VISIBLE PARTS AND DIMENSIONS

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Abstract. We study the visible parts of subsets of n-dimensional Euclidean space: a point a of a compact set A is visible from an affine subspace K of $\mathbb{R}^n$, if the line segment joining $P_K(a)$ to a only intersects A at a (here P_K denotes orthogonal projection onto K). The set of all such points visible from a given subspace K is called the visible part of A from K.

We prove that if the Hausdorff dimension of a compact set is at most $n-1$, then the Hausdorff dimension of a visible part is almost surely equal to the Hausdorff dimension of the set. On the other hand, provided that the set has Hausdorff dimension larger than $n-1$, we have the almost sure lower bound $n-1$ for the Hausdorff dimensions of visible parts.

We also investigate some examples of planar sets with Hausdorff dimension bigger than 1. In particular, we prove that for quasi-circles in the plane all visible parts have Hausdorff dimension equal to 1.

1. Introduction

The visible part of a set A from an affine subspace of $\mathbb{R}^n$ consists of those points of A that one can see from the subspace when looking perpendicularly away from it (for the exact definition, see Definition 2.1). In particular, in the extreme case when the affine subspace is just a point, the visible part consists of those points of the set that one sees when turning around at that point. This is similar to what we see when looking at stars whilst standing on the earth.

Visibility has been used in convex analysis for the study of star-like sets (see for example [B], [Ce], and [F]). There one is interested in the dimension of the visibility kernel, that is, the size of the set of points from which all the points of a given set can be seen. It has also been studied in the context of lattices (see for example [AC]). There is also a measure-theoretic definition for visibility (see [Cs]).

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The paper is organized as follows. In Section 2, we introduce some basic notation and describe a general projection theorem of Peres and Schlag (Theorem 2.6) which is our tool for studying small sets. In Section 3, we use it to prove that if a set has Hausdorff dimension no bigger than $n-1$, then the Hausdorff dimensions of its visible parts are almost surely equal to the Hausdorff dimension of the original set (Theorem 3.1). For sets that have dimension strictly greater than $n-1$, we show that the visible parts are almost surely at least $n-1$ dimensional (Proposition 3.2). For such sets the visible parts have almost surely zero measure for any measure which has dimension greater than $n-1$ and which gives positive measure to the original set (Proposition 3.3). In the last section, we investigate some examples of plane sets with dimension bigger than $n-1$. In particular, we show that the visible parts of a quasi-circle are always one-dimensional (Proposition 4.4). Finally, we observe that if there exists an almost sure value for the dimension of the visible parts for all sets with dimension larger than $n-1$, then this value must be equal to $n-1$ (Remark 4.2).

Our interest in the subject is partially motivated by a problem from cosmology: namely the basic assumption in cosmological models of the homogeneity and isotropy of galaxy distribution. This is usually interpreted as saying that the (spatial) dimension of galaxy distribution is three. To justify this assumption, one would like to measure the dimension of the Universe (see for example [D] and [PMS]). The aim of this paper is to find out whether one can conclude something about the dimension of a set only using information obtained via direct measurements from a given point or plane. This corresponds to the fact that our measurements are taken from a single base point; our solar system.

Our results suggest that if the dimension of the Universe is less than 2, then direct measurement from the earth should give the correct answer. However, if the dimension of the universe is larger than 2 and the examples in Section 4 correctly indicate the general behaviour of sets with dimension larger than 2, then it is impossible to measure the dimension of the Universe using only direct methods. And if, instead, our examples do not illustrate the general behaviour of sets, then Remark 4.2 suggests that direct measurements can only ever give lower bounds for dimension and it will be impossible to decide whether this lower bound is a good approximation or not (unless it is equal to 3).

2. Preliminaries and notation

We start this section by giving the definition of the visible part of a set from an affine plane and by making some basic observations that we will need later.

Let $0 \le k \le n-1$ be integers. We use the notations $A(n,k)$ and $G(n,k)$ for the space of affine k-dimensional subspaces of $\mathbb{R}^n$ and for the Grassmann manifold of linear k-dimensional subspaces of $\mathbb{R}^n$, respectively. By a 0-plane we mean a point. For $V \in G(n,k)$, let $V^\perp \in G(n,n-k)$ be the orthogonal complement of V, and let $\mathrm{P}_V : \mathbb{R}^n \to V$ be the orthogonal projection onto V.

Let K, $L \in A(n,k)$. Using the unique representations $K = V + a$ and $L = W + b$ where V, $W \in G(n,k)$, $a \in V^\perp$, and $b \in W^\perp$, we metricize $A(n,k)$ as follows:

$$d(K,L) = \| \mathrm{P}_V - \mathrm{P}_W \| + |a-b|.$$

The standard Radon measures on $G(n,k)$ and $A(n,k)$ are denoted by $\gamma_{n,k}$ and $\Gamma_{n,k}$, respectively (see [Mat 3.9]). For all Borel sets $A \subset A(n,k)$ we have [Mat, 3.16]

$$\Gamma_{n,k}(A) = \int \mathcal{H}^{n-k}(\{a \in V^{\perp} \mid V + a \in A\} \, d\gamma_{n,k}(V)$$

where $\mathcal{H}^{n-k}$ is the $n-k$-dimensional Hausdorff measure (for the definition see [Mat, 4.3]). Since $A(n,k)$ is a countable union of sets with finite $\Gamma_{n,k}$-measure, after a suitable restriction we may assume that $\Gamma_{n,k}$ is a finite Radon measure.

2.1. Definition. *Let $0 \le k \le n-1$ be integers. Let $A \subset \mathbb{R}^n$ be compact and let $K \in A(n,k)$ with $A \cap K = \emptyset$. The visible part of A from the affine subspace K is*

$$V_K(A) = \{x \in A \mid [\mathrm{P}_K(x), x] \cap A = \{x\}\}$$

where $\mathrm{P}_K(x)$ is the closest point to x on K (meaning that $\mathrm{P}_K(x) = \mathrm{P}_V(x) + a$ is the projection of x to the affine plane $K = V + a$, where $V \in G(n,k)$ and $a \in V^{\perp}$) and $[x,y]$ is the line segment between x and y.

2.2. Remarks. *(a) For any compact set $A \subset \mathbb{R}^n$ and for any affine subspace $K \in A(n,k)$ with $A \cap K = \emptyset$, the visible part $V_K(A)$ is a Borel set as the graph of a lower semi-continuous function.*

(b) In Section 4 Definition 2.1 will be used for subsets of affine subspaces in the following sense: Let $A \subset \mathbb{R}^n$ be compact, $K \in A(n,k)$, and $S \subset K$. The visible part of A from S is

$$V_S(A) = \{x \in A \mid \mathrm{P}_K(x) \in S \textit{ and } [\mathrm{P}_K(x), x] \cap A = \{x\}\}.$$

It is possible to extend the definition of visibility so that it makes sense for all affine subspaces including those that intersect A, namely by defining

$$V_K(A) = (A \cap K) \cup \{x \in A \mid [P_K(x), x] \cap (A \setminus K) = \{x\}\}.$$

For simplicity of the exposition, we shall assume throughout that $A \cap K = \emptyset$.

For the purpose of adapting the general formulation of the projection theorem due to Peres and Schlag [PS], we recall the following notation from [PS].

2.3. Definition. *Let (X,d) be a compact metric space, $Q \subset \mathbb{R}^n$ an open connected set, and $\Pi : Q \times X \to \mathbb{R}^m$ a continuous map with $n \ge m$. For any multi-index $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$, let $|\eta| = \sum_{i=1}^n \eta_i$ denote its length, and*

$$\partial^{\eta} = \frac{\partial^{|\eta|}}{(\partial\lambda_1)^{\eta_1} \ldots (\partial\lambda_n)^{\eta_n}}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in Q$. Let L be a positive integer and $\delta \in [0,1)$. We say that $\Pi \in C^{L,\delta}(Q)$ if for any compact set $Q' \subset Q$ and for any multi-index η with $|\eta| \le L$ there exist constants $C_{\eta,Q'}$ and $C_{\delta,Q'}$ such that

$$|\partial^{\eta}\Pi(\lambda,x)| \le C_{\eta,Q'} \textit{ and } \sup_{|\eta'|=L} |\partial^{\eta'}\Pi(\lambda,x) - \partial^{\eta'}\Pi(\lambda',x)| \le C_{\delta,Q'}|\lambda - \lambda'|^{\delta}$$

for all $\lambda, \lambda' \in Q'$ and $x \in X$.

We continue by defining a subclass of $C^{L,\delta}(Q)$ consisting of regular functions on a region of transversality of certain order.

2.4. Definition. *Let $\Pi \in C^{L,\delta}(Q)$ for some L and δ. Define for all $x \neq y \in X$*

$$\Phi_{x,y}(\lambda) = \frac{\Pi(\lambda, x) - \Pi(\lambda, y)}{d(x, y)}.$$

Let $\beta \in [0, 1)$. The set Q is a region of transversality of order β for Π if there exists a constant C_β such that for all $\lambda \in Q$ and for all $x \neq y \in X$ the condition $|\Phi_{x,y}(\lambda)| \leq C_\beta d(x, y)^\beta$ implies

$$\det(D\Phi_{x,y}(\lambda)(D\Phi_{x,y}(\lambda))^T) \geq C_\beta^2 d(x, y)^{2\beta}.$$

Here the derivative with respect to λ is denoted by D and A^T is the transpose of a matrix A. Further, Π is (L, δ)-regular on Q if there exists a constant $C_{\beta,L,\delta}$ and for all multi-indices η with $|\eta| \leq L$ there exists a constant $C_{\beta,\eta}$ such that, for all $\lambda, \lambda' \in Q$ and for all distinct $x, y \in X$,

$$|\partial^\eta \Phi_{x,y}(\lambda)| \leq C_{\beta,\eta} d(x, y)^{-\beta|\eta|}$$

and

$$\sup_{|\eta'|=L} |\partial^{\eta'} \Phi_{x,y}(\lambda) - \partial^{\eta'} \Phi_{x,y}(\lambda')| \leq C_{\beta,L,\delta} |\lambda - \lambda'|^\delta d(x, y)^{-\beta(L+\delta)}.$$

After defining Sobolev norms and Sobolev dimension of a measure we are ready to state the result from [PS] that we will need in Section 3.

2.5. Definition. *Let μ be a finite, compactly supported Borel measure on X and $\alpha \in \mathbb{R}$. The α-energy of μ is*

$$\mathcal{E}_\alpha(\mu) = \int_X \int_X d(x, y)^{-\alpha} d\mu(x) d\mu(y).$$

In the case $X = \mathbb{R}^m$ we define the Sobolev norm $\|\cdot\|_{2,\gamma}$ for $\gamma \geq 0$ by

$$\|\mu\|_{2,\gamma}^2 = \int_{\mathbb{R}^m} |\hat{\mu}(\xi)|^2 |\xi|^{2\gamma} d\mathcal{H}^m(\xi)$$

where

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^m} e^{-i\xi\cdot x} d\mu(x)$$

is the Fourier transform of μ. The Sobolev dimension of μ is

$$\dim_S(\mu) = \sup\{\alpha \in \mathbb{R} \mid \int_{\mathbb{R}^m} |\hat{\mu}(\xi)|^2 (1 + |\xi|)^{\alpha - m} d\mathcal{H}^m(\xi) < \infty\}.$$

The following theorem from [PS] gives a relation between energies of a given measure and Sobolev-norms of image measures under $C^{L,\delta}(Q)$-mappings. We use the notation $f_*\mu$ for the image of a measure μ under a map $f : X \to Y$, that is, $f_*\mu(A) = \mu(f^{-1}(A))$ for all $A \subset Y$.

2.6. Theorem. *Assume that $Q \subset \mathbb{R}^n$ and $\Pi \in C^{L,\delta}(Q)$ such that $L+\delta > 1$. Let $\beta \in [0,1)$. Assume that Q is a region of transversality of order β for Π and that Π is (L,δ)-regular on Q. Let μ be a finite Borel measure on X such that $\mathcal{E}_\alpha(\mu) < \infty$ for some $\alpha > 0$. Then there exists a constant a_0 depending only on m, n, and δ such that for any compact $Q' \subset Q$*

$$\int_{Q'} \|(\Pi_\lambda)_*\mu\|_{2,\gamma}^2 \, d\mathcal{H}^n(\lambda) \le C_\gamma \mathcal{E}_\alpha(\mu) \tag{2.1}$$

for some constant C_γ provided that $0 < (m+2\gamma)(1+a_0\beta) \le \alpha$ and $2\gamma < L+\delta-1$. Moreover, for any $\sigma \in (0, \min\{\alpha, m\}]$ we have

$$\dim_{\mathrm{H}}\{\lambda \in Q \mid \dim_{\mathrm{S}}((\Pi_\lambda)_*\mu) \le \sigma\} \le n + \sigma - \min\{\frac{\alpha}{1+a_0\beta}, L+\delta\}. \tag{2.2}$$

Here the Hausdorff dimension is denoted by $\dim_{\mathrm{H}}$ *(for the definition see* [Mat, 4.8]*) and $\Pi_\lambda = \Pi(\lambda, \cdot)$.*

Proof. See [PS, Theorem 7.3]. □

3. Visible parts and general projection formalism

We begin this section by considering visible parts of compact sets in $\mathbb{R}^n$ with Hausdorff dimension at most $n-1$. According to the following theorem, from typical affine subspaces visible parts have in this case the same Hausdorff dimension as the set itself.

3.1. Theorem. *Let $0 \le k \le n-1$. Assume that $A \subset \mathbb{R}^n$ is a compact set with* $\dim_{\mathrm{H}}(A) \le n-1$. *Then*

$$\dim_{\mathrm{H}}(V_K(A)) = \dim_{\mathrm{H}}(A)$$

for $\Gamma_{n,k}$-almost all $K \in A(n,k)$ not intersecting A.

Proof. Since for all $r > 0$ we have $V_{W+ra}(rA) = r(V_{W+a}(A))$ for all $W \in G(n,k)$ and $a \in W^\perp$ with $A \cap (W+a) = \emptyset$ and since Hausdorff dimension is preserved under homotheties and translations, we may assume that $A \subset B(0, \frac{1}{2})$.

For all $\delta > 0$ and $K \in A(n,k)$, define

$$A(\delta) = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) < \delta\}$$

and

$$S_\delta(K) = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, K) = \delta\}.$$

Here the distance of a point $x \in \mathbb{R}^n$ from a set $B \subset \mathbb{R}^n$ is denoted by $\operatorname{dist}(x, B)$.

Fix $0 < c < \frac{1}{2}$. Let $\widetilde{K} \in A(n,k)$ such that $\operatorname{dist}(\widetilde{K}, A) \ge 3c$. Clearly it is sufficient to prove the claim for $\Gamma_{n,k}$-almost all $K \in N_c(\widetilde{K})$ where

$$N_c(\widetilde{K}) = \{K \in A(n,k) \mid d(K, \widetilde{K}) < c\}.$$

For this purpose, we construct for all $K \in N_c(\widetilde{K})$ a map $g(K, \cdot) : A(\frac{c}{10}) \to S_c(K)$ such that

$$g(K, A) = g(K, V_K(A)) \tag{3.1}$$

for all $K \in N_c(\widetilde{K})$, and furthermore,

$$\dim_{\mathrm{H}}(g(K, A)) = \dim_{\mathrm{H}} A \tag{3.2}$$

for $\Gamma_{n,k}$-almost all $K \in N_c(\widetilde{K})$.

For all $K \in N_c(\widetilde{K})$ and $x \in A(\frac{c}{10})$ define

$$g(K, x) = y \text{ where } \{y\} = [\mathrm{P}_K(x), x] \cap S_c(K).$$

Now g is a well-defined Lipschitz function since $\mathrm{dist}(K, A) \geq 2c$ for all $K \in N_c(\widetilde{K})$. Moreover, property (3.1) follows directly from the definition.

For the purpose of proving (3.2), we may assume that $\mathrm{diam}(A) < \frac{c}{10}$ where the diameter of the set A is denoted by $\mathrm{diam}(A)$. To adapt Theorem 2.6 involves technical arguments using the mapping $\Pi = f \circ g : N_c(\widetilde{K}) \times A(\frac{c}{10}) \to \mathbb{R}^{n-1}$ where f (which depends smoothly on K) is a smooth diffeomorphism from a neighbourhood of $g(K, A(\frac{c}{10}))$ on $S_c(K)$ to $\mathbb{R}^{n-1}$. Note that f is well-defined since $\mathrm{diam}(A) < \frac{c}{10}$. After having proved that for $\Gamma_{n,k}$-almost all $K \in N_c(\widetilde{K})$ we have

$$\dim_{\mathrm{H}}(\Pi(K, A)) = \dim_{\mathrm{H}}(A), \tag{3.3}$$

equality (3.2) follows since f preserves Hausdorff dimension.

To verify (3.3) we first show that the function Π satisfies the assumptions of Theorem 2.6 with $L = \infty$ and $\beta = 0$. Indeed, the smoothness assumptions are clear. Note that δ plays no role in the case $L = \infty$. What is left is to show that Π satisfies the following transversality condition: there exists $\varepsilon > 0$ such that for all $K \in N_c(\widetilde{K})$ and $y \neq z \in A(\frac{c}{10})$

$$|\Phi_{y,z}(K)| \leq \varepsilon \implies \det(D\Phi_{y,z}(K)(D\Phi_{y,z}(K))^T) \geq \varepsilon^2, \tag{3.4}$$

where

$$\Phi_{y,z}(K) = \frac{\Pi(K, y) - \Pi(K, z)}{d(y, z)}.$$

Taking $\varepsilon > 0$ sufficiently small, we may assume that the piece of $S_c(K)$ we are considering is almost flat, that is, the difference between the projection to the cylinder and to a hyper-plane is negligible. Including this error in the term $\mathcal{O}(\varepsilon^2)$ below, we conclude that it is sufficient to study the orthogonal projection to a hyper-plane. Given two points $y, z \in \mathbb{R}^n$, the problem reduces to the study of the projection of $x = (0, \ldots, 0, x_n)$ to hyperplanes close to the orthogonal complement of $(0, \ldots, 0, 1)$, since projection is linear. Here $x_n = d(y, z)$. We parametrize the hyper-planes by their orthogonal complements $\epsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}, \sqrt{1 - \sum_{i=1}^{n-1} \varepsilon_i^2})$ where $|\varepsilon_1|, \ldots, |\varepsilon_{n-1}| < \varepsilon$ for some small ε. Thus

$$\Phi_{y,z}(\epsilon) = -(\varepsilon_1, \ldots, \varepsilon_{n-1}) + \mathcal{O}(\varepsilon^2)$$

giving

$$\det(D\Phi_{y,z}(\epsilon) D\Phi_{y,z}(\epsilon)^T) = 1 + \mathcal{O}(\varepsilon).$$

This completes the proof of (3.4).

Now assume that $\dim_{\mathrm{H}}(A) = s \le n-1$. Let $\alpha < s$. By Frostman's lemma [Mat, Theorem 8.8] there is a Radon measure μ on A with $\mathcal{E}_\alpha(\mu) < \infty$. According to (2.2) we have for $\Gamma_{n,k}$-almost all $K \in N_c(\widetilde{K})$

$$\dim_{\mathrm{S}}((\Pi_K)_*\mu) \ge \alpha$$

where $\Pi_K = \Pi(K,\cdot)$. (Note that $\Gamma_{n,k}$ is a uniformly distributed Radon measure, and hence it is a constant multiple of the d-dimensional Hausdorff measure for $d = \dim A(n,k)$.) Observing that for any compactly supported Radon measure ν on $\mathbb{R}^m$ and $\beta < m$ there exist constants C and D such that

$$\mathcal{E}_\beta(\nu) = C\int_{\mathbb{R}^m} |\hat{\nu}(\xi)|^2|\xi|^{\beta-m}d\mathcal{H}^m(\xi) \le D\int_{\mathbb{R}^m} |\hat{\nu}(\xi)|^2(1+|\xi|)^{\beta-m}d\mathcal{H}^m(\xi)$$

(for the equality see [Mat, Lemma 12.12]), [Mat, Theorem 8.9] implies that

$$\dim_{\mathrm{H}}(\Pi(K,A)) \ge \alpha$$

for $\Gamma_{n,k}$-almost all $K \in N_c(\tilde{K})$. Taking an increasing sequence $\alpha_i \to s$ we conclude that (3.3) holds since Π cannot increase the dimension.

To complete the proof, (3.1) and the fact that the map $g(K,\cdot)$ does not increase dimension give

$$\dim_{\mathrm{H}}(g(K,A)) \le \dim_{\mathrm{H}}(V_K(A)) \le \dim_{\mathrm{H}}(A). \tag{3.5}$$

This implies the claim by (3.2). $\square$

The same method gives the almost sure lower bound $n-1$ for visible parts of compact sets in $\mathbb{R}^n$ having Hausdorff dimension greater than $n-1$.

3.2. Proposition. *Let $A \subset \mathbb{R}^n$ be compact with $\dim_{\mathrm{H}}(A) > n-1$ and let $0 \le k \le n-1$. Then*

$$\dim_{\mathrm{H}}(V_K(A)) \ge n-1$$

for $\Gamma_{n,k}$-almost all $K \in A(n,k)$ not intersecting A.

Proof. We proceed as in the proof of Theorem 3.1; the only difference being that under the assumption $\dim_{\mathrm{H}}(A) > n-1$ we have

$$\dim_{\mathrm{H}}(g(K,A)) = n-1 \tag{3.6}$$

for $\Gamma_{n,k}$-almost all $K \in N_c(\widetilde{K})$. Indeed, Frostman's lemma and (2.1) imply that $(\Pi_K)_*\mu$ is absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure for $\Gamma_{n,k}$-almost all $K \in N_c(\widetilde{K})$, and has therefore Hausdorff dimension equal to $n-1$. This in turn gives (3.6) since $\dim_{\mathrm{H}}(\Pi(K,A)) = \dim_{\mathrm{H}}(g(K,A))$ for all $K \in N_c(\widetilde{K})$. Finally, the claim follows from the first inequality in (3.5). $\square$

We continue studying visible parts of compact sets in $\mathbb{R}^n$ having Hausdorff dimension greater than $n-1$. Given such a set A, there is a compactly supported Radon measure μ on A with $0 < \mu(A) < \infty$ and $\dim_{\mathrm{H}}(\mu) > n-1$ [Cu, Theorem 1.5]. Recall that

$$\dim_{\mathrm{H}}(\mu) = \mu\text{-}\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \liminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r}$$

where $B(x,r)$ is the closed ball with radius r centred at x. In a forthcoming paper [JJN] we will prove the following proposition. According to it, the visible parts of A are almost surely smaller than the set A itself, that is, almost all visible parts have μ-measure zero. The proof is based on an extension of the results of [JM] concerning dimensional properties of sliced measures.

3.3. Proposition. *Assume that* $A \subset \mathbb{R}^n$ *is a compact set and* μ *is a Radon measure with compact support such that* $\mu(A) > 0$ *and* $\dim_{\mathrm{H}}(\mu) > n-1$. *Then* $\mu(V_K(A)) = 0$ *for* $\Gamma_{n,k}$*-almost all* $K \in A(n,k)$ *where* $0 \le k < n$.

3.4. Remark. *A consequence of Proposition 3.3 is that any attempt to construct a measure with Hausdorff dimension greater than* $n-1$, *giving non-trivial values for positively many visible parts, fails. More precisely, let* $D \subset A(n, n-1)$ *with* $\Gamma_{n,n-1}(D) > 0$, $A \subset \mathbb{R}^n$ *be a compact set, and* $B = \cup_{K \in D} V_K(A)$. *Proposition 3.3 implies that for any compactly supported Radon measure* μ *with* $\mu(B) > 0$ *we have either* $\dim_{\mathrm{H}}(\mu) \le n-1$ *or* $\mu(V_K(A)) = 0$ *for* $\Gamma_{n,n-1}$*-almost all* $K \in D$.

4. Examples: Quasi-circles and Cantor sets

In this section we concentrate on plane sets and visible parts from affine lines. We begin by constructing a set for which the Hausdorff dimension of the visible part from countably many lines has any value between 1 and 2. The construction is based on the following example.

4.1. Example. *Let* $A \subset \mathbb{R}^2$ *be the attractor of the iterated function system* $f_i : [0,1] \times [0,1] \to [0,1] \times [0,1]$, $i = 1,2,3,4$, *where* $f_1(x,y) = (x/4, y/2)$, $f_2(x,y) = (x/4 + 1/4, y/2 + 1/2)$, $f_3(x,y) = (x/4 + 1/2, y/2)$, *and* $f_4(x,y) = (x/4 + 3/4, y/2 + 1/2)$. *Then* $\dim_{\mathrm{H}}(V_L(A)) = 3/2$ *for the affine line* $L = \{(t,-1) \mid t \in \mathbb{R}\} \in A(2,1)$.

Proof. Noting that any vertical line not having quadratic rational x-coordinate intersects the set A in at most one point, we obtain

$$\dim_{\mathrm{H}}(V_L(A)) = \dim_{\mathrm{H}}(A).$$

Since at the n^{th} level of the construction there are 4^n rectangles of width 4^{-n} and of height 2^{-n} one needs $4^n 2^{-n}/4^{-n}$ squares of side length 4^{-n} to cover the set A. Thus $\dim_{\mathrm{H}}(V_L(A)) = \dim_{\mathrm{H}}(A) = 3/2$. □

One can easily modify the above construction to obtain an example where the visible part from a fixed affine line has any dimension between 1 and 2. For fixed $1 \le s \le 2$ this in turn can be modified to show the existence of a compact set $A \subset \mathbb{R}^2$ with $\dim_{\mathrm{H}}(A) = s$ such that the Hausdorff dimensions of the visible parts from countably many (even dense set of) lines are equal to $\dim_{\mathrm{H}}(A)$.

In [DF] Davies and Fast constructed a compact set $A \subset \mathbb{R}^n$ such that its Hausdorff dimension equals n and the set of directions of lines which intersect A at most at one point is a dense G_δ-set. Clearly this set has the property that the visible parts from uncountably many hyperplanes have Hausdorff dimension n. The question whether there can be positively many planes (or lines in the plane) with this property remains open. We do not know the affirmative answer but we give some examples which all support the negative answer.

4.2. Remark. *Note that* $n-1$ *is the only possible constant value for Hausdorff dimensions of visible sets of large sets. More precisely, let* $0 \le k \le n-1$. *If for all compact sets* $A \subset \mathbb{R}^n$ *with* $\dim_{\mathrm{H}}(A) > n-1$ *there exists* $c = c(A,k)$ *such that* $\dim_{\mathrm{H}}(V_K(A)) = c(A,k)$ *for* $\Gamma_{n,k}$*-almost all* $K \in A(n,k)$, *then* $c(A,k) = n-1$.

Proof. Assume to the contrary that $c(A,k) > n-1$ for some k and $A \subset \mathbb{R}^n$. Let B be the union of A and a piece of a suitable hyper-plane disjoint from A. Then $\dim_{\mathrm{H}}(B) > n-1$. Moreover, there are E, $F \subset A(n,k)$ with $\Gamma_{n,k}(E) > 0$ and $\Gamma_{n,k}(F) > 0$ such that $\dim_{\mathrm{H}}(V_K(A)) = c(A,k)$ for all $K \in E$ and $\dim_{\mathrm{H}}(V_K(A)) = n-1$ for all $K \in F$. This leads to a contradiction. □

4.3. Example. *Let A be the graph of a continuous function $f : [0,1] \to [0,1]$. Then $\dim_{\mathrm{H}}(V_L(A)) = 1$ for all affine lines L which do not intersect A with the possible exception of one direction.*

Proof. This follows easily from the fact that at every point of $V_L(A)$ the open cone determined by the line perpendicular to L and the line parallel to y-axis is in the complement of $V_L(A)$. According to [Mat, Lemma 15.13] the set $V_L(A)$ is 1-rectifiable. $\square$

Note that unlike the previous example, visible parts are not necessarily rectifiable. For example, the von Koch curve is a purely 1-unrectifiable set having purely 1-unrectifiable visible parts. The following proposition implies that all of its visible parts have Hausdorff dimension 1. In fact, Proposition 4.4 gives a stronger conclusion according to which all visible parts of any quasi-circle have upper box counting dimension 1. We say that $A \subset \mathbb{R}^2$ is a quasi-circle of distortion $M \geq 1$ if A is a homeomorphic image of the unit circle into the plane such that for all $x, y \in A$ with $d(x,y) < M^{-1}\operatorname{diam}(A)$, we can join x to y by a (necessarily unique) sub-arc

$$\operatorname{arc}(x,y) \subset A \cap B(x, Md(x,y)).$$

Recall that the upper box counting dimension, $\overline{\dim}_{\mathrm{B}}$, is defined as follows

$$\overline{\dim}_{\mathrm{B}}(A) = \limsup_{r\to 0} \frac{\log P(A,r)}{-\log r}$$

where $P(A,r)$ is the greatest number of disjoint r-balls with centres in A (see [Mat, 5.3]).

4.4. Proposition. *Let $A \subset \mathbb{R}^2$ be a quasi-circle of distortion $M \geq 1$. Then $\overline{\dim}_{\mathrm{B}}(V_L(A)) = 1$ for all affine lines L not meeting A.*

Proof. Since A is a quasi-circle, we easily deduce that $\overline{\dim}_{\mathrm{B}}(V_L(A)) \geq 1$ for all affine lines L not meeting A. Without loss of generality we may assume that L is the x-axis, A lies in the upper half-plane, and that A projects vertically onto $[-1,1]$. We let P_x and P_y denote the orthogonal projection onto the x- and y-axis, respectively.

Given $0 < r < \min\{1, \operatorname{diam}(A)/(2M)\}$, we will show that if $z_1, \ldots, z_N \in V_L(A)$ with

$$d(z_i, z_j) > r \text{ for } i \neq j \tag{4.1}$$

then

$$N \leq -\frac{c\log r}{r} \tag{4.2}$$

where c is a constant depending on M. This immediately implies the theorem.

From now on we fix $0 < r < \min\{1, \operatorname{diam}(A)/(2M)\}$ and points $z_1, \ldots, z_N \in V_L(A)$ satisfying (4.1). We make a series of observations which will ultimately let us count how many points z_j there can be.

4.5. Observation. *Suppose that $z, w \in V_L(A)$. Then for $u \in V_L(A) \cap \operatorname{arc}(z, w)$ such that $\mathrm{P}_x(u)$ is between $\mathrm{P}_x(z)$ and $\mathrm{P}_x(w)$, we find*

$$\min\{\mathrm{P}_y(z), \mathrm{P}_y(w)\} - \mathrm{P}_y(u) \leq M|\,\mathrm{P}_x(z) - \mathrm{P}_x(w)|.$$

Proof. Without loss of generality we may assume that $\mathrm{P}_y(u) \leq \mathrm{P}_y(z) \leq \mathrm{P}_y(w)$. Since $z, w \in V_L(A)$ the arc $\operatorname{arc}(u, w)$ cannot intersect the vertical lines $x = \mathrm{P}_x(z)$ and $x = \mathrm{P}_x(w)$ below $\mathrm{P}_y(z)$ and $\mathrm{P}_y(w)$, respectively. Thus there exists $v \in \operatorname{arc}(u, w)$ with $\mathrm{P}_y(v) = \mathrm{P}_y(z)$ and $|\,\mathrm{P}_x(v) - \mathrm{P}_x(z)| \leq |\,\mathrm{P}_x(z) - \mathrm{P}_x(w)|$. This implies the claim, since $u \in \operatorname{arc}(z, v) \subset B(v, M|\,\mathrm{P}_x(v) - \mathrm{P}_x(z)|)$. □

To each z_j we associate the closed vertical half-line H_j consisting of a half-line descending vertically downwards from z_j together with a 'head' of height $r/2$ extending vertically upwards from z_j. Set

$$\mathcal{H} = \bigcup_{j=1}^{N} H_j \quad \text{and} \quad \mathcal{G} = \mathcal{H} \cup \{x = -2\} \cup \{x = 2\}.$$

The extra lines are introduced for technical convenience only. Define I_j to be the closed horizontal line segment in the plane whose endpoints lie in $\mathcal{G}$ and for which

$$\mathcal{G} \cap \operatorname{Int}(I_j) = \{z_j\}, \tag{4.3}$$

where $\operatorname{Int}(I_j)$ is the open horizontal line segment obtained from I_j by removing its endpoints. Clearly $\operatorname{diam}(I_j) \leq 4$ for all j. (This is the reason why we added the lines $x = -2$ and $x = 2$ to $\mathcal{G}$.)

4.6. Observation. *For all $j = 1, \ldots, N$,*

$$\operatorname{diam}(I_j) \geq \frac{r}{2M}.$$

Proof. Suppose, instead, that there is a j for which $\operatorname{diam}(I_j) < r/(2M)$. Then the endpoints of I_j must both lie in $\mathcal{H}$ and for $i \neq j$ $z_i \notin B(z_j, r)$ by (4.1). Thus the endpoints of I_j must belong to $H_m, H_n \in \mathcal{H}$, say, where both z_m and z_n lie above z_j, and where, without loss of generality, we may assume that $\mathrm{P}_y(z_m) \leq \mathrm{P}_y(z_n)$. By our assumption on the length of I_j

$$|\,\mathrm{P}_x(z_m) - \mathrm{P}_x(z_n)| < \frac{r}{2M}$$

and so

$$|\,\mathrm{P}_y(z_m) - \mathrm{P}_y(z_j)| \geq r(1 - 1/(4M^2))^{1/2} > r/2 \geq M|\,\mathrm{P}_x(z_m) - \mathrm{P}_x(z_n)|$$

contradicting Observation 4.5. □

We define R_j to be the open rectangle whose top edge is $\operatorname{Int}(I_j)$ and whose height is $r/2$. Notice that for $i \neq j$, $z_i \notin R_j$. We also let S_j be the open half-strip whose top edge is also $\operatorname{Int}(I_j)$. Notice that

$$\text{if } z_i \in S_j \text{ then } R_i \cap R_j = \emptyset. \tag{4.4}$$

Finally, we let $h_j = \mathrm{P}_y(I_j)$, the height of I_j above the x-axis, and $\mathrm{P}_j = \mathrm{P}_x(\operatorname{Int}(I_j))$, the projection of $\operatorname{Int}(I_j)$ onto the x-axis.

4.7. Observation. *If $i \neq j$ and $h_i \leq h_j$ then either*
(a) H_i contains an endpoint of I_j,
(b) $S_i \subset S_j$, or
(c) $S_i \cap S_j = \emptyset$.

Proof. We denote the closure of a set B by $\overline{B}$ and its boundary by ∂B. If $z_i \notin \overline{S}_j$ then $S_i \cap S_j = \emptyset$. If $z_i \in S_j$ then $S_i \subset S_j$. And if $z_i \in \partial S_j$ then $z_i \notin \operatorname{Int}(I_j)$. Hence z_i is in one of the vertical edges of S_j and thus H_i contains an endpoint of I_j. □

We note that at most two of the half-lines H_i can satisfy (a) for a given j.

4.8. Observation. *We have*

$$\sum_{j=1}^{N} \mathbf{1}_{R_j} \leq 2,$$

where the characteristic function of a set B is denoted by $\mathbf{1}_B$.

Proof. Without loss of generality suppose that $z \in R_1 \cap R_2 \cap R_3$ and that $h_1 \geq \max\{h_2, h_3\}$. As $z \in R_1 \cap R_2 \subset S_1 \cap S_2$, we conclude that $S_1 \cap S_2 \neq \emptyset$. If $S_2 \subset S_1$, then $z_2 \in S_1$ and thus, by (4.4), $R_1 \cap R_2 = \emptyset$ which is impossible. Hence, by Observation 4.7, H_2 contains an endpoint of I_1. Similarly, H_3 contains an endpoint of I_1. We may assume that H_2 contains the left endpoint and H_3 the right one. But then I_2 lies to the left of H_1 and I_3 to the right of H_1 which implies $R_2 \cap R_3 = \emptyset$ leading to a contradiction. □

We set for all integers k

$$\mathcal{I}_k = \{I_j \mid \tfrac{1}{2}8^{-k} < \operatorname{diam}(I_j) \leq \tfrac{1}{2}8^{1-k}\}. \tag{4.5}$$

Then $\mathcal{I}_0$ contains those intervals with endpoints on the lines $x = \pm 2$ and for $k > 0$ the condition $I \in \mathcal{I}_k$ means that the endpoints of I lie in $\mathcal{H}$. Note that by Observation 4.6

$$\mathcal{I}_k = \emptyset \text{ if } k < 0 \text{ or } k \geq -\frac{\log \frac{r}{8M}}{\log 8}. \tag{4.6}$$

We now estimate the cardinality of the sets $\mathcal{I}_k$, as $N = \operatorname{card}(\cup_{k=0}^{\infty} \mathcal{I}_k)$.

4.9. Observation. *We have*

$$\operatorname{card}(\mathcal{I}_0) \leq 40 \operatorname{diam}(A) r^{-1}.$$

Proof. For each $I \in \mathcal{I}_0$ the associated rectangle R has area at least $\frac{r}{2} \cdot \frac{1}{2} = \frac{r}{4}$. Since the quasi-circle A lies in a horizontal strip of height at most $\operatorname{diam}(A)$, the rectangles all lie in a region of area at most $4(\operatorname{diam}(A) + \frac{r}{2}) < 5 \operatorname{diam}(A)$. Observation 4.8 thus allows us to estimate that

$$\operatorname{card}(\mathcal{I}_0)\tfrac{r}{4} \leq 2 \cdot 5 \operatorname{diam}(A),$$

from which the result follows. □

The corresponding bound on the cardinality of $\mathcal{I}_k$ for $k > 0$ is slightly trickier to obtain.

4.10. Observation. *If $k > 0$, $I_i, I_j \in \mathcal{I}_k$, and $\mathrm{P}_i \cap \mathrm{P}_j \neq \emptyset$, then*

$$|h_i - h_j| \leq 8 \cdot 8^{-k} M.$$

Proof. Without loss of generality we assume that $h_j \geq h_i$. Suppose that

$$h_j - h_i > 8 \cdot 8^{-k} M$$

and thus, as $I_i, I_j \in \mathcal{I}_k$,

$$h_j - h_i > r$$

by Observation 4.6. In particular H_i does not contain endpoints of I_j. Since we also know that $\mathrm{P}_i \cap \mathrm{P}_j \neq \emptyset$, we deduce that $S_i \cap S_j \neq \emptyset$ and hence, by Observation 4.7, $S_i \subset S_j$. Consequently $z_i \in S_j$.

Since $k > 0$, the endpoints of I_j are in $\mathcal{H}$ and we can find m and n for which the endpoints of I_j lie in H_m and H_n. Since

$$\min\{\mathrm{P}_y(z_m), \mathrm{P}_y(z_n)\} - \mathrm{P}_y(z_i) \geq h_j - \frac{r}{2} - h_i$$

and Observation 4.5 implies that

$$\min\{\mathrm{P}_y(z_m), \mathrm{P}_y(z_n)\} - \mathrm{P}_y(z_i) \leq M \operatorname{diam}(I_j),$$

we conclude by the fact $I_j \in \mathcal{I}_k$ and Observation 4.6 that

$$h_j - h_i \leq \frac{r}{2} + 4M8^{-k} \leq 8M8^{-k}$$

which is a contradiction. $\square$

4.11. Observation. *There exists $c > 0$ such that for all $k \geq 0$*

$$\sum_{I \in \mathcal{I}_k} \mathbf{1}_{\mathrm{P}_x(I)} \leq c8^{-k} r^{-1}.$$

Proof. Without loss of generality suppose that $u \in \mathrm{P}_1 \cap \cdots \cap \mathrm{P}_m$ where $I_1, \ldots, I_m \in \mathcal{I}_k$. It follows that $I_1 \cup \cdots \cup I_m$ lies in a vertical strip of width at most 8^{1-k}. Observation 4.10 implies that $I_1 \cup \cdots \cup I_m$ lies in a horizontal strip of height at most $8M8^{-k}$. Using Observation 4.6 we see that $R_1 \cup \cdots \cup R_m$ lies in a rectangle of area $8^{1-k}(8M8^{-k} + \frac{r}{2}) \leq 96M8^{-2k}$. But, by Observation 4.8, the rectangles R_i can overlap at most twice and so

$$m\tfrac{1}{2}8^{-k}\tfrac{r}{2} \leq 192M8^{-2k}.$$

We conclude that

$$m \leq 768M8^{-k}r^{-1}$$

as required. $\square$

We now estimate the cardinality of $\mathcal{I}_k$ for $k > 0$.

4.12. Observation. *There exists* $C > 0$ *such that for* $k > 0$

$$\mathrm{card}(\mathcal{I}_k) \le Cr^{-1}.$$

Proof. Integrating over the interval $[-2, 2]$ both sides of the inequality in Observation 4.11 yields

$$\sum_{I \in \mathcal{I}_k} \mathrm{diam}(\mathrm{P}_x(I)) \le 4c8^{-k} r^{-1}.$$

But $\mathrm{diam}(\mathrm{P}_x(I)) = \mathrm{diam}(I) \ge \frac{1}{2} 8^{-k}$ as $I \in \mathcal{I}_k$ and so

$$\mathrm{card}(\mathcal{I}_k) \tfrac{1}{2} 8^{-k} \le 4c8^{-k} r^{-1}$$

which rearranges to give the estimate we require. □

Observation 4.12 and (4.6) immediately imply (4.2) and thus prove Proposition 4.4. □

Next we will consider λ-Cantor sets with $\lambda \le 1/2$. Such sets are generated by four similitudes which contract the unit square $Q_0 = [0, 1] \times [0, 1]$ by λ and then translate these smaller squares to the corners of Q_0.

4.13. Example. *Let* C *be the* λ*-Cantor set in the plane with* $1/3 \le \lambda \le 1/2$. *Then* $\dim_{\mathrm{H}}(V_L(C)) \le 1$ *for all affine lines* L *which do not meet* C.

Proof. We parametrize the lines $l_{\alpha'}$ through the origin by the angle $0 \le \alpha' < \pi$ that they make with the negative x-axis. It is clearly enough to consider affine lines $L_{\alpha'}$ that are parallel to $l_{\alpha'}$ with $\pi/2 \le \alpha' \le 3\pi/4$ and do not meet Q_0. To all such affine lines we associate half-lines R_α starting from $L_{\alpha'}$ and perpendicular to it (that is $\alpha = \alpha' - \pi/2$).

Let α_1 be the angle determined by the negative x-axis and the line which goes through points $(1, \lambda)$ and $(0, 1 - \lambda)$. Consider a half-line R_α with $\alpha_1 \le \alpha \le \pi/4$. If Q_n is a square at the n^{th} level of the construction such that $R_\alpha \cap Q_n \ne \emptyset$, then

$$R_\alpha \cap C \cap Q_n \ne \emptyset. \tag{4.7}$$

For, assuming that $R_\alpha \cap Q_n \ne \emptyset$, the choice of α_1 implies the existence of squares $Q_n \supset Q_{n+1} \supset \dots$ such that Q_i is a square at the i^{th} level of the construction and $R_\alpha \cap Q_i \ne \emptyset$ for all i. Since $R_\alpha \cap C$ is closed, one obtains a Cauchy sequence which converges to a point in $x \in R_\alpha \cap C$ giving (4.7). Applying this argument to Q_0 we find

$$\mathrm{P}_{L_{\alpha'}}(C) = \mathrm{P}_{L_{\alpha'}}(Q_0).$$

We show that for some constant c,

$$\mathcal{H}^1(V_{L_{\alpha'}}(C)) \le c\mathcal{H}^1(\mathrm{P}_{L_{\alpha'}}(C)). \tag{4.8}$$

Let

$$\mathrm{P}_{L_{\alpha'}}(C) \subset \bigcup_{I \in \mathcal{I}} I$$

where $\mathcal{I}$ is a collection of pairwise disjoint subintervals of $L_{\alpha'}$. For fixed $I \in \mathcal{I}$, let k be the largest integer with $\lambda^k \ge (\sqrt{2}\cos(\pi/4 - \alpha))^{-1} \mathrm{diam}(I)$. Let Q_k be the

k^{th} level square that is closest to I among those k^{th} level squares Q'_k for which $R_\alpha \cap Q'_k \neq \emptyset$ for R_α starting from I. Note that the uniqueness of Q_k follows from the choice of α'. We divide I into two parts $I = I^1 \cup I^2$ such that for half-lines R_α starting from I^1 we have $R_\alpha \cap Q_k \neq \emptyset$ and those starting from I^2 we have $R_\alpha \cap Q_k = \emptyset$. If $I^2 = \emptyset$, then by (4.7)

$$V_I(C) = V_{I^1}(C) \subset Q_k,$$

where $V_J(C)$ is the set that is visible from $J \subset L_{\alpha'}$. If $I^2 \neq \emptyset$, then all the rays R_α starting from I^2 pass Q_k either below the lower left corner or above the upper right corner. This is implied by the choice of k since

$$\mathcal{H}^1(\mathrm{P}_{L_{\alpha'}}(Q_k)) \geq \lambda^k \sqrt{2} \cos(\pi/4 - \alpha) \geq \operatorname{diam}(I).$$

Now $V_{I^1}(C)$ can be covered by either a square in the lower left corner or one in the upper right corner with side length at most $((1-\lambda)\sin\alpha)^{-1} \operatorname{diam}(I^1)$. Note that this is true also in the case that I^1 is a single point. Applying the same procedure for I^2 we see that $V_I(C)$ can be covered by a countable family of cubes C_l such that

$$\sum_l \operatorname{diam}(C_l) \leq \sqrt{2}((1-\lambda)\sin\alpha)^{-1} \operatorname{diam}(I)$$

implying (4.8). Observe that for each I there are at most three points which are covered by a square of zero side length.

Now consider angles $\alpha_2 \leq \alpha \leq \alpha_1$ where α_2 is the angle determined by the negative x-axis and the line which goes through points $(1, \lambda^2)$ and $(0, \lambda - \lambda^2)$. For a fixed i^{th} level square Q_i we say that a half-line R_α starting from $L_{\alpha'}$ hits Q_i if it is closest to $L_{\alpha'}$ among those i^{th} level squares Q'_i for which $Q'_i \cap R_\alpha \neq \emptyset$. Let $H_{\alpha'}(Q_i)$ be the subinterval of $L_{\alpha'}$ such that all half-lines R_α starting from $H_{\alpha'}(Q_i)$ hit Q_i. The choice of α_2 guarantees that

$$V_{H_{\alpha'}(Q_i)}(C) \subset A_1(Q_i)$$

where $A_1(Q_i)$ is the first ancestor of Q_i, that is, the $(i-1)^{\text{th}}$ level square containing Q_i. This follows from the fact that if R_α hits Q_i, then there exists a square $Q_{i+1} \subset A_1(Q_i)$ at the $(i+1)^{\text{th}}$ level such that R_α hits Q_{i+1}. The above reasoning guarantees now the validity of (4.8) with a different constant. We continue by considering a sequence of angles α_k determined by the negative x-axis and the line which goes through points $(1, \lambda^k)$ and $(0, \lambda^{k-1} - \lambda^k)$ and by proving that for $\alpha_k \leq \alpha \leq \alpha_{k-1}$ and for a fixed i^{th} level square Q_i we have

$$V_{H_{\alpha'}(Q_i)}(C) \subset A_{k-1}(Q_i) \tag{4.9}$$

where $A_{k-1}(Q_i)$ is the $(k-1)^{\text{th}}$ ancestor of Q_i, that is, the $(i-(k-1))^{\text{th}}$ level square containing Q_i. In this way one goes through all angles except zero for which the visible part is the one dimensional λ-Cantor set. □

One can also apply the method of Example 4.13 to Cantor sets for $1/4 < \lambda < 1/3$ but it covers only positively many lines L, not all of them. The reason is that the proof of (4.9) above depends on the choice of λ.

4.14. Example. *Let $Q_0 = [0,1] \times [0,1]$. Consider a connected self-similar set A generated by finitely many contracting similitudes $f_i = Q_0 \to Q_0$ none of which contains a rotation. Then $\dim_{\mathrm{H}}(V_L(A)) = 1$ for all affine lines L not meeting A (unless A is an interval or one point).*

Proof. Fix an affine line L which does not meet A. Let $Q_i = f_i(Q_0)$ for all i. Let

$$\alpha = \min_i \operatorname{diam}(\mathrm{P}_L(Q_i) \setminus \cup_{j \neq i} \mathrm{P}_L(Q_j)) / \operatorname{diam}(Q_i)$$

where the minimum is taken over such i for which the denominator is not zero. Let $\beta = \min_i |f_i'|$ and $\gamma = \operatorname{diam}(\mathrm{P}_L(A))/\sqrt{2}$ where $|f_i'|$ is the norm of the derivative of f_i. Note that, since A is connected, $\mathrm{P}_L(A)$ is an interval. We may assume that $\gamma > 0$. Cover $\mathrm{P}_L(A)$ by disjoint intervals and fix one of them, say I.

Let Q be the smallest construction square such that $\operatorname{diam}(\mathrm{P}_L(A \cap Q)) \geq \operatorname{diam}(I)$ and such that Q minimizes the distance

$$\operatorname{dist}_L(Q, I) = \inf \operatorname{dist}(R \cap I, R \cap Q),$$

where the infimum is taken over all half-lines R starting from I and being perpendicular to L such that $R \cap Q \neq \emptyset$. Then either $V_I(A) \subset Q$ or a part of the half-lines starting from I pass Q either above or below it (but not both). In the first case we have

$$\operatorname{diam}(I) \geq \beta \operatorname{diam}(\mathrm{P}_L(A \cap Q)) \geq \beta\gamma \operatorname{diam}(Q).$$

Suppose that a part of the half-lines starting from I pass Q above it. If the remaining half-lines hit more than one sub-square of Q then

$$\operatorname{diam}(\mathrm{P}_L(V_I(A) \cap Q)) \geq \alpha\beta \operatorname{diam}(Q).$$

Otherwise, let Q' be the largest sub-square of Q such that the half-lines R starting from I with $V_{R \cap I}(A) \in Q$ intersect more than one sub-square of Q'. Then again

$$\operatorname{diam}(\mathrm{P}_L(V_I(A) \cap Q)) = \operatorname{diam}(\mathrm{P}_L(V_I(A) \cap Q')) \geq \alpha\beta \operatorname{diam}(Q').$$

Note that Q' exists unless the intersection is only one point (see Example 4.13). Applying the same procedure to those rays which pass Q we conclude that

$$\mathcal{H}^1(\mathrm{P}_L(A)) \leq (\alpha\beta)^{-1} \operatorname{diam}(\mathrm{P}_L(A))$$

where we used the fact that $\alpha \leq \gamma$. $\square$

4.15. Remark. *One can replace the unit square in Example 4.14 by any other reasonable seed set. One can also allow rational rotations in f_i's since these induce only finitely many directions and the definition of positive α is possible. Irrational rotations induce directions such that α becomes zero. This phenomenon is best demonstrated by projecting an interval onto line L. As L tends to the line perpendicular to the interval, α tends to zero.*

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