}(R_1H(o_4-o_1)) \tag{207}$$

and finally by (189)

$$v(o_1)-v(\theta_1)\in B_{c_6\kappa\frac{m_0}{1024}\epsilon}(R_1H(o_1-\theta_1)). \tag{208}$$

Now by adding together (206), (207) and (208) we have

$$v(z)\in B_{42c_7\kappa\frac{m_0}{1024}\epsilon}(R_1H(z-\theta_1)+v(\theta_1)) \tag{209}$$

and this establishes (197).

The cases, $z\in[o_1,o_4]\backslash B_{\sqrt{\kappa}\epsilon}(\theta_4)$, $z\in[o_2,o_4]\backslash B_{\sqrt{\kappa}\epsilon}(\theta_2)$, $z\in[o_2,o_3]\backslash B_{\sqrt{\kappa}\epsilon}(\theta_3)$ and can be argued in the same way

Step 8

Now we assume $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ have been chosen big enough so that if $\tau_i\in\triangle_\epsilon$ is the element of the triangulation that contains a, then $\tau_i\subset\mathfrak{D}$. Let $\{\omega_1,\omega_2,\omega_3\}$ denote the corners of τ_i. Now by assumption, none of the edges of τ_i are close to being parallel to the axis ϕ_1, ϕ_2. So we have that

$$\mathrm{Card}\left(\{\omega_1,\omega_2,\omega_3\}\cap\left(N_{\sqrt{\kappa}\epsilon}(a+\langle\phi_1\rangle)\cup N_{\sqrt{\kappa}\epsilon}(a+\langle\phi_2\rangle)\right)\right)\le 1.$$

If $\{\omega_1,\omega_2,\omega_3\}\cap\left(N_{\sqrt{\kappa}\epsilon}(a+\langle\phi_1\rangle)\cup N_{\sqrt{\kappa}\epsilon}(a+\langle\phi_2\rangle)\right)=\emptyset$ then the situation is even simpler, so we will argue the case where the interaction is non-empty.

So without loss of generality assume

$$\{\omega_1\}=\{\omega_1,\omega_2,\omega_3\}\cap N_{\sqrt{\kappa}\epsilon}(a+\langle\phi_1\rangle). \tag{210}$$

Let $\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3,\tilde\gamma_4>0$ be numbers we decide on later and let $\tilde\theta_1:=a+\tilde\gamma_1\phi_1$, $\tilde\theta_2:=a-\tilde\gamma_2\phi_1$, $\tilde\theta_3:=a+\tilde\gamma_3\phi_2$ and $\tilde\theta_4:=a+\tilde\gamma_4\phi_2$. By (175) and (177) we can assume $\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3,\tilde\gamma_4$ have been chosen so that

- $\tilde\theta_1\in P^{-1}_{\Diamond_1^\perp}(x_{k_5})$ for some $k_5\in Q_2$, $\tilde\theta_2\in P^{-1}_{\Diamond_1^\perp}(x_{k_6})$ for some $k_6\in Q_2$, $\tilde\theta_3\in P^{-1}_{\Diamond_2^\perp}(z_{k_7})$ for some $k_7\in P_2$ and $\tilde\theta_4\in P^{-1}_{\Diamond_2^\perp}(z_{k_8})$ for some $k_8\in P_2$.
- $\omega_1\in N_{20c_1\kappa}\left(P^{-1}_{\Diamond_1^\perp}\left(P_{\Diamond_1^\perp}\left(\tilde\theta_1\right)\right)\right)$, $\omega_2\in N_{20c_1\kappa}\left(P^{-1}_{\Diamond_2^\perp}\left(P_{\Diamond_2^\perp}\left(\tilde\theta_3\right)\right)\right)$ and $\omega_3\in N_{20c_1\kappa}\left(P^{-1}_{\Diamond_2^\perp}\left(P_{\Diamond_2^\perp}\left(\tilde\theta_4\right)\right)\right)$.

Note that this last condition together with (210) implies $\omega_1\in N_{\sqrt{\kappa}}\left(\tilde\theta_1\right)$. Let $\widetilde{\mathfrak{D}}$ denote the region enclosed by

$$\left\{\left(\tilde\theta_1+\langle\Diamond_1\rangle\right),\left(\tilde\theta_2+\langle\Diamond_1\rangle\right),\left(\tilde\theta_3+\langle\Diamond_2\rangle\right),\left(\tilde\theta_4+\langle\Diamond_2\rangle\right)\right\}.$$

Either by arguments entirely analogues to those we used to establish (197) or by using (197) directly to show that $v\left(\widetilde{\mathfrak{D}}\right)$ isn't mapped onto the "wrong parallelogram" (the one that isn't $H\left(\widetilde{\mathfrak{D}}\right)$) we can establish the following:

There exits $R \in SO(2)$ such that for any

$$x \in \partial\widetilde{\mathfrak{D}} \backslash \bigcup_{j=1}^{4} B_{\sqrt{\kappa}\epsilon}\left(\tilde{\theta}_j\right)$$

that

$$v(x) \in B_{42c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(v\left(\tilde{\theta}_1\right) + RH\left(x - \tilde{\theta}_1\right)\right) \subset B_{44c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(v(\omega_1) + RH(x - \omega_1)\right).$$

This of course implies that

$$v(\omega_2) \in B_{44c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(v(\omega_1) + RH(\omega_2 - \omega_1)\right) \tag{211}$$

$$v(\omega_3) \in B_{44c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(v(\omega_1) + RH(\omega_3 - \omega_1)\right). \tag{212}$$

Now let A be the affine map we get by interpolating v on the corners $\{\omega_1, \omega_2, \omega_3\}$. So $A = L + \zeta$ where $L \in M^{2\times 2}$ and $\zeta \in \mathbb{R}^2$. Thus we have $v(\omega_2) - v(\omega_1) = L(\omega_2 - \omega_1)$ and $v(\omega_3) - v(\omega_1) = L(\omega_3 - \omega_1)$. As from (211) and (212) we have

$$L(\omega_2 - \omega_1) = v(\omega_2) - v(\omega_1) \in B_{44c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(RH(\omega_2 - \omega_1)\right)$$

$$L(\omega_3 - \omega_1) = v(\omega_3) - v(\omega_1) \in B_{44c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(RH(\omega_3 - \omega_1)\right).$$

This implies

$$\begin{aligned} L\left(\frac{\omega_2 - \omega_1}{|\omega_2 - \omega_1|}\right) &\in \left\{\frac{z}{|\omega_2 - \omega_1|} : z \in B_{44c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(RH(\omega_2 - \omega_1)\right)\right\} \\ &\subset B_{c_{12}\kappa^{\frac{m_0}{1024}}}\left(RH\left(\frac{\omega_2 - \omega_1}{|\omega_2 - \omega_1|}\right)\right) \end{aligned} \tag{213}$$

and

$$\begin{aligned} L\left(\frac{\omega_3 - \omega_1}{|\omega_3 - \omega_1|}\right) &\in \left\{\frac{z}{|\omega_3 - \omega_1|} : z \in B_{44c_7\kappa^{\frac{m_0}{1024}}\epsilon}\left(RH(\omega_3 - \omega_1)\right)\right\} \\ &\subset B_{c_{12}\kappa^{\frac{m_0}{512}}}\left(RH\left(\frac{\omega_3 - \omega_1}{|\omega_3 - \omega_1|}\right)\right). \end{aligned} \tag{214}$$

So $\left|(L - RH)\left(\frac{\omega_3 - \omega_1}{|\omega_3 - \omega_1|}\right)\right| \leq c_{12}\kappa^{\frac{m_0}{1024}}$ and $\left|(L - RH)\left(\frac{\omega_2 - \omega_1}{|\omega_2 - \omega_1|}\right)\right| \leq c_{12}\kappa^{\frac{m_0}{1024}}$.

From this it follows easily that $|L - RH| \leq c_{13}\kappa^{\frac{m_0}{1024}}$. And so for Case 1, where $\mathrm{Card}\left(G_0^{(2)}\right) \geq \mathrm{Card}\left(G_0^{(1)}\right)$ we have established the lemma.

Case 2 where $\mathrm{Card}\left(G_0^{(1)}\right) \geq \mathrm{Card}\left(G_0^{(2)}\right)$ follows via entirely analogous arguments, in fact its is even easier. In this case we take a chain of points $\left\{z_{k_j}\right\} \in P_{n_1^{\perp}}(S)$ and $\left\{x_{k_j}\right\} \in P_{n_2^{\perp}}(S)$ so we gain control of v on the lines $\left\{P_{n_1^{\perp}}^{-1}\left(z_{k_j}\right) \cap S\right\}$ and $\left\{P_{n_2^{\perp}}^{-1}\left(x_{k_j}\right) \cap S\right\}$. Exactly analogous to what we have done in Case 1, we will show a parallelogram with sides parallel to n_1 and n_2 is mapped (roughly) to a rotated version of some parallelogram. From this we harvest the approximate control on the corners of τ_i and we are done.

$\square$

9. Counting the Oscillation

This last lemma is one of the most crucial in the whole proof. The basic idea has been explained in section 4.3 of the introduction. As noted in the introduction, the "coarea alternative" we actually need is more subtle than the one sketched there. The basic idea behind the improved "coarea alternative" is explained in the introduction to section 7; as described we must argue the "coarea alternative" in the thin columns running up through S *in directions* ϕ_1 *and* ϕ_2.
Given a thin rectangular region C *of width* w *parallel to* ϕ_i, *inside* S *the basic rule of thumb for the relationship between the "coarea integral" of* v *over* $v(C)$ *and the surface energy of* v *over* C *is (assuming the bulk energy of* v *over* C *is small enough).*

$$\frac{\int_{v(C)} J(x)\left|D\Theta_i(x)\right| dL^2x}{w} \leq c\int_C \left|D^2 v(x)\right| dL^2x$$

And the proof of this is basically as described in section 4.3 of the introduction. The rest of the proof is just arithmetic to show that the quantities add up to be what we expect.

Lemma 10. *Let* $v \in \mathcal{A}_F(\Omega)$. *Given skew cube* $S := P(a, \phi_1, \phi_2, \mathfrak{c}_9\epsilon) \subset \Omega$. *Let* $m_0 \in \mathbb{N}$ *be a big integer whose value we will decide on later.*

Suppose for some $p \in \{1,2\}$ *we have;*

- *Let* $\{C_k^p : k \in \{1,2,\dots[\kappa^{-m_0}]\}\}$ *denote the set of columns of width* $\kappa^{m_0}\epsilon$ *going through* S, *parallel to* ϕ_p. *Let* $a_k^{(p)}$ *denote the center point in* $C_k^{(p)}$. *Let* Θ_k^p *denote the level set function defined with respect to line* $\left\{a_k^{(p)} + \langle\phi_k\rangle\right\}$. *Let* $E_k^{(p)} := N_{\mathfrak{c}_5\kappa^{m_0}}\left(C_k^{(p)}\right) \cap S$
  Let $\{k_1, k_2, \dots k_{Q_0}\} \subset \{1,2,\dots[\kappa^{-m_0}]\}$ *be a subset of distinct numbers such that*
  $$\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\, dL^2x \leq \kappa^{3m_0+7}\epsilon^2 \tag{215}$$
  for each $j \in \{1,2,\dots Q_0\}$ *and* $Q_0 \geq \left(1-\kappa^{\frac{m_0}{2}}\right)[\kappa^{-m_0}]$
- $$\sum_{j=1}^{Q_0}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\left|D\Theta_k^p(x)\right| dL^2x \geq \kappa^{m_0+1}\epsilon^2$$

then

$$\int_S \left|D^2 v(x)\right| dL^2x \geq \mathfrak{c}_0\kappa\epsilon.$$

Proof. Firstly we note that we have

$$\sum_{j=1}^{Q_0}\epsilon^{-1}\kappa^{-m_0}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\left|D\Theta_k^p(x)\right| dL^2x > \kappa\epsilon.$$

Let

$$G_0 := \left\{j \in \{1,2,\dots Q_0\} : \epsilon^{-1}\kappa^{-m_0}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\left|D\Theta_k^p(x)\right| dL^2x \geq \kappa^{m_0+2}\epsilon\right\}$$

and let $B_0 := \{1,2,\dots Q_0\}\backslash G_0$.

As

$$\sum_{j\in G_0}\epsilon^{-1}\kappa^{-m_0}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\left|D\Theta_k^p(x)\right| dL^2x + \sum_{j\in B_0}\kappa^{m_0+2}\epsilon \geq \kappa\epsilon$$

we have

$$\sum_{j\in G_0}\epsilon^{-1}\kappa^{-m_0}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\left|D\Theta_k^p(x)\right| dL^2x \geq (1-\kappa)\kappa\epsilon.$$

Let

$$\vartheta_j := \epsilon^{-1}\kappa^{-m_0}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\left|D\Theta_k^p(x)\right| dL^2x \tag{216}$$

for $j\in G_0$. So

$$\sum_{j\in G_0}\vartheta_j \geq (1-\kappa)\,\kappa\epsilon \tag{217}$$

and

$$\theta_j \geq \kappa^{m_0+2}\epsilon \tag{218}$$

for each $j\in G_0$.

Claim 2. We will show we can find $z_0\in P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right)$ such that

$$\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z_0)\right)} J(x)\left|D\Theta_k^p(x)\right| dH^1x \geq \sigma^4\theta_j. \tag{219}$$

And

$$\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z_0)\right)} J(x)\, dH^1x \leq \kappa^{m_0+4}\sigma^{-4}\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z_0)\right)} J(x)\left|D\Theta_k^p(x)\right| dH^1x$$

Proof of Claim
Let

$$G_2 := \left\{ \begin{array}{ll} z\in P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right): & \int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z)\right)} J(x)\left|D\Theta_{k_j}^p(x)\right|\left|D\Psi_p(x)\right| dH^1x \\ & \geq \kappa^{-m_0-4}\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z)\right)} J(x)\left|D\Psi_p(x)\right| dH^1x \end{array}\right\}$$

and let

$$G_3 := \left\{ z\in P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right): \int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z)\right)} J(x)\left|D\Theta_{k_j}^p(x)\right|\left|D\Psi_p(x)\right| dH^1x \geq \sigma^4\theta_j\right\}.$$

Suppose the claim is not true and so $G_2\cap G_3=\emptyset$. Thus $P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right)\subset G_2^c\cup G_3^c$. By the Coarea formula

$$\begin{aligned}
\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\left|D\Theta_{k_j}^p(x)\right| dL^2x &= \int_{z\in P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right)}\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z)\right)} J(x)\left|D\Theta_{k_j}^p(x)\right|\left|D\Psi_p(x)\right| dH^1x\,dL^1z \\
&\leq \int_{z\in P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right)\backslash G_2}\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z)\right)} J(x)\left|D\Theta_{k_j}^p(x)\right|\left|D\Psi_p(x)\right| dH^1x\,dL^1z \\
&\quad + \int_{z\in P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right)\backslash G_3}\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z)\right)} J(x)\left|D\Theta_{k_j}^p(x)\right|\left|D\Psi_p(x)\right| dH^1x\,dL^1z \\
&\leq \int_{z\in P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right)}\left(\kappa^{-m_0-4}\int_{v\left(E_{k_j}^{(p)}\cap P^{-1}_{\phi_p^\perp}(z)\right)} J(x)\left|D\Psi_p(x)\right| dH^1x+\sigma^4\theta_j\right) dL^1z \\
&\leq \kappa^{-m_0-4}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\, dL^2x+\sigma^4\theta_j L^1\left(P_{\phi_p^\perp}\left(E_{k_j}^{(p)}\right)\right).
\end{aligned} \tag{220}$$

Recall definition of θ_j, (216), rearranging (220) we have by (215) and (218) that

$$\begin{aligned}
\epsilon\kappa^{m_0}\theta_j-\sigma^4\theta_j c_0\kappa^{m_0}\epsilon &\leq \kappa^{-m_0-4}\int_{v\left(E_{k_j}^{(p)}\right)} J(x)\, dL^2x \\
&\leq \kappa^{2m_0+3}\epsilon^2 \\
&\leq \kappa^{m_0+1}\theta_j\epsilon
\end{aligned}$$

and as $\sigma^4 << c_0$ we have a contradiction. So we have established the claim. $\diamondsuit$

So we can find $z_0 \in G_2 \cap G_3$ such that

$$\int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x \geq \sigma^4 \theta_j$$

and (by the fact that $\sigma^2 \leq |D\Psi_p(x)| \leq \sigma^{-2}$)

$$\int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x)\, dH^1 x \leq \kappa^{m_0+4} \sigma^{-4} \int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x. \tag{221}$$

Claim 3. We can find subset $U_0 \subset v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)$ such that

$$H^1(U_0) \leq \kappa^{m_0+3} \epsilon \tag{222}$$

and

$$\int_{U_0} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x \geq \frac{\sigma^4}{2} \theta_j \tag{223}$$

Proof of Claim.

Note, most of the points of $v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)$ are much less than average: Formally, let

$$U_0 := \left\{ x \in v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right) : J(x) > \kappa \sigma^{-4} \frac{\int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x}{H^1\left(v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)\right)} \right\}$$

so using (221) for the first inequality

$$\begin{aligned} \kappa^{m_0+4} \sigma^{-4} \int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x &\geq \int_{U_0} J(x)\, dH^1 x \\ &\geq \kappa \sigma^{-4} \frac{\int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x}{H^1\left(v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)\right)} H^1(U_0). \end{aligned}$$

So we have

$$\kappa^{m_0+3} H^1\left(v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)\right) \geq H^1(U_0).$$

Let

$$D_0 := v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right) \backslash U_0.$$

So

$$\int_{D_0} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x \leq \kappa \sigma^{-4} \frac{\int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x}{H^1\left(v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)\right)} \int_{D_0} \left|D\Theta_{k_j}^p(x)\right| dH^1 x.$$

As by Lemma 1 $\int_{D_0} \left|D\Theta_{k_j}^p(x)\right| dH^1 x \leq \sigma^{-2} \epsilon$. So

$$\begin{aligned} \int_{D_0} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x &\leq \kappa \sigma^{-6} \frac{\epsilon}{H^1\left(v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)\right)} \int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x \\ &\leq \kappa \sigma^{-8} \int_{v\left(E_{k_j}^{(p)} \cap P_{\phi_p^{\perp}}^{-1}(z_0)\right)} J(x) \left|D\Theta_{k_j}^p(x)\right| dH^1 x \end{aligned}$$

and so as $\kappa << \sigma^8$ we have

$$\int_{U_0} J(x)\left|D\Theta^p_{k_j}(x)\right| dH^1 x \geq \frac{1}{2}\int_{v\left(E^{(p)}_{k_j}\cap P^{-1}_{\phi^\perp_p}(z_0)\right)} J(x)\left|D\Theta^p_{k_j}(x)\right| dH^1 x \geq \frac{\sigma^4\theta_j}{2}$$

and this establishes the claim. $\diamondsuit$

Note in particular

$$\int_{U_0}\left|D\Theta^p_{k_j}(x)\right| dH^1 x \geq \frac{\sigma^6\theta_j}{2} \tag{224}$$

Now for any point $x \in v(l_t)$, let $t(x)$ be the tangent to the curve $v(l_t)$ at x, as Θ^p_k is increasing up the curve $v(l_t)$, we have that

$$t(x)\cdot D\Theta^p_k(x) > 0.$$

On the other hand if we let $v_x \in t(x)^\perp$ we claim

$$v_x \cdot D\Theta^p_{k_j}(x) = 0 \tag{225}$$

to prove (225) we argue as follows.

Take function $X:[0,\infty)\to \mathbb{R}^2$ solving the ODE

$$X(0) = x \quad \frac{dX}{dt}(t_0) = D\Psi_p(X(t_0))$$

then by definition of we know Θ^p_a is constant on $\{X(t): t>0\}$. So

$$D\Theta^p_{k_j}(x)\cdot\frac{dX}{dt}(0) = 0. \tag{226}$$

From what we have previously calculated (see (9)), we know

$$\frac{dX}{dt}(0) = Dv^{-T}\left(v^{-1}(x)\right)n_p.$$

So

$$\begin{aligned}
\frac{dX}{dt}(0)\cdot t(x) &= Dv^{-T}\left(v^{-1}(x)\right)n_p\cdot Dv\left(v^{-1}(x)\right)\phi_p\\
&= n_p\cdot Dv^{-1}\left(v^{-1}(x)\right)Dv\left(v^{-1}(x)\right)\phi_p\\
&= n_p\cdot\phi_p\\
&= 0,
\end{aligned}$$

so $\frac{dX}{dt}(0)\in t(x)^\perp$. Now as we are in $\mathbb{R}^2$ so $v_x \in t(x)^\perp$ will be of the form $v_x = \lambda\frac{dX}{dt}(0)$ for some $\lambda\in\mathbb{R}$ and so by (226) we have that for any $v_x\in t(x)^\perp$, (225) is true.

So in fact $v(l_{t_0})$ is an integral curve for vector field $D\Theta^p_{k_j}$. Formally; for any $x\in v(l_t)$ we have

$$t(x)\,\|D\Theta^p_{k_j}(x)\,,\ t(x)\cdot D\Theta^p_{k_j}(x) > 0. \tag{227}$$

For convenience, at this point we fixed $p=1$. Now U_0 is an open set in $v\left(E^{(p)}_{k_j}\cap P^{-1}_{\phi^\perp_p}(z_0)\right)$ so $v^{-1}(U_0)$ is an open set in $l_{t_0}\cap S$. So $v^{-1}(U_0)$ is a countable union of intervals in $E^{(p)}_{k_j}\cap P^{-1}_{\phi^\perp_p}(z_0)$, i.e. $v^{-1}(U_0) = \bigcup_n (a_n,b_n)$ where $a_n, b_n \in E^{(p)}_{k_j}\cap P^{-1}_{\phi^\perp_p}(z_0)$. We have from (222)

$$\begin{aligned}
\sum_{n\in\mathbb{N}}|a_n-b_n| &\leq H^1\left(v^{-1}(U_0)\right)\\
&\leq \sigma^{-2}H^1(U_0)\\
&\leq \sigma^{-2}\kappa^{m_0+3}\epsilon.
\end{aligned}$$

Take $n \in \mathbb{N}$, firstly we solve the ODEs

$$X^{(1)}(0) = v(a_n) \qquad \frac{dX^{(1)}}{dt}(t_0) = D\Psi_1\left(X^{(1)}(t_0)\right)$$
$$X^{(2)}(0) = v(b_n) \qquad \frac{dX^{(2)}}{dt}(t_0) = D\Psi_1\left(X^{(2)}(t_0)\right)$$

By Lemma 1 there exists unique numbers $t_1, t_2 \in \mathbb{R}$ such that $X^{(1)}(t_1) \in \left\{a_p^k + \langle \phi_1 \rangle\right\}$ and $X^{(2)}(t_2) \in \left\{a_p^k + \langle \phi_1 \rangle\right\}$ then by definition of $\Theta^1_{k_j}$ we have that

$$\begin{aligned} \Theta^1_{k_j}(v(a_n)) &= X^{(1)}(t_1) \cdot \phi_1 \\ \Theta^1_{k_j}(v(b_n)) &= X^{(1)}(t_2) \cdot \phi_1. \end{aligned}$$

Now let $M := \left\{n \in \mathbb{N} : \left|\Theta^1_{k_j}(v(a_n)) - \Theta^1_{k_j}(v(b_n))\right| > 2|a_n - b_n|\right\}$. So by (227) letting $t(x)$ denote the tangent to curve $v(l_{t_0})$ at point x, we have

$$\begin{aligned} \sum_{\mathbb{N}\setminus M} \int_{v((a_n,b_n))} \left|D\Theta^1_{k_j}(x)\right| dH^1 x &= \sum_{\mathbb{N}\setminus M} \int_{v((a_n,b_n))} D\Theta^1_{k_j}(x) \cdot t(x)\, dH^1 x \\ &= \sum_{\mathbb{N}\setminus M} \Theta^1_{k_j}(v(b_n)) - \Theta^1_{k_j}(v(a_n)) \\ &\leq \sum_{n\in\mathbb{N}} 2|a_n - b_n| \\ &\leq 2\sigma^{-2}\kappa^{m_0+3}\epsilon. \end{aligned} \tag{228}$$

So from (224), (228) the fact that $U_0 = \bigcup_{n\in\mathbb{N}} v((a_n, b_n))$ and the fact that by definition of G_0, $\theta_j \geq \kappa^{m_0+2}\epsilon$ we have

$$\begin{aligned} \sum_{n\in M} \int_{v((a_n,b_n))} \left|D\Theta^1_{k_j}(x)\right| dH^1 x &\geq \frac{\sigma^6\theta_j}{2} - 2\sigma^{-2}\kappa^{m_0+3}\epsilon \\ &\geq \frac{\sigma^6\theta_j}{4}. \end{aligned} \tag{229}$$

Recall we know from Lemma 1, $Y^{(1)}(t) := v^{-1}\left(X^{(1)}(t)\right)$ and $Y^{(2)}(t) := v^{-1}\left(X^{(2)}(t)\right)$ travel in cones. Let $i \in M$, $k \in \{1,2\}$ and let $X_i^{(k)}$ be the solution of

$$X_i^{(1)}(0) = v(a_i) \qquad \frac{dX_i^{(1)}}{dt}(t_0) = D\Psi_1\left(X_i^{(1)}(t_0)\right)$$
$$X_i^{(2)}(0) = v(b_i) \qquad \frac{dX_i^{(2)}}{dt}(t_0) = D\Psi_1\left(X_i^{(2)}(t_0)\right)$$

Let $Y_i^{(k)}(t) = v^{-1}\left(X_i^{(k)}(t)\right)$ and let $t_i^k \in \mathbb{R}$ be the unique number such that $Y_i^{(k)}\left(t_i^k\right) \in \{a + \langle \phi_1 \rangle\}$.

Denoted by $Q_i^{(k)} := \left\{Y_i^{(k)}(t) : t \in \left[0, t_i^k\right]\right\}$ and define function $\vartheta_i^k : P_{\langle \phi_i^\perp \rangle}\left(Q_i^{(k)}\right) \to \mathbb{R}$ by

$$\vartheta_i^k(x) := \left(P^{-1}_{\langle \phi_1^\perp \rangle}(x) \cap Q_i^{(k)}\right) \cdot \phi_1.$$

So by Lemma 1, ϑ_i^1 and ϑ_i^2 are well defined Lipschitz functions. See figure 40.

At this point the indices have been changed and fixed so often we chose to take a moment to recall where we are. We are in column $E^1_{k_j}$, U_0 is a subset of a line running parallel to ϕ_1 up through $E^1_{k_j}$. U_0 is the countable union of intervals $\{(a_n, b_n) : n \in \mathbb{N}\}$, ϑ_i^1 is the function from $\phi_i^\perp$ to $\mathbb{R}^2$ whose graph is the pullback of part of an integral curve that runs from $v(a_i)$ to $v(a + \langle \phi_1 \rangle)$ and ϑ_i^2 is the function from $\phi_i^\perp$ to $\mathbb{R}^2$ whose graph is the pullback of part of an integral curve that runs from $v(b_i)$ to $v(a + \langle \phi_1 \rangle)$.

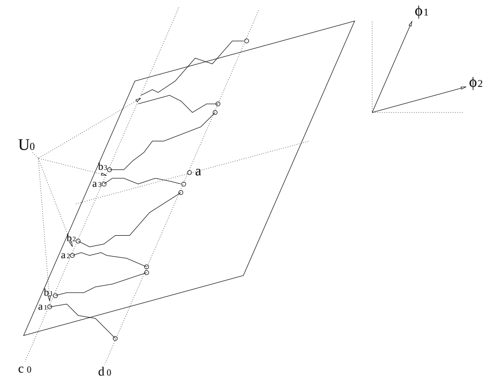

FIGURE 40

From this point on to simplify notation we will take i to be fixed. So let $p_k := \vartheta_i^k$. Now let functions $q_1 : P_{\langle\phi_1^\perp\rangle}\left(Q_i^{(1)}\right) \to \mathbb{R}$ and $q_2 : P_{\langle\phi_1^\perp\rangle}\left(Q_i^{(2)}\right) \to \mathbb{R}$ be defined as follows. Let c_0, d_0 be the end points of interval $P_{\langle\phi_1^\perp\rangle}\left(Q_i^{(1)}\right)$ (which also, as can be seen from fig 40, are the endpoints of $P_{\langle\phi_1^\perp\rangle}\left(Q_i^{(2)}\right)$). For $x \in [c_0, d_0]$ let

$$q_k(x) := \int_{c_0}^{x} p_k'(z)\, dL^1 z \text{ for } k = 1, 2. \tag{230}$$

So

$$\begin{aligned} |q_1(d_0) - q_2(d_0)| &= \left|\int_{c_0}^{d_0} p_1'(z) - p_2'(z)\, dL^1 z\right| \\ &\leq \int_{c_0}^{d_0} |p_1'(z) - p_2'(z)|\, dL^1 z. \end{aligned} \tag{231}$$

So from (230) and since by definition of $\Theta_{k_j}^1$, $\Theta_{k_j}^1(v(a_i)) = \vartheta_i^1(d_0) = p_1(d_0)$ and $\Theta_{k_j}^1(v(b_i)) = \vartheta_i^2(d_0) = p_2(d_0)$ and as $p_1(c_0) = a_i$, $p_2(c_0) = b_i$ we have

$$\begin{aligned} |q_1(d_0) - q_2(d_0)| &= |(p_1(d_0) - p_2(d_0)) - (p_1(c_0) - p_2(c_0))| \\ &\geq \left|\left|\Theta_{k_j}^1(v(a_i)) - \Theta_{k_j}^1(v(b_i))\right| - |p_1(c_0) - p_2(c_0)|\right| \\ &\geq \left|\Theta_{k_j}^1(v(a_i)) - \Theta_{k_j}^1(v(b_i))\right| - |a_i - b_i| \\ &\geq \frac{\left|\Theta_{k_j}^1(v(a_i)) - \Theta_{k_j}^1(v(b_i))\right|}{2}. \end{aligned}$$

So by (231)

$$\frac{\left|\Theta_{k_j}^1(v(a_i)) - \Theta_{k_j}^1(v(b_i))\right|}{2} \leq \int_{c_0}^{d_0} |p_1'(z) - p_2'(z)|\, dL^1 z. \tag{232}$$

Let $t_1 \in (0, t_0)$. Now if we let $v_{t_1}^{(k)}$ denote the tangent to the path $\left\{Y_i^{(k)}(t) : t \in [c_0, d_0]\right\}$ at point $Y_i^{(k)}(t_1)$ then as we have already calculated

$$v_{t_1}^{(k)} := \left[S^{-1}\left(Y_i^{(k)}(t_1)\right) S^{-1}\left(Y_i^{(k)}(t_1)\right)\right] n_1. \tag{233}$$

As vector $p_k'(t_1)\,\phi_1 + n_1$ is parallel to and pointing in the same direction as $v_{t_1}^{(k)}$ we have

$$\frac{p_k'(t_1)\,\phi_1 + n_1}{\left(p_k'(t_1)^2 + 1\right)^{\frac{1}{2}}} = v_{t_1}^{(k)} \quad \text{for } k = 1, 2.$$

Claim 4. We will show

$$\left|v_{t_1}^{(1)} - v_{t_1}^{(2)}\right| \geq \frac{2c_{15}\left|p_2'(t_1) - p_1'(t_2)\right|}{\sqrt{\sigma^{-2}+1}} \tag{234}$$

for some small constant $c_1 > 0$.

Unfortunately we will have to consider three cases.
Firstly the trivial case.
Case 0. If $p_1'(t_1)$ and $p_2'(t_1)$ has the opposite sign, then

$$\begin{aligned}
\left|\frac{p_1'(t_1)}{\sqrt{(p_1'(t_1))^2+1}} - \frac{p_2'(t_1)}{\sqrt{(p_2'(t_1))^2+1}}\right| &= \frac{|p_1'(t_1)|}{\sqrt{(p_1'(t_1))^2+1}} + \frac{|p_2'(t_1)|}{\sqrt{(p_2'(t_1))^2+1}} \\
&\geq \frac{|p_1'(t_1) - p_2'(t_1)|}{2\sqrt{(\sigma^{-2}+1)}}
\end{aligned} \tag{235}$$

and we are done.
Case 1 If $\max\{|p_1(t_1)|, |p_2(t_1)|\} \leq c_1$ where $c_1 > 0$ is a small constant we will decide on later.
Now

$$\begin{aligned}
\left|v_{t_1}^{(1)} - v_{t_1}^{(2)}\right| &\geq \left|\left(v_{t_1}^{(1)} - v_{t_1}^{(2)}\right)\cdot\phi_1\right| \\
&= \left|\frac{p_1'(t_1)}{\sqrt{(p_1'(t_1))^2+1}} - \frac{p_2'(t_1)}{\sqrt{(p_2'(t_1))^2+1}}\right| \\
&\geq \left|\frac{p_1'(t_1) - p_2'(t_1)}{\sqrt{(p_1'(t_1))^2+1}}\right| - \left|p_2'(t_1)\left(\frac{1}{\sqrt{(p_1'(t_1))^2+1}} - \frac{1}{\sqrt{(p_1'(t_2))^2+1}}\right)\right|
\end{aligned} \tag{236}$$

So as

$$\begin{aligned}
\left|\frac{1}{\sqrt{(p_1'(t_1))^2+1}} - \frac{1}{\sqrt{(p_1'(t_2))^2+1}}\right| &= \left|\frac{\sqrt{(p_1'(t_2))^2+1} - \sqrt{(p_1'(t_1))^2+1}}{\sqrt{(p_1'(t_1))^2+1}\sqrt{(p_1'(t_2))^2+1}}\right| \\
&\leq 2\left(\sigma^{-4}+1\right)^{-1}\left|\sqrt{(p_1'(t_2))^2+1} - \sqrt{(p_1'(t_1))^2+1}\right|
\end{aligned} \tag{237}$$

Now observe fig 41.

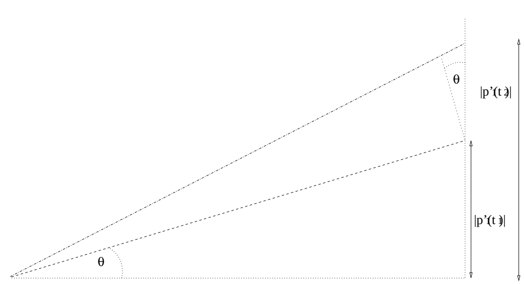

FIGURE 41

Now as can be seen from fig 41, assuming constant c_1 has been chosen small enough we have that

$$\left|\sqrt{1+\left(p_1'\left(t_1\right)\right)^2}-\sqrt{1+\left(p_1'\left(t_2\right)\right)^2}\right| \leq 2\left|\left|p_1'\left(t_1\right)\right|-\left|p_1'\left(t_2\right)\right|\right|\sin\theta_1 .$$

And for some enough c_1 we also know $\theta_1 \leq 2\left|p_1'\left(t_1\right)\right|$. This gives

$$\left|\sqrt{1+\left(p_1'\left(t_1\right)\right)^2}-\sqrt{1+\left(p_1'\left(t_2\right)\right)^2}\right| \leq 4\left|\left|p_1'\left(t_1\right)\right|-\left|p_1'\left(t_2\right)\right|\right|\left|p_1'\left(t_2\right)\right| .$$

Putting this together with (237) and recalling the assumption the assumption that $\max\left\{\left|p_1\left(t_1\right)\right|,\left|\left|p_2\left(t_1\right)\right|\right\} \leq c_1$ for some small constant c_1 we have

$$\left|\left(1+\left(p_1'\left(t_1\right)\right)^2\right)^{-\frac{1}{2}}-\left(1+\left(p_1'\left(t_2\right)\right)^2\right)^{-\frac{1}{2}}\right| \leq 8c_1\left(\sigma^{-4}+1\right)\left|\left|p_1'\left(t_1\right)\right|-\left|p_1'\left(t_2\right)\right|\right| ,$$

Applying this to (236) gives

$$\begin{aligned}\left|v_{t_1}^{(1)}-v_{t_1}^{(2)}\right| &\geq \frac{\left|p_1'\left(t_1\right)-p_2'\left(t_1\right)\right|}{\left(\sigma^{-2}+1\right)^2}-8c_1^2\left(\sigma^{-4}+1\right)^{-1}\left|\left|p_1'\left(t_1\right)\right|-\left|p_1'\left(t_2\right)\right|\right| \\ &\geq \frac{\left|p_1'\left(t_1\right)-p_2'\left(t_1\right)\right|}{2\left(\sigma^{-2}+1\right)^2}\end{aligned}$$

which is what we want.

Case 2 $\max\left\{\left|p_1'\left(t_1\right)\right|,\left|p_2'\left(t_1\right)\right|\right\} \geq c_1$.

$$\begin{aligned}\left|v_{t_1}^{(1)}-v_{t_1}^{(2)}\right| &\geq \left|\left(v_{t_1}^{(1)}-v_{t_1}^{(2)}\right)\cdot n_1\right| \\ &\geq \frac{\left|\left(p_2'\left(t_1\right)^2+1\right)^{\frac{1}{2}}-\left(p_1'\left(t_1\right)^2+1\right)^{\frac{1}{2}}\right|}{\left(p_1'\left(t_1\right)^2+1\right)^{\frac{1}{2}}\left(p_2'\left(t_1\right)^2+1\right)^{\frac{1}{2}}} \\ &\geq \frac{\left|\left(p_2'\left(t_1\right)^2+1\right)^{\frac{1}{2}}-\left(p_1'\left(t_1\right)^2+1\right)^{\frac{1}{2}}\right|}{\left(\sigma^{-2}+1\right)}.\end{aligned}$$

Now we need to use the following identity

$$\sqrt{\left(a^2+1\right)}-\sqrt{\left(b^2+1\right)}=\frac{\left(a-b\right)\left(a+b\right)}{\sqrt{\left(a^2+1\right)}+\sqrt{\left(b^2+1\right)}}.$$

So we have

$$\begin{aligned}&\left|\sqrt{\left(p_2'\left(t_1\right)^2+1\right)}-\sqrt{\left(p_1'\left(t_1\right)^2+1\right)}\right| \\ &\geq \frac{\left|p_2'\left(t_1\right)-p_1'\left(t_1\right)\right|\left|p_2'\left(t_1\right)+p_1'\left(t_1\right)\right|}{2\sqrt{\left(\sigma^{-2}+1\right)}} \\ &\geq \frac{2c_1\left|p_1'\left(t_1\right)-p_2'\left(t_1\right)\right|}{2\sqrt{\sigma^{-2}+1}}.\end{aligned}$$

Thus we have established the claim. $\diamondsuit$

More notation, for a point $z \in Q_i^{(k)}$ let $t_k\left(z\right) \in S^1$ denote the tangent to graph $Q_i^{(k)}$ for $k \in \{1,2\}$, then inequality (234) in our notation becomes; for any $z_0 \in P_{\langle\phi_1^{\perp}\rangle}\left(Q_i^{(k)}\right)$

$$\left|t_1\left(p_1\left(z_0\right)\right)-t_2\left(p_2\left(z_0\right)\right)\right| \geq \frac{2c_1}{\left(\sigma^{-2}+1\right)^{\frac{1}{2}}}\left|p_2'\left(z_0\right)-p_1'\left(z_0\right)\right| .$$

So inequality (232) becomes

$$\frac{\left|\Theta^1_{k_j}\left(v\left(a_i\right)\right)-\Theta^1_{k_j}\left(v\left(b_i\right)\right)\right|}{2}\leq\frac{\left(\sigma^{-2}+1\right)^{\frac{1}{2}}}{2c_1}\int_{c_0}^{d_0}\left|t_1\left(p_1\left(z\right)\right)-t_2\left(p_2\left(z\right)\right)\right|dL^1z. \tag{238}$$

Now as we have already calculated $t_i\left(p_1\left(z\right)\right)=\left[S^{-1}\left(p_1\left(z\right)\right)S^{-1}\left(p_1\left(z\right)\right)\right]n_i$, (recall the notation, $Dv\left(\cdot\right)=:R\left(\cdot\right)S\left(\cdot\right)$ where $R\left(\cdot\right)\in SO\left(2\right)$ and $S\left(\cdot\right)$ is a positive symmetric matrix). Since the map $Dv\left(\cdot\right)\rightarrow S\left(\cdot\right)$ is a projection and $\left|S\left(\cdot\right)\right|>\sigma$ we have

$$\begin{aligned}\left|Dv\left(p_1\left(z\right)\right)-Dv\left(p_2\left(z\right)\right)\right| &> \left|S\left(p_1\left(z\right)\right)-S\left(p_2\left(z\right)\right)\right|\\ &\geq \frac{1}{2}\|S\left(p_1\left(z\right)\right)-S\left(p_2\left(z\right)\right)\|\\ &\geq \frac{\sigma^2}{2}\|\left(S\left(p_1\left(z\right)\right)-S\left(p_2\left(z\right)\right)\right)\left(S\left(p_1\left(z\right)\right)S\left(p_2\left(z\right)\right)\right)^{-1}\|\\ &= \frac{\sigma^2}{2}\|S\left(p_1\left(z\right)\right)^{-1}-S\left(p_2\left(z\right)\right)^{-1}\|\end{aligned}$$

Now

$$\begin{aligned}\frac{\left|Dv\left(p_1\left(z\right)\right)-Dv\left(p_2\left(z\right)\right)\right|}{\sigma} &\geq \frac{\sigma^2}{2}\|S\left(p_2\left(z\right)\right)^{-1}\|\|S\left(p_1\left(z\right)\right)^{-1}-S\left(p_2\left(z\right)\right)^{-1}\|\\ &\geq \frac{\sigma^2}{2}\|S\left(p_2\left(z\right)\right)^{-2}-S\left(p_1\left(z\right)\right)^{-1}S\left(p_2\left(z\right)\right)^{-1}\|\end{aligned}$$

and similarly

$$\frac{\left|Dv\left(p_1\left(z\right)\right)-Dv\left(p_2\left(z\right)\right)\right|}{\sigma}\geq\frac{\sigma^2}{2}\|S\left(p_1\left(z\right)\right)^{-2}-S\left(p_1\left(z\right)\right)^{-1}S\left(p_2\left(z\right)\right)^{-1}\|.$$

Putting these things together we have

$$\begin{aligned}\frac{\left|Dv\left(p_2\left(z\right)\right)-Dv\left(p_1\left(z\right)\right)\right|}{\sigma} &\geq \frac{\sigma^2}{2}\|S\left(p_2\left(z\right)\right)^{-2}-S\left(p_1\left(z\right)\right)^{-2}\|\\ &\geq \frac{\sigma^2}{2}\left|\left(S\left(p_2\left(z\right)\right)^{-2}-S\left(p_1\left(z\right)\right)^{-2}\right)n_1\right|\\ &\geq \frac{\sigma^2}{2}\left|t_2\left(p_2\left(z\right)\right)-t_1\left(p_1\left(z\right)\right)\right|.\end{aligned}$$

Inserting this into equation (238) gives us

$$\begin{aligned}\frac{\left|\Theta^1_{k_j}\left(v\left(a_i\right)\right)-\Theta^1_{k_j}\left(v\left(b_i\right)\right)\right|}{2} &\leq \frac{4\left(\sigma^{-2}+1\right)^{\frac{1}{2}}}{c_1\sigma^3}\int_{c_0}^{d_0}\left|Dv\left(p_1\left(z\right)\right)-Dv\left(p_2\left(z\right)\right)\right|dL^1z\\ &= \frac{4\left(\sigma^{-2}+1\right)^{\frac{1}{2}}}{c_1\sigma^3}\int_{c_0}^{d_0}\left|\int_{p_2(z)}^{p_1(z)}\frac{\partial}{\partial\phi_1}Dv\left(x\right)dL^1x\right|dL^1z\\ &\leq \frac{4\left(\sigma^{-2}+1\right)^{\frac{1}{2}}}{c_1\sigma^3}\int_{c_0}^{d_0}\int_{p_2(z)}^{p_1(z)}\left|D^2v\left(x\right)\right|dL^1xdL^1z. \end{aligned}\tag{239}$$

Now let V_i denote the region enclosed by the two graphs $Q_i^{(1)}$, $Q_i^{(2)}$ and the lines l_{t_0}, $l_{P_{\langle\phi_1^{\perp}\rangle}(a)}$. Formally $V_i:=\bigcup_{z\in[c_0,d_0]}\left[p_1\left(z\right),p_2\left(z\right)\right]$. The set $\{E_i:i\in M\}$ is pairwise disjoint and the equation (239) in this notation is by Fubini

$$\frac{\left|\Theta^1_{k_j}\left(v\left(a_i\right)\right)-\Theta^1_{k_j}\left(v\left(b_i\right)\right)\right|}{2}\leq\frac{4\left(\sigma^{-2}+1\right)^{\frac{1}{2}}}{c_1\sigma^3}\int_{V_i}\left|D^2v\left(x\right)\right|dL^2x.$$

So from (229) and from the fact that $v(l_{t_1})$ is an integral curve for $\Theta^1_{k_j}$ we have by summing over M

$$\begin{aligned}\frac{\sigma^6\epsilon}{4} &\leq \sum_{i\in M}\int_{v((a_i,b_i))} D\Theta^1_{k_j}(x)\, t_x dH^1 x\\ &= \sum_{i\in M}\left|\Theta^1_{k_j}(v(a_i)) - \Theta^1_{k_j}(v(b_i))\right|\\ &\leq \frac{4\left(\sigma^{-2}+1\right)^{\frac{1}{2}}}{c_1\sigma^3}\left(\sum_{i\in M}\int_{V_i}\left|D^2 v(x)\right| dL^2 x\right).\end{aligned}$$

Now $\bigcup_{i\in M} V_i \subset E^{(p)}_{k_j}$ so finally we have

$$\int_{E^{(p)}_{k_j}} \left|D^2 v(x)\right| dL^2 x > \frac{c_1\sigma^9\theta_j}{16}.$$

So for each $j\in G_0$ we have that

$$\int_{E^{(p)}_{k_j}} \left|D^2 v(x)\right| dL^2 x \geq \frac{c_1\sigma^9}{16}\theta_j.$$

Thus recall (217)

$$\sum_{j\in G_0}\int_{E^{(p)}_{k_j}} \left|D^2 v(x)\right| dL^2 x \geq \frac{c_1\sigma^9}{16}\left(\sum_{j\in G_0}\vartheta_j\right) \geq \frac{c_1\sigma^9}{16}(1-\kappa)\kappa\epsilon.$$

However $\left\{E^{(p)}_{k_j} : j\in G_0\right\}$ are overlapping, recall $E^{(p)}_k := N_{\mathfrak{c}_5\kappa^{m_0}\epsilon}\left(C^{(p)}_k\right)\cap S$, where $\left\{C^{(p)}_{k_j} : j\in G_0\right\}$ are disjoint columns of width $\kappa^{m_0}\epsilon$ going through S. So $\left\{E^{(p)}_{k_j} : j\in G_0\right\}$ can not overlap by more than $2[\mathfrak{c}_5]$ times.

Thus must be able to find a subset $G_1\subset G_0$ such that

- $\sum_{j\in G_0}\int_{E^{(p)}_{k_j}} \left|D^2 v(x)\right| dL^2 x \geq \frac{c_2\sigma^9\kappa\epsilon}{64}$.
- $\left\{E^{(p)}_{k_j} : j\in G_1\right\}$ are disjoint.
- $\left\{E^{(p)}_{k_j} : j\in G_1\right\}\subset S$.

So this implies

$$\int_S \left|D^2 v(x)\right| dL^2 x \geq \frac{c_2\sigma^9\kappa\epsilon}{64}.$$

$\square$

10. Proof of Theorem 2

The proof of Theorem 2 is just a matter of collecting everything together.

Proof. Recall, we have triangulation $\triangle_\epsilon$ of Ω, with triangulation size ϵ. Let $v \in \mathcal{A}_F(\Omega)$ and we have skewcube $S := P(a, \phi_1, \phi_2, c\epsilon)$ such that $N_{\frac{\epsilon}{\sigma^2}}(S) \subset \Omega$. In addition we have following inequalities

$$\int_S d(Dv(x), K)\, dL^2x \leq \kappa^{\frac{7m_0}{2}+8}\epsilon^2 \tag{240}$$

and

$$\int_S \left|D^2 v(x)\right| dL^2 x \leq \mathfrak{c}_0 \kappa\epsilon. \tag{241}$$

Let $\left\{C_k^{(p)} : k \in \{1, \dots [\kappa^{-m_0}] + 1\}\right\}$ denote the set of columns width $\kappa^{m_0}\epsilon$ going through S, parallel to ϕ_p for $p = 1, 2$. Let $E_k^{(p)} := N_{\mathfrak{c}_5\kappa^{m_0}\epsilon}\left(C_k^{(p)}\right) \cap S$ for $k = 1, 2, \dots [\kappa^{-m_0}] + 1$. Let

$$L_p := \left\{k \in \{1, \dots \left[\kappa^{-m_0}\right] + 1\} : \int_{v\left(E_k^{(p)}\right)} J(x)\, dL^2 x \geq \kappa^{3m_0+7}\epsilon^2\right\}.$$

So as the set $\left\{E_k^{(p)} : k \in \{1, \dots [\kappa^{-m_0}] + 1\}\right\}$ does not overlap more than $2\mathfrak{c}_5\kappa^{-m_0}$ times

$$\begin{aligned}
\frac{1}{2\mathfrak{c}_5}\kappa^{m_0}\kappa^{3m_0+7}\epsilon^2 \mathrm{Card}(L_p) &\leq \left(2\mathfrak{c}_5\kappa^{-m_0}\right)^{-1}\left(\sum_{k\in L_p}\int_{v\left(E_k^{(p)}\right)} J(x)\, dL^2 x\right) \\
&\leq \int_{v(S)} J(x)\, dL^2 x \\
&\geq \sigma^{-2}\kappa^{\frac{7m_0}{2}+8}\epsilon^2
\end{aligned}$$

Which implies

$$\mathrm{Card}(L_p) \leq 2\mathfrak{c}_5\kappa^{-m_0}\kappa^{\frac{m_0}{2}+1}.$$

So we must be able to find distinct numbers $\left\{k_1, k_2, \dots k_{Q_0^p}\right\} \subset \{1, 2, \dots [\kappa^{-m_0}] + 1\}$ such that $Q_0^p \geq \left(1 - \kappa^{\frac{m_0}{2}}\right)\kappa^{-m_0}$ and

$$\int_{v\left(E_k^{(p)}\right)} J(x)\, dL^2 x \leq \kappa^{3m_0+7}\epsilon^2$$

for $j \in \{1, 2, \dots Q_0^p\}$. Now if we have that for $p \in \{1, 2\}$

$$\sum_{j=1}^{Q_0^p}\int_{v\left(E_k^{(p)}\right)} J(x)\left|D\Theta_k^{(p)}(x)\right| dL^2 x \geq \kappa^{m_0+1}\epsilon^2$$

we can apply Lemma 10 to conclude

$$\int_S \left|D^2 v(x)\right| dL^2 x \geq \mathfrak{c}_0\kappa\epsilon$$

and this contradicts (241).

So we must have

$$\sum_{j=1}^{Q_0^p}\int_{v\left(E_k^{(p)}\right)} J(x)\left|D\Theta_k^{(p)}(x)\right| dL^2 x \leq \kappa^{m_0+1}\epsilon^2. \tag{242}$$

So now, by (240), (241) and (242) can invoke Proposition 1 which gives the following conclusion;

Let

$$\left\{\mathfrak{L}_{i,j} := P\left(a_{i,j}, \phi_1, \phi_2, \kappa^{m_0}\epsilon\right) : i, j \in \left\{1, 2, \dots \left[\kappa^{-m_0}\right] + 1\right\}\right\}$$

be a set of pairwise disjoint skewcubes such that $S \subset \cup_{i,i\in\{1,\ldots[\kappa^{-m_0}]+1\}} \mathfrak{L}_{i,j}$. Let

$$S_{i,j} := P\left(a_{i,j}, \phi_1, \phi_2, c_2\kappa^{m_0}\epsilon\right)$$

for some constant $c_2 > 1$ we will decide on later.

There exists a set $G_0 \subset \{S_{i,j} : i,j \in \{1,\ldots[\kappa^{-m_0}]+1\}\}$ such that

- $$L^2\left(\left\{x \in S_{i,j} : d\left(Dv\left(x\right), R_{i,j}T_{i,j}\right) > \kappa^{\frac{m_0}{16}}\right\}\right) < 20\mathfrak{c}_7^2\tilde{\mathfrak{c}}_4\sigma^{-8}\kappa^{\frac{m_0}{64}}\left(\epsilon\kappa^{m_0}\right)^2$$
  for some $R_{i,j} \in SO\left(2\right)$, $T_{i,j} \in \{Id, H\}$
- $$\mathrm{Card}\left(G_0\right) \geq \frac{1 - 16\sigma^{-2}\kappa^{\frac{m_0}{8}}}{\kappa^{2m_0}}$$

Now this, together with (240) and (241) gives us all we need to invoke Lemma 9. So if τ_i is a triangle in $\triangle_\epsilon$ that contains a and L is the linear part of the affine map we get from interpolating v on the corners of τ_i, we have the following inequality

$$d\left(L, SO\left(2\right) \cup SO\left(2\right)H\right) < \kappa^{\frac{m_0}{1024}}$$

and this complete the proof of Theorem 2. $\square$

11. The Proof of Theorem 1

First note that we have the following trivial lower bound for for the finite element approximation of I. Now consider a triangulation of Ω with triangles size of α. Let

$$A_\alpha := \left\{ v : \Omega \to \mathbb{R}^2 : \begin{array}{c} v \text{ satisfies the affine boundary condition and piecewise affine on} \\ \{\tau_i : i = 1, \dots M_1\} \end{array} \right\}$$

Note that its trivial that for any $u \in A_\alpha$ we must have as least $\frac{\alpha^{-1}}{100}$ triangles $\tau_{i,j}$ with

$$d\left(D\tau_{i,j}, SO\left(2\right) \cup SO\left(2\right) H\right) > \sigma^4.$$

So

$$\inf_{u \in A_\alpha} I\left(u\right) \geq \frac{\sigma^4}{100}\alpha.$$

We can use Theorem 2 in the following way:

Suppose

$$\int_\Omega d\left(Dv\left(x\right), SO\left(2\right) \cup SO\left(2\right) H\right) dL^2 x \leq \epsilon^\zeta$$

and

$$\int_\Omega \left|D^2 v\left(x\right)\right| dL^2 x \leq \epsilon^{-\beta}.$$

Let $\left\{\tau_i : 1 = 1, \dots \left[\epsilon^{-2}\right] + 2\right\}$ be a triangulation of with triangle size ϵ. Let $\kappa := \epsilon^{1-\beta}$. Let $\tilde{\kappa} := \kappa^{\frac{1024}{m_0}}$. Let $B_1 := \left\{\tau_i : \int_{\tau_i} T\left(x\right) dL^2 x \geq \epsilon^{2+\gamma}\right\}$. So $\mathrm{Card}\left(B_1\right) \epsilon^{2+\gamma} \leq \epsilon^\zeta$ which implies $\mathrm{Card}\left(B_1\right) \leq \epsilon^{\zeta - \gamma - 2}$. Let

$$B_2 := \left\{ \tau_i : \begin{array}{c} \text{The linear part of the affine interpolation of } S \text{ on } \tau_i \text{ is distance } > \kappa \text{ from} \\ SO\left(2\right) \cup SO\left(2\right) H \end{array} \right\}$$

So by Theorem 1 we have

$$\begin{aligned} \mathrm{Card}\left(B_2\right) \tilde{\kappa}\epsilon &= \mathrm{Card}\left(B_2\right) \epsilon^{1 + \frac{1024(1-\beta)}{m_0}} \\ &\leq \epsilon^{-\beta} \end{aligned}$$

which implies

$$\mathrm{Card}\left(B_2\right) \leq \epsilon^{-1 - \frac{1024}{m_0} - \beta}.$$

Now let $\tilde{v}_\epsilon$ be the triangulation of v on $\left\{\tau_i : i = 1, \dots \epsilon^{-2}\right\}$. We have the following estimate

$$\begin{aligned} I\left(\tilde{v}_\epsilon\right) &= \int_\Omega d\left(D\tilde{v}_\epsilon\left(x\right), K\right) dL^2 x \\ &= \sum_{\tau_i \in B_1} \int_{\tau_i} d\left(D\tilde{v}_\epsilon\left(x\right), K\right) dL^2 x + \sum_{\tau_i \in B_2} \int_{\tau_i} d\left(D\tilde{v}_\epsilon\left(x\right), K\right) dL^2 x + \sum_{\tau_i \notin (B_1 \cup B_2)} \int_{\tau_i} d\left(D\tilde{v}_\epsilon\left(x\right), K\right) dL^2 x \\ &\leq \sigma^{-2}\epsilon^2 \left(\mathrm{Card}\left(B_1\right) + \mathrm{Card}\left(B_2\right)\right) + \sum_{\tau_i \notin (B_1 \cup B_2)} \int_{\tau_i} d\left(D\tilde{v}_\epsilon\left(x\right), K\right) dL^2 x \\ &\leq \sigma^{-2}\epsilon^2 \left(\epsilon^{\zeta - \gamma - 2} + \epsilon^{-1 - \frac{1024}{m_0} - \beta}\right) + \epsilon^{1-\beta} L^2\left(\Omega\right) \end{aligned}$$

So by letting $\zeta := \frac{7m_0}{2} + 8$, $\gamma := \frac{m_0}{2}$, $\beta := \frac{1024}{m_0}$ we have

$$I\left(\tilde{v}_\epsilon\right) \leq c\epsilon^{1 - \frac{2048}{m_0}}.$$

Now letting $m_1 = 4m_0$ completes the proof of Theorem 1.

12. Proof of Corollary 1

So if we have a function v such that $\frac{I_\epsilon(v)}{\epsilon} \leq \epsilon^{-\frac{2048}{m_1^2}}$. Recall $h := \epsilon^{\frac{m_1^2-2048}{m_1^3}}$

$$\begin{aligned}\int_\Omega d\left(Dv\left(x\right), SO\left(2\right) \cup SO\left(2\right) H\right) dL^2 x &\leq \epsilon^{\frac{m_1^2-2048}{m_1^2}} \\ &= h^{m_1}.\end{aligned}$$

And

$$\begin{aligned}\int_\Omega \left|D^2 v\left(x\right)\right| dL^2 x &\leq \frac{I_\epsilon\left(v\right)}{\epsilon} \\ &\leq h^{-\left(\frac{m_1^3}{m_1^2-2048}\right)\frac{2048}{m_1^2}} \\ &= h^{\frac{-2048 m_1}{m_1^2-2048}}\end{aligned}$$

As we know $m_1 \geq 2048$ so we have $m_1^2 - 2048 = m_1\left(m_1 - \frac{2048}{m_1}\right) \geq \frac{m_1^2}{2}$ so $\frac{2048 m_1}{m_1^2-2048} \leq \frac{4096}{m_1}$ and so

$$\begin{aligned}\int_\Omega \left|D^2 v\left(x\right)\right| dL^2 x &\leq h^{\frac{-2048 m_1}{m_1^2-2048}} \\ &\leq h^{-\frac{4096}{m_1}}. \qquad (243)\end{aligned}$$

So by Theorem 1 if $\tilde{v}_h$ denotes the F.E. approximation on the triangulation $\triangle_h$ we have

$$I\left(\tilde{v}_h\right) \leq c h^{1-\frac{8192}{m_1}}.$$

13. Appendix

13.1. $H^{-1}H^{-1}n_i = \Diamond_i$ **for** $i = 1, 2$. We begin by calculating $\Diamond_1$ and $\Diamond_2$. Firstly we know $\Diamond_i \in \Xi_i$ for $i = 1, 2$. Recall from section 4.1 in the introduction (in particular see fig 2), that $\Diamond_1$ is the optimal direction for the path W that joins two lines l_1, l_2 (that are parallel to ϕ_1), but that minimizes the integral $\int_W |Ht(x)|\, dH^1 x$ (where $t(x)$ is the tangent to the path W at point x). Now let $\psi_0 \in S^1$ be the vector "in between" $-\phi_1$ and $-\phi_2$ (formally $\psi_0 := \frac{-\phi_1 - \phi_2}{|\phi_1 + \phi_2|}$), ψ_0 is the vector that is most shrunk under the action of H (see for example fig 14). However as we will see, ψ_0 is not the optimal angle for W because it is at too flat an angle to $-\phi_1$ (see fig 2) so any path joining l_1 and l_2 that is parallel to ψ_0 will have to be so long it cancels the shrinking effects of ψ_0. For the same reason it is clear that the optimal vector $\Diamond_1$ must be in the "half" of the shrink directions that lie between $-\phi_2$ and ψ_0, i.e. paths that are parallel to vectors in the other "half" will be too long.

Hence $\Diamond_1$ (as is shown on fig 42) points above the x-axis. By absolutely identical considerations we see that $\Diamond_2$ also points above the x-axis.

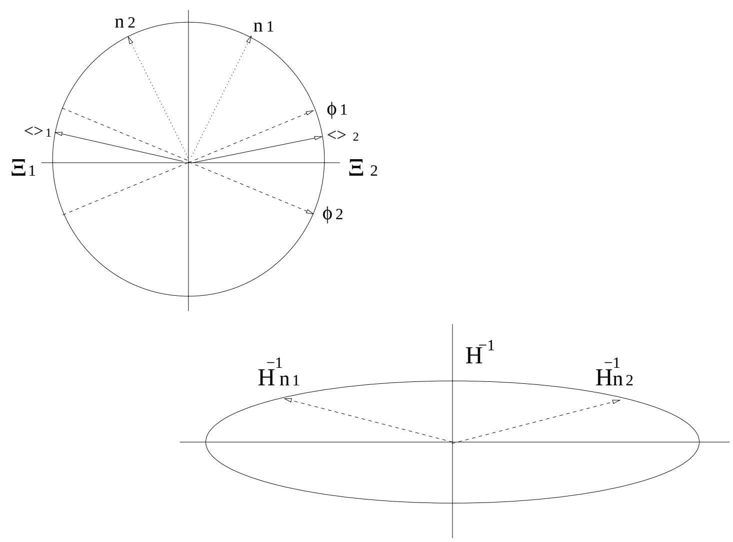

Figure 42

Using this initial (crude) information about $\Diamond_1$, $\Diamond_2$ we will now calculate the coordinate of $\Diamond_2$. Firstly recall from (54) that for the unique $\tilde{a} \in [0, \pi)$ such that $\tan 2\tilde{a} = \frac{2\tilde{\sigma}^3}{(1-\tilde{\sigma}^6)}$ we have $\Diamond_2 := \begin{pmatrix} \cos \tilde{a} \\ \sin \tilde{a} \end{pmatrix}$. Note that

$$\begin{aligned} \sqrt{\left((2\tilde{\sigma}^3)^2 + (1-\tilde{\sigma}^6)^2 \right)} &= \sqrt{(4\tilde{\sigma}^6 + 1 - 2\tilde{\sigma}^6 + \tilde{\sigma}^{12})} \\ &= \sqrt{(1 + 2\tilde{\sigma}^6 + \tilde{\sigma}^{12})} \\ &= \left(1 + \tilde{\sigma}^6\right) \end{aligned}$$

So

$$\sin 2\tilde{a} = \frac{2\tilde{\sigma}^3}{(1+\tilde{\sigma}^6)} \text{ and } \cos 2\tilde{a} = \frac{\left(1 - \tilde{\sigma}^6\right)}{(1+\tilde{\sigma}^6)}.$$

Hence

$$\begin{aligned}(\sin\tilde{a})^2 &= \frac{1}{2}-\frac{\left(1-\tilde{\sigma}^6\right)}{2\left(1+\tilde{\sigma}^6\right)}\\ &= \frac{1}{2}\left(\frac{\left(1+\tilde{\sigma}^6\right)}{\left(1+\tilde{\sigma}^6\right)}-\frac{\left(1-\tilde{\sigma}^6\right)}{\left(1+\tilde{\sigma}^6\right)}\right)\\ &= \frac{1}{2}\left(\frac{2\tilde{\sigma}^6}{\left(1+\tilde{\sigma}^6\right)}\right)\\ &= \frac{\tilde{\sigma}^6}{\left(1+\tilde{\sigma}^6\right)}.\end{aligned}$$

And thus

$$\sin\tilde{a}=\pm\frac{\tilde{\sigma}^3}{\sqrt{\left(1+\tilde{\sigma}^6\right)}}. \tag{244}$$

$$\begin{aligned}(\cos\tilde{a})^2 &= \frac{1}{2}+\frac{\left(1-\tilde{\sigma}^6\right)}{2\left(1+\tilde{\sigma}^6\right)}\\ &= \frac{1}{\left(1+\tilde{\sigma}^6\right)}\end{aligned}$$

Hence

$$\cos\tilde{a}=\pm\frac{1}{\sqrt{\left(1+\tilde{\sigma}^6\right)}} \tag{245}$$

Now from (244) and (245) and what we have established about $\Diamond_2$ pointing above the x-axis and the fact that it belongs to Ξ_2, we must have that

$$\sin\tilde{a}=\frac{\tilde{\sigma}^3}{\sqrt{\left(1+\tilde{\sigma}^6\right)}},\quad \cos\tilde{a}=\frac{1}{\sqrt{\left(1+\tilde{\sigma}^6\right)}}.$$

Now we need to calculate $\Diamond_1$. We have to start from scratch. So for $\psi:=\left(\begin{smallmatrix}\cos a\\ \sin a\end{smallmatrix}\right)$, let

$$\begin{aligned}g(a) &:= |H\psi|^2-(\psi\cdot n_1)^2\\ &= \tilde{\sigma}^2\cos^2 a+\frac{\sin^2 a}{\tilde{\sigma}^2}-\left(-\frac{\tilde{\sigma}\cos a}{\sqrt{\left(1+\tilde{\sigma}^2\right)}}+\frac{\sin a}{\sqrt{\left(1+\tilde{\sigma}^2\right)}}\right)^2\\ &= \tilde{\sigma}^2\cos^2 a+\frac{\sin^2 a}{\tilde{\sigma}^2}-\frac{\tilde{\sigma}^2\cos^2 a}{\left(1+\tilde{\sigma}^2\right)}-\frac{\sin^2 a}{\left(1+\tilde{\sigma}^2\right)}+\frac{2\tilde{\sigma}\sin\tilde{a}\cos\tilde{a}}{\left(1+\tilde{\sigma}^2\right)}.\end{aligned}$$

So

$$\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)g(a):=\tilde{\sigma}^6\cos^2 a+\sin^2 a+2\tilde{\sigma}\cos a\sin a.$$

And as in the calculation of $\tilde{a}$, by standard trigonometric identities this reduces to

$$\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)g(a)=\tilde{\sigma}^6\left(\frac{1+\cos 2a}{2}\right)+\frac{1-\cos 2a}{2}+\tilde{\sigma}^2\sin 2a.$$

so

$$2\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)g(a)=\left(\tilde{\sigma}^6-1\right)\cos 2a+2\tilde{\sigma}^3\sin 2a+\tilde{\sigma}^6+1.$$

Thus

$$2\tilde{\sigma}^2\left(\tilde{\sigma}^2+1\right)g'(a)=-2\left(\tilde{\sigma}^6-1\right)\sin 2a+4\tilde{\sigma}^3\cos 2a.$$

So

$$\begin{aligned}g'\left(\tilde{b}\right)=0 \quad&\Leftrightarrow\quad 2\left(\sigma^6-1\right)\sin 2\tilde{b}=4\tilde{\sigma}^3\cos 2\tilde{b}\\ &\Leftrightarrow\quad \tan 2\tilde{b}=\frac{-2\tilde{\sigma}^3}{\left(1-\tilde{\sigma}^6\right)}\end{aligned}$$

Hence

$$\tan 2\tilde{b} = -\tan 2\tilde{a}. \tag{246}$$

Now as we know $\lozenge_2 = \binom{\cos\tilde{a}}{\sin\tilde{a}}$ points upwards and to the right (as shown in fig 42) so $\tilde{a} \in \left(0, \frac{\pi}{2}\right)$.

So (246) implies either $\tilde{b} = -\tilde{a}$ or $\tilde{b} = \pi - \tilde{a}$. Now as the former possibility implies that $\lozenge_1 = \binom{\cos\tilde{b}}{\sin\tilde{b}}$ is not in Ξ_1 so we must have that $\tilde{b} = \pi - \tilde{a}$.

Thus

$$\lozenge_1 = \binom{\cos\pi - \tilde{a}}{\sin\pi - \tilde{a}} = \binom{-\cos\tilde{a}}{\sin\tilde{a}} = \overline{\lozenge_2}. \tag{247}$$

We claim

$$H^{-1}H^{-1}n_2 = \lozenge_2. \tag{248}$$

From fig 42 it should seem reasonable that $H^{-1}H^{-1}n_2$ is sent into the shrink directions, specifically into Ξ_1. And from what we have calculated $\lozenge_2 := \begin{pmatrix} \sqrt{\left(\frac{1-\tilde{\sigma}^2}{\lambda^2-\tilde{\sigma}^2}\right)} \\ \sqrt{\left(\frac{\lambda^2-1}{\lambda^2-\tilde{\sigma}^2}\right)} \end{pmatrix}$. Now from (35) we have $n_2 := \begin{pmatrix} \sqrt{\frac{1-\tilde{\sigma}^2}{\lambda^2-\tilde{\sigma}^2}} \\ \sqrt{\frac{\lambda^2-1}{\lambda^2-\tilde{\sigma}^2}} \end{pmatrix}$ for $\lambda = \tilde{\sigma}^{-1}$. So by writing this out more carefully we have $n_2 := \begin{pmatrix} \frac{\tilde{\sigma}}{\sqrt{(1+\tilde{\sigma}^2)}} \\ \frac{1}{\sqrt{(1+\tilde{\sigma}^2)}} \end{pmatrix}$. Now

$$\begin{aligned} H^{-1}H^{-1}n_2 &= \begin{pmatrix} \tilde{\sigma}^{-2} & 0 \\ 0 & \tilde{\sigma}^2 \end{pmatrix} \begin{pmatrix} \frac{\tilde{\sigma}}{\sqrt{(1+\tilde{\sigma}^2)}} \\ \frac{1}{\sqrt{(1+\tilde{\sigma}^2)}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\tilde{\sigma}\sqrt{(1+\tilde{\sigma}^2)}} & \frac{\tilde{\sigma}^2}{\sqrt{(1+\tilde{\sigma}^2)}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\tilde{\sigma}\sqrt{1+\tilde{\sigma}^2}} \\ \frac{\tilde{\sigma}^2}{\sqrt{1+\tilde{\sigma}^2}} \end{pmatrix}. \end{aligned} \tag{249}$$

Now we normalize the vector; so

$$\begin{aligned} \sqrt{\left(\frac{1}{\tilde{\sigma}^2(1+\tilde{\sigma}^2)} + \frac{\tilde{\sigma}^4}{(1+\tilde{\sigma}^2)}\right)} &= \sqrt{\frac{\tilde{\sigma}^6+1}{\tilde{\sigma}^2(1+\tilde{\sigma}^2)}} \\ &= \frac{\sqrt{(\tilde{\sigma}^6+1)}}{\tilde{\sigma}\sqrt{(1+\tilde{\sigma}^2)}} \end{aligned}$$

And hence

$$\begin{aligned} \frac{H^{-1}H^{-1}n_2}{|H^{-1}H^{-1}n_2|} &= \frac{\tilde{\sigma}\sqrt{(1+\tilde{\sigma}^2)}}{\sqrt{(\tilde{\sigma}^6+1)}} H^{-1}H^{-1}n_2 \\ &= \begin{pmatrix} \frac{\tilde{\sigma}\sqrt{1+\tilde{\sigma}^2}}{\sqrt{(\tilde{\sigma}^6+1)}} \frac{1}{\tilde{\sigma}\sqrt{1+\tilde{\sigma}^2}} \\ \frac{\tilde{\sigma}\sqrt{1+\tilde{\sigma}^2}}{\sqrt{(\tilde{\sigma}^6+1)}} \frac{\tilde{\sigma}^2}{\tilde{\sigma}\sqrt{1+\tilde{\sigma}^2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{(\tilde{\sigma}^6+1)}} \\ \frac{\tilde{\sigma}^3}{\sqrt{(\tilde{\sigma}^6+1)}} \end{pmatrix} \\ &= \lozenge_2 \end{aligned}$$

which establishes the claim.

From the fact that $\overline{\lozenge_2} = \lozenge_1$ and $\overline{n_1} = n_2$, using (248) we get

$$H^{-1}H^{-1}n_1 = H^{-1}H^{-1}\overline{n_1} = \overline{H^{-1}H^{-1}n_2} = \overline{\lozenge_2} = \lozenge_1$$

and this completes the calculation.

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