Mappings of finite distortion: Capacity and modulus inequalities

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Abstract

We establish capacity and modulus inequalities for mappings of finite distortion under minimal regularity assumptions.

1 Introduction

In 1966, Reshetnyak introduced in [24] the class of mappings $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ for which

$$|Df(x)|^n \leq KJ(x,f) \qquad \text{a.e.} \tag{1}$$

for some fixed $1 \leq K < \infty$, and called them mappings of bounded distortion. Here $\Omega \subset \mathbb{R}^n$, $Df(x)$ is the formal differential of f, $|Df(x)|$ is the operator norm of $Df(x)$ and $J(\cdot, f)$ is the determinant of $Df(x)$. In the same paper he proved that each such a mapping has a representative which is Hölder-continuous with exponent $1/K$. Subsequently, in 1967, he proved in [25] the remarkable result that a mapping of bounded distortion, defined in an open and connected set, is either constant or both open and discrete. This opened up the way for the study of local properties of these mappings and of the value distribution theory. One of the key points here was that the

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so-called path lifting can be applied to mappings that are both open and discrete. Relatively soon after the result on openness and discreteness Poletsky established in 1970 (cf. [23]), relying on the path lifting, the general principle that the modulus of curve families - a tool crucial in the work on value distribution - decreases under mappings of bounded distortion. Analogous capacity inequalities and improvements on the Poletsky inequality were given by the Finnish school of Martio, Rickman and Väisälä, cf. [22], [21], [28]; they called mappings of bounded distortion quasiregular mappings. Such inequalities served as the principal tool in Rickman's deep work on the value distribution (cf. [27]), including a version of the Picard theorem, and they also lead to a number of local results.

The assumptions in Reshetnyak's theorem have recently been relaxed. First of all, under the requirement that $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, it suffices to assume that

$$|Df(x)|^n \leq K(x)J(x,f) \qquad \text{a.e.} \tag{2}$$

with $1 \leq K \in L^p_{loc}(\Omega)$ for some $p > n-1$ when $n \geq 3$ and that $1 \leq K \in L^1_{loc}(\Omega)$ in the planar case. For these results see [11], [19], [20], and also [6]. The standing conjecture is that the borderline case $L^{n-1}_{loc}(\Omega)$ (or even something slightly weaker, cf. [10]) suffices in all dimensions. An example in [2] shows that the exponent of integrability of K cannot be decreased to any number strictly less than $n-1$. Under an additional a priori topological assumption the sufficiency of $K \in L^{n-1}_{loc}(\Omega)$ has been very recently verified in [7]. Secondly, there have been attempts to relax the a priori assumption that $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. There are serious obstacles here, because the Jacobians of the smooth approximations to f do not then necessarily converge in $L^1_{loc}(\Omega)$ to the Jacobian of f. Partially motivated by nonlinear elasticity one is tempted to only assume that

$$f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n),$$

that

$$J(\cdot, f) \in L^1_{loc}(\Omega),$$

and that (2) holds for some measurable $1 \leq K(x)$, finite almost everywhere. Mappings like this are called mappings of finite distortion in [8], [12], [14]. The example referred to above demonstrates that also an integrability condition should be imposed on K. The essentially sharp condition is by now known (cf. [14]) to be

(A-0) $\exp(\mathcal{A}(K)) \in L^1_{loc}(\Omega)$

with an Orlicz-function $\mathcal{A}$ so that

(A-1) $\displaystyle\int_1^\infty \frac{\mathcal{A}'(t)}{t}\,dt = \infty,$

(A-2) there exists a positive number t_0 such that $\mathcal{A}'(t)t$ increases to infinity for $t \geq t_0$.

We call an infinitely differentiable on $(0,\infty)$ and strictly increasing function $\mathcal{A} : [0,\infty) \to [0,\infty)$ with $\mathcal{A}(0) = 0$ and $\lim\limits_{s\to\infty} \mathcal{A}(s) = \infty$ an Orlicz function. For further reference, let us say that a mapping f of finite distortion satisfies (A) if (A-0), (A-1) and (A-2) hold.

Mappings of finite distortion satisfying (A) have been shown in [14] to enjoy further regularity conditions. In particular,

(P-0) $P(|Df|^n) \in L^1_{loc}(\Omega)$,

where P is an Orlicz-function that satisfies the conditions:

(P-1) $\displaystyle\int_1^\infty \frac{P(s)}{s^2}\,ds = \infty,$

(P-2) there exists $s_0 \in (0,\infty)$ such that the function $s^{-\frac{n}{n+1}}P(s)$ is increasing on (s_0,∞).

Let us say that a mapping f of finite distortion satisfies (P) if (P-0), (P-1) and (P-2) hold, with an Orlicz-function P, and if, in addition, $K \in L^p_{loc}(\Omega)$ for some $p > n-1$. Based on [14], we know that (P) is sufficient for Reshetnyak's theorem, i.e. a mapping f of finite distortion that satisfies P in a domain Ω is continuous and either constant or both open and discrete.

Based on the above extensions of Reshetnyak's theorem it is then natural to aim for capacity and modulus inequalities analogous to those in the case of bounded distortion. Somewhat surprisingly, up to now, such results seem to have been out of reach.

The original inequalities were based on the facts that the definition of mappings of bounded distortion implies (and is even equivalent to) a metric condition that can be used to control the behavior of one-sided inverses of f and that the inverse mapping of a quasiconformal mapping (i.e. homeomorphic mapping of bounded distortion) is quasiconformal. Thus one uses analytic information to conclude metric properties, and, in turn, proves further analytic consequences of this. In fact, all the proofs we have found in the literature for inequalities of this type rely on an inequality from [22] that strongly uses the boundedness of K.

In our more general setting, one can still prove a metric property for our mappings but this condition is not uniform and not powerful enough to act as a substitute for the metric condition of quasiconformal mappings. Moreover, integrability conditions on the distortion of a homeomorphic mapping of finite distortion do not result in good bounds on the distortion of the inverse mapping. Thus this approach seems doomed and instead of this we opt for a direct analytic argument, mimicking the ideas in [1] for the case of homeomorphic mappings of bounded distortion.

In this paper we establish capacity and modulus inequalities that will apparently result in a number of applications in the field. At this point we would like to stress that our approach is new even in the case of mappings of bounded distortion or quasiregular mappings. To illustrate our results, we formulate here a special case of our main estimate, an immediate corollary to it and a consequence of the resulting capacity inequalities. The actual capacity and modulus inequalities are given in Section 3 and in Section 4.

Theorem 1.1 *Suppose that f is a non-constant mapping of finite distortion that satisfies either (A) or (P). Let $U \subset\subset \Omega$, and $u \in C_0^\infty(U)$; if U is a normal domain, then it suffices that $u \in C_0^\infty(\Omega)$. Set*

$$g_U(y) = \sum_{x \in f^{-1}(y) \cap U} i(x,f)u(x),$$

for $y \in f(U)$. Then $g_U \in W^{1,n}(f(U))$ and

$$\int_{f(U)} |\nabla g_U(y)|^n \, dy \leq N(f,U)^{n-1} \int_U |\nabla u(x)|^n K^{n-1}(x,f) \, dx.$$

Above, $i(x,f)$ is the local topological index of f at x and $N(f,U) = \sup \sharp\{f^{-1}(y) \cap U : y \in \mathbb{R}^n\} < \infty$ is the maximal multiplicity of f in U. A reader familiar with capacity inequalities should immediately notice that this estimate leads to such inequalities.

Corollary 1.2 *Suppose that f is a homeomorphism of finite distortion that satisfies the assumptions of the preceding theorem. Then $f^{-1} \in W^{1,n}_{loc}(f(\Omega), \mathbb{R}^n)$.*

The simple example $f(x_1, \cdots, x_n) = (x_1, \cdots, x_{n-1}, x_n|x_n|^s)$, $s > 0$, shows that (P) cannot be substantially relaxed even when f is Lipschitz. Indeed, $f^{-1} \in W^{1,n}_{loc}(\mathbb{R}^n)$ exactly when $s < 1/(n-1)$. On the other hand, $K(x, f) \leq x_n^{-s}$ when $|x_n| < 1$, and $K(x, f)$ is locally bounded when $|x_n| \geq 1$, and thus $K \in L^p_{loc}(\mathbb{R}^n)$, exactly when $p < 1/s$.

Corollary 1.3 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping of finite distortion that satisfies either (A) or (P) locally. If f omits a set of positive n-capacity, $\int_1^\infty r^{-1} \left(\fint_{B(0,r)} K^{n-1} \right)^{-\frac{1}{n-1}} dr = \infty$, and $\int_{B(0,2r)} K^{n-1} \leq C \int_{B(0,r)} K^{n-1}$, for some $C \geq 1$ and all $r \geq 1$, then f is constant.*

This conclusion is a kind of a Picard theorem. Again, the assumptions are very sharp. Indeed, an example from [15] shows that, given any $\mathcal{A}$ so that the integral at (A-1) converges, one can construct a bounded, continuous, non-constant mapping of finite distortion that satisfies condition (A) for this $\mathcal{A}$ with $K \equiv 1$ outside a compact set. Moreover, in Section 6 we show that given any increasing ϱ with $\varrho(r) > 0$ and $\varrho(2r) \leq C_1\, \varrho(r)$, for $r \geq 1$, and so that $\int_1^\infty \frac{dr}{r\varrho(r)} < \infty$, one can construct a Lipschitz homeomorphism $f : \mathbb{R}^n \to \Omega' \subset\subset \mathbb{R}^n$ that has locally bounded distortion K satisfying $\left(\fint_{B(0,r)} K^{n-1} \right)^{\frac{1}{n-1}} \geq C_2 \varrho(r)$.

It would be very interesting to know whether the size of the omitted set in Corollary 1.3 can be reduced. In the forthcoming paper [17] we establish, in dimensions $n \geq 3$, a bound on the injectivity radius of a locally homeomorphic mapping of finite distortion that satisfies (A). This relies on the capacity estimates from this paper, and the assumption (A) turns out to be crucial. Yet another consequence of the capacity inequalities is given in Section 5 below, where we prove a local modulus of continuity estimate for mappings of finite distortion that takes the local index into account.

The paper is organized as follows. We prove our main estimate in Section 2. The basic capacity inequalities are given in Section 3 and the modulus inequalities in Section 4. Theorem 1.1 and Corollary 1.2 are proven in Section 5. Finally, in Section 6, we prove Corollary 1.3.

2 The main estimate

In this section we will prove our main estimate. In order to prove the capacity inequalities we will use the fact that functions $u \in C_0^\infty(\Omega)$ can be pushed forward to a function $g : f(\Omega) \to \mathbb{R}$. If f is a homeomorphism, then this is easy: $g : f(\Omega) \to \mathbb{R}$ is given as $g = u \circ f^{-1}$. The situation is more complicated when f is only assumed to be discrete and open. Openness means that f maps open sets to open sets and discreteness that the set of preimages of any point in $\mathbb{R}^n$ is finite in each compact subset of Ω.

Suppose that $f : \Omega \to \mathbb{R}^n$ is continuous, discrete and open mapping, and let $U \subset\subset \Omega$ be a domain, i.e. $\overline{U}$ is compact subset of Ω. Assume that $u \in C_0^\infty(U)$; in the case that U is a normal domain, i.e. when $f(\partial U) = \partial f(U)$, we only assume that $u \in C_0^\infty(\Omega)$. In fact, $u \in C^\infty(\overline{U})$ suffices in this case. We define a function $g_U : f(U) \to \mathbb{R}$ by setting

$$g_U(y) = \sum_{x \in f^{-1}(y) \cap U} i(x, f) u(x). \tag{3}$$

Here $i(x, f)$ is the local topological index of f at x, see [27], [5], or [26]. Our function g_U is continuous, see [22, Lemma 7.6 (4)], [21, Lemma 5.4 (3)] or [27, proof of Lemma 5.3]. Notice that $N(f, U) = \sup \sharp\{f^{-1}(y) \cap U : y \in \mathbb{R}^n\} < \infty$ because f is discrete and open and $U \subset\subset \Omega$.

Theorem 2.1 *Assume that an Orlicz function P satisfies the conditions (P-1) and (P-2). Let $f : \Omega \to \mathbb{R}^n$ be a non-constant mapping of finite distortion such that $P(|Df|^n) \in L^1_{loc}(\Omega)$ and $K_I(\cdot, f) \in L^1_{loc}(\Omega)$. Suppose that f is continuous, discrete, open and sense-preserving, the measure of the branch set B_f is zero and f satisfies the conditions (N) and (N^{-1}). Let U, u, and g_U be as above. Then $g_U \in W^{1,n}(f(U)) \cap C(f(U))$ and*

$$\int_{f(U)} |\nabla g_U(y)|^n \, dy \le N(f, U)^{n-1} \int_U |\nabla u(x)|^n K_I(x, f) \, dx. \tag{4}$$

That the mapping f satisfies the condition (N) means that f maps sets of Lebesgue measure zero to sets of measure zero. We say that the mapping f satisfies condition (N^{-1}) if $|f^{-1}(E)| = 0$ whenever $|E| = 0$ for a measurable set $E \subset \mathbb{R}^n$. The branch set B_f of f is defined as the set of all $x \in \Omega$ such that f is not a local homeomorphism at x. The inner distortion K_I is defined below in (6).

Let $D^\sharp f(x)$ denote the $n \times n$-matrix of cofactors of $Df(x)$. Inequality (2) yields

$$|D^\sharp f(x)|^n \le \tilde{K}(x) J(x,f)^{n-1} \qquad \text{a.e.} \tag{5}$$

where $1 \le \tilde{K}(x) < \infty$. On the other hand, notice that (5) does not imply the inequality (2); consider e.g $f(x_1, ..., x_n) = (x_1, 0, ..., 0)$. The smallest $\tilde{K} \ge 1$ for which (5) holds will be denoted by $K_I(x,f)$ and called inner distortion function:

$$K_I(x,f) = \begin{cases} \frac{|D^\sharp f(x)|^n}{J(x,f)^{n-1}}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } J(x,f) = 0 \text{ and } |D^\sharp f(x)| = 0 \\ \infty, & \text{if } J(x,f) = 0 \text{ and } |D^\sharp f(x)| \neq 0. \end{cases} \tag{6}$$

We have the point-wise inequality $K_I(x,f) \le K(x)^{n-1}$; see [10, Section 6] for a detailed discussion.

Using the point-wise inequality $K_I(x,f) \le K^{n-1}(x)$, we can replace the assumption $K_I \in L^1_{loc}(\Omega)$ by $K \in L^{n-1}_{loc}(\Omega)$ and conclude with inequality (4). Furthermore, applying [14, Proposition 2.5] (see also [12] and [13]) and [15, Theorem 1.1], we see that all the topological and analytic assumptions of Theorem 2.1 are fulfilled, provided that the distortion function K is L^p-integrable for some $p > n - 1$.

Corollary 2.2 *Assume that an Orlicz function P satisfies conditions (P-1) and (P-2). Let $f : \Omega \to \mathbb{R}^n$ be a non-constant mapping of finite distortion such that $P(|Df|^n) \in L^1_{loc}(\Omega)$ and $K \in L^p_{loc}(\Omega)$, for some $p > n-1$. Let U, u, and g_U be as above. Then $g_U \in W^{1,n}(f(U)) \cap C(f(U))$ and*

$$\int_{f(U)} |\nabla g_U(y)|^n \, dy \le N(f,U)^{n-1} \int_U |\nabla u(x)|^n K_I(x,f) \, dx. \tag{7}$$

Using [13, Lemma 3.2] and [15, Theorem 1.2] we see that the regularity condition $P(|Df|^n) \in L^1_{loc}(\Omega)$ together with the assumption $K_I \in L^1_{loc}(\Omega)$ suffice to imply the conditions (N) and (N^{-1}) for homeomorphic mappings of finite distortion.

Corollary 2.3 *Assume that an Orlicz function P satisfies conditions (P-1) and (P-2). Let $f : \Omega \to \Omega'$ be homeomorphic and a mapping of finite distortion such that $P(|Df|^n) \in L^1_{loc}(\Omega)$ and $K_I \in L^1_{loc}(\Omega)$. Then the inverse map f^{-1} is of finite distortion and it belongs to the Sobolev class $W^{1,n}(\Omega', \mathbb{R}^n)$.*

Proof. Fix $\Omega' \subset\subset \Omega$ and $k \in \{1, ..., n\}$. Choosing $u(x) = x_k$ in Theorem 2.1, we have

$$\int_{f(\Omega')} |\nabla f_k^{-1}(y)|^n \, dy \leq \int_{\Omega'} K_I(x, f) \, dx. \tag{8}$$

Combining this with the well-known result (cf. [26, Corollary 1, p. 182]) that homeomorphisms in $W^{1,n}$ satisfy the condition (N), allows us to use the change of variables formula to f^{-1} (see Lemma 2.6) to conclude that

$$\int_{\Omega'} K_I(x, f) \, dx = \int_{f(\Omega')} K_I(f^{-1}(y), f) J(y, f^{-1}) \, dy. \tag{9}$$

Inequalities (8) and (9) imply that

$$\fint_{B(y,r)} |Df^{-1}(z)|^n \, dz \leq n \fint_{B(y,r)} K_I(f^{-1}(z), f) J(z, f^{-1}) \, dz \tag{10}$$

for each $y \in f(\Omega)$ and all sufficiently small $r > 0$. Because $f^{-1} \in W^{1,n}_{loc}(f(\Omega), \mathbb{R}^n)$, it follows from Hadamard's inequality that $J(\cdot, f^{-1}) \in L^1_{loc}(f(\Omega))$. Thus the claim follows from (10) by using the Lebesgue differentiation theorem and (9) which shows that $K_I(f^{-1}(\cdot), f)J(\cdot, f^{-1}) \in L^1_{loc}(f(\Omega))$.

In the proof of Theorem 2.1 we will use the following auxiliary result. Notice that in the case $P(t) = t$, i.e. when $|Df| \in L^n_{loc}(\Omega)$, Proposition 2.4 is an immediate consequence of Stokes' theorem and a standard approximation argument.

Proposition 2.4 *Assume that an Orlicz function P satisfies conditions (P-1) and (P-2). Let $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ satisfy $J(x, f) \geq 0$ a.e. in Ω with*

$$P(|Df|^n) \in L^1_{loc}(\Omega).$$

If $\eta \in C^\infty(\Omega)$ and $\varphi \in C^\infty(f(\Omega))$ are so that spt$(\eta(\varphi \circ f)) \subset\subset \Omega$, then

$$\int_\Omega J(x, f_1, ..., f_{i-1}, \eta(\varphi \circ f), f_{i+1}, ..., f_n)\, dx = 0 \tag{11}$$

for all $i = 1, ..., n$.

The proof of Proposition 2.4 will be based on the following estimate.

Lemma 2.5 *Let $f \in W^{1,\frac{n^2}{n+1}}(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\left| \int_{\{M(|Df|^{\frac{n^2}{n+1}})(x) \le 2t\}} J(x, f)\, dx \right| \le C(n) t^{\frac{1}{n}} \int_{\{|Df(x)|^{\frac{n^2}{n+1}} > t\}} |Df(y)|^{\frac{n^2}{n+1}}\, dy \tag{12}$$

for almost every $t > 0$.

Here the notation $M(|Df|^{\frac{n^2}{n+1}})(x)$ refers to the usual Hardy-Littlewood maximal function of $|Df|^{\frac{n^2}{n+1}}$, defined by the formula

$$M(|Df|^{\frac{n^2}{n+1}})(x) = \sup \Big\{ \fint_{B(y,r)} |Df|^{\frac{n^2}{n+1}} : x \in B(y, r) \Big\}.$$

We refer to [3, Proposition 5.1] for the proof of Lemma 2.5.

Proof of Proposition 2.4 Without loss of generality we can assume that $i = 1$. Fix $\Psi \in C_0^\infty(\Omega)$ so that $\Psi \equiv 1$ on the support of $\eta(\varphi \circ f)$. By (P-2), it is clear that (the zero extension of) the mapping

$$\tilde{f} = (\eta(\varphi \circ f), \Psi f_2, ..., \Psi f_n) \tag{13}$$

lies in the Sobolev space $W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ for all $p \in [1, \frac{n^2}{n+1}]$. Furthermore

$$P(|D\tilde{f}|^n) \in L^1(\mathbb{R}^n). \tag{14}$$

Using the language of differential forms we can write

$$\begin{aligned} J(x, \tilde{f})\, dx &= d(\eta\varphi(f)) \wedge d(\Psi f_2) \wedge ... \wedge d(\Psi f_n) \\ &= d(\eta\varphi(f)) \wedge d(f_2) \wedge ... \wedge d(f_n) \\ &= \varphi(f) d\eta \wedge d(f_2) \wedge ... \wedge d(f_n) + \eta \sum_{k=1}^n \frac{\partial \varphi}{\partial y_k} df_k \wedge df_2 \wedge ... \wedge df_n \\ &= \varphi(f) d\eta \wedge d(f_2) \wedge ... \wedge d(f_n) + \eta \frac{\partial \varphi}{\partial y_1} J(x, f)\, dx. \end{aligned} \tag{15}$$

In the last identity we used the fact that $f \in W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$. By [3, Theorem 1.3], we see that $J(\cdot, f) \in L^1_{loc}(\Omega)$ and so we conclude that

$$J(\cdot, \tilde{f}) \in L^1(\mathbb{R}^n).$$

Lemma 2.5 gives us the estimate

$$\left| \int_{\{M(|D\tilde{f}|^{\frac{n^2}{n+1}})(x) \leq 2t\}} J(x, \tilde{f})\, dx \right| \leq C(n) t^{\frac{1}{n}} \int_{\{|D\tilde{f}(x)|^{\frac{n^2}{n+1}} > t\}} |D\tilde{f}(y)|^{\frac{n^2}{n+1}}\, dy \quad (16)$$

for almost every $t > 0$.

If we can show that the right hand side of inequality (16) tends to zero as t increases to infinity through a sequence in the set $[0, \infty) \setminus E$, where $\mathcal{H}^1(E) = 0$, Proposition 2.4 follows from (13) and the Lebesgue Dominated Convergence theorem. Assume that such a sequence cannot be found. Then there exists $\delta > 0$ and $t_0 \in (0, \infty) \setminus E$ such that

$$t^{\frac{1}{n}} \int_{\{|D\tilde{f}(x)|^{\frac{n^2}{n+1}} > t\}} |D\tilde{f}(y)|^{\frac{n^2}{n+1}}\, dy \geq \delta$$

for all $t \in (t_0, \infty) \setminus E$. We define an auxiliary function Φ by setting

$$\Phi(t) = t^{-\frac{1}{n}} \frac{d}{dt}[t^{-1} P(t^{\frac{n+1}{n}})]$$

for all $t > 0$. First we observe that

$$\begin{aligned} \int_{t_0}^{\infty} \Phi(t)\, dt &= \frac{P(t^{\frac{n+1}{n}})}{t^{\frac{n+1}{n}}}\Big|_{t_0}^{\infty} + \frac{1}{n} \int_{t_0}^{\infty} \frac{P(t^{\frac{n+1}{n}})}{t^{\frac{n+1}{n}+1}} \\ &\geq -\frac{P(t_0^{\frac{n+1}{n}})}{t_0^{\frac{n+1}{n}}} + \frac{1}{n+1} \int_{t_0^{\frac{n+1}{n}}}^{\infty} \frac{P(s)}{s^2} ds = \infty. \end{aligned} \quad (17)$$

Using Fubini's Theorem, we have

$$\begin{aligned} \delta \int_{t_0}^{\infty} \Phi(t) dt &\leq \int_{t_0}^{\infty} \Phi(t) t^{\frac{1}{n}} \int_{\{|D\tilde{f}(x)|^{\frac{n^2}{n+1}} > t\}} |D\tilde{f}(y)|^{\frac{n^2}{n+1}}\, dy\, dt \\ &= \int_{\{|D\tilde{f}(x)|^{\frac{n^2}{n+1}} > t_0\}} |D\tilde{f}(y)|^{\frac{n^2}{n+1}} \int_{t_0}^{|D\tilde{f}(x)|^{\frac{n^2}{n+1}}} t^{\frac{1}{n}} \Phi(t) dt dy \\ &= \int_{\{|D\tilde{f}(x)|^{\frac{n^2}{n+1}} > t_0\}} |D\tilde{f}(y)|^{\frac{n^2}{n+1}} \left(\frac{P(|D\tilde{f}(y)|^n)}{|D\tilde{f}(y)|^{\frac{n^2}{n+1}}} - \frac{P(t_0^{\frac{n+1}{n}})}{t_0} \right) \\ &\leq \int_{\mathbb{R}^n} P(|D\tilde{f}(x)|^n)\, dy. \end{aligned} \quad (18)$$

Because the right hand side of (18) is finite by (14) and the left hand side of (18) is infinite, we have arrived at a contradiction. The claim follows.

We will also need the following version of the change of variables formula, for a proof see e.g. [18].

Lemma 2.6 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$. Suppose that f satisfies the condition (N). Let $E \subset \Omega$ be measurable and $u : f(E) \to \mathbb{R}$ a measurable function. Then*

$$\int_E u(f(x))|J(x,f)|\,dx = \int_{\mathbb{R}^n} u(y)N(y,f,E)\,dy \tag{19}$$

provided that at least one of the integrals makes sense.

Here and what follows we use the standard notation $N(y,f,E) = \sharp\{x \in E : f(x) = y\}$.

Proof of Theorem 2.1 We will show that

$$\begin{aligned}\left|\int_{f(U)} g_U(y)\mathrm{div}\varphi(y)\,dy\right| \quad &\leq \quad N(f,U)^{\frac{n-1}{n}}\left(\int_U |\nabla u|^n K_I(x,f)\,dx\right)^{\frac{1}{n}} \\ &\qquad \left(\int_{f(U)} |\varphi(y)|^{\frac{n}{n-1}}\,dy\right)^{\frac{n-1}{n}}\end{aligned} \tag{20}$$

for every test function $\varphi \in C_0^\infty(f(U), \mathbb{R}^n)$. Using the Hahn-Banach theorem we see that, if (20) holds for all test functions φ, then the claim follows. Here $\mathrm{div}\varphi$ denotes the divergence of φ, i.e $\mathrm{div}\varphi = \sum_{i=1}^n \frac{\partial \varphi_i}{\partial y_i}$.

Fix a point $y_0 \in \mathrm{spt} g_U \setminus f(\mathrm{spt} u \cap B_f \cap U)$. By [27, I Proposition 4.10], the set $f^{-1}(y_0) \cap \mathrm{spt} u \cap U$ has only a finite number of elements, say

$$f^{-1}(y_0) \cap \mathrm{spt} u \cap U = \{x_1, ..., x_{k_0}\}.$$

Of course the set $f^{-1}(y_0) \cap \mathrm{spt} u \cap U$ is non-empty. We will use the notation of [27]. According to [27, Lemma I 4.9], the x_i-component $U(x_i, f, r_0)$ of $f^{-1}(B(y_0, r_0))$ is a normal domain when $r_0 > 0$ is sufficiently small. Moreover, the diameter of $U(x_i, f, r_0)$ tends to zero as $r_0 \to 0$, by [27, I Lemma 4.9], and so we can choose $r_0 > 0$ so that $f|_{\overline{U(x_i,f,r_0)}}$ is injective, for all

$i \in \{1, ..., k_0\}$. Then the domains $U(x_i, f, r_0)$ are pairwise disjoint. Choose a positive number $\tilde{r}_0 \leq r_0$ such that

$$B(y_0, \tilde{r}_0) \cap f(\mathrm{spt} u \cap U \setminus \bigcup_{j=1}^{k_0} U(x_j, f, r_0)) = \emptyset \tag{21}$$

and denote $U(x_i, f, \tilde{r}_0)$ by U_i for all $i \in \{1, ..., k_0\}$.

Let G be a component of $f^{-1}(B(y_0, \tilde{r}_0))$ so that $G \cap \mathrm{spt} u \cap U \neq \emptyset$. Then $G \cap U_j \neq \emptyset$, for a (unique) $j \in \{1, ..., k_0\}$ by (21). Since $f\big|_{\overline{U}_j}$ is injective, we have $B(y_0, \tilde{r}_0) \cap f(\partial U_j) = \emptyset$ and hence $D \cap \partial U_j = \emptyset$. This implies $D \subset U_j$ and so

$$f^{-1}(B(y_0, \tilde{r}_0)) \cap \mathrm{spt} u \cap U = \bigcup_{j=1}^{k_0} U_j \cap \mathrm{spt} u \cap U. \tag{22}$$

By Vitali Covering Theorem, we find pairwise disjoint balls $B_i = B_i(y_i, r_i)$ such that

$$|(\mathrm{spt} g_U \setminus f(B_f \cap \mathrm{spt} u \cap U)) \setminus \bigcup_{i=1}^{\infty} B_i| = 0 \tag{23}$$

and satisfying the conclusions obtained for $B(y_0, \tilde{r}_0)$. Especially, f is homeomorphism from $U_{i,j} = U(x_{i,j}, f, r_i)$ onto B_i, where $j = 1, ..., k_i$, $i = 1, 2, ...$ and

$$f^{-1}(y_i) \cap \mathrm{spt} u \cap U = \{x_{i,1}, ..., x_{i,k_i}\}. \tag{24}$$

We denote the inverse map of f from B_i onto $U_{i,j}$ by $h_{i,j}$.

Fix $\varphi \in C_0^\infty(f(U), \mathbb{R}^n)$. Combining the equation (23) with the assumptions that the measure of the branch set of f is zero and that the mapping f satisfies the condition (N), we have

$$\int_{f(U)} g_U(y) \mathrm{div}\, \varphi(y)\, dy = \int_{\cup_{i=1}^{\infty} B_i} g_U(y) \mathrm{div}\, \varphi(y)\, dy. \tag{25}$$

Since $i(x, f) = 1$ for $x \in \Omega \setminus B_f$ (see [27, p. 18]), we find that

$$\int_{\cup_{i=1}^{\infty} B_i} g_U(y) \mathrm{div}\, \varphi(y)\, dy = \sum_{i=1}^{\infty} \int_{B_i} \left(\sum_{x \in f^{-1}(y) \cap U} u(x) \right) \mathrm{div}\, \varphi(y)\, dy. \tag{26}$$

Here we also used the fact that the balls B_i are pairwise disjoint. Applying the equations (22) and (24) we conclude that

$$\sum_{i=1}^{\infty} \int_{B_i} \left(\sum_{x \in f^{-1}(y) \cap U} u(x) \right) \mathrm{div}\, \varphi(y)\, dy = \sum_{i=1}^{\infty} \sum_{j=1}^{k_i} \int_{B_i} u(h_{i,j}(y)) \mathrm{div}\, \varphi(y)\, dy. \tag{27}$$

Using the change of variables formula (19) applied to the functions $(u \circ h_{i,j})\,\mathrm{div}\varphi$, we notice that

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{k_i}\int_{B_i} u(h_{i,j}(y))\mathrm{div}\,\varphi(y)\,dy &= \sum_{i=1}^{\infty}\sum_{j=1}^{k_i}\int_{U_{i,j}\cap U} u(x)\mathrm{div}\,\varphi(f(x))J(x,f)\,dx \\ &= \int_{f^{-1}(\cup_{i=1}^{\infty}B_i)\cap U} u(x)\mathrm{div}\,\varphi(f(x))J(x,f)\,dx. \qquad (28)\end{aligned}$$

The last equation follows from the facts that the sets $U_{i,j}$ are pairwise disjoint and that the equation (22) is valid. Combining the equation (23) with the assumptions that $|B_f| = 0$ and that the mapping f satisfies the conditions (N) and (N^{-1}), we find that

$$\int_{f^{-1}(\cup_{i=1}^{\infty}B_i)\cap U} u(x)\mathrm{div}\,\varphi(f(x))J(x,f)\,dx = \int_U u(x)\mathrm{div}\,\varphi(f(x))J(x,f)\,dx. \quad (29)$$

Combining the equations (25)-(29) we arrive at

$$\int_{f(U)} g_U(y)\mathrm{div}\,\varphi(y)\,dy = \int_U u(x)\mathrm{div}\,\varphi(f(x))J(x,f)\,dx. \qquad (30)$$

Next we will estimate the right hand side term of the equation (30). Fix $l \in \{1, ..., n\}$. The product rule together with the chain rule yield

$$\nabla(u \cdot \varphi_l \circ f)(x) = \varphi_l(f(x))\nabla u(x) + u(x)\nabla\varphi_l(f(x))Df(x). \qquad (31)$$

Multiplying both sides by the matrix $D^\sharp f(x)$ and using Cramer's rule, for the case $J(x,f) \neq 0$, we find that

$$\nabla(u \cdot \varphi_l \circ f)(x)D^\sharp f(x) = \varphi_l(f(x))\nabla u(x)D^\sharp f(x) + u(x)\nabla\varphi_l(f(x))J(x,f)\mathbf{I}. \qquad (32)$$

Notice that here we used the assumption that f has finite distortion, which implies that $Df(x) = 0$ and so also $D^\sharp f(x) = 0$ for almost every x such that $J(x,f) = 0$. Especially we obtain the identity

$$\begin{aligned}J(x, f_1, ..., f_{l-1}, u \cdot \varphi_l \circ f, f_{l+1}, ..., f_n) &= \left\langle \nabla(u \cdot \varphi_l \circ f)D^\sharp f | e_l \right\rangle \\ &= \left\langle \varphi_l(f(x))\nabla u(x)D^\sharp f(x) | e_l \right\rangle + u(x)\partial_l\varphi_l(f(x))J(x,f). \qquad (33)\end{aligned}$$

Summing over l we find that

$$\sum_{l=1}^{n} J(x, f_1, ..., f_{l-1}, u \cdot \varphi_l \circ f, f_{l+1}, ..., f_n) =$$
$$\big\langle \varphi(f(x)) | \nabla u(x) D^\sharp f(x) \big\rangle + u(x)(\mathrm{div}\varphi)(f(x))J(x, f). \tag{34}$$

The function $u \cdot (\varphi_l \circ f)$ vanishes on the boundary of U (when U is a normal domain it suffices to assume that $u \in C_0^\infty(\Omega)$, because the function $\varphi_l \circ f$ vanishes on the boundary of U) and so Proposition 2.4 implies that

$$\begin{aligned}\left| \int_U u(x) \mathrm{div}\, \varphi(f(x)) J(x, f)\, dx \right| &= \left| \int_U \big\langle \varphi(f(x)) | \nabla u(x) D^\sharp f(x) \big\rangle\, dx \right| \\ &\leq \int_U |\varphi(f(x))| |\nabla u(x)| |D^\sharp f(x)|\, dx. \end{aligned} \tag{35}$$

Combining this inequality with the inequality $|D^\sharp f(x)|^n \leq K_I(x, f) J(x, f)^{n-1}$, where the inner dilatation function $K_I(\cdot, f)$ is given by the rule (6), we arrive at

$$\left| \int_U u(x) \mathrm{div}\, \varphi(f(x)) J(x, f)\, dx \right| \leq \int_U |\varphi(f(x))| |\nabla u(x)| K_I(x, f)^{\frac{1}{n}} J(x, f)^{\frac{n-1}{n}}\, dx. \tag{36}$$

Applying Hölder's inequality we finally conclude that

$$\begin{aligned}\left| \int_U u(x) \mathrm{div}\, \varphi(f(x)) J(x, f)\, dx \right| &\leq \left(\int_U |\nabla u(x)|^n K_I(x, f)\, dx \right)^{\frac{1}{n}} \\ &\quad \times \left(\int_U |\varphi(f(x))|^{\frac{n}{n-1}} J(x, f) \right)^{\frac{n-1}{n}} \\ &\leq N(f, U)^{\frac{n-1}{n}} \left(\int_U |\nabla u(x)|^n K_I(x, f)\, dx \right)^{\frac{1}{n}} \\ &\quad \times \left(\int_{f(U)} |\varphi(y)|^{\frac{n}{n-1}} dy \right)^{\frac{n-1}{n}}. \end{aligned} \tag{37}$$

Here we also used the change of variables formula (19). The desired inequality (20) follows from (30) and (37).

3 Capacity inequalities

Following [22], we call a pair (G, C) a condenser if $G \subset \mathbb{R}^n$ is a domain and C is non-empty, compact, and contained in G. Let $f : \Omega \to \mathbb{R}^n$ be a continuous, open and discrete mapping. A condenser (G, C) is called a normal condenser of f provided that G is a normal domain of f. The image $(f(G), f(C))$ of a condenser (G, C) in Ω is also a condenser, because f is continuous and open.

Let $0 \le \omega \in L^1(G)$. We define the w-weighted capacity of (G, C) by setting

$$\mathrm{Cap}_\omega(G, C) = \inf \left\{ \int_D |\nabla u(x)|^n \omega(x)\, dx : u \in C_0^\infty(G) \text{ and } u \ge 1 \text{ on } C \right\}.$$

Furthermore, when $\omega \equiv 1$, we write $\mathrm{Cap}(G, C)$ instead of $\mathrm{Cap}_1(G, C)$.

Suppose that $f : \Omega \to \mathbb{R}^n$ is a continuous, open and discrete mapping. If C is a non-empty and compact subset of Ω and $y \in f(C)$, then we set

$$M(y, f, C) = \sum_{x \in f^{-1}(y) \cap C} i(x, f). \tag{38}$$

The sum in (38) only has a finite number of terms as f is discrete and $C \subset \Omega$ is compact. The number $M(f, C) = \inf\{M(y, f, C) : y \in f(C)\}$ is called the minimal multiplicity of f on C. If U is a normal domain of f, then $M(f, \overline{U}) = N(f, U) = \sup\{N(y, f, U) : y \in \mathbb{R}^n\} = \sup \sharp\{x \in U : f(x) = y\}$. We refer the reader to [21, Lemma 3.7] for the proof of these facts.

Theorem 3.1 *Let (G, C) be a normal condenser in Ω with $G \subset\subset \Omega$. Under the assumptions of Theorem 2.1 or Corollary 2.2, we have*

$$Cap(f(G), f(C)) \le \frac{Cap_{K_I(\cdot, f)}(G, C)}{N(f, G)}. \tag{39}$$

Proof. Fix $0 \le u \in C_0^\infty(G)$ so that $u \ge 1$ on C. Define $v : f(G) \to \mathbb{R}$ by setting

$$v(y) = \frac{1}{N(f, G)} \sum_{x \in f^{-1}(y) \cap G} i(x, f) u(x). \tag{40}$$

Let $y \in f(C)$. Since $M(f, C) = N(f, G)$ and f is sense-preserving (i.e. $i(x, f) \ge 1$), $u(x) \ge 1$ for each $x \in f^{-1}(u) \cap C$ and $u \ge 0$, we have

$$v(y) \ \ge \ \frac{1}{N(f, G)} \sum_{x \in f^{-1}(y) \cap C} i(x, f) u(x) \ge \frac{1}{M(f, C)} \sum_{x \in f^{-1}(y) \cap C} i(x, f) \ge 1. \tag{41}$$

and so $v(y) \geq 1$ for all $y \in f(C)$. Let us show that $f(\mathrm{spt}u) = \mathrm{spt}v$. It is clear that $f(\{x \in \Omega : u(x) \neq 0\}) = \{y \in f(\Omega) : v(y) \neq 0\}$. Since f is continuous, we have

$$\begin{aligned} f(\mathrm{spt}u) &= f(\overline{\{x \in \Omega : u(x) \neq 0\}}) = \overline{f(\{x \in \Omega : u(x) \neq 0\})} \\ &= \overline{\{y \in f(\Omega) : v(y) \neq 0\}} = \mathrm{spt}v. \end{aligned} \tag{42}$$

Thus $\mathrm{spt}v \subset f(G)$. Combining this with Theorem 2.1 (or Corollary 2.2) we conclude that $v \in W_0^{1,n}(f(G)) \cap C(f(G))$. Fix $\epsilon > 0$ and multiply v by $1+\epsilon$. Using a standard approximation argument we find $\tilde{v}_\epsilon \in C_0^\infty(f(G))$ so that

$$\int_{f(G)} |\nabla \tilde{v}_\epsilon|^n \leq (1+\epsilon)^n \int_{f(G)} |\nabla v|^n + \epsilon$$

and

$$\tilde{v}_\epsilon \geq 1 \text{ on } f(C).$$

By Theorem 2.1 (or Corollary 2.2) we find that

$$\begin{aligned} \mathrm{Cap}(f(G), f(C)) &\leq (1+\epsilon)^n \int_{f(G)} |\nabla v(x)|^n dx + \epsilon \\ &\leq \frac{(1+\epsilon)^n}{N(f,G)} \int_G |\nabla u(x)|^n K_I(x,f) dx + \epsilon \\ &\leq \frac{(1+\epsilon)^n}{N(f,G)} \mathrm{Cap}_{K_I(\cdot,f)}(G,C) + \epsilon. \end{aligned} \tag{43}$$

Letting $\epsilon \to 0$, Theorem 3.1 follows.

It also turns out to be important to have capacity inequalities for more general condensers than normal condensers. The following result provides us with such estimates.

Theorem 3.2 *Under the assumptions of Theorem 2.1 or Corollary 2.2 and assuming that (G, C) is a condenser so that $G \subset\subset \Omega$ we have*

$$Cap(f(G), f(C)) \leq \frac{Cap_{K_I(\cdot,f)}(G,C)}{M(f,C)}. \tag{44}$$

We do not know how to prove Theorem 3.2 directly from our main estimate. Instead of this we will deduce it from an analogous modulus estimate given in Section 4. and we will thus postpone its proof to the end of Section 4.

4 Modulus inequalities

In this section we establish a very general version of the Poletsky inequality using our main estimate in Section 2. It gives as a special case both the Poletsky inequality and the strongest extension of this inequality that we have been able to find in the literature, the Väisälä inequality.

Theorem 4.1 *Suppose that the assumptions of Theorem 2.1 are fulfilled. Assume that $J(x,f) > 0$ for a.e. $x \in \Omega$. Let Γ be a path family in Ω, Γ' be a path family in $\mathbb{R}^n$, and m be a positive integer such that the following is true. For every path $\beta : I \to \mathbb{R}^n$ in Γ' there are paths $\alpha_1, ..., \alpha_m$ in Γ such that $f \circ \alpha_j \subset \beta$ for all j and such that for every $x \in \Omega$ and $t \in I$ the equality $\alpha_j(t) = x$ holds for at most $i(x,f)$ indices j. Then*

$$M(\Gamma') \leq \frac{M_{K_I(\cdot,f)}(\Gamma)}{m}. \tag{45}$$

Here

$$M(\Gamma') = \inf\Big\{ \int_{\mathbb{R}^n} \varrho^n(x)dx : \varrho : \mathbb{R}^n \to [0,\infty) \text{ is a Borel function such that } \int_\gamma \varrho \geq 1 \text{ for each } \gamma \in \Gamma' \Big\}$$

and

$$M_{K_I(\cdot,f)}(\Gamma) = \inf\Big\{ \int_{\mathbb{R}^n} \varrho^n(x) K_I(x,f)dx : \varrho : \mathbb{R}^n \to [0,\infty) \text{ is a Borel function such that } \int_\gamma \varrho \geq 1 \text{ for each } \gamma \in \Gamma \Big\}.$$

Using the point-wise inequality $K_I(x,f) \leq K^{n-1}(x)$ we can replace the assumption $K_I \in L^1_{loc}(\Omega)$ with $K \in L^{n-1}_{loc}(\Omega)$. Furthermore, applying [14, Proposition 2.5] (also see [12], [13]) and [15, Theorem 1.1], we see that the topological and analytic assumptions of Theorem 2.1 are satisfied provided that f satisfies (P).

Corollary 4.2 *Assume that an Orlicz function P satisfies conditions (P-1) and (P-2). Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion such that $P(|Df|^n) \in L^1_{loc}(\Omega)$ and $K \in L^p_{loc}(\Omega)$, for some $p > n-1$. Let Γ and Γ' be as in Theorem 4.1. Then*

$$M(\Gamma') \leq \frac{M_{K_I(\cdot,f)}(\Gamma)}{m}. \tag{46}$$

We shall not give all details of the proof of Theorem 4.1, because the inequality is a rather direct consequence of Theorem 2.1 and the proof of Poletsky's inequality given in [27]. We will follow the proof in [27] and only verify the parts which are different in our case. Our terminology and notation in this section are as in [27] and we also assume that $f : \Omega \to \mathbb{R}^n$ satisfies the hypothesis of Theorem 2.1. First, we will prove a generalization of Poletsky's lemma [27, Poletskii's lemma 5.1]. The reasons behind this result are essentially the fact that the local "inverse" maps of f belong to the Sobolev class $W^{1,n}_{loc}(f(\Omega), \mathbb{R}^n)$ and Fuglede's theorem [27, Theorem 2.2], according to which (continuous) Sobolev mappings are absolutely continuous on modulus a.e. curve. Before we formulate our version of Poletsky's lemma, let us recall what it means for f to be absolutely precontinuous on a path (cf. [27, p. 24]). Let $\beta : I_0 \to \mathbb{R}^n$ be a closed rectifiable path, and let $\alpha : I \to \Omega$ be a path such that $f \circ \alpha \subset \beta$. This means that $f \circ \alpha$ is the restriction of β to some subinterval of I_0. If the length function $s_\beta : I_0 \to [0, l(\beta)]$ is constant on some interval $J \subset I$, β is also constant on J, and the discreteness of f implies that also α is constant on J. It follows that there is a unique mapping $\alpha^* : s_\beta(I) \to \Omega$ such that $\alpha = \alpha^* \circ (s_\beta\big|_I)$. We say that f is absolutely precontinuous on α if α^* is absolutely continuous.

Lemma 4.3 *Assume that the hypotheses of Theorem 2.1 are fulfilled. Suppose that Γ is a family of paths γ in Ω such that $f \circ \gamma$ is locally rectifiable and there is a closed subpath α of γ on which is not absolutely precontinuous. Then $M(f\Gamma) = 0$.*

In order to prove Lemma 4.3 it suffices to establish Lemma 7.2 in [27] under our assumptions. This will be Lemma 4.4 below. Indeed, combining this generalization of Lemma 7.2 with [27, proof of 5.1, p. 47], one then concludes with Lemma 4.3.

Following notations in [27], we fix a domain $D \subset\subset \Omega$ and set $B_k = \{x \in D : i(x, f) = k\}$, $k \geq 1$. We can choose pairwise disjoint open cubes Q_j, $j \in \mathbb{N}$, such that $\frac{4}{3}Q_j \subset D \setminus B_f$, $f\big|_{\frac{4}{3}Q_j}$ is injective, $D \setminus B_f \subset \bigcup_{j=1}^{\infty} \overline{Q}_j$, and $\sum_{j=1}^{\infty} \chi_{\frac{4}{3}Q_j}(x) \leq C(n)$. We have the homeomorphic inverse mappings $h_j : f(\frac{4}{3}Q_j) \to \frac{4}{3}Q_j$. By Corollary 2.3, we know that

$$h_j \in W^{1,n}(f(\frac{4}{3}Q_j), \mathbb{R}^n) \quad \text{for all } j \in \mathbb{N}. \tag{47}$$

Using the fact (cf. [27, VI Lemma 4.4]) that a homeomorphism $g \in W^{1,p}_{loc}(G, \mathbb{R}^n)$, $p > n-1$, is differentiable a.e in the open set G, we conclude that h_j is differentiable in a set $A_j \subset f(\frac{4}{3}Q_j)$ with $|f(\frac{4}{3}Q_j) \setminus A_j| = 0$. We denote the classical differential of h_j by h_j'. We set $h_j' = 0$ for $y \in \mathbb{R}^n \setminus A_j$ and define a Borel function ρ by setting

$$\rho = \sup \Big\{ |h_j'| \chi_{f(\frac{4}{3}Q_j)}(x) : j \in \mathbb{N} \Big\}.$$

For each point $x \in B_k$ there exists a normal neighborhood [27, I Lemma 4.9] $U \subset D$ of x. We cover B_k by such normal neighborhoods U_{ki}, $i \in \mathbb{N}$, and let

$$g_{ki} = \left(\sum_{x \in f^{-1}(y) \cap U_{ki}} i(x,f) x_1, \cdots, \sum_{x \in f^{-1}(y) \cap U_{ki}} i(x,f) x_n \right)$$

for all $y \in f(U_{ki})$. By Theorem 2.1 we have that

$$g_{ki}(y) \in W^{1,n}(f(U_{ki}), \mathbb{R}^n). \tag{48}$$

As $f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega, \mathbb{R}^n)$ (by the condition (P-0) and (P-2)) and f is open, f is differentiable a.e. in Ω (cf. [27, VI Lemma 4.4]). Because also $|B_f| = 0$, we can find a set $C \subset \Omega$ such that $|C| = 0$, $B_f \subset C$ and f is differentiable a.e. on the set $\Omega \setminus C$. Finally we can fix a set $F \subset \mathbb{R}^n$ of zero measure which contains all the points where at least one of h_j is not differentiable and which also contains the set $f(C)$; this is possible because the mapping f satisfies the condition (N).

Lemma 4.4 *Assume that the assumptions of Theorem 2.1 are fulfilled. Let Γ_0 be a family paths γ in D such that either $f \circ \gamma$ is unrectifiable or $f \circ \gamma$ is rectifiable and at least one of the following conditions is not true:*

1. $\int_{f \circ \gamma} \chi_F ds = 0$.
2. $\int_{f \circ \gamma} \rho ds < \infty$.
3. *If α is a closed subpath of γ and if $|\alpha| \subset \frac{4}{3}Q_j$, h_j is absolutely continuous on $f \circ \alpha$.*
4. *If α is a closed subpath of γ and if $|\alpha| \subset U_{ki}$, g_{ki} is absolutely continuous on $f \circ \alpha$.*

Then $M(f\Gamma_0) = 0$.

Proof. Since the family of nonrectifiable paths in $\mathbb{R}^n$ is of modulus zero [29, 6.10], we may assume that $f \circ \gamma$ is rectifiable for all $\gamma \in \Gamma_0$. Let Γ_q, $q = 1, ..., 4$, be the family of paths $\gamma \in \Gamma_0$ for which the condition q. is not true. Now $M(f\Gamma_1) = 0$ holds by [29, 33.1] because $|F| = 0$. Next $M(f\Gamma_3) = 0 = M(f\Gamma_4)$ follows from (47),(48) and Fuglede's theorem [27, II Theorem 2.3]. Using Theorem 2.1 (with $u(x) = x_k, k = 1, ..., n$) we notice that

$$\begin{aligned}\int_{\mathbb{R}^n} [\rho(x)]^n \, dx &\leq \sum_{j=1}^{\infty} \int_{f(\frac{4}{3}Q_j)} |h_j'(y)|^n dy \leq C(n) \sum_{j=1}^{\infty} \int_{\frac{4}{3}Q_j} K_I(x, f) \, dx \\ &\leq C(n) \int_D K_I(x, f) \, dx < \infty,\end{aligned}$$

and so $M(f\Gamma_2) = 0$.

Proof of Lemma 4.3 Following the proof given in [27, proof of 5.1, p.47] line by line and replacing the conditions (1)-(4) in [27, Lemma 7.2] by the conditions 1.-4. in Lemma 4.4, we conclude with Lemma 4.3.

Proof of Theorem 4.1 Let $C \subset \Omega$ be a set of measure zero as given before Lemma 4.4. As $J(x, f) > 0$ a.e. in Ω, we may assume that $J(x, f) > 0$ for each $x \in \Omega \setminus C$. Since f satisfies condition (N), we find a Borel set B of measure zero, containing $f(C)$, so that $B_f \subset f^{-1}(B) = \emptyset$ and so that f is differentiable at each $x \in \Omega \setminus f^{-1}(B)$ with $J(x, f) > 0$.

We may clearly assume that each $\beta \in \Gamma'$ is locally rectifiable, and, because $|B| = 0$, we may further assume that

$$\int_{\beta} \chi_B = 0$$

for each $\beta \in \Gamma'$. Owing to Lemma 4.3 we may also assume that if α is a path in Ω with $f \circ \alpha \subset \beta \in \Gamma'$, then f is locally absolutely precontinuous on α.

Let $\varrho \geq 0$ be a Borel function with $\int_{\beta} \varrho \geq 1$ for each $\gamma \in \Gamma$. Define

$$\sigma(x) = \varrho(x) / \min_{|h|=1} |Df(x)h|$$

when $x \in \Omega \setminus f^{-1}(B)$, and extend σ as zero to the rest of $\mathbb{R}^n$. Set

$$\varrho'(y) = \frac{1}{m}\chi_{f(\Omega)}(y) \sup_{x \in A} \sigma(x),$$

where A runs through over all subset of $f^{-1}(y)$ such that $\#A \leq m$. Then $\varrho' \geq 0$ is a Borel function (cf. [27, pp. 49–51]). Moreover, the argument in [27, pp. 51–52] applies verbatim to yield that

$$\int_\beta \varrho' \geq 1$$

for each $\beta \in \Gamma'$.

Let (Ω_i) be an exhaustion of Ω, and set $\varrho_i = \varrho\chi_{\overline{\Omega}_i}$, $\sigma_i = \sigma\chi_{\overline{\Omega}_i}$, and $\varrho'_i = \varrho'\chi_{f(\overline{\Omega}_i)}$. Suppose $y_0 \in f(\overline{\Omega}_i) \setminus f(\Omega_i \cap B_f)$. Then there is a connected neighborhood V of y_0 and k inverse mappings $g_\mu : V \to D_\mu$ with

$$\overline{\Omega}_i \cap f^{-1}(V) = \bigcup \{\overline{\Omega}_i \cap D_\mu : \ 1 \leq \mu \leq k\}.$$

For each $y \in V$, we define a set $L_y \subset J := \{1, \cdots, k\}$ as follows. If $k \leq m$, then $L_y = J$. If $k > m$, then $\#L_y = m$, and for each $\mu \in L_y$, $\nu \in J \setminus L_y$, either $\sigma_i(g_\mu(y)) > \sigma_i(g_\nu(y))$ or $\sigma_i(g_\mu(y)) = \sigma_i(g_\nu(y))$ and $\mu > \nu$. Then

$$\varrho'_i(y) = \frac{1}{m} \sum_{\mu \in L_y} \sigma_i(g_\mu(y))$$

for $y \in V$. Furthermore, for $L \subset J$, the sets $V_L = \{y \in V : \ L_y = L\}$ are pairwise disjoint Borel sets. By Hölder's inequality for series,

$$[\varrho'_i(y)]^n \leq \frac{1}{m} \sum_{\mu \in L_y} \sigma_i(g_\mu(y))^n.$$

Now

$$\int_{V_L} [\varrho'_i(y)]^n \, dy \leq \frac{1}{m} \sum_{\mu \in L} \int_{V_L} (\sigma_i \circ g_\mu)^n. \tag{49}$$

The change of variables formula (19) yields

$$\int_{V_L} [\varrho'_i(y)]^n \, dy \leq \frac{1}{m} \sum_{\mu \in L} \int_{g_\mu(V_L)} \sigma_i^n(x) J(x, f) \, dx. \tag{50}$$

In the set $g_\mu(V_L)$ the Jacobian determinant of f is strictly positive a.e. and so

$$K_I(x,f) = \frac{|D^\sharp f|^n}{J(x,f)^{n-1}} = \frac{J(x,f)}{\min_{|h|=1|} |Df(x)h|^n},$$

for almost every $x \in g_\mu(V_L)$. Thus

$$\int_{V_L} [\varrho_i'(y)]^n \, dy \le \frac{1}{m} \sum_{\mu \in L} \int_{g_\mu V_L} \varrho_i^n(x) K_I(x,f) \, dx. \tag{51}$$

As in [27, pp. 51-52] we conclude that

$$\int_{\mathbb{R}^n} [\varrho_i'(y)]^n \, dy \le \frac{1}{m} \int_{\mathbb{R}^n} \varrho_i^n(x) K_I(x,f) \, dx. \tag{52}$$

Letting $i \to \infty$, we obtain Theorem 4.1.

We close this section by giving a proof for Theorem 3.2.

Proof of Theorem 3.2 Let (G,C) be a condenser with $G \subset\subset \Omega$. Then $(f(G), f(C))$ is a condenser in $f(\Omega)$. Now

$$\mathrm{Cap}(f(G), f(C)) = M(\Gamma'), \tag{53}$$

where Γ' consists of all paths that join $f(C)$ to $\partial f(G)$ in $f(G)$, see [27, II Proposition 10.2]. Let $\beta : [a,b) \to f(G)$ be a path in Γ' so that $C \cap f^{-1}(\beta(a))$ contains points $x_1, ..., x_k$ and

$$\sum_{j=1}^{k} i(x_j, f) \ge M(f, C).$$

The theory of path lifting (see [27, I Theorem 3.2]) provides us with a maximal sequence of liftings $\alpha_l : [a, c_l) \to G$ of β, $1 \le l \le \sum_{j=1}^k i(x_j, f)$, starting at the points $x_1, ..., x_k$ so that $\alpha([0, c_j)$ is not compactly contained G. Denote the family of all maximal liftings by Γ. It follows that Γ and Γ' satisfy the conditions in Theorem 4.1 and Corollary 4.2, and thus we have

$$M(\Gamma') \le \frac{M_{K_I(\cdot,f)}(\Gamma)}{M(f,C)}. \tag{54}$$

The claim follows, because

$$M_{K_I(\cdot,f)}(\Gamma) \le \mathrm{Cap}_{K_I(\cdot,f)}(G,C), \tag{55}$$

as in easily seen by considering $\varrho(x) = |\nabla u(x)|$ for a given test function u for $\mathrm{Cap}_{K_I(\cdot,f)}(G,C)$.

5 Proofs of Theorem 1.1 and Corollary 1.2

We will first reduce Theorem 1.1 to Corollary 2.2. We begin with a result from [14], see lemmas 2.1 and 2.3 there.

Proposition 5.1 *Assume that $\mathcal{A}$ is an Orlicz function satisfying (A-1) and (A-2). Then we have the point-wise inequality*

$$P(KJ) \leq J + exp(\mathcal{A}(K)) - 1 \tag{56}$$

for all $K, J \geq 0$, where the Orlicz function P satisfies the integrability condition

$$\int_1^\infty \frac{P(s)}{s^2} = \infty \tag{57}$$

and also the technical condition that for every $\epsilon > 0$ we have

$$(t^{-1}P(t))' \leq 0 \leq (t^{\epsilon-1}P(t))' \tag{58}$$

for all $t \geq t_1(\epsilon, \mathcal{A})$.

Theorem 1.1 (and Corollary 1.2) follows from Corollary 2.2 (and Corollary 2.3) by noticing that (57) is (P-1) and that (P-2) follows from (58) and that (A-0) and (A-2) easily guarantee that $K \in L^p_{loc}(\Omega)$ for all $p < \infty$.

The above observation and the results in Section 3 and in Section 4 immediately yield capacity and modulus inequalities. For simplicity, we only formulate the following result.

Corollary 5.2 *Suppose that $\mathcal{A}$ is an Orlicz function satisfying (A-1) and (A-2). Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion with $exp(\mathcal{A}(K)) \in L^1_{loc}(\Omega)$. If (G, C) is condenser in Ω such that $G \subset\subset \Omega$, then*

$$Cap(f(G), f(C)) \leq \frac{Cap_{K_I(\cdot,f)}(G, C)}{M(f, C)}.$$

Next we establish an upper bound on our weighted capacity. Fix an Orlicz function $\mathcal{A}$ so that $\mathcal{A}$ satisfies the conditions (A-1) and (A-2). Pick a positive number $b = b(n, \mathcal{A})$ such that the function $t \to \exp(\mathcal{A}(t^{\frac{1}{n-1}}))$ is convex on (b, ∞). This is possible by the assumption (A-2), see [16, Lemma 2.4].

Theorem 5.3 *Suppose that*

$$I = \int_{B_R} exp(\mathcal{A}(K(x)))\, dx < \infty \tag{59}$$

where $B_R = B(x_0, R)$*, and let* $0 < r < R/6$*. Then*

$$Cap_{K^{n-1}}(B_R, \overline{B}_r) \leq \omega_{n-1}\, g(\epsilon) \left(\int_{2r}^{R/2} \varphi_\epsilon(s)\, ds \right)^{1-n} \tag{60}$$

where the function $g : (0, \frac{1}{n}) \to (1, \infty)$ *satisfies* $\lim_{\epsilon \to 0} g(\epsilon) = 1$*. Furthermore the function* φ_ϵ *is given by the rule*

$$\varphi_\epsilon(s) = \frac{1}{s\mathcal{A}^{-1}\left(\log\left(\frac{exp(\mathcal{A}(b))I}{(2^{\epsilon n}-1)\frac{\omega_{n-1}}{n}s^n}\right)\right)}. \tag{61}$$

Notice that $I \geq |B_R| = \frac{\omega_{n-1}}{n} R^n$ and so $\left(\frac{\exp(\mathcal{A}(b))I}{(2^{\epsilon n}-1)\frac{\omega_{n-1}}{n}s^n}\right) > 1$ for all $s \in (0, R]$. Moreover,

$$\int_0 \frac{ds}{s\mathcal{A}^{-1}(\log \frac{C}{s^n})} = \frac{1}{n} \int^\infty \frac{\mathcal{A}'(t)}{t} dt = \infty, \tag{62}$$

and thus $\mathrm{Cap}_{K^{n-1}}(B_R, \overline{B}_r)$ tends to zero when R is fixed and r approaches zero.

Proof. Without loss of generality, we can assume that $x_0 = 0$. Fix $\epsilon \in (0, \frac{1}{n})$ and write $a_{i,\epsilon} = \frac{R}{2^{\epsilon i}}$ for all $i \in \mathbb{N}$. Choose $k \in \mathbb{N}$ so that $r \in (a_{k+1,\epsilon}, a_{k,\epsilon}]$. We set

$$\tilde{K}(x) = \begin{cases} K(x), & K(x) > b \\ b, & K(x) \leq b. \end{cases} \tag{63}$$

Assumption (A-2) implies that there exists $t_0 = t_0(n, \mathcal{A}) \in (0, \infty)$ (see [16, Lemma 2.3]) such that the function $t \to [t\mathcal{A}^{-1}(\log t^{-n})]$ is decreasing on $(0, t_0)$. Using this fact we see the function $h : (0, \frac{1}{n}) \to (0, \infty)$ given by the rule

$$\sup_{i \in \{1,\ldots,k-1\}} \max_{a_{i+1,\epsilon} \leq t \leq a_{i,\epsilon}} \frac{\varphi_\epsilon(t)}{\varphi_\epsilon(a_{i+1,\epsilon})} = 1 + h(\epsilon) \tag{64}$$

has the property

$$\lim_{\epsilon \to 0} h(\epsilon) = 0. \tag{65}$$

Define

$$u(x)=\begin{cases}1-\frac{\int_{a_{k,\epsilon}}^{|x|}\varphi_\epsilon(s)ds}{\int_{a_{k,\epsilon}}^{R/2^\epsilon}\varphi_\epsilon(s)ds}, & a_{k,\epsilon}<|x|<R/2^\epsilon\\ 1, & |x|\le a_{k,\epsilon}\\ 0, & |x|\ge R/2^\epsilon.\end{cases} \tag{66}$$

Now

$$\int_{B_R}|\nabla u(x)|^n[K(x)]^{n-1}\,dx\le\sum_{i=1}^{k-1}\int_{B_{a_{i,\epsilon}}\setminus B_{a_{i+1,\epsilon}}}|\nabla u(x)|^n[\tilde{K}(x)]^{n-1}\,dx. \tag{67}$$

By (64), we find that

$$\begin{aligned}\int_{B_R}|\nabla u(x)|^n[K(x)]^{n-1}\,dx \;\;&\le\;\; \left(\frac{1+h(\epsilon)}{\int_{a_{k,\epsilon}}^{R/2^\epsilon}\varphi_\epsilon(s)ds}\right)^n\sum_{i=1}^{k-1}[\varphi_\epsilon(a_{i+1,\epsilon})]^n\\ &\quad\times|B_{a_{i,\epsilon}}\setminus B_{a_{i+1,\epsilon}}|\fint_{B_{a_{i,\epsilon}}\setminus B_{a_{i+1,\epsilon}}}[\tilde{K}(x)]^{n-1}\,dx.\end{aligned}$$

Jensen's inequality applied to the convex function $(b,\infty)\to(0,\infty):\tau\to\exp(\mathcal{A}(\tau^{\frac{1}{n-1}}))$ yields

$$\fint_{B_{a_{i,\epsilon}}\setminus B_{a_{i+1,\epsilon}}}[\tilde{K}(x)]^{n-1}\,dx\le\Big[\mathcal{A}^{-1}(\log(\fint_{B_{a_{i,\epsilon}}\setminus B_{a_{i+1,\epsilon}}}\exp\mathcal{A}(\tilde{K}(x))))dx\Big]^{n-1} \tag{68}$$

and computations show that

$$
\begin{aligned}
\int_{B_R} |\nabla u(x)|^n [K(x)]^{n-1}\, dx \;&\leq\; \left(\frac{1+h(\epsilon)}{\int_{a_{k,\epsilon}}^{R/2^\epsilon} \varphi_\epsilon(s) ds} \right)^n \frac{\omega_{n-1}}{n} (2^{\epsilon n} - 1) \\
&\quad \times \sum_{i=1}^{k-1} [\varphi_\epsilon(a_{i+1,\epsilon})]^n a_{i+1,\epsilon}^n \\
&\quad \times \left[\mathcal{A}^{-1} \left(\log \left(\frac{\exp(\mathcal{A}(b)) I}{(2^{\epsilon n} - 1) \frac{\omega_{n-1}}{n} a_{i+1,\epsilon}^n} \right) \right) \right]^{n-1} \\
&= \left(\frac{1+h(\epsilon)}{\int_{a_{k,\epsilon}}^{R/2^\epsilon} \varphi_\epsilon(s) ds} \right)^n \frac{\omega_{n-1}}{n} (2^{\epsilon n} - 1) \\
&\quad \times \sum_{i=1}^{k-1} \frac{1}{\mathcal{A}^{-1} \left(\log \left(\frac{\exp(\mathcal{A}(b)) I}{(2^{\epsilon n} - 1) \frac{\omega_{n-1}}{n} a_{i+1,\epsilon}^n} \right) \right)} \\
&\leq \left(\frac{1+h(\epsilon)}{\int_{a_{k,\epsilon}}^{R/2^\epsilon} \varphi_\epsilon(s) ds} \right)^n \frac{\omega_{n-1}}{n} \frac{2^{\epsilon n} - 1}{2^\epsilon - 1} \int_{a_{k,\epsilon}}^{R/2^\epsilon} \varphi_\epsilon(s)\, ds.
\end{aligned}
$$

Using the elementary inequalities $\epsilon a \log 2 \leq 2^{a\epsilon} - 1 \leq \epsilon a 2^{a\epsilon} \log 2$, for $a, \epsilon \geq 0$, we conclude that

$$
\begin{aligned}
\int_{B_R} |\nabla u(x)|^n [K(x)]^{n-1}\, dx \;&\leq\; \omega_{n-1} (1+h(\epsilon))^n 2^{\epsilon n} \left(\int_{a_{k,\epsilon}}^{R/2^\epsilon} \varphi_\epsilon(s)\, ds \right)^{1-n} \\
&\leq\; \omega_{n-1} (1+h(\epsilon))^n 2^{\epsilon n} \left(\int_{2r}^{R/2} \varphi_\epsilon(s)\, ds \right)^{1-n} \qquad (69)
\end{aligned}
$$

as desired.

We close this section by giving an essentially sharp local version of a modulus of continuity for mappings of subexponentially integrable distortion. This result is an immediate consequence of Corollary 5.2, Theorem 5.3, and the following lemma of F. W. Gehring's [4], and it improves on the corresponding estimate in [16].

Lemma 5.4 *Let (G, C) be a condenser so that G is bounded and C is connected. Then*

$$
Cap(G, C) \geq \frac{\omega_{n-1}}{\left(\log \left(\frac{C(n)\, diam G}{diam C} \right) \right)^{n-1}}. \qquad (70)
$$

Corollary 5.5 *Assume that an Orlicz function $\mathcal{A}$ satisfies (A-1) and (A-2). Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion whose distortion function satisfies the integrability condition*

$$I = \int_{B_R} exp(\mathcal{A}(K(x)))\, dx < \infty,$$

where $B_R = B(x_0, R) \subset\subset \Omega$. Then, for all small $\epsilon > 0$, we have

$$\frac{|f(x) - f(y)|}{diam f(B_R)} \le C(n) exp\Big[- \frac{M(f, \overline{B}_R)^{\frac{1}{n-1}}}{1+\epsilon} \int_{2|x-y|}^{\frac{R}{2}} \frac{ds}{s\mathcal{A}^{-1}\left(\log C_{\mathcal{A},n}(\epsilon) \frac{nI}{\omega_{n-1} s^n}\right)}\Big] \tag{71}$$

whenever $x, y \in B(x_0, \frac{R}{10})$.

In the special case $\mathcal{A}(t) = \lambda t$, for some $\lambda > 0$, the modulus of continuity estimate (71) reads as

$$\frac{|f(x) - f(y)|}{\operatorname{diam} f(B_R)} \le C_{I,n,\lambda}(\epsilon) \left[\frac{\log\left(\frac{nI}{\omega_{n-1} R^n}\right)}{\log\left(\frac{nI}{\omega_{n-1}|x-y|^n}\right)} \right]^{\frac{\lambda M(f, \overline{B}_R)^{\frac{1}{n-1}}}{n+\epsilon}}. \tag{72}$$

6 Proof of Corollary 1.3

Let us begin by showing that the divergence condition in Corollary 1.3 cannot be relaxed.

Suppose that we are given a (continuous) function $\varrho : [1, \infty) \to [\delta, \infty)$ where $\delta > 0$ so that $\varrho(2r) \le C\varrho(r)$ for all $r \ge 1$ and

$$\int_1^\infty \frac{dr}{r\,\varrho(r)} < \infty.$$

Define

$$f(x) = \frac{x}{|x|} \int_0^{|x|} \frac{dr}{r\,\tilde{\varrho}(r)}, \tag{73}$$

where $\tilde{\varrho}(r) = \varrho(r)$ when $r \ge 1$ and $\tilde{\varrho}(r) = r^{-\frac{3}{2}}$ for $0 < r < 1$. Then f is a Lipschitz continuous homeomorphism of $\mathbb{R}^n$ onto a bounded domain and of locally bounded distortion K with

$$\left(\fint_{B_r} K^{n-1} \right)^{\frac{1}{n-1}} \ge \frac{\varrho(r)}{C} \quad \text{for } \; r \ge 1. \tag{74}$$

Indeed, a simple computation (see [10, Section 6.5]) shows that

$$K(x) = \tilde{\varrho}(|x|) \int_0^{|x|} \frac{dr}{r\,\tilde{\varrho}(r)}. \tag{75}$$

Now we prove Corollary 1.3. Suppose that f is non-constant. Let $R > 1$ and $B_s = B(0,s)$. By Theorem 3.2 and Corollary 5.2 we know that

$$\mathrm{Cap}(f(B_R), f(\overline{B}_1)) \le \mathrm{Cap}_{K_I(\cdot,f)}(B_R, \overline{B}_1). \tag{76}$$

Because f is open, $f(B(0,1))$ contains a ball. As $\mathrm{Cap}(\mathbb{R}^n \setminus f(\mathbb{R}^n)) > 0$, it follows that

$$\mathrm{Cap}(f(B_R), f(\overline{B}_1)) \ge \delta > 0, \quad \text{where } \delta \text{ is independent of } R \tag{77}$$

see [27, III Lemma 2.6]. Thus a contradiction follows if we show that there is a sequence of functions $u_i \in C_0^\infty(\mathbb{R}^n)$ so that $u_i \equiv 1$ on $\overline{B}_1$ and

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} |\nabla u_i(x)|^n K^{n-1}(x)\, dx = 0.$$

It clearly suffices to find Lipschitz continuous functions like this. To this end, let $i \in \mathbb{N}$ be large, and define

$$v_i(x) = \max\left\{0, \frac{\int_1^{2^i} r^{-1} \left(⨍_{B_r} K^{n-1}\right)^{\frac{-1}{n-1}} dr - \int_1^{|x|} r^{-1} \left(⨍_{B_r} K^{n-1}\right)^{\frac{-1}{n-1}} dr}{\int_1^{2^i} r^{-1} \left(⨍_{B_r} K^{n-1}\right)^{\frac{-1}{n-1}} dr}\right\}$$

and

$$u_i(x) = \min\{1, v_i(x)\}.$$

Write $a = \int_1^{2^i} r^{-1} \left(⨍_{B_r} K^{n-1}\right)^{\frac{-1}{n-1}} dr$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u_i|^n K^{n-1} &\le a^{-n} \int_{B_{2^i}\setminus B_1} K^{n-1}(x)|x|^{-n} \left(⨍_{B_{|x|}} K^{n-1}\right)^{-\frac{n}{n-1}} dx \\
&\le a^{-n} \sum_{k=1}^{i} \int_{B_{2^k}\setminus B_{2^{k-1}}} K^{n-1}(x)|x|^{-n} \left(⨍_{B_{|x|}} K^{n-1}\right)^{-\frac{n}{n-1}} dx \\
&\le a^{-n}\, C_n \sum_{k=1}^{i} \left(⨍_{B_{2^{k-1}}} K^{n-1}\right)^{-\frac{n}{n-1}} \left(⨍_{B_{2^k}} K^{n-1}\right).
\end{aligned} \tag{78}$$

Using the assumption $\int_{B_{2^k}} K^{n-1} \leq C \int_{B_{2^{k-1}}} K^{n-1}$, we conclude that

$$\int_{\mathbb{R}^n} |\nabla u_i|^n K^{n-1} \leq CC_n \left[\int_1^{2^i} r^{-1} \left(\fint_{B_r} K^{n-1} \right)^{\frac{-1}{n-1}} dr \right]^{1-n}, \qquad (79)$$

and the claim follows by letting i tend to infinity.

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