MAPPING PROPERTIES OF THE MAXIMAL AVERAGING OPERATOR ASSOCIATED TO THE 2-PLANE TRANSFORM

THEMIS MITSIS

Abstract. We extend Christ's estimate for the 2-plane transform to a maximal operator setting.

1. Introduction

Let $\mathcal{G}_{n,2}$ be the Grassmannian manifold of all 2-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the unique invariant probability measure $\gamma_{n,2}$. For a function f satisfying the appropriate integrability conditions, the 2-plane transform $T_{n,2}f$ is defined by

$$T_{n,2}f(\Pi, y) = \int_{\Pi} f(x+y)d\mathcal{L}^2(x),$$

where $\mathcal{L}^2$ is 2-dimensional Lebesgue measure on the plane $\Pi \in \mathcal{G}_{n,2}$.

The following mixed-norm estimate was proved by Christ [2].

$$\|T_{n,2}f\|_{L^q(L^r)} \leq C_{n,p,q,r}\|f\|_p \tag{1.1}$$

where

$$\frac{n}{p} - \frac{n-2}{r} = 2, \quad 1 \leq p \leq \frac{n+1}{3}, \quad q \leq (n-2)p',$$

and

$$\|T_{n,2}f\|^q_{L^q(L^r)} = \int_{\mathcal{G}_{n,2}} \left(\int_{\Pi^\perp} |T_{n,2}f(\Pi, y)|^r d\mathcal{L}^{n-2}(y)\right)^{q/r} d\gamma_{n,2}(\Pi),$$

$\Pi^\perp$ being the orthogonal complement of Π, $\mathcal{L}^{n-2}$ $(n-2)$-dimensional Lebesgue measure on $\Pi^\perp$, and p' the conjugate exponent of p.

It is unlikely that this estimate is sharp. For example, it does not imply full Hausdorff dimension for $(n, 2)$-sets, a fact which was proved by the author in [3]. Actually, Christ conjectured that (1.1) should hold with

$$\frac{n}{p} - \frac{n-2}{r} = 2, \quad 1 \leq p < \frac{n}{2}, \quad q \leq (n-2)p'.$$

Notice that in the above conjectured range of boundedness, r approaches ∞ as p approaches the endpoint $n/2$. It is therefore natural to consider the corresponding maximal averaging operator which is analogous to the

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Kakeya maximal function introduced by Bourgain [1]. To do this, we need some further notation. For $\Pi \in \mathcal{G}_{n,2}$, $a \in \mathbb{R}^n$, $\delta \ll 1$, we denote by $\Pi^\delta(a)$ the $\delta/2$-neighborhood of the intersection of $\Pi + a$ with the ball of radius $1/2$ centered at a, and define

$$\mathcal{M}_\delta f : \mathcal{G}_{n,2} \to \mathbb{R}$$

by

$$\mathcal{M}_\delta f(\Pi) = \sup_{a\in\mathbb{R}^n} \frac{1}{|\Pi^\delta(a)|} \int_{\Pi^\delta(a)} |f(y)| dy,$$

where $|\Pi^\delta(a)|$ is the volume of $\Pi^\delta(a)$, and $f : \mathbb{R}^n \to \mathbb{R}$ any locally integrable function.

We are interested in proving $L^p \to L^q(\mathcal{G}_{n,2}, \gamma_{n,2})$ estimates for this operator. To find the optimal range for p and q we argue as follows.

If f is the characteristic function of a ball of radius δ, then $\|f\|_p$ is comparable to $\delta^{n/p}$, and $\|\mathcal{M}_\delta\|_{L^q(\mathcal{G}_{n,2},\gamma_{n,2})}$ is comparable to δ^2. Therefore, the best possible bound is

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\gamma_{n,2})} \leq C_{n,p,q} \delta^{2-n/p} \|f\|_p, \; p < n/2, \; q \geq 1.$$

On the other hand, if f is the characteristic function of a rectangle of dimensions $1 \times 1 \times \delta \times \cdots \times \delta$, then $\|f\|_p = \delta^{(n-2)/p}$ and $\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\gamma_{n,2})}$ is, up to a multiplicative constant, greater than $\delta^{2(n-2)/q}$. It follows that if the above estimate is true, we must have

$$\delta^{2(n-2)/q} \leq C_{n,p,q} \delta^{2-n/p} \delta^{(n-2)/p},$$

which forces q to be less than $(n-2)p'$.

These examples suggest the following conjecture which, if true, would imply the result in [3].

For every $\varepsilon > 0$ there exists a constant $C_{\varepsilon,p,q} > 0$ such that

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\gamma_{n,2})} \leq C_{\varepsilon,p,q} \delta^{2-n/p-\varepsilon} \|f\|_p,$$

where

$$1 \leq p \leq \frac{n}{2}, \quad q \leq (n-2)p'.$$

The purpose of this paper is to give a geometric proof of the following partial result, which may be thought of as a stronger version of Christ's estimate.

Theorem 1.1. *For every $\varepsilon > 0$ there exists a constant $C_{\varepsilon,p,q} > 0$ such that*

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\gamma_{n,2})} \leq C_{\varepsilon,p,q} \delta^{2-n/p-\varepsilon} \|f\|_p, \tag{1.2}$$

where

$$1 \leq p \leq \frac{n+1}{3}, \quad q \leq (n-2)p'.$$

In an attempt to prove an estimate like (1.2) which is insensitive to $\delta^{-\varepsilon}$ factors, it would seem reasonable to try to modify or refine the nearly optimal argument of Mitsis [3]. However, that argument is based, in part, on a discretization of the sharp bound for the Radon transform due to Oberlin and Stein [4], which yields a distributional-type inequality with the "wrong" exponent for the λ-parameter. This defect is of no consequence, as far as the geometric problem considered in [3] is concerned, but makes the approach of that paper inapplicable in the present context. Therefore, we have to use a more direct, but less efficient, high-low multiplicity argument. It is the author's impression that any improvement on (1.2) would require a complete understanding of the geometry of the Radon transform, and so, an alternative, purely geometric proof of the result in [4] would be a valuable contribution.

2. Preliminaries

Throughout this paper, the capital letter C, subscripted or otherwise, will denote various constants whose values may change from line to line. $x \lesssim y$ means $x \leq Cy$, and similarly with $x \gtrsim y$ and $x \simeq y$. Also, we will use the notation Π^δ for any set $\Pi^\delta(a)$, since the basepoint a is irrelevant in all our arguments. Further notational conventions follow below.

S^{n-1} is the $(n-1)$-dimensional sphere.
$B(a, r)$ is the ball of radius r centered at a.
$A(a, r)$ is the annulus $B(a, 2r) \setminus B(a, r)$.
$L_e(a)$ is the line in the direction $e \in S^{n-1}$ passing through the point a, i.e.

$$L_e(a) = \{a + te : t \in \mathbb{R}\}.$$

$T_e^{(r)(\beta)}(a)$ is the tube of length r, cross-section radius β, centered at a, and with axis in the direction $e \in S^{n-1}$, i.e.

$$T_e^{(r)(\beta)}(a) = \{x \in \mathbb{R}^n : \text{dist}(x, L_e(a)) \leq \beta \text{ and } |\text{proj}_{L_e(a)}(x) - a| \leq r/2\},$$

where $\text{proj}_{L_e(a)}(x)$ is the orthogonal projection of x onto $L_e(a)$.
χ_E is the characteristic function of the set E.
$|\cdot|$ denotes Lebesgue measure or cardinality, depending on the context.
If $\Pi_1, \Pi_2 \in \mathcal{G}_{n,2}$, then their distance θ is defined by

$$\theta(\Pi_1, \Pi_2) = \|\text{proj}_{\Pi_1} - \text{proj}_{\Pi_2}\|,$$

where $\|\cdot\|$ is the operator norm. $\gamma_{n,2}$ is a $2(n-2)$-dimensional regular measure with respect to this distance, in the sense that

$$\gamma_{n,2}(\{\Pi \in \mathcal{G}_{n,2} : \theta(\Pi, \Pi_0) \leq r\}) \simeq r^{2(n-2)}, \ \forall \Pi_0 \in \mathcal{G}_{n,2}, r < 1.$$

A finite subset of $\mathcal{G}_{n,2}$ is called δ-separated if the distance between any two of its elements is at least δ. So, if $\mathcal{B}$ is a maximal δ-separated subset of $\mathcal{A} \subset \mathcal{G}_{n,2}$, then

$$\gamma_{n,2}(\mathcal{A}) \lesssim |\mathcal{B}|\delta^{2(n-2)}.$$

Moreover, if $\mathcal{A} \subset \mathcal{G}_{n,2}$ is δ-separated, and $\mathcal{B}$ is a maximal η-separated subset of $\mathcal{A}$ with $\eta \geq \delta$, then

$$|\mathcal{B}| \gtrsim |\mathcal{A}|(\delta/\eta)^{2(n-2)}.$$

For technical reasons, we introduce the following subsets of $\mathcal{G}_{n,2}$.

$$\mathcal{A}_{n,2} := \{\Pi \in \mathcal{G}_{n,2} : \theta(\Pi, x_1x_2\text{-plane}) \leq 1/4\},$$

$$\mathcal{B}_{n,2} := \{\Pi \in \mathcal{G}_{n,2} : \theta(\Pi, x_1x_2\text{-plane}) \leq 1/2\}.$$

Notice that by invariance, it is enough to prove Theorem 1.1 for $\mathcal{M}_\delta$ restricted to $\mathcal{A}_{n,2}$.

We finally note the following fact which can be proved by fairly elementary arguments.

Lemma 2.1. *Let $\Pi \in \mathcal{B}_{n,2}$. Then there exist unique $\overline{u}, \overline{v} \in \mathbb{R}^{n-2}$ with $|u|, |v| \lesssim 1$ such that $\Pi = \{(s, t, s\overline{u} + t\overline{v}) : s, t \in \mathbb{R}\}$. Further, for any $\Pi_j = \{(s, t, s\overline{u}_j + t\overline{v}_j) : s, t \in \mathbb{R}\} \in \mathcal{B}_{n,2}$, $j = 1, 2$, we have $\theta(\Pi_1, \Pi_2) \simeq |\overline{u}_1 - \overline{u}_2| + |\overline{v}_1 - \overline{v}_2|$.*

3. Geometric Lemmas

In this section we prove two technical results that will allow us to control the cardinality and the intersection properties and of a family of sets Π^δ containing a fixed line segment.

Lemma 3.1. *Let $\{\Pi_j\}_{j=1}^M$ be a δ-separated set in $\mathcal{A}_{n,2}$. Suppose that there exist points $a, b \in \mathbb{R}^n$ and a number $\rho \geq 4\delta$ such that $|a - b| \geq \rho$, and for each j there is a Π_j^δ with $a, b \in \Pi_j^\delta$. Further, suppose that $\mathcal{B}$ is a maximal ζ-separated subset of $\{\Pi_j\}_{j=1}^M$ with $\zeta \geq C\delta/\rho$. Then $|\mathcal{B}| \gtrsim M(\rho\delta/\zeta)^{n-2}$.*

Proof. After rotating and translating, we may assume that Π_1 is the x_1x_2-plane, $\{\Pi_j\} \subset \mathcal{B}_{n,2}$, $a = \mathbf{0}$ and $b = (b_1, 0, \overline{b})$, for some $\overline{b} \in \mathbb{R}^{n-2}$. Then, by Lemma 2.1, there exist unique $u_j, v_j \in \mathbb{R}^{n-2}$ such that

$$\Pi_j = \{(s, t, su_j + tv_j) : s, t \in \mathbb{R}\}$$

and

$$\theta(\Pi_j, \Pi_k) \simeq |u_j - u_k| + |v_j - v_k|.$$

Therefore, we can think of $\{\Pi_j\}_{j=1}^M$ as a set of points in $B(0, C) \subset \mathbb{R}^{2(n-2)}$ under the identification $\Pi_j \leftrightarrow (u_j, v_j)$. We claim that $\{\Pi_j\}_{j=1}^M$ is contained in a rectangle R of sidelength, up to constants, δ/ρ in $n - 2$ dimensions and C in the remaining $n - 2$ dimensions. To see this, note that since $b \in \Pi_j^\delta$, there exists

$$w_j = (s_j, t_j, s_ju_j + t_jv_j) \in \Pi_j$$

with

$$|w_j - b| = |(s_j - b_1, t_j, s_ju_j + t_jv_j - \overline{b})| \leq \delta.$$

Therefore

$$|s_j - b_1| \leq \delta, \quad |t_j| \leq \delta, \quad |s_ju_j + t_jv_j - \overline{b}| \leq \delta.$$

In particular $|\overline{b}| \leq \delta$, and so

$$|s_j| \geq |b_1| - \delta \geq |b| - 2\delta \geq \rho - 2\delta \geq \rho/2.$$

On the other hand

$$|s_j u_j| \le \delta + |t_j v_j| + |\overline{b}| \lesssim \delta.$$

Consequently $|u_j| \lesssim \delta/\rho$, proving the claim.

Now, for appropriately chosen constants C_1, C_2, C_3, we have

$$\{B(\Pi_j, C_1^{-1}\delta)\}_{j=1}^M \quad \text{is a disjoint family,}$$

$$B(\Pi_j, C_1^{-1}\delta) \subset C_2 R \quad \text{(the dilate of } R \text{ around its center),}$$

$$\bigcup_{j=1}^M B(\Pi_j, C_1^{-1}\delta) \subset \bigcup_{\Pi_j \in \mathcal{B}} (B(\Pi_j, C_3\zeta) \cap C_2 R),$$

and $B(\Pi_j, C_3\zeta) \cap C_2 R$ is contained in a rectangle of sidelength, up to constants, δ/ρ in $n-2$ dimensions and ζ in the remaining $n-2$ dimensions. Therefore, by volume counting

$$M\delta^{2(n-2)} \lesssim |\mathcal{B}|(\delta/\rho)^{n-2}\zeta^{n-2}.$$

We conclude that

$$|\mathcal{B}| \gtrsim M(\delta\rho/\zeta)^{n-2}.$$

□

Lemma 3.2. *Let $\Pi_1, \Pi_2 \in \mathcal{G}_{n,2}$ be such that $\theta(\Pi_1, \Pi_2) \le 1/2$. Suppose that there exist $a, b \in \Pi_1^\delta \cap \Pi_2^\delta$, $\rho > 0$ with $\rho \le |a-b| \le 2\rho$. Then*

$$\Pi_1^\delta \cap \Pi_2^\delta \cap B(a, 2\rho) \subset T_e^{(4\rho)(\beta)}(a),$$

where $e = (a-b)/|a-b|$ and $\beta = C\delta/\theta(\Pi_1, \Pi_2)$.

Proof. Let $\theta = \theta(\Pi_1, \Pi_2)$. If $\rho \le C_1\delta/\theta$ then $B(a, 2\rho) \subset T_e^{(4\rho)(\beta)}(a)$, so we may assume that $\rho \ge C_1\delta/\theta$. We can also assume that Π_2 is the x_1x_2-plane, $a = \mathbf{0}$, $b = (b_1, 0, \overline{b})$, $\overline{b} \in \mathbb{R}^{n-2}$. Since $\theta \le 1/2$, by Lemma 2.1, we can write $\Pi_1 = \{(s, t, s\overline{u} + t\overline{v}) : s, t \in \mathbb{R}\}$, where $\overline{u}, \overline{v} \in \mathbb{R}^{n-2}$, $|\overline{u}|, |\overline{v}| \lesssim 1$, and $\theta \simeq |\overline{u}| + |\overline{v}|$. Since $b \in \Pi_1^\delta \cap \Pi_2^\delta$, there exists $(s, t, s\overline{u} + t\overline{v}) \in \Pi_1$ such that

$$|b_1 - s| \lesssim \delta, \ |t| \lesssim \delta, \ |s\overline{u} + t\overline{v} - \overline{b}| \lesssim \delta, \ |\overline{b}| \lesssim \delta.$$

Hence

$$\begin{aligned} |\overline{u}| &\le \frac{|s\overline{u} + t\overline{v} - \overline{b}| + |t\overline{u}| + |\overline{b}|}{|s|} \lesssim \frac{\delta}{|b| - |\overline{b}| - |b_1 - s|} \\ &\le \frac{\delta}{\rho - C\delta} \le \frac{\delta}{\rho(1 - CC_1^{-1})} \lesssim \frac{\delta}{\rho}, \end{aligned}$$

for C_1 sufficiently large. Consequently

$$|\overline{v}| = |\overline{v}| + |\overline{u}| - |\overline{u}| \ge C^{-1}\theta - C\delta/\rho \ge (C^{-1} - CC_1^{-1})\theta \gtrsim \delta,$$

for C_1 large enough.

Now let $y = (y_i) \in \Pi_1^\delta \cap \Pi_2^\delta \cap B(a, 2\rho)$. Then there exist $z_1 = (s_1, t_1, s_1\overline{u} + t_1\overline{v}) \in \Pi_1$, $z_2 = (s_2, t_2, \overline{\mathbf{0}}) \in \Pi_2$ such that $|y - z_1| \lesssim \delta$, $|y - z_2| \lesssim \delta$. Therefore

$$|s_1\overline{u} + t_1\overline{v}| \lesssim \delta, \ |s_1| \le |z_1| \le |z_1 - y| + |y| \lesssim \delta + \rho \lesssim \rho.$$

It follows that

$$|t_1| \le \frac{|s_1\overline{u} + t_1\overline{v}| + |s_1\overline{u}|}{|\overline{v}|} \lesssim \frac{\delta + \rho(\delta/\rho)}{\theta} \lesssim \frac{\delta}{\theta}.$$

Hence

$$\begin{aligned}\operatorname{dist}(y, x_1\text{-axis}) &\le |y - (s_1, 0, \overline{\mathbf{0}})| \le |y - z_1| + |t_1| + |s_1\overline{u} + t_1\overline{v}| \\ &\lesssim \delta + \delta/\theta + \delta \lesssim \delta/\theta.\end{aligned}$$

We conclude that

$$\begin{aligned}\operatorname{dist}(y, L_e(\mathbf{0})) &\le |y - y_1 b_1^{-1} b| \le \operatorname{dist}(y, x_1\text{-axis}) + |y_1||b_1|^{-1}|\overline{b}| \\ &\lesssim \delta/\theta + \delta \lesssim \delta/\theta.\end{aligned}$$

□

4. Proof of Theorem 1.1

Let $E \subset \mathbb{R}^n$, $0 < \lambda \le 1$, and

$$A_\lambda = \{\Pi \in \mathcal{A}_{n,2} : \mathcal{M}_\delta \chi_E(\Pi) \ge \lambda\}.$$

By the standard interpolation theorems, it is enough to prove the following restricted weak-type estimate at the endpoint.

$$\gamma_{n,2}(A_\lambda) \le C_\varepsilon \left(\frac{1}{\delta}\right)^\varepsilon \left(\frac{|E|}{\lambda^{(n+1)/3}\delta^{(n-2)/3}}\right)^3. \tag{4.1}$$

Now, let $\{\Pi_j\}_{j=1}^M$ be a maximal δ-separated subset of A_λ. Then proving (4.1) amounts to proving

$$|E| \ge C_\varepsilon^{-1}\delta^\varepsilon \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}. \tag{4.2}$$

Since $\Pi_j \in A_\lambda$, there exists Π_j^δ such that

$$|\Pi_j^\delta \cap E| \ge \frac{3}{4}\lambda|\Pi_j^\delta|. \tag{4.3}$$

Put $\gamma = \lambda^{1/2}(\log(1/\delta))^{-1/2}$ and note that (4.2) is trivial if $4\delta \ge \gamma$. Indeed, (4.3) implies

$$\begin{aligned}|E| &\gtrsim \lambda\delta^{n-2} = \lambda^{(n+1)/3}\lambda^{-(n-2)/3}\delta^{n-2} \\ &\gtrsim \lambda^{(n+1)/3}(\delta^2\log(1/\delta))^{-(n-2)/3}\delta^{n-2} \\ &= (\log(1/\delta))^{-(n-2)/3}\lambda^{(n+1)/3}((1/\delta)^{2(n-2)})^{1/3}\delta^{n-2} \\ &\ge C_\varepsilon^{-1}\delta^\varepsilon\lambda^{(n+1)/3}M^{1/3}\delta^{n-2}.\end{aligned}$$

We may therefore assume that $4\delta \le \gamma$.

Now, let δ_0 be a small constant to be determined later. Then for $\delta \ge \delta_0$, we have $M \lesssim 1$, and so (4.3) trivially implies (4.2) as before. Hence we can also assume that $\delta \le \delta_0$.

After these preliminary reductions, we can proceed with the proof of (4.2). First, we find a large number of sets Π_j^δ so that the measure of their intersection with E is concentrated in annuli of fixed dimensions. More precisely, we claim that there exist a number $\rho \geq \gamma$ and a set $\mathcal{C} \subset \{\Pi_j^\delta\}_{j=1}^M$ with

$$|\mathcal{C}| \gtrsim (\log(C/\gamma))^{-2} M, \tag{4.4}$$

so that for each $\Pi_j^\delta \in \mathcal{C}$ there is a set $P_j \subset \Pi_j^\delta$ of measure

$$|P_j| \gtrsim (\log(C/\gamma))^{-2} \lambda |\Pi_j^\delta| \tag{4.5}$$

such that for each $z \in P_j$

$$\left| \Pi_j^\delta \cap E \cap B(z, 2\rho) \cap (T_e^{(4r)(\gamma^2/r)}(z))^{\complement} \right| \gtrsim \lambda |\Pi_j^\delta|, \tag{4.6}$$

for all $e \in S^{n-1}$, $\gamma \leq r \leq 1$, and

$$|\Pi_j^\delta \cap E \cap A(z, \rho)| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\delta|. \tag{4.7}$$

To see this, note that for all $z \in \mathbb{R}^n$, $e \in S^{n-1}$ and $r > 0$ with $\gamma \leq r \leq 1$ we have

$$\begin{aligned} \left| \Pi_j^\delta \cap E \cap (T_e^{(4r)(\gamma^2/r)}(z))^{\complement} \right| &= |\Pi_j^\delta \cap E| - |\Pi_j^\delta \cap E \cap T_e^{(4r)(\gamma^2/r)}(z)| \\ &\geq \frac{3}{4} \left(\lambda |\Pi_j^\delta| - Cr \frac{\gamma^2}{r} |\Pi_j^\delta| \right) \\ &= \frac{3}{4} \lambda (1 - C(\log(1/\delta))^{-1}) |\Pi_j^\delta| \\ &\geq \frac{\lambda}{2} |\Pi_j^\delta|, \end{aligned}$$

for δ_0 small enough.

Now, for each $1 \leq j \leq M$, $z \in \Pi_j^\delta \cap E$, $i \in \mathbb{N}$, consider the quantity

$$Q(j, z, i) = \inf_{\substack{r : \gamma \leq r \leq 1 \\ e \in S^{n-1}}} \left| \Pi_j^\delta \cap E \cap B(z, \gamma 2^i) \cap (T_e^{(4r)(\gamma^2/r)}(z))^{\complement} \right|.$$

Then

$$Q(j, z, 0) \leq C\gamma^2 \delta^{n-2} = C\lambda (\log(1/\delta))^{-1} \delta^{n-2} \leq \frac{\lambda}{10} |\Pi_j^\delta|,$$

provided that δ_0 has been chosen small enough. On the other hand

$$Q(j, z, \log(C/\gamma)) \geq \frac{\lambda}{2} |\Pi_j^\delta|.$$

Therefore, there exists $i_{j,z}$ with $1 \leq i_{j,z} \leq \log(C/\gamma)$, such that

$$Q(j, z, i_{j,z}) \geq \frac{\lambda}{4} |\Pi_j^\delta|, \quad \text{and} \quad Q(j, z, i_{j,z} - 1) < \frac{\lambda}{4} |\Pi_j^\delta|.$$

Since there are at most $\log(C/\gamma)$ possible $i_{j,z}$, there is an i_j and a set $P'_j \subset \Pi_j^\delta \cap E$ of measure

$$|P'_j| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\delta|$$

such that for each $z \in \Pi'_j$

$$Q(j, z, i_j) \geq \frac{\lambda}{4} |\Pi_j^\delta|, \quad \text{and} \quad Q(j, z, i_j - 1) < \frac{\lambda}{4} |\Pi_j^\delta|.$$

Since there are M sets Π_j^δ and at most $\log(C/\gamma)$ possible i_j, there is an i_0 and a subset $C' \subset \{\Pi_j^\delta\}_{j=1}^M$ such that

$$|C'| \gtrsim (\log(C/\gamma))^{-1} M,$$

and for each $\Pi_j^\delta \in C'$ and each $z \in \Pi_j'$

$$\left|\Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0}) \cap (T_e^{(4r)(\gamma^2/r)}(z))^\complement\right| \geq \frac{\lambda}{4}|\Pi_j^\delta|,$$

for all $e \in S^{n-1}$, $\gamma \leq r \leq 1$, and

$$\left|\Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0 - 1}) \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement\right| < \frac{\lambda}{4}|\Pi_j^\delta|,$$

for some $e_{j,z} \in S^{n-1}$, $\gamma \leq r_{j,z} \leq 1$. It follows that

$$\begin{aligned}
\frac{\lambda}{4}|\Pi_j^\delta| &\leq \left|\Pi_j^\delta \cap E \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement\right| \\
&\quad - \left|\Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0-1}) \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement\right| \\
&= \left|\Pi_j^\delta \cap E \cap (B(z, \gamma 2^{i_0-1}))^\complement \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement\right| \\
&\leq \left|\Pi_j^\delta \cap E \cap (B(z, \gamma 2^{i_0-1}))^\complement\right| \\
&= \sum_{k=0}^{\log(C/\gamma)} |\Pi_j^\delta \cap E \cap A(z, \gamma 2^{i_0+k-1})|.
\end{aligned}$$

Therefore, there is a $k_{j,z}$ such that

$$|\Pi_j^\delta \cap E \cap A(z, \gamma 2^{i_0 + k_{j,z} - 1})| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\delta|.$$

Repeatedly using the pigeonhole principle as before, we conclude that there is a number $\rho = \gamma 2^{i_0 + k_0 - 1}$ and a set $C \subset C'$ with

$$|C| \gtrsim (\log(C/\gamma))^{-2} M,$$

so that for each $\Pi_j^\delta \in C$, there is a subset $P_j \subset P_j'$ of measure

$$|P_j| \gtrsim (\log(C/\gamma))^{-2} \lambda |\Pi_j^\delta|$$

such that for each $z \in P_j$

$$\left|\Pi_j^\delta \cap E \cap B(z, 2\rho) \cap (T_e^{(4r)(\gamma^2/r)}(z))^\complement\right| \gtrsim \lambda |\Pi_j^\delta|,$$

for all $e \in S^{n-1}$, $\gamma \leq r \leq 1$, and

$$|\Pi_j^\delta \cap E \cap A(z, \rho)| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\delta|,$$

proving the claim.

This construction will allow us to carry out a "high-low multiplicity segment" argument as follows. We fix a number N and consider two cases.

CASE I. $\forall a \in \mathbb{R}^n \ \ |\{j : a \in P_j\}| \leq N$.
CASE II. $\exists a \in \mathbb{R}^n \ \ |\{j : a \in P_j\}| \geq N$.

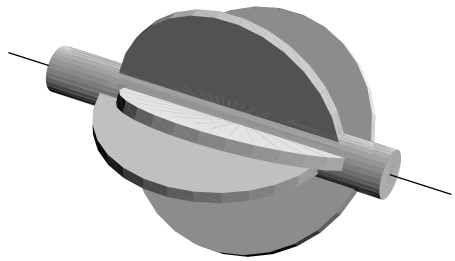

FIGURE 1. High multiplicity line segment

In case I we have

$$
\begin{aligned}
|E| &\geq \Big| \bigcup_{j:\Pi_j^\delta \in C} P_j \Big| \geq \frac{1}{N} \sum_{j:\Pi_j^\delta \in C} |P_j| \\
&\gtrsim \frac{1}{N} |C| \, (\log(C/\gamma))^{-2} \, \lambda \delta^{n-2} \\
&\gtrsim \frac{M}{N} \, (\log(C/\gamma))^{-4} \, \lambda \delta^{n-2}, \qquad (4.8)
\end{aligned}
$$

where we have used (4.4) and (4.5).

In case II, we fix a number μ and consider two subcases.

$(\mathrm{II})_1$. $\forall b \in A(a,\rho)$ $|\{j : a \in P_j, b \in \Pi_j^\delta\}| \leq \mu$.
$(\mathrm{II})_2$. $\exists b \in A(a,\rho)$ $|\{j : a \in P_j, b \in \Pi_j^\delta\}| \geq \mu$ (see Figure 1).

In subcase $(\mathrm{II})_1$ we have

$$
\begin{aligned}
|E| &\geq \Big| \bigcup_{j:a \in P_j} \Pi_j^\delta \cap E \cap A(a,\rho) \Big| \\
&\geq \frac{1}{\mu} \sum_{j:a \in P_j} |\Pi_j^\delta \cap E \cap A(a,\rho)| \\
&\gtrsim \frac{N}{\mu} \, (\log(C/\gamma))^{-1} \, \lambda \delta^{n-2}, \qquad (4.9)
\end{aligned}
$$

where the last inequality follows from (4.7).

In subcase $(\mathrm{II})_2$ let $\mathcal{B}$ be a maximal $C_1\rho\delta/\gamma^2$-separated subset of $\{\Pi_j :$ $a \in P_j, b \in \Pi_j^\delta\}$. Then for C_1 large enough, $C_1\rho\delta/\gamma^2 \geq C\delta/\rho$. Therefore by

Lemma 3.1

$$|\mathcal{B}| \gtrsim \mu\gamma^{2(n-2)}.$$

Note that if $\Pi_j, \Pi_k \in \mathcal{B}$ then by Lemma 3.2

$$\Pi_j^\delta \cap \Pi_k^\delta \cap B(a, 2\rho) \subset T_e^{(4\rho)(CC_1^{-1}\gamma^2/\rho)}(a) \subset T_e^{(4\rho)(\gamma^2/\rho)}(a),$$

where $e = (a-b)/|a-b|$, provided that C_1 has been chosen large enough. Therefore the family

$$\left\{\Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^\complement : \Pi_j \in \mathcal{B}\right\}$$

is disjoint. Consequently

$$\begin{aligned}
|E| &\geq \Big| \bigcup_{j:\Pi_j \in \mathcal{B}} \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^\complement \Big| \\
&= \sum_{j:\Pi_j \in \mathcal{B}} \left|\Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^\complement\right| \\
&\gtrsim |\mathcal{B}|\lambda\delta^{n-2} \\
&\gtrsim \gamma^{2(n-2)}\mu\lambda\delta^{n-2}, \qquad (4.10)
\end{aligned}$$

where we have used (4.6). So, in case II we see that choosing

$$\mu = N^{1/2}\left(\log(C/\gamma)\right)^{-1/2}\gamma^{-(n-2)},$$

(4.9) and (4.10) imply that

$$|E| \gtrsim \left(\log(C/\gamma)\right)^{-1/2}\gamma^{n-2}\lambda N^{1/2}\delta^{n-2}. \qquad (4.11)$$

Choosing

$$N = M^{2/3}\left(\log(C/\gamma)\right)^{-7/3}\gamma^{-2(n-2)/3},$$

(4.8) and (4.11) yield

$$\begin{aligned}
|E| &\gtrsim \left(\log(C/\gamma)\right)^{-5/3}\gamma^{2(n-2)/3}\lambda M^{1/3}\delta^{n-2} \\
&= \left(\log\frac{C}{\lambda^{1/2}(\log(1/\delta))^{-1/2}}\right)^{-5/3}(\lambda^{1/2}(\log(1/\delta))^{-1/2})^{2(n-2)/3}\lambda M^{1/3}\delta^{n-2} \\
&\gtrsim C_\varepsilon^{-1}\delta^\varepsilon\lambda^{(n+1)/3}M^{1/3}\delta^{n-2},
\end{aligned}$$

which is (4.2). The proof is complete.

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Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland

E-mail address: mitsis@maths.jyu.fi