MAPPING PROPERTIES OF THE MAXIMAL AVERAGING OPERATOR ASSOCIATED TO THE 2-PLANE TRANSFORM

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Abstract. We extend Christ’s estimate for the 2-plane transform to a maximal operator setting.

1. Introduction

Let \( G_{n,2} \) be the Grassmannian manifold of all 2-dimensional linear subspaces of \( \mathbb{R}^n \) equipped with the unique invariant probability measure \( \gamma_{n,2} \). For a function \( f \) satisfying the appropriate integrability conditions, the 2-plane transform \( T_{n,2}f \) is defined by

\[
T_{n,2}f(\Pi, y) = \int_{\Pi} f(x + y) d\mathcal{L}^2(x),
\]

where \( \mathcal{L}^2 \) is 2-dimensional Lebesgue measure on the plane \( \Pi \in G_{n,2} \).

The following mixed-norm estimate was proved by Christ [2].

\[
\|T_{n,2}f\|_{L^q(L^r')} \leq C_{n,p,q,r}\|f\|_p
\]

where

\[
\frac{n}{p} - \frac{n - 2}{r} = 2, \quad 1 \leq p \leq \frac{n + 1}{3}, \quad q \leq (n - 2)p',
\]

and

\[
\|T_{n,2}f\|_{L^q(L^r')}^q = \int_{G_{n,2}} \left( \int_{\Pi} |T_{n,2}f(\Pi, y)|^r d\mathcal{L}^{n-2}(y) \right)^{q/r} d\gamma_{n,2}(\Pi),
\]

\( \Pi^\perp \) being the orthogonal complement of \( \Pi \), \( \mathcal{L}^{n-2} \) \((n - 2)\)-dimensional Lebesgue measure on \( \Pi^\perp \), and \( p' \) the conjugate exponent of \( p \).

It is unlikely that this estimate is sharp. For example, it does not imply full Hausdorff dimension for \((n, 2)\)-sets, a fact which was proved by the author in [3]. Actually, Christ conjectured that (1.1) should hold with

\[
\frac{n}{p} - \frac{n - 2}{r} = 2, \quad 1 \leq p < \frac{n}{2}, \quad q \leq (n - 2)p'.
\]

Notice that in the above conjectured range of boundedness, \( r \) approaches \( \infty \) as \( p \) approaches the endpoint \( n/2 \). It is therefore natural to consider the corresponding maximal averaging operator which is analogous to the

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Kakeya maximal function introduced by Bourgain [1]. To do this, we need some further notation. For $\Pi \in \mathcal{G}_{n,2}$, $a \in \mathbb{R}^n$, $\delta \ll 1$, we denote by $\Pi^\delta(a)$ the $\delta/2$-neighborhood of the intersection of $\Pi + a$ with the ball of radius $1/2$ centered at $a$, and define

$$M_\delta f : \mathcal{G}_{n,2} \to \mathbb{R}$$

by

$$M_\delta f(\Pi) = \sup_{a \in \mathbb{R}^n} \frac{1}{|\Pi^\delta(a)|} \int_{\Pi^\delta(a)} |f(y)| dy,$$

where $|\Pi^\delta(a)|$ is the volume of $\Pi^\delta(a)$, and $f : \mathbb{R}^n \to \mathbb{R}$ any locally integrable function.

We are interested in proving $L^p \to L^q(\mathcal{G}_{n,2}, \gamma_{n,2})$ estimates for this operator. To find the optimal range for $p$ and $q$ we argue as follows.

If $f$ is the characteristic function of a ball of radius $\delta$, then $\|f\|_p$ is comparable to $\delta^{n/p}$, and $\|M_\delta f\|_{L^q(\mathcal{G}_{n,2}, \gamma_{n,2})}$ is comparable to $\delta^2$. Therefore, the best possible bound is

$$\|M_\delta f\|_{L^q(\mathcal{G}_{n,2}, \gamma_{n,2})} \leq C_{n,p,q} \delta^{2-n/p} \|f\|_p, \quad p < n/2, \quad q \geq 1.$$

On the other hand, if $f$ is the characteristic function of a rectangle of dimensions $1 \times 1 \times \delta \times \cdots \times \delta$, then $\|f\|_p = \delta^{(n-2)/p}$ and $\|M_\delta f\|_{L^q(\mathcal{G}_{n,2}, \gamma_{n,2})}$ is, up to a multiplicative constant, greater than $\delta^{2(n-2)/q}$. It follows that if the above estimate is true, we must have

$$\delta^{2(n-2)/q} \leq C_{n,p,q} \delta^{2-n/p} \delta^{(n-2)/p},$$

which forces $q$ to be less than $(n-2)p'$.

These examples suggest the following conjecture which, if true, would imply the result in [3].

For every $\varepsilon > 0$ there exists a constant $C_{p,q} > 0$ such that

$$\|M_\delta f\|_{L^q(\mathcal{G}_{n,2}, \gamma_{n,2})} \leq C_{\varepsilon,p,q} \delta^{2-n/p-\varepsilon} \|f\|_p,$$

where

$$1 \leq p \leq \frac{n}{2}, \quad q \leq (n-2)p'.$$

The purpose of this paper is to give a geometric proof of the following partial result, which may be thought of as a stronger version of Christ’s estimate.

**Theorem 1.1.** For every $\varepsilon > 0$ there exists a constant $C_{\varepsilon,p,q} > 0$ such that

$$\|M_\delta f\|_{L^q(\mathcal{G}_{n,2}, \gamma_{n,2})} \leq C_{\varepsilon,p,q} \delta^{2-n/p-\varepsilon} \|f\|_p,$$

where

$$1 \leq p \leq \frac{n+1}{3}, \quad q \leq (n-2)p'.$$
In an attempt to prove an estimate like (1.2) which is insensitive to \( \delta^{-e} \) factors, it would seem reasonable to try to modify or refine the nearly optimal argument of Mitsis [3]. However, that argument is based, in part, on a discretization of the sharp bound for the Radon transform due to Oberlin and Stein [4], which yields a distributional-type inequality with the “wrong” exponent for the \( \lambda \)-parameter. This defect is of no consequence, as far as the geometric problem considered in [3] is concerned, but makes the approach of that paper inapplicable in the present context. Therefore, we have to use a more direct, but less efficient, high-low multiplicity argument. It is the author’s impression that any improvement on (1.2) would require a complete understanding of the geometry of the Radon transform, and so, an alternative, purely geometric proof of the result in [4] would be a valuable contribution.

2. Preliminaries

Throughout this paper, the capital letter \( C \), subscripted or otherwise, will denote various constants whose values may change from line to line. \( x \lesssim y \) means \( x \leq Cy \), and similarly with \( x \gtrsim y \) and \( x \approx y \). Also, we will use the notation \( \Pi^\delta \) for any set \( \Pi^\delta(a) \), since the basepoint \( a \) is irrelevant in all our arguments. Further notational conventions follow below.

\( S^{n-1} \) is the \((n-1)\)-dimensional sphere.
\( B(a, r) \) is the ball of radius \( r \) centered at \( a \).
\( A(a, r) \) is the annulus \( B(a, 2r) \setminus B(a, r) \).
\( L_e(a) \) is the line in the direction \( e \in S^{n-1} \) passing through the point \( a \), i.e. \( L_e(a) = \{ a + te : t \in \mathbb{R} \} \).
\( T_e^{(r)\beta}(a) \) is the tube of length \( r \), cross-section radius \( \beta \), centered at \( a \), and with axis in the direction \( e \in S^{n-1} \), i.e. \( T_e^{(r)\beta}(a) = \{ x \in \mathbb{R}^n : \text{dist}(x, L_e(a)) \leq \beta \text{ and } |\text{proj}_{L_e(a)}(x) - a| \leq r/2 \} \),

where \( \text{proj}_{L_e(a)}(x) \) is the orthogonal projection of \( x \) onto \( L_e(a) \).
\( \chi_E \) is the characteristic function of the set \( E \).
\( | \cdot | \) denotes Lebesgue measure or cardinality, depending on the context.
If \( \Pi_1, \Pi_2 \in \mathcal{G}_{n,2} \), then their distance \( \theta \) is defined by
\[
\theta(\Pi_1, \Pi_2) = ||\text{proj}_{\Pi_1} - \text{proj}_{\Pi_2}||,
\]
where \( || \cdot || \) is the operator norm. \( \gamma_{n,2} \) is a \( 2(n-2) \)-dimensional regular measure with respect to this distance, in the sense that
\[
\gamma_{n,2}(\{ \Pi \in \mathcal{G}_{n,2} : \theta(\Pi, \Pi_0) \leq r \}) \approx r^{2(n-2)}, \ \forall \Pi_0 \in \mathcal{G}_{n,2}, r < 1.
\]

A finite subset of \( \mathcal{G}_{n,2} \) is called \( \delta \)-separated if the distance between any two of its elements is at least \( \delta \). So, if \( \mathcal{B} \) is a maximal \( \delta \)-separated subset of \( \mathcal{A} \subset \mathcal{G}_{n,2} \), then
\[
\gamma_{n,2}(\mathcal{A}) \lesssim |\mathcal{B}| \delta^{2(n-2)}.
\]
Moreover, if \( \mathcal{A} \subset \mathcal{G}_{n,2} \) is \( \delta \)-separated, and \( \mathcal{B} \) is a maximal \( \eta \)-separated subset of \( \mathcal{A} \) with \( \eta \geq \delta \), then
\[
|\mathcal{B}| \geq |\mathcal{A}|(\delta/\eta)^{2(n-2)}.
\]

For technical reasons, we introduce the following subsets of \( \mathcal{G}_{n,2} \).
\[
\mathcal{A}_{n,2} := \{ \Pi \in \mathcal{G}_{n,2} : \theta(\Pi, x_1x_2\text{-plane}) \leq 1/4 \},
\]
\[
\mathcal{B}_{n,2} := \{ \Pi \in \mathcal{G}_{n,2} : \theta(\Pi, x_1x_2\text{-plane}) \leq 1/2 \}.
\]

Notice that by invariance, it is enough to prove Theorem 1.1 for \( \mathcal{M}_\delta \) restricted to \( \mathcal{A}_{n,2} \).

We finally note the following fact which can be proved by fairly elementary arguments.

**Lemma 2.1.** Let \( \Pi \in \mathcal{B}_{n,2} \). Then there exist unique \( \overline{u}, \overline{v} \in \mathbb{R}^{n-2} \) with \( |u|, |v| \leq 1 \) such that \( \Pi = \{(s, t, s\overline{u} \pm \overline{v}) : s, t \in \mathbb{R} \} \). Further, for any \( \Pi_j = \{(s, t, s\overline{u}_j \pm t\overline{v}_j) : s, t \in \mathbb{R} \} \in \mathcal{B}_{n,2}, \ j = 1, 2 \), we have \( \theta(\Pi_1, \Pi_2) \approx |\overline{u}_1 - \overline{u}_2| + |\overline{v}_1 - \overline{v}_2| \).

3. **Geometric Lemmas**

In this section we prove two technical results that will allow us to control the cardinality and the intersection properties of a family of sets \( \Pi^\delta \) containing a fixed line segment.

**Lemma 3.1.** Let \( \{\Pi_j\}_{j=1}^M \) be a \( \delta \)-separated set in \( \mathcal{A}_{n,2} \). Suppose that there exist points \( a, b \in \mathbb{R}^n \) and a number \( \rho \geq 4\delta \) such that \( |a - b| \geq \rho \), and for each \( j \) there is a \( \Pi^\delta_j \) with \( a, b \in \Pi^\delta_j \). Further, suppose that \( \mathcal{B} \) is a maximal \( \zeta \)-separated subset of \( \{\Pi_j\}_{j=1}^M \) with \( \zeta \geq C\delta/\rho \). Then \( |\mathcal{B}| \geq M(\rho\delta/\zeta)^{n-2} \).

**Proof.** After rotating and translating, we may assume that \( \Pi_1 \) is the \( x_1x_2 \)-plane, \( \Pi_j \subset \mathcal{B}_{n,2}, a = 0 \) and \( b = (b_1, 0, \overline{b}) \), for some \( \overline{b} \in \mathbb{R}^{n-2} \). Then, by Lemma 2.1, there exist unique \( u_j, v_j \in \mathbb{R}^{n-2} \) such that
\[
\Pi_j = \{(s, t, su_j + tv_j) : s, t \in \mathbb{R} \}
\]
and
\[
\theta(\Pi_j, \Pi_k) \approx |u_j - u_k| + |v_j - v_k|.
\]
Therefore, we can think of \( \{\Pi_j\}_{j=1}^M \) as a set of points in \( \mathcal{B}(0, C) \subset \mathbb{R}^{2(n-2)} \) under the identification \( \Pi_j \leftrightarrow (u_j, v_j) \). We claim that \( \{\Pi_j\}_{j=1}^M \) is contained in a rectangle \( R \) of sidelength, up to constants, \( \delta/\rho \) in \( n - 2 \) dimensions and \( C \) in the remaining \( n - 2 \) dimensions. To see this, note that since \( b \in \Pi^\delta_j \), there exists
\[
w_j = (s_j, t_j, sju_j + tv_j) \in \Pi_j
\]
with
\[
|w_j - b| = |(s_j - b_1, t_j, sju_j + tv_j - \overline{b})| \leq \delta.
\]
Therefore
\[
|s_j - b_1| \leq \delta, \quad |t_j| \leq \delta, \quad |sju_j + tv_j - \overline{b}| \leq \delta.
\]
In particular \( |\overline{b}| \leq \delta \), and so
\[
|s_j| \geq |b_1| - \delta \geq |b| - 2\delta \geq \rho - 2\delta \geq \rho/2.
\]
On the other hand

$$|s_j u_j| \leq \delta + |t_j v_j| + |\overline{b}| \leq \delta.$$ 

Consequently $|u_j| \leq \delta / \rho$, proving the claim.

Now, for appropriately chosen constants $C_1, C_2, C_3$, we have

$$\{B(\Pi_j, C_1^{-1} \delta)\}_{j=1}^M$$

is a disjoint family,

$$B(\Pi_j, C_1^{-1} \delta) \subset C_2 R$$ (the dilate of $R$ around its center),

$$\bigcup_{j=1}^M B(\Pi_j, C_1^{-1} \delta) \subset \bigcup_{\Pi \in B} (B(\Pi_j, C_3 \zeta) \cap C_2 R),$$

and $B(\Pi_j, C_3 \zeta) \cap C_2 R$ is contained in a rectangle of sidelength, up to constants, $\delta / \rho$ in $n - 2$ dimensions and $\zeta$ in the remaining $n - 2$ dimensions. Therefore, by volume counting

$$M \delta^{2(n-2)} \leq |B|(\delta / \rho)^{n-2} \zeta^{n-2}.$$ 

We conclude that

$$|B| \geq M(\delta / \rho)^{n-2}.$$  

\[ \square \]

**Lemma 3.2.** Let $\Pi_1, \Pi_2 \in G_{n,2}$ be such that $\theta(\Pi_1, \Pi_2) \leq 1/2$. Suppose that there exist $a, b \in \Pi_1^i \cap \Pi_2^j$, $\rho > 0$ with $\rho \leq |a - b| \leq 2 \rho$. Then

$$\Pi_1^i \cap \Pi_2^j \cap B(a, 2 \rho) \subset T_{e}^{(4 \rho)}(a),$$

where $e = (a - b)/|a - b|$ and $\beta = C \delta / \theta(\Pi_1, \Pi_2)$.

**Proof.** Let $\theta = \theta(\Pi_1, \Pi_2)$. If $\rho \leq C \delta / \theta$ then $B(a, 2 \rho) \subset T_{e}^{(4 \rho)}(a)$, so we may assume that $\rho \geq C \delta / \theta$. We can also assume that $\Pi_2$ is the $x_1, x_2$-plane, $a = 0$, $b = (b_1, 0, b_2)$, $\overline{b} \in \mathbb{R}^{n-2}$. Since $\theta \leq 1/2$, by Lemma 2.1, we can write $\Pi_1 = \{(s, t, \overline{s} \overline{u} + \overline{v}) : s, t \in \mathbb{R}\}$, where $\overline{s}, \overline{v} \in \mathbb{R}^{n-2}$, $|\overline{s}|, |\overline{v}| \leq 1$, and $\theta \geq |\overline{s}| + |\overline{v}|$ . Since $b \in \Pi_1^i \cap \Pi_2^j$, there exists $(s, t, \overline{s} \overline{u} + \overline{v}) \in \Pi_1$ such that

$$|b_1 - s| \leq \delta, |t| \leq \delta, |s \overline{u} + v| \leq \delta, |\overline{b}| \leq \delta.$$

Hence

$$|\overline{u}| \leq \frac{|s \overline{u} + \overline{v} - \overline{b}| + |s \overline{u}| + |\overline{b}|}{|s|} \leq \frac{\delta}{|b| - |\overline{b}| - |b_1 - s|} \leq \frac{\delta}{\rho - C \delta} \leq \frac{\delta}{\rho(1 - C C^{-1})} \leq \frac{\delta}{\rho},$$

for $C_1$ sufficiently large. Consequently

$$|\overline{v}| = |\overline{u}| + |\overline{b}| - |\overline{u}| \geq C^{-1} \theta - C \delta / \rho \geq (C^{-1} - C C^{-1}) \theta \geq \delta,$$

for $C_1$ large enough.

Now let $y = (y_i) \in \Pi_1^i \cap \Pi_2^j \cap B(a, 2 \rho)$. Then there exist $z_1 = (s_1, t_1, s_1 \overline{u} + t_1 \overline{v}) \in \Pi_1$, $z_2 = (s_2, t_2, \overline{b}) \in \Pi_2$ such that $|y - z_1| \leq \delta$, $|y - z_2| \leq \delta$. Therefore

$$|s_1 \overline{u} + t_1 \overline{v}| \leq \delta, |s_1| \leq |z_1| \leq |z_1 - y| + |y| \leq \delta + \rho \leq \rho.$$
It follows that
\[ |t_1| \leq \frac{|s_1 u + t_1 v| + |s_1 u|}{|v|} \leq \frac{\delta + \rho(\delta/\rho)}{\theta} \leq \frac{\delta}{\theta}. \]

Hence
\[ \text{dist}(y, x_1\text{-axis}) \leq |y - (s_1, 0, \vec{u})| \leq |y - z_1| + |t_1| + |s_1 u + t_1 v| \leq \delta + \delta/\theta + \delta \leq \delta/\theta. \]

We conclude that
\[ \text{dist}(y, L_\alpha(0)) \leq |y - y_1 b_1^{-1} b| \leq \text{dist}(y, x_1\text{-axis}) + |y_1||b_1|^{-1}|\vec{b}| \leq \delta/\theta + \delta \leq \delta/\theta. \]

\[ \square \]

4. Proof of Theorem 1.1

Let \( E \subset \mathbb{R}^n \), \( 0 < \lambda \leq 1 \), and
\[ A_\lambda = \{ \Pi \in \mathcal{A}_{n,2} : M_{b_\lambda}E(\Pi) \geq \lambda \}. \]
By the standard interpolation theorems, it is enough to prove the following restricted weak-type estimate at the endpoint.
\[ \gamma_{n,2}(A_\lambda) \leq C\varepsilon \left( \frac{1}{\delta} \right)^{\frac{n}{3}} \left( \frac{|E|}{\lambda^{(n+1)/3} \delta^{(n-2)/3}} \right)^{\frac{3}{n}}. \] (4.1)

Now, let \( \{\Pi_j\}_{j=1}^M \) be a maximal \( \delta \)-separated subset of \( A_\lambda \). Then proving (4.1) amounts to proving
\[ |E| \geq C^{-1}_\varepsilon \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}. \] (4.2)

Since \( \Pi_j \in A_\lambda \), there exists \( \varepsilon \Pi_j^\delta \) such that
\[ ||\varepsilon \Pi_j^\delta \cap E|| \geq \frac{3}{4} \lambda ||\Pi_j^\delta||. \] (4.3)

Put \( \gamma = \lambda^{1/2} (\log(1/\delta))^{-1/2} \) and note that (4.2) is trivial if \( 4\delta \geq \gamma \). Indeed, (4.3) implies
\[ |E| \geq \lambda_0 \delta^{n-2} = \lambda^{(n+1)/3} \lambda^{-(n-2)/3} \delta^{n-2} \geq \lambda^{(n+1)/3} \delta^2 \log(1/\delta) \lambda^{-(n-2)/3} \delta^{n-2} = (\log(1/\delta))^{-(n-2)/3} \lambda^{(n+1)/3} \delta^{2(n-2)/3} \delta^{n-2} \geq C^{-1}_\varepsilon \delta \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}. \]

We may therefore assume that \( 4\delta \leq \gamma \).

Now, let \( \delta_0 \) be a small constant to be determined later. Then for \( \delta \geq \delta_0 \), we have \( M \leq 1 \), and so (4.3) trivially implies (4.2) as before. Hence we can also assume that \( \delta \leq \delta_0 \).

After these preliminary reductions, we can proceed with the proof of (4.2). First, we find a large number of sets \( \Pi_j^\delta \) so that the measure of their intersection with \( E \) is concentrated in annuli of fixed dimensions. More
precisely, we claim that there exist a number \( \rho \geq \gamma \) and a set \( C \subset \{ \Pi_j \}_{j=1}^M \) with
\[
|C| \gtrsim (\log(C/\gamma)^{-2}M,
\]
so that for each \( \Pi_j^\rho \in C \) there is a set \( P_j \subset \Pi_j^\rho \) of measure
\[
|P_j| \gtrsim (\log(C/\gamma))^{-2} \lambda |\Pi_j^\rho|,
\]
such that for each \( z \in P_j \)
\[
|\Pi_j^\rho \cap E \cap B(z, 2\rho) \cap (T_e^{(4r)(\gamma^2/r)}(z))C| \gtrsim \lambda |\Pi_j^\rho|,
\]
for all \( e \in S^{n-1}, \gamma \leq r \leq 1, \) and
\[
|\Pi_j^\rho \cap E \cap A(z, \rho)| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\rho|.
\]
To see this, note that for all \( z \in \mathbb{R}^n, e \in S^{n-1} \) and \( r > 0 \) with \( \gamma \leq r \leq 1 \) we have
\[
\left| \Pi_j^\rho \cap E \cap (T_e^{(4r)(\gamma^2/r)}(z))C \right|
\]
\[
\geq \frac{3}{4} \lambda |\Pi_j^\rho| \geq C \lambda |\Pi_j^\rho|
\]
for \( \delta_0 \) small enough.
Now, for each \( 1 \leq j \leq M, z \in \Pi_j^\rho \cap E, i \in \mathbb{N}, \) consider the quantity
\[
Q(j, z, i) = \inf_{r, \gamma \leq r \leq 1} \left| \Pi_j^\rho \cap E \cap B(z, 2^i) \cap (T_e^{(4r)(\gamma^2/r)}(z))C \right|.
\]
Then
\[
Q(j, z, 0) \leq C \gamma^2 \delta_0 \delta_0 \leq C \lambda (\log(1/\delta))^{-1} \delta_0 \leq \frac{\lambda}{10} |\Pi_j^\rho|
\]
provided that \( \delta_0 \) has been chosen small enough. On the other hand
\[
Q(j, z, \log(C/\gamma)) \geq \frac{\lambda}{2} |\Pi_j^\rho|
\]
Therefore, there exists \( i_{jz} \) with \( 1 \leq i_{jz} \leq \log(C/\gamma) \), such that
\[
Q(j, z, i_{jz}) \geq \frac{\lambda}{4} |\Pi_j^\rho|, \quad \text{and} \quad Q(j, z, i_{jz} - 1) < \frac{\lambda}{4} |\Pi_j^\rho|
\]
Since there are at most \( \log(C/\gamma) \) possible \( i_{jz} \), there is an \( i_j \) and a set \( P_j^\prime \subset \Pi_j^\rho \cap E \) of measure
\[
|P_j^\prime| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\rho|
\]
such that for each \( z \in P_j^\prime \)
\[
Q(j, z, i_j) \geq \frac{\lambda}{4} |\Pi_j^\rho|, \quad \text{and} \quad Q(j, z, i_j - 1) < \frac{\lambda}{4} |\Pi_j^\rho|
\]
Since there are $M$ sets $\Pi_j^\varepsilon$ and at most $\log(C/\gamma)$ possible $i_j$, there is an $i_0$ and a subset $C' \subseteq \{\Pi_j^\varepsilon\}_{j=1}^M$ such that

$$|C'| \geq (\log(C/\gamma))^{-1} M,$$

and for each $\Pi_j^\varepsilon \in C'$ and each $z \in \Pi_j^\varepsilon$

$$|\Pi_j^\varepsilon \cap E \cap B(z, \gamma 2^{i_0}) \cap (T_{e^{(4r)}}(\gamma^2/r)^{(z)})^C| \geq \frac{\lambda}{4} |\Pi_j^\varepsilon|,$$

for all $e \in S^{n-1}$, $\gamma \leq r \leq 1$, and

$$|\Pi_j^\varepsilon \cap E \cap B(z, \gamma 2^{i_0-1}) \cap (T_{e^{(4r)}}(\gamma^2/r)^{(z)})^C| < \frac{\lambda}{4} |\Pi_j^\varepsilon|,$$

for some $e_{jz} \in S^{n-1}$, $\gamma \leq r_{jz} \leq 1$. It follows that

$$\frac{\lambda}{4} |\Pi_j^\varepsilon| \leq |\Pi_j^\varepsilon \cap E \cap (T_{e^{(4r)}}(\gamma^2/r)^{(z)})^C|$$

$$- |\Pi_j^\varepsilon \cap E \cap B(z, \gamma 2^{i_0-1}) \cap (T_{e^{(4r)}}(\gamma^2/r)^{(z)})^C|$$

$$= |\Pi_j^\varepsilon \cap E \cap (B(z, \gamma 2^{i_0-1}))^C \cap (T_{e^{(4r)}}(\gamma^2/r)^{(z)})^C|$$

$$\leq |\Pi_j^\varepsilon \cap E \cap (B(z, \gamma 2^{i_0-1}))^C|$$

$$= \sum_{k=0}^{|\log(C/\gamma)|} |\Pi_j^\varepsilon \cap E \cap A(z, \gamma 2^{i_0+k-1})|.$$ 

Therefore, there is a $k_{jz}$ such that

$$|\Pi_j^\varepsilon \cap E \cap A(z, \gamma 2^{i_0+k_{jz}-1})| \geq (\log(C/\gamma))^{-1} \lambda |\Pi_j^\varepsilon|.$$ 

Repeatedly using the pigeonhole principle as before, we conclude that there is a number $\rho = \gamma 2^{i_0+k_{jz}-1}$ and a set $C \subseteq C'$ with

$$|C| \geq (\log(C/\gamma))^{-2} M,$$

so that for each $\Pi_j^\varepsilon \in C$, there is a subset $P_j \subset P_j^\varepsilon$ of measure

$$|P_j| \geq (\log(C/\gamma))^{-2} \lambda |\Pi_j^\varepsilon|$$

such that for each $z \in P_j$

$$|\Pi_j^\varepsilon \cap E \cap B(z, \gamma 2\rho) \cap (T_{e^{(4r)}}(\gamma^2/r)^{(z)})^C| \geq \lambda |\Pi_j^\varepsilon|,$$

for all $e \in S^{n-1}$, $\gamma \leq r \leq 1$, and

$$|\Pi_j^\varepsilon \cap E \cap A(z, \rho)| \geq (\log(C/\gamma))^{-1} \lambda |\Pi_j^\varepsilon|,$$

proving the claim.

This construction will allow us to carry out a “high-low multiplicity segment” argument as follows. We fix a number $N$ and consider two cases.

CASE I. $\forall a \in \mathbb{R}^n \ |\{j : a \in P_j\}| \leq N$.

CASE II. $\exists a \in \mathbb{R}^n \ |\{j : a \in P_j\}| \geq N$. 

8
In case I we have
\[|E| \geq \left| \bigcup_{j \in \Pi_j \subseteq C} P_j \right| \geq \frac{1}{N} \sum_{j \in \Pi_j \subseteq C} |P_j| \]
\[\geq \frac{1}{N} |C| (\log(C/\gamma))^{-2} \lambda \delta^{n-2} \]
\[\geq \frac{M}{N} (\log(C/\gamma))^{-4} \lambda \delta^{n-2}, \quad (4.8)\]
where we have used (4.4) and (4.5).

In case II, we fix a number \(\mu\) and consider two subcases.

(II) \(1\). \( \forall b \in A(a, \rho) \ || \{j : a \in P_j, b \in \Pi_j^b\} \| \leq \mu. \)

(II) \(2\). \( \exists b \in A(a, \rho) \ || \{j : a \in P_j, b \in \Pi_j^b\} \| \geq \mu \) (see Figure 1).

In subcase (II) \(1\) we have
\[|E| \geq \left| \bigcup_{j \in \Pi_j} \Pi_j^b \cap E \cap A(a, \rho) \right| \]
\[\geq \frac{1}{\mu} \sum_{j \in \Pi_j} |\Pi_j^b \cap E \cap A(a, \rho)| \]
\[\geq \frac{N}{\mu} (\log(C/\gamma))^{-1} \lambda \delta^{n-2}, \quad (4.9)\]
where the last inequality follows from (4.7).

In subcase (II) \(2\) let \(B\) be a maximal \(C_1 \rho \delta/\gamma^2\)-separated subset of \(\{\Pi_j : a \in P_j, b \in \Pi_j^b\}\). Then for \(C_1\) large enough, \(C_1 \rho \delta/\gamma^2 \geq C \delta/\rho\). Therefore by
Lemma 3.1

\[ |B| \geq \mu \gamma^{2(n-2)}. \]

Note that if \( \Pi_j, \Pi_k \in B \) then by Lemma 3.2

\[ \Pi_j^\delta \cap \Pi_k^\delta \cap B(a, 2\rho) \subset T_{e^{(4\rho)(4\gamma^2/\rho)}}(a) \cap T_{e^{(4\rho)(\gamma^2/\rho)}}(a), \]

where \( e = (a - b)/|a - b| \), provided that \( C_1 \) has been chosen large enough.

Therefore the family

\[ \left\{ \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_{e^{(4\rho)(\gamma^2/\rho)}}(a))^\complement : \Pi_j \in B \right\} \]

is disjoint. Consequently

\[ |E| \geq \sum_{j: \Pi_j \in B} \left| \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_{e^{(4\rho)(\gamma^2/\rho)}}(a))^\complement \right| \]

\[ \geq |B| \lambda \delta^{n-2} \]

\[ \geq \gamma^{2(n-2)} \mu \lambda \delta^{n-2}, \quad (4.10) \]

where we have used (4.6). So, in case II we see that choosing

\[ \mu = N^{1/2} (\log(C/\gamma))^{-1/2} \gamma^{-(n-2)}, \]

(4.9) and (4.10) imply that

\[ |E| \geq (\log(C/\gamma))^{-1/2} \gamma^{-n} \lambda N^{1/2} \delta^{n-2}. \quad (4.11) \]

Choosing

\[ N = M^{2/3} (\log(C/\gamma))^{-7/3} \gamma^{-2(n-2)/3}, \]

(4.8) and (4.11) yield

\[ |E| \geq (\log(C/\gamma))^{-5/3} \gamma^{2(n-2)/3} \lambda M^{1/3} \delta^{n-2} \]

\[ = \left( \frac{C}{\lambda^{1/2} (\log(1/\delta))^{-1/2}} \right)^{-5/3} \left( \lambda^{1/2} (\log(1/\delta))^{-1/2} \right)^{2(n-2)/3} \lambda M^{1/3} \delta^{n-2} \]

\[ \geq C^{-1} \delta^\epsilon \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}, \]

which is (4.2). The proof is complete.

REFERENCES