# Lipschitz continuity of Cheeger-harmonic functions in metric measure spaces

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**Abstract**

We establish the Lipschitz continuity of Cheeger-harmonic functions in certain metric measure spaces. Examples are given to illustrate the necessity of our assumptions on these spaces.

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## 1 Introduction

Suppose that $X$ is a pathwise connected metric space equipped with a doubling, Borel regular measure $\mu$. For simplicity, we will assume that $X$ is proper: each closed ball in $X$ is compact. Given $u : \Omega \to \overline{\mathbb{R}}$, where $\Omega \subset X$ is a domain, we call a Borel function $g : \Omega \to [0, \infty]$ an *upper gradient* of $u$ on $\Omega$ provided

$$|u(x) - u(y)| \le \int_\gamma g \, ds \tag{1}$$

for all $x, y \in \Omega$ and each rectifiable curve $\gamma : [0, l] \to \Omega$ that joins $x$ and $y$. This concept was introduced in [16] as a substitute for the length of the gradient. In order to have control, in average, of functions in terms of upper gradients, it is then natural to require a Poincaré inequality. Following [16], we say that the metric measure space $(X, \mu)$ supports a (weak) $p$-Poincaré

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inequality if there exist $C > 0$ and $\lambda \geq 1$ such that for every ball $B(x, r) \subseteq X$ and for all continuous functions $u$ and all upper gradients $g$ of $u$ on $B(x, \lambda r)$,

$$\fint_{B(x,r)} |u(y) - u_B| \, d\mu(y) \leq Cr \left( \fint_{B(x,\lambda r)} g(y)^p \, d\mu(y) \right)^{\frac{1}{p}} . \tag{2}$$

Here and throughout we will use the notation

$$u_B = \fint_B u(y) \, d\mu(y) = \mu(B)^{-1} \int_B u(y) \, d\mu(y).$$

Upper gradients allow us to define the first order Sobolev spaces on $X$. We define the Newtonian (generalized Sobolev) space $N^{1,p}(X) = N^{1,p}(X, \mu)$ to consist of those $p$-integrable functions on $X$ for which there exists a $p$-integrable upper gradient, see [27] and Section 2 below. The amazing fact is that this setting allows one to assign a differential $Du$ to each function $u \in N^{1,p}(X, \mu)$. The differential $Du$ that we call the Cheeger derivative (cf. [6]) will be further discussed in Section 2. This derivative satisfies the usual Leibniz rule, and $Du(x) \in \mathbb{R}^N$ for almost all $x$, where $N$ only depends on the data of the space $X$ (including the constants in the $p$-Poincaré inequality above). Further, $Du$ is naturally connected with the upper gradients in the sense that, for Lipschitz functions, the length of $Du$ is comparable to a suitably defined minimal upper gradient.

Armed with a Poincaré inequality and the concepts of a differential and a Sobolev class, it is then natural to define harmonic functions and to try and establish their basic properties. To this end, we assume that the 2-Poincaré inequality holds in $X$. We will give more precise definitions of Newtonian spaces in Section 2. Especially the "Newtonian spaces with zero boundary values" ($N_0^{1,p}$), which are needed in order to assign the Dirichlet problem, will be defined there.

Let $\Omega \subseteq X$ be a domain. We say that a function $u \in N_{loc}^{1,2}(\Omega)$ is *Cheeger-harmonic* in $\Omega$ if, for each relatively compact open subset $V \subseteq \Omega$, and for all $v \in N_0^{1,2}(V)$,

$$\int_V |Du(x)|^2 \, d\mu(x) \leq \int_V |D(u + v)(x)|^2 \, d\mu(x). \tag{3}$$

Equivalently, $u$ is harmonic in $\Omega$ if for all Lipschitz functions $\phi$ with compact support in $\Omega$,

$$\int_\Omega Du \cdot D\phi = 0. \tag{4}$$

The equivalence is shown by the same argument that is used in the Euclidean setting, see [17].

Cheeger-harmonic functions have been considered in [6] where many of the basic properties have been established. One can show the existence of solutions to the Dirichlet problem (cf. [6], [28]), and harmonic functions are known to be locally Hölder continuous under the above assumptions, assuming the Poincaré inequality for $p = 2$ (cf. [4], [6]): the usual iteration arguments apply even in this generality. To be precise, a seemingly stronger Poincaré inequality, the $(2, 2)$-Poincaré inequality, is required in [4]. However this is equivalent to the 2-Poincaré inequality under the above assumptions, see [14], [15]. It is then natural to further inquire for conditions under which better regularity of harmonic functions could be obtained.

Smoothness does not make sense in our abstract setting, and thus local Lipschitz continuity seems to be what one should aim for. Indeed, it is immediate from the definitions that the function $u$, defined by $u(x, y) = y$ when $y \leq 0$ and $u(x, y) = 2y$ when $y \geq 0$, is harmonic in $(\mathbb{R}^2, |.|, \mu)$ when $d\mu(x) = dx$ for $y < 0$ and $d\mu(x) = \frac{1}{2}dx$ for $y \geq 0$. Here $dx$ refers to the Lebesgue measure.

Our next example shows that even when the underlying space is $\mathbb{R}^n$ equipped with the Euclidean metric, the doubling property of the measure $\mu$ is not sufficient to guarantee the Lipschitz continuity of Cheeger-harmonic functions.

Consider $X = \mathbb{R}^2$ and the domain $\Omega = (-1, 1) \times (-1, 1)$. Let $w(x, y) = \sqrt{|x|}$ when $|x| \leq 1$, and $w(x, y) = 1$ otherwise, and set $d\mu = w dx$. Then, by [17], $(\mathbb{R}^2, \mu)$ supports a 2-Poincaré inequality and $\mu$ is a doubling measure. The function $u(x, y) = \operatorname{sgn}(x)\sqrt{|x|}$ when $|x| \leq 1$, $u(x, y) = 1$ when $x > 1$ and $u(x, y) = -1$ when $x < -1$ is harmonic on $\Omega$, but is not locally Lipschitz on $\Omega$. However, $u$ is indeed locally Hölder continuous.

Motivated by the previous example, we will from now on assume that $\mu$ is (Ahlfors) $Q$-regular for some $Q > 1$; there exist constants $Q > 1$ and $C > 0$ such that for every $x \in X$ and all $r > 0$,

$$\frac{1}{C} r^Q \leq \mu\big(B(x, r)\big) \leq C r^Q.$$

At first sight, it does not seem unreasonable that the Poincaré inequality and the $Q$-regularity could be sufficient for the Lipschitz continuity of Cheeger-harmonic functions. This hope turns out to futile. Indeed, let $\pi < \alpha < 2\pi$ and consider the space $(X_\alpha, |.|, dx)$, where

$$X_\alpha = \left\{(x, y) = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \,:\, \varphi \in [0, \alpha],\, r \geq 0\right\}.$$

Then the function

$$u(r,\theta) = r^{\frac{\pi}{\alpha}} \cos\left(\frac{\pi}{\alpha}\theta\right),$$

defined in polar coordinates, is harmonic in our sense, because $\Delta u(x,y) = 0$ in the interior of $X_\alpha$, the partial derivatives of $u$ are integrable and the normal derivatives on the boundary vanish. However, $u$ fails to be Lipschitz continuous at $(0,0)$.

Our additional assumption will be given in terms of the heat kernel $p(t,x,y)$ associated with the Dirichlet form $\mathcal{E}$, defined by $\mathcal{E}(f,g) = \int_X Df(y) \cdot Dg(y)\, d\mu(y)$, $Df$ being the Cheeger derivative of $f$. Let us now state our main theorem.

**Theorem 1.1.** *Let $(X,\mu)$ be a pathwise connected, proper metric measure space endowed with a $Q$-regular measure $\mu$, $Q > 1$. Assume that $\mu$ supports a 2-Poincaré inequality. Furthermore, assume that there exist constants $C > 0$ and $t_0 > 0$ such that for each $0 < t < t_0$ and every $f \in N^{1,2}(X)$ we have*

$$\begin{aligned} (5) \quad \int_X f(y)^2\, p(t,x,y)\, d\mu(y) &\leq (2t + Ct^2) \int_X |Df(y)|^2 p(t,x,y)\, d\mu(y) \\ &+ \left(\int_X f(y)\, p(t,x,y)\, d\mu(y)\right)^2 \end{aligned}$$

*for almost every $x \in X$. Then, every function $u \in N^{1,2}_{loc}(X)$ that is harmonic on a domain $\Omega \subseteq X$ is locally Lipschitz continuous on $\Omega$.*

Notice that the Sobolev inequality (5) holds for the space $(X_\alpha, |.|, dx)$ when $\alpha = \pi$ (see Section 5) and fails for $\pi < \alpha < 2\pi$. Thus inequality (5) appears to be a rather natural assumption. It holds in many situations, for example, when the curvature of $X$ is bounded from below in the sense of Bakry and Emery [3], [2]; in particular for those Riemannian manifolds whose Ricci curvature is bounded from below. For this see the discussion in Section 5. For these manifolds, the Lipschitz continuity follows from the Cheng-Yau gradient estimate [8]. We do not know if inequality (5) is stable under Gromov-Hausdorff convergence; see [7] for interesting results regarding eigenfunctions on Gromov-Hausdorff limits of manifolds whose Ricci curvature is bounded from below. Inequality (5) is also guaranteed by the logarithmic Sobolev inequality

$$\begin{aligned} (6) \quad & \int_X f(x)^2 \log\left(\frac{f(x)^2}{||f||^2_{L^2(X,p(t,x_0,x)\, d\mu)}}\right) p(t,x_0,x)\, d\mu(x) \\ \leq\ & (4t + 2Ct^2) \int_X |\nabla f(x)|^2\, p(t,x_0,x)\, d\mu(x), \end{aligned}$$

see for instance [2]. By Gross's Theorem [12], the logarithmic Sobolev inequality (6) with $C = 0$ is equivalent to the *hypercontractivity* of the semigroup $(\tilde{T}_t)_{t>0}$, which is the semigroup corresponding to the Dirichlet form $\tilde{\mathcal{E}}$ defined by

$$\tilde{\mathcal{E}}(u,v) = \int_X Du(x) \cdot Dv(x)\, p(t,x_0,x)\, d\mu(x).$$

Hypercontractivity properties of heat semigroups and relations to logarithmic Sobolev inequalities have been widely studied, see for instance [1], [2], [3], [9], [12], [13], [33], and [34].

How does one then prove the Lipschitz continuity? In the Euclidean setting, the $C^\infty$-regularity of a harmonic function $u$ is proven by using mollifications or by using harmonic approximations $\frac{1}{h}\big(u(x+e_i h) - u(x)\big)$ to $\nabla u$, see for instance [11]. Neither technique is available in the general setting considered in Theorem 1.1. We instead adapt a very recently established technique of Caffarelli and Kenig [5]. Naturally, our abstract setting causes new difficulties. These get handled using the abstract theory of Dirichlet forms, some old ideas of Moser and Serrin, and recent results by Sturm [29], [30], [31].

The paper is organized as follows. In Section 2, we give a brief introduction to Newtonian spaces and define the Cheeger derivative. In Section 3, we recall the basic theory of Dirichlet forms and the associated abstract heat equation. We also prove some auxiliary results that are needed in order to prove Theorem 1.1. Section 4 contains the proof of Theorem 1.1. Finally, in Section 5, we discuss some situations where the assumptions of Theorem 1.1 can be verified.

## 2 Cheeger-harmonic functions

Let $(X,\mu)$ be a metric measure space, i.e. a set $X$ with metric $d$ and a Borel regular measure $\mu$. A measure $\mu$ is *doubling* if there exists a $C_d > 0$ such that for every $x \in X$ and all $r > 0$,

$$\mu\big(B(x,2r)\big) \le C_d \mu\big(B(x,r)\big).$$

Note that $Q$-regular measures are always doubling.

Following [27], we define the Newtonian (generalized Sobolev) space $N^{1,p}(X)$ in a metric measure space $(X,\mu)$ to be the class of those $p$-integrable (equivalence classes of) functions for which there exists a $p$-integrable upper gradient. We equip $N^{1,p}(X)$ with the pseudonorm

$$||u||_{N^{1,p}(X)} = ||u||_{L^p(X)} + \inf_g ||g||_{L^p(X)},$$

where the infimum is taken over all upper gradients of $u$. We recall from [27] that, under the $p$-Poincaré inequality and doubling assumptions, Lipschitz functions are dense in $N^{1,p}(X)$. Newtonian spaces of open subsets, as well as local Newtonian spaces, are now defined in an obvious manner. We follow [18] and define for a domain $\Omega$ the "Newtonian space with zero boundary values", $N_0^{1,p}(\Omega)$, to be the class of those Newtonian functions $u$ for which $u\chi_{X\setminus\Omega}$ vanishes $p$-quasieverywhere. See, for instance, [16], [18] and [27] for the definition of capacity in metric measure spaces. Cheeger [6] defines Sobolev spaces based on upper gradients in a different way, but his function classes coincide with the corresponding Newtonian spaces for $p > 1$. This was proven in [27].

For a function $u$ there exists a minimal upper gradient $g_u$ (or a "minimal generalized upper gradient", see [6]), minimal in a sense that $g_u \leq g$ almost everywhere for every upper gradient $g$ of $u$. Cheeger showed in [6], assuming the doubling property and the $p$-Poincaré inequality for some $p > 1$, that for Lipschitz functions the minimal upper gradient coincides with the pointwise Lipschitz constant lip defined by

$$\operatorname{lip} u(x) = \liminf_{r\to 0} \sup_{d(x,y)\leq r} \frac{|u(x)-u(y)|}{r}.$$

In this paper, we will use the so-called Cheeger derivatives as our generalization of Euclidean gradients in metric spaces. The important existence results and properties of such derivatives are proved in [6]. The following theorem, proved in [6], gives the most essential information about the differential structure we are dealing with.

**Theorem 2.1.** *Assume that $(X,\mu)$ supports a weak $p$-Poincaré inequality for some $p > 1$ and that $\mu$ is doubling. Then there exists $N > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that the following holds. There exists a countable collection of measurable sets $U_\alpha$, $\mu(U_\alpha) > 0$ for all $\alpha$, and Lipschitz functions $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha : U_\alpha \to \mathbb{R}$ with $1 \leq k(\alpha) \leq N$ such that*

$$\mu\left(X \setminus \cup_{\alpha=1}^{\infty} U_\alpha\right) = 0,$$

*and for all $\alpha$ and $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha$ the following holds:*
*For $f : X \to \mathbb{R}$ Lipschitz, there exists $V_\alpha(f) \subseteq U_\alpha$ such that $\mu(U_\alpha \setminus V_\alpha(f)) = 0$, and Borel functions $b_1^\alpha(x,f), \dots, b_{k(\alpha)}^\alpha(x,f)$ of class $L^\infty$ such that if $x \in V_\alpha$, then*

$$g_{f-a_1X_1^\alpha-\cdots-a_{k(\alpha)}X_{k(\alpha)}^\alpha}(x) = 0$$

*if and only if* $(a_1, \dots, a_{k(\alpha)}) = \big(b_1^\alpha(x,f), \dots, b_{k(\alpha)}^\alpha(x,f)\big)$. *Moreover, for almost every* $x \in U_{\alpha_1} \cap U_{\alpha_2}$, *the "coordinate functions"* $X_i^{\alpha_2}$ *are linear combinations of the* $X_i^{\alpha_1}$*:s.*

By Theorem 2.1, we now have a finite-dimensional $L^\infty$ vector bundle $T^*X$, and for each Lipschitz function $u$ there exists a corresponding $L^\infty$ section of this bundle. We call this section the Cheeger derivative of $u$ and denote it by $Du$. We will repeatedly use the fact that the Cheeger derivative satisfies the Leibniz rule, i.e.

$$D(uv)(x) = u(x)Dv(x) + v(x)Du(x).$$

In addition, the Euclidean norm $|Du|$ of $Du$ is comparable to the minimal upper gradient $g_u$, see [6]. Thus Cheeger-harmonic functions are quasiminimizers in the sense of [19]. Cheeger also proved that the differential operator $D$ can be extended to all functions of the associated Sobolev space. In particular, this holds for the Newtonian space $N^{1,p}(X)$, which coincides with the space considered by Cheeger.

In proving Theorem 1.1, we will use the fact that Cheeger-harmonic functions satisfy the Caccioppoli inequality: there exists $C > 0$, not depending on the harmonic function $u$, such that

$$\int_{B(x,r)} |Du(x)|^2 \, d\mu(x) \le \frac{C}{(R-r)^2} \int_{B(x,R)} u(x)^2 \, d\mu(x) \tag{7}$$

whenever $B(x,r) \subset\subset B(x,R) \subset\subset \Omega$. This estimate is obtained by the same argument as in the Euclidean setting (cf. [17]); apply (4) to $\phi = u\eta^2$, where $\eta$ is a suitable Lipschitz cut-off function and use Hölder's inequality.

# 3 Dirichlet forms and heat kernels

Let $\mathcal{E} : L^2(X,\mu) \times L^2(X,\mu) \to [-\infty, \infty]$ be the bilinear form defined by

$$\mathcal{E}(f,g) = \int_X Df(x) \cdot Dg(x) \, d\mu(x)$$

if $f, g \in N^{1,2}(X)$ and, if $f$ or $g$ does not belong to the class $N^{1,2}(X)$, then

$$\mathcal{E}(f,g) = \infty.$$

Such a bilinear operator is an example of a regular and strongly local Dirichlet form; see [10] and [20]. Corresponding to such a form there exists an infinitesimal generator $A$ which acts on a dense subspace $\mathbf{D}(A)$ of $N^{1,2}(X)$ so that for all $f \in \mathbf{D}(A)$ and each $g \in N^{1,2}(X)$,

$$\int_X g(x)Af(x)\,d\mu(x) = -\mathcal{E}(g,f). \tag{8}$$

Note that if $X = \mathbb{R}^n$ with Lebesgue measure and the operator $D$ is the classical gradient $\nabla$, then $A$ is the Laplacian operator $\triangle$. The following lemma indicates that, even in the abstract setting considered in this paper, the operator $A$ behaves like the Laplacian in the sense that it satisfies the Leibniz theorem of calculus.

**Lemma 3.1.** *If $u$, $v \in N^{1,2}(X)$, and $\phi \in N^{1,2}(X)$ is a bounded Lipschitz function, then*

$$\mathcal{E}(\phi, uv) = \mathcal{E}(\phi u, v) + \mathcal{E}(\phi v, u) + 2\int_X \phi\, Du(x)\cdot Dv(x)\,d\mu(x). \tag{9}$$

*Moreover, if $u, v \in \mathbf{D}(A)$, then we can unambiguously define the measure $A(uv)$ by setting*

$$A(uv) = uAv + vAu + 2Du\cdot Dv. \tag{10}$$

*Proof.* Equation (9) follows from the Leibniz rule obeyed by the Cheeger derivative, and equation (10) is seen to be consistent (that is, it does not create inconsistencies in the event that $uv \in \mathbf{D}(A)$ ) by combining equation (9) with (8). ☐

Also, associated with the Dirichlet form $\mathcal{E}$, there is a semigroup $(T_t)_{t>0}$, acting on $L^2(X)$, with the following properties (see [10], Chapter 1):

1. $T_t \circ T_s = T_{t+s} \qquad \forall\, t,\, s > 0,$
2. $\int_X |T_t f(x)|^2\,d\mu(x) \le \int_X f(x)^2\,d\mu(x) \qquad \forall\, f \in L^2(X,\mu) \quad \forall\, t > 0,$
3. $T_t f \to f$ in $L^2(X,\mu)$ when $t \to 0$,
4. if $f \in L^2(X,\mu)$ satisfies $0 \le f \le C$, then $0 \le T_t f \le C$ for all $t > 0$,
5. if $f \in \mathbf{D}(A)$, then $\frac{1}{t}\big(T_t f - f\big) \to Af$ in $L^2(X,\mu)$ as $t \to 0$, and
6. $AT_t f = \frac{\partial}{\partial t} T_t f \qquad \forall\, t > 0, \quad \forall\, f \in L^2(X,\mu).$

By the above properties, $T_t f$ is the unique solution to the heat problem

$$\begin{cases} u : X \times [0,\infty) \to \mathbb{R}, \\ Au(x,t) = \frac{\partial}{\partial t} u(x,t), \\ u(\cdot,t) \to f(\cdot) \text{ in } L^2(X,\mu) \text{ as } t \to 0. \end{cases}$$

See also [31], Proposition 1.2, for uniqueness of such solutions.
In the Euclidean case, Moser proved a parabolic version of the Harnack inequality for weak solutions of the heat equation

$$\triangle u = \frac{\partial}{\partial t} u.$$

Using the technique developed by Moser in [22], the parabolic Harnack inequality has been proven by many people in different settings; Saloff-Coste in the setting of manifolds ([23] and [24]), Kuwae, Machigashira and Shioya in the setting of Alexandrov spaces ([21]), and Sturm in the setting of complete metric spaces endowed with a doubling measure supporting a 2-Poincaré inequality; see [29], [30], [31] and [32].
A measurable function $p : \mathbb{R} \times X \times X \to [0,\infty]$ is said to be a heat kernel on $X$ if

$$T_t f(x) = \int_X f(y)\, p(t,x,y)\, d\mu(y)$$

for every $f \in L^2(X,\mu)$ and all $t > 0$, and $p(t,x,y) = 0$ for every $t \leq 0$. Sturm proves the existence of the heat kernel under the assumption that the measure on $X$ is doubling and supports a 2-Poincaré inequality; see [31], Proposition 2.3. Under the doubling assumption, it is true, by [29], that the heat kernel is a probability measure; for each $x \in X$ and every $t > 0$

$$\int_X p(t,x,y)\, d\mu(y) = 1.$$

Sturm also proves a Gaussian estimate for the heat kernel in [31], Corollary 4.10 and inequality 4.4; there exist $C$, $C_1$ and $C_2 > 0$ such that

$$p(t,x,y) \leq C \Big(\mu\big(B(x,\sqrt{t})\big)\Big)^{-\frac{1}{2}} \Big(\mu\big(B(y,\sqrt{t})\big)\Big)^{-\frac{1}{2}} e^{\frac{-d(x,y)^2}{C_1 t}}$$

and

$$p(t,x,y) \geq \frac{1}{C} \Big(\mu\big(B(x,\sqrt{t})\big)\Big)^{-\frac{1}{2}} \Big(\mu\big(B(y,\sqrt{t})\big)\Big)^{-\frac{1}{2}} e^{\frac{-d(x,y)^2}{C_2 t}}.$$

Under our assumption of $Q$-regularity of the measure $\mu$, we thus have the heat kernel estimate

$$C^{-1} t^{-\frac{Q}{2}} e^{\frac{-d(x,y)^2}{C_2 t}} \leq p(t,x,y) \leq C t^{-\frac{Q}{2}} e^{\frac{-d(x,y)^2}{C_1 t}}. \tag{11}$$

In what follows, constants denoted by $C_1$ or $C_2$ will refer to the corresponding constants in (11).

It is also seen from the argument of Moser in [22] (while the argument there is phrased for Euclidean spaces, the proof holds in this generality), that for every $x \in X$,

$$\begin{aligned}(12)\qquad &\int_{T_0}^{T_1} \int_{B(x,R_1)} |D_y p(t,x,y)|^2 \, d\mu(y)\, dt \\ &\le C\left[\frac{1}{(R_2-R_1)^2} + \frac{1}{(T_2-T_1)}\right] \int_{T_0}^{T_2} \int_{B(x,R_2)} p(t,x,y)^2 \, d\mu(y)\, dt\end{aligned}$$

whenever $0 < R_1 < R_2$ and $0 \le T_0 < T_1 < T_2$, where C is a constant independent of $R_1$, $R_2$, $T_0$, $T_1$, $T_2$ and $x$.

We will use the estimates (11) and (12) in our proof of Theorem 1.1. In particular, we use inequality (12) to obtain the estimates in the following two lemmas. In what follows, we will make the assumptions of Theorem 1.1, except that the inequality (5) will not be used before Section 4.

**Lemma 3.2.** *For $\mu$-almost every $x \in X$,*

$$\int_0^T \int_X |D_y p(t,x,y)|^2 \, d\mu(y)\, dt \le C_{T,x}$$

*and hence $|D_y p(\cdot,x,\cdot)| \in L^2([0,T]\times X)$ whenever $0 < T$.*

*Proof.* Letting $R_2 = 2R_1$ and $T_2 = 2T_1$, we obtain from inequality (12) the following inequality:

$$\begin{aligned}&\int_0^{T_1} \int_{B(x,R_1)} |D_y p(t,x,y)|^2 \, d\mu(y)\, dt \\ &\le C\left[\frac{1}{R_1^2} + \frac{1}{T_1}\right] \int_0^{2T_1} \int_{B(x,2R_1)} p(t,x,y)^2 \, d\mu(y)\, dt.\end{aligned}$$

Letting $R_1 \to \infty$, we see that

$$\int_0^{T_1} \int_X |D_y p(t,x,y)|^2 \, d\mu(y)\, dt \le C\frac{1}{T_1} \int_0^{2T_1} \int_X p(t,x,y)^2 \, d\mu(y)\, dt.$$

By [31], Proposition 2.3,

$$\int_0^{2T_1} \int_X \int_X p(t,x,y)^2 \, d\mu(y)\, d\mu(x)\, dt < \infty.$$

Hence for $\mu$-a.e. $x \in X$,

$$\int_0^{2T_1} \int_X p(t,x,y)^2 \, d\mu(y) \, dt = C'_{T_1,x} < \infty.$$

Hence

$$\int_0^{T_1} \int_X |D_y p(t,x,y)|^2 \, d\mu(y) \, dt \le C'_{T_1,x}.$$

Letting $T_1 = T$ we obtain the desired inequality. $\square$

**Lemma 3.3.** *If* $0 < T < r^3 < 1$, *then there exists a constant* $C > 0$, *independent of* $T$ *and* $r$, *such that for every* $x \in X$,

$$\int_0^T \int_{B(x,2r)\setminus B(x,r)} |D_y p(t,x,y)|^2 d\mu(y) \, dt \le e^{-CT^{\frac{-1}{3}}}.$$

*Proof.* Let $A$ denote the annulus $B(x,2r) \setminus B(x,r)$. Because the measure is doubling, we can cover $A$ by a fixed finite number of balls $B_i$ of radius $\frac{r}{4}$ so that the balls $\{2B_i\}$ (balls with the same centers and twice the radii) have bounded overlap with the bound independent of $r$; see [25].

Now, by inequality (12),

$$\int_0^T \int_{B_i} |D_y p(t,x,y)|^2 \, d\mu(y) \, dt \le C \left[ \frac{1}{r^2} + \frac{1}{T} \right] \int_0^{2T} \int_{2B_i} p(t,x,y)^2 \, d\mu(y) \, dt.$$

Therefore,

$$\begin{aligned} \int_0^T \int_A |D_y p(t,x,y)|^2 \, d\mu(y) \, dt &\le \sum_i \int_0^T \int_{B_i} |D_y p(t,x,y)|^2 \, d\mu(y) \, dt \\ &\le C \sum_i \left[ \frac{1}{r^2} + \frac{1}{T} \right] \int_0^{2T} \int_{2B_i} p(t,x,y)^2 \, d\mu(y) \, dt \\ &\le C \left[ \frac{1}{r^2} + \frac{1}{T} \right] \int_0^{2T} \int_{A_0} p(t,x,y)^2 \, d\mu(y) \, dt, \end{aligned}$$

where $A_0 = B(x,3r) \setminus B(x, \frac{r}{2})$. By the pointwise estimate (11), this is no more than

$$C \left[ \frac{1}{r^2} + \frac{1}{T} \right] \int_0^{2T} \int_{A_0} t^{-Q} e^{-\frac{r^2}{C_1 t}} \, d\mu(y) \, dt \le C \frac{r^Q}{T} \int_0^{2T} t^{-Q} e^{-Ct^{\frac{-1}{3}}} \, dt, \tag{13}$$

since $t^{\frac{2}{3}} \le T^{\frac{2}{3}} < r^2$, and $\mu$ is $Q$-regular. But now

$$T^{-1} t^{-Q} e^{-Ct^{\frac{-1}{3}}} \le C T^{-Q-1} e^{-CT^{\frac{-1}{3}}} \le e^{-C'T^{\frac{-1}{3}}}$$

for some constant $C' > 0$ and for every $t < 2T$. Thus, since $r < 1$, the right hand side in (13) is no more than $e^{-CT^{\frac{-1}{3}}}$. The claim follows. $\square$

We next prove that the heat kernel is, at least in the weak sense, a fundamental solution to the heat equation. We define test functions as continuous functions $\phi : [0,T] \times X \to \mathbb{R}$ such that for every fixed $t > 0$, $\phi(t,\cdot) = \phi_t(\cdot) \in N^{1,2}(X)$, $|D_y\phi(t,y)| \in L^2([0,T]\times X)$, and for $\mu$-almost every $x \in X$, $\phi(\cdot,x)$ is absolutely continuous on $[0,T]$. Moreover, we assume that there is a constant $\delta_0 = \delta_0(x) > 0$ such that the following Hölder continuity property holds for $\phi$ and the "center point" $x$ of the heat kernel $p(\cdot,x,\cdot)$: there exist $C$ and $\alpha > 0$ such that for every $\delta < \delta_0$ and for all $(t,y) \in [0,\delta]\times B(x,\delta)$,

$$|\phi(t,y) - \phi(0,x)| \le C\delta^\alpha. \tag{14}$$

**Proposition 3.4.** *There exists a constant $K$, independent of $x$ and $\phi$, such that for every test function $\phi_t(y)$ and for almost every $x \in X$,*

$$\begin{aligned} &\int_0^T \int_X \phi_t(y)\, A_y p(t,x,y)\, d\mu(y)\, dt \\ := &-\int_0^T \int_X D\phi_t(y)\cdot D_y p(t,x,y)\, d\mu(y)\, dt \\ = &\int_0^T \int_X \phi_t(y)\, \frac{\partial}{\partial t} p(t,x,y)\, d\mu(y)\, dt + K\phi_0(x). \end{aligned} \tag{15}$$

*Proof.* By [31], for all $t_2 > t_1 > 0$,

$$\int_{t_1}^{t_2} \int_X \phi_t(y)\, A_y p(t,x,y)\, d\mu(y)\, dt = \int_{t_1}^{t_2} \int_X \phi_t(y)\, \frac{\partial}{\partial t}\, p(t,x,y)\, d\mu(y)\, dt. \tag{16}$$

Observe the difference between equation (15) and equation (16); since the distributional measure of $-\frac{\partial p}{\partial t} + A_y p(\cdot,x,\cdot)$ is supported in $\{(0,x,x)\}$, equation (16) does not detect it while equation (15), based on the integral over the region $[0,T]\times X$, does detect the measure $-\frac{\partial p}{\partial t} + A_y p(\cdot,x,\cdot)$. We follow an argument of Serrin [26] in proving equation (15), where the constant $K$ is independent of $x, T$ and $\phi$.

Let $\theta_1$ and $\theta_2$ be two test functions on $[0,T]\times X$ so that both are identically 1 in a neighborhood of the point $(0,x)$. Then $\theta_1 - \theta_2$ vanishes in a neighborhood of $(0,x)$, and hence by equation (16),

$$\begin{aligned} &\int_0^T \int_X \theta_1(t,y) \left(A_y - \frac{\partial}{\partial t}\right) p(t,x,y)\, d\mu(y)\, dt \\ = &\int_0^T \int_X \theta_2(t,y) \left(A_y - \frac{\partial}{\partial t}\right) p(t,x,y)\, d\mu(y)\, dt. \end{aligned}$$

Hence, there exists a constant $K$ so that

$$\int_0^T \int_X \theta(t,y) \left( A_y - \frac{\partial}{\partial t} \right) p(t,x,y) \, d\mu(y) \, dt = K \tag{17}$$

for every test function $\theta : [0,T] \times X \to \mathbb{R}$ which is identically equal to 1 in a neighborhood of $(0,x)$.

Now let $\phi(t,y)$ be any test function. Choose $\theta = \theta_r$ to be a Lipschitz function whose support lies in $[0,2r) \times B(x,2r)$ so that $0 \le \theta_r \le 1$, $|D\,\theta_r| \le \frac{C}{r}$ (recall that $|D\,\theta_r|$ is comparable to the Lipschitz constant of $\theta_r$) and $\theta_r \equiv 1$ on $[0,r) \times B(x,r)$. We split $\phi$ by writing

$$\phi = \theta\phi + (1-\theta)\phi.$$

By equation (16),

$$\int_0^T \int_X \big(1 - \theta_r(t,y)\big)\phi(t,y) \left( A_y - \frac{\partial}{\partial t} \right) p(t,x,y) \, d\mu(y) \, dt = 0$$

i.e.

$$\begin{aligned} & \int_0^T \int_X \phi(t,y) \left( A_y - \frac{\partial}{\partial t} \right) p(t,x,y) \, d\mu(y) \, dt \\ = \; & \int_0^T \int_X \theta_r \phi(t,y) \left( A_y - \frac{\partial}{\partial t} \right) p(t,x,y) \, d\mu(y) \, dt. \end{aligned}$$

Therefore, by equation (17),

$$\begin{aligned} & \left| \int_0^T \int_X \phi(t,y) \left( A_y - \frac{\partial}{\partial t} \right) p(t,x,y) \, d\mu(y) \, dt - K\phi(0,x) \right| \\ = \; & \left| \int_0^T \int_X \big(\phi(t,y) - \phi(0,x)\big) \, \theta_r(t,y) \left( A_y - \frac{\partial}{\partial t} \right) p(t,x,y) \, d\mu(y) \, dt \right| \\ = \; & \left| \int_0^T \int_X D\Big( \big(\phi(t,y) - \phi(0,x)\big) \, \theta_r(t,y) \Big) \cdot D_y p(t,x,y) \, d\mu(y) \, dt \right. \\ + \; & \left. \int_0^T \int_X \big(\phi(t,y) - \phi(0,x)\big) \, \theta_r(t,y) \, \frac{\partial}{\partial t} p(t,x,y) \, d\mu(y) \, dt \right| \\ \le \; & \left| \int_0^T \int_X D\Big( \big(\phi(t,y) - \phi(0,x)\big) \, \theta_r(t,y) \Big) \cdot D_y p(t,x,y) \, d\mu(y) \, dt \right| \\ + \; & \left| \int_0^T \int_X \big(\phi(t,y) - \phi(0,x)\big) \, \theta_r(t,y) \, \frac{\partial}{\partial t} p(t,x,y) \, d\mu(y) \, dt \right|. \end{aligned}$$

Now we estimate the two terms separately. Taking $T > 2r$ and applying the Leibniz rule and Hölder inequality, we see that

$$\begin{aligned}
&\left| \int_0^T \int_X D\Big( \big(\phi(t,y) - \phi(0,x)\big)\, \theta_r(t,y) \Big) \cdot D_y p(t,x,y)\, d\mu(y)\, dt \right| \\
\leq\ & \frac{C}{r} ||\phi - \phi(0,x)||_{L^\infty(B(x,2r)\times[0,2r])} \int_0^{2r} \int_{B(x,2r)} |D_y p(t,x,y)|\, d\mu(y)\, dt \\
+\ & \int_0^{2r} \int_{B(x,2r)} |D_y \phi(t,y)|\, |D_y p(t,x,y)|\, d\mu(y)\, dt \\
\leq\ & \frac{C}{r} \Big( 2r\mu\big(B(x,2r)\big)\Big)^{\frac{1}{2}} \left( \int_0^T \int_{B(x,2r)} |D_y p(t,x,y)|^2\, d\mu(y)\, dt \right)^{\frac{1}{2}} +
\end{aligned}$$

$$+ \left( \int_0^{2r} \int_{B(x,2r)} |D_y \phi(t,y)|^2\, d\mu(y)\, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{B(x,2r)} |D_y p(t,x,y)|^2\, d\mu(y)\, dt \right)^{\frac{1}{2}}.$$

By Lemma 3.2, $Dp(\cdot, x, \cdot) \in L^2([0,T] \times X)$ for almost every $x \in X$. Since $\phi$ is a test function and $r^{-1}\big(r\mu(B(x,2r))\big)^{\frac{1}{2}} < C^{\frac{1}{2}}$ for $r < 1$ ($C$ is the $Q$-regularity constant of $\mu$), both terms tend to 0 as $r \to 0$, for almost every $x \in X$.

In [30], Sturm proved an estimate for $\frac{\partial}{\partial t} p(t,x,y)$ as well:

$$\left| \frac{\partial}{\partial t} p(t,x,y) \right| \leq C t^{-\frac{Q}{2}-1} e^{\frac{-d(x,y)^2}{C_1 t}}.$$

Combining this with the estimate (14) yields (we may assume that $2r < \delta_0$, where $\delta_0$ is as in (14))

$$\begin{aligned}
&\left| \int_0^T \int_X \big(\phi(t,y) - \phi(0,x)\big)\, \theta_r(t,y)\, \frac{\partial}{\partial t} p(t,x,y)\, d\mu(y)\, dt \right| \\
\leq\ & C \int_0^{2r} \int_{B(x,2r)} |\phi(t,y) - \phi(0,x)|\, t^{-\frac{Q}{2}-1} e^{\frac{-d(x,y)^2}{C_1 t}}\, d\mu(y)\, dt \\
=\ & C \int_0^{2r} \int_{B(x,2r)\setminus B(x,t^{\frac{1}{3}})} |\phi(t,y) - \phi(0,x)|\, t^{-\frac{Q}{2}-1} e^{\frac{-d(x,y)^2}{C_1 t}}\, d\mu(y)\, dt \\
+\ & C \int_0^{2r} \int_{B(x,t^{\frac{1}{3}})} |\phi(t,y) - \phi(0,x)|\, t^{-\frac{Q}{2}-1} e^{\frac{-d(x,y)^2}{C_1 t}}\, d\mu(y)\, dt \\
\leq\ & C\mu(B(x,2r))\, ||\phi||_{L^\infty([0,2r]\times B(x,2r))} \int_0^{2r} t^{-\frac{Q}{2}-1} e^{-Ct^{\frac{-1}{3}}}\, dt \\
+\ & C \int_0^{2r} \int_{B(x,t^{\frac{1}{3}})} t^{-\frac{Q}{2}-1+\frac{\alpha}{3}} e^{\frac{-d(x,y)^2}{C_2(lt)}}\, d\mu(y)\, dt
\end{aligned}$$

$$
\begin{aligned}
&\leq && C\mu(B(x,2r))\,||\phi||_{L^\infty([0,2r]\times B(x,2r))}\int_0^{2r} t^{-\frac{Q}{2}-1}e^{-Ct^{\frac{-1}{3}}}\,dt \\
&+ && C\int_0^{2r} t^{-1+\frac{\alpha}{3}}\int_{B(x,t^{\frac{1}{3}})} p(lt,x,y)\,d\mu(y)\,dt \\
&\leq && C\mu(B(x,2r))\,||\phi||_{L^\infty([0,2r]\times B(x,2r))}\int_0^{2r} t^{-\frac{Q}{2}-1}e^{-Ct^{\frac{-1}{3}}}\,dt \\
&+ && C\int_0^{2r} t^{-1+\frac{\alpha}{3}}\,dt \leq C_r||\phi||_{L^\infty([0,r]\times B(x,2r))}+C_r',
\end{aligned}
$$

where $l = C_1/C_2$ and $C_r$, $C_r' \to 0$ as $r \to 0$. Now, letting $r \to 0$, we obtain

$$
\left|\int_0^T\int_X \phi(t,y)\left(A_y - \frac{\partial}{\partial t}\right)p(t,x,y)\,d\mu(y)\,dt - K\phi(0,x)\right| = 0.
$$

Thus equation (15) holds true for every test function $\phi$, for almost every $x \in X$. □

# 4 Proof of Theorem 1.1

*Proof.* Our proof is based on an adaptation of an argument from [5]. Let $u$ be harmonic on a domain $\Omega \subseteq X$ and let $B = B(y_0, 10r) \subset\subset \Omega$. Our proof will bound $|Du(x_0)|$ for every $x_0 \in B(y_0, r) \setminus A$, where $A$ is a set of measure zero depending on $u$ and containing the set of measure zero on which the previous statements about the heat kernel are not valid.

Fix $T < r^3 < 1$. Consider a Lipschitz function $\phi$ on $X$ such that $\phi \equiv 1$ on $B(x_0, r)$, $\operatorname{spt}\phi \subseteq B(x_0, 2r)$, and $|D\phi| \leq \frac{C}{r}$. Let $v = u\phi$ and for every $T > t > 0$, let

$$
w(t,x) := u(x)\phi(x) - T_t(u\phi)(x_0).
$$

Note that $Dw = Dv$ and $Aw = Av$, and by Lemma 3.1,

$$
|Dw|^2 = \frac{1}{2}Aw^2 - w\big(uA\phi + \phi Au + 2Du\cdot D\phi\big)
$$

in the weak sense of measures. Here and in what follows we extend $A$ formally to all of $N^{1,2}(X)$ by defining

$$
\int_X vAu = -\int_X Du\cdot Dv = \int_X uAv. \tag{18}
$$

Notice that this is consistent with (8).

Let $m(t) = T_t(u\phi)(x_0)$. Then $\frac{\partial}{\partial t}w^2 = 2w\frac{\partial}{\partial t}w = -2w\,m'(t)$ (we know from [10] that $m(t)$ is differentiable in the $t$- variable, and hence $m'(t)$ makes sense here). Therefore,

$$|Dw|^2 = \frac{1}{2}\left(A + \frac{\partial}{\partial t}\right) w^2 - w\big(uA\phi + \phi Au + 2Du \cdot D\phi - m'(t)\big)$$

in the weak sense. Hence,

$$\int_0^t \int_X |Dw|^2(s,x)\, p(s,x_0,x)\, d\mu(x)\, ds \tag{19}$$

$$\begin{aligned} = \;& \frac{1}{2}\int_0^t \int_X \left(A + \frac{\partial}{\partial s}\right) w^2(s,x)\, p(s,x_0,x)\, d\mu(x)\, ds \\ - \;& \int_0^t \int_X w(s,x)\big(u(x)A\phi(x) + \phi(x)Au(x) + 2Du(x)\cdot D\phi(x) - m'(s)\big) \\ & p(s,x_0,x)\, d\mu(x)\, ds. \end{aligned}$$

Observe that

$$\begin{aligned} & \int_0^t \int_X w(s,x)\, m'(s)\, p(s,x_0,x)\, d\mu(x)\, ds \\ = \;& \int_0^t m'(s) \int_X \big(u(x)\phi(x) - T_s(u\phi)(x_0)\big)\, p(s,x_0,x)\, d\mu(x)\, ds \\ = \;& \int_0^t m'(s)\, T_s(u\phi)(x_0) \left(1 - \int_X p(s,x_0,x)\, d\mu(x)\right) ds. \end{aligned}$$

Recall that for every $s > 0$ and $x_0 \in X$,

$$\int_X p(s,x_0,x)\, d\mu(x) = 1.$$

Hence

$$\int_0^t \int_X w(s,x)\, m'(s)\, p(s,x_0,x)\, d\mu(x)\, ds = 0. \tag{20}$$

Thus (19) holds without $m'(s)$. We will estimate the remaining terms in (19) separately.

First, integrating by parts and using (18), we see that

$$
\begin{aligned}
& \int_0^t \int_X \left(A + \frac{\partial}{\partial s}\right) w^2(s,x)\, p(s,x_0,x)\, d\mu(x)\, ds \\
= & \int_0^t \int_X w^2(s,x)\, A_x p(s,x_0,x)\, d\mu(x)\, ds \\
- & \int_0^t \int_X w^2(s,x)\, \frac{\partial}{\partial s} p(s,x_0,x)\, d\mu(x)\, ds \\
+ & \int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) - \int_X w^2(0,x)\, p(0,x_0,x)\, d\mu(x).
\end{aligned}
$$

Next we wish to use Proposition 3.4 for $\phi_t(y) = w^2(t,y)$. In order to do this, we have to show that $w^2(t,y)$ can be used as a test function. Our function $w^2(t,\cdot)$ may not belong to $L^2(X,\mu)$ for a fixed $t$, but this clearly does not cause any problems. The other test function properties, required in Proposition 3.4, hold for $w^2(t,y)$, and only the Hölder continuity estimate (14) is not obvious from the definition of $w^2(t,y)$. We use the local Hölder continuity of $u$ to show that (14) holds for $w^2(t,y)$. Note that $w^2(0,x_0) = 0$.
Let $\delta_0$ be chosen so that $u$ is Hölder continuous, with exponent $\alpha$, in $B(x_0, \delta_0^{1/3})$. Fix $\delta < \delta_0$ and suppose that $(t,x) \in [0,\delta] \times B(x_0,\delta)$. Then

$$
\begin{aligned}
|w(t,x)| &= |u(x)\phi(x) - T_t(u\phi)(x_0)| \\
&= |u(x)\phi(x) - u(x_0)\phi(x_0) + u(x_0)\phi(x_0) - T_t(u\phi)(x_0)| \qquad (21) \\
&\le C\delta^\alpha + \int_X |u(x_0)\phi(x_0) - u(x)\phi(x)|\, p(t,x_0,x)\, d\mu(x) \\
&= C\delta^\alpha + \int_{B(x_0,t^{\frac{1}{3}})} |u(x_0)\phi(x_0) - u(x)\phi(x)|\, p(t,x_0,x)\, d\mu(x) \\
&+ \int_{X\setminus B(x_0,t^{\frac{1}{3}})} |u(x_0)\phi(x_0) - u(x)\phi(x)|\, p(t,x_0,x)\, d\mu(x) \le C\delta^\alpha \\
&+ Ct^{\frac{\alpha}{3}} + C||u||_{L^\infty(B(x_0,8r))} \int_{X\setminus B(x_0,t^{\frac{1}{3}})} t^{\frac{-Q}{2}} e^{\frac{-d(x,x_0)^2}{2C_1 t}} e^{\frac{-d(x,x_0)^2}{2C_1 t}}\, d\mu(x) \\
&\le C\delta^{\frac{\alpha}{3}} + C||u||_{L^\infty(B(x_0,8r))} e^{-Ct^{\frac{-1}{3}}} \int_{X\setminus B(x_0,t^{\frac{1}{3}})} (lt)^{\frac{-Q}{2}} e^{\frac{-d(x,x_0)^2}{C_2(lt)}}\, d\mu(x) \\
&\le C\delta^{\frac{\alpha}{3}} + C||u||_{L^\infty(B(x_0,8r))} e^{-Ct^{\frac{-1}{3}}} \int_X p(lt,x_0,x)\, d\mu(x) \le C\delta^{\frac{\alpha}{3}}.
\end{aligned}
$$

Here $l = 2C_1/C_2$. Thus, (14) holds for $w$ as a test function, which implies that it holds for $w^2$ as a test function as well. Now we can use equation (15) for $w^2(t,y)$, and we have

$$\begin{aligned} &\int_0^t \int_X \left(A + \frac{\partial}{\partial s}\right) w^2(s,x)\, p(s,x_0,x)\, d\mu(x)\, ds \\ = \;& K w^2(0,x_0) + \int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) \\ - \;& \int_X w^2(0,x)\, p(0,x_0,x)\, d\mu(x). \end{aligned}$$

Now

$$\begin{aligned} (22) \qquad & \int_X w^2(s,x)\, p(s,x_0,x)\, d\mu(x) = \int_X (u\phi)^2(x)\, p(s,x_0,x)\, d\mu(x) \\ + \;& m(s)^2 \int_X p(s,x_0,x)\, d\mu(x) - 2\int_X m(s)\, (u\phi)(x)\, p(s,x_0,x)\, d\mu(x) \\ = \;& T_s(u\phi)^2(x_0) + m(s)^2 - 2m(s)\, T_s(u\phi)(x_0) = T_s(u\phi)^2(x_0) - m(s)^2, \end{aligned}$$

and thus

$$\lim_{s\to 0} \int_X w^2(s,x)\, p(s,x_0,x)\, d\mu(x) = \lim_{s\to 0} T_s(u\phi)^2(x_0) - \big(T_s(u\phi)(x_0)\big)^2,$$

provided the last limit exists. On the other hand, the Hölder continuity estimate (21) implies that

$$\lim_{s\to 0} T_s(u\phi)^2(x_0) = \big(u(x_0)\phi(x_0)\big)^2.$$

Similarly

$$\lim_{s\to 0} \big(T_s(u\phi)(x_0)\big)^2 = u(x_0)^2\phi(x_0)^2.$$

Hence

$$\lim_{s\to 0} \int_X w^2(s,x)\, p(s,x_0,x)\, d\mu(x) = 0.$$

Therefore, as $w(0,x_0) = 0$, we see that

(23)

$$\int_0^t \int_X \left(A + \frac{\partial}{\partial s}\right) w^2(s,x)\, p(s,x_0,x)\, d\mu(x)\, ds = \int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x).$$

Secondly, since $u$ is harmonic on $\operatorname{spt}\phi$,

$$\begin{aligned} (24) \qquad & \int_0^t \int_X w(s,x)\, \phi(x)\, Au(x)\, p(s,x_0,x)\, d\mu(x)\, ds \\ = \;& -\int_0^t \int_X D\big(w(s,x)\, \phi(x)\, p(s,x_0,x)\big) \cdot Du\, d\mu(x)\, ds = 0. \end{aligned}$$

Note that (24) is well-defined, since $w(s,\cdot)\,\phi(\cdot)\,p(s,x_0,\cdot)$ is in $N^{1,2}(X)$. This can be seen as follows: let $(h_i)$ and $(v_i)$ be sequences of Lipschitz-functions converging in $N^{1,2}(E)$ to $w(s,\cdot)$ and $p(s,x_0,\cdot)$, respectively, where $E$ is a bounded domain. Then, by the Leibniz rule and the continuity of $w(s,\cdot)$ and $p(s,x_0,\cdot)$, $(h_i\phi v_i)$ is a Cauchy sequence in $N^{1,2}(E)$ converging to $w\phi p$ in $L^2(E)$. Now the $N^{1,2}(E)$-limit of $(h_i\phi v_i)$ exists and has to be $w\phi p$.

Thirdly, by the fact that $\phi \equiv 1$ on $B(x_0,r)$ and $\operatorname{spt}\phi \subseteq B(x_0,2r)$,

$$\begin{aligned}
&\left|\int_0^t \int_X w(s,x)\,Du(x)\cdot D\phi(x)\,p(s,x_0,x)\,d\mu(x)\,ds\right| \\
\leq\ & C_r\,||w||_{L^\infty([0,t]\times B(x_0,2r))}\int_0^t\int_{B(x_0,2r)\setminus B(x_0,r)} |Du(x)|^2\,p(s,x_0,x)\,d\mu(x)\,ds \\
\leq\ & C_r\,||w||_{L^\infty([0,t]\times B(x_0,2r))}\int_0^t\int_{B(x_0,2r)\setminus B(x_0,r)} |Du(x)|^2\,s^{\frac{-Q}{2}}e^{\frac{-Cr^2}{s}}\,d\mu(x)\,ds \\
\leq\ & C_r\,||u||_{L^\infty(B(x_0,2r))}e^{-Ct^{\frac{-1}{3}}}\int_{B(x_0,2r)\setminus B(x_0,r)} |Du(x)|^2\,d\mu(x),
\end{aligned}$$

where we used the fact that $s^{\frac{-Q}{2}}e^{\frac{-Cr^2}{s}} \leq Ce^{-Ct^{\frac{-1}{3}}}$ for all $s<t<T<r^3$, and the fact that by the Markov property (property 4 in Section 3) of the heat semigroup $(T_t)_{t>0}$, $||w||_{L^\infty([0,t]\times B(x_0,2r))} \leq 2||u||_{L^\infty(B(x_0,2r))}$. Now, by the Caccioppoli inequality (7) for harmonic functions,

$$\begin{aligned}
(25)\qquad &\left|\int_0^t \int_X w(s,x)\,Du(x)\cdot D\phi(x)\,p(s,x_0,x)\,d\mu(x)\,ds\right| \\
\leq\ & C_r||u||^2_{L^\infty(B(x_0,4r))}e^{-Ct^{\frac{-1}{3}}}.
\end{aligned}$$

We finally estimate the remaining term of equation (19). Using Lemma 3.3,

$$\begin{aligned}
(26)\qquad &\left|\int_0^t \int_X w(s,x)\,u(x)\,A\phi(x)\,p(s,x_0,x)\,d\mu(x)\,ds\right| \\
=\ & \left|\int_0^t \int_X D\big(w(s,x)\,u(x)\,p(s,x_0,x)\big)\cdot D\phi(x)\,d\mu(x)\,ds\right| \\
\leq\ & \left|\int_0^t \int_X u(x)\,p(s,x_0,x)\,Dw(s,x)\cdot D\phi(x)\,d\mu(x)\,ds\right| \\
+\ & \left|\int_0^t \int_X w(s,x)\,p(s,x_0,x)\,Du(x)\cdot D\phi(x)\,d\mu(x)\,ds\right|
\end{aligned}$$

$$\begin{aligned}
&+ \left|\int_0^t \int_X w(s,x)\, u(x)\, Dp(s,x_0,x) \cdot D\phi(x)\, d\mu(x)\, ds\right| \\
&\leq C_r ||u||_{L^\infty(B(x_0,2r))} e^{-Ct^{\frac{-1}{3}}} \left(\int_{B(x_0,2r)\setminus B(x_0,r)} |Du(x)|^2\, d\mu(x)\right)^{\frac{1}{2}} \\
&+ C_r ||u||_{L^\infty(B(x_0,2r))} e^{-Ct^{\frac{-1}{3}}} \left(\int_{B(x_0,2r)\setminus B(x_0,r)} |Du(x)|^2\, d\mu(x)\right)^{\frac{1}{2}} \\
&+ C_r ||u||_{L^\infty(B(x_0,2r))} e^{-Ct^{\frac{-1}{3}}} \left(\int_0^t \int_{B(x_0,2r)\setminus B(x_0,r)} |Dp(s,x_0,x)|^2\, d\mu(x)\, dt\right)^{\frac{1}{2}} \\
&\leq C_r ||u||_{L^\infty(B(x_0,4r))} e^{-Ct^{\frac{-1}{3}}} + C_r ||u||_{L^\infty(B(x_0,2r))} e^{-Ct^{\frac{-1}{3}}} \\
&\leq C_r ||u||_{L^\infty(B(x_0,4r))} e^{-Ct^{\frac{-1}{3}}}.
\end{aligned}$$

From [4] we know that there exists a constant $C$, independent of $r, x_0$ and $u$, such that

$$||u||_{L^\infty(B(x_0,4r))} \leq Cr^{-Q} ||u||_{L^2(B(x_0,8r))}. \tag{27}$$

Combining inequalities (19), (20), (23), (24), (25), (26) and (27), we see that

$$\begin{aligned}
0 &\leq \int_0^t \int_X |Dw(s,x)|^2\, p(s,x_0,x)\, d\mu(x) ds \\
&\leq \frac{1}{2} \int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) \\
&+ C_r ||u||^2_{L^2(B(x_0,8r))} e^{-Ct^{\frac{-1}{3}}}.
\end{aligned} \tag{28}$$

For $T > t > 0$, let

$$\begin{aligned}
J(t) &:= \frac{1}{t} \int_0^t \int_X |D(u\phi)(x)|^2\, p(s,x_0,x)\, d\mu(x)\, ds \\
&= \frac{1}{t} \int_0^t \int_X |Dw(s,x)|^2\, p(s,x_0,x)\, d\mu(x)\, ds.
\end{aligned}$$

Then, by inequality (28) (here we use again the fact that $t^{-\alpha} e^{-Ct^{\frac{-1}{3}}} \leq e^{-C' t^{\frac{-1}{3}}}$ for $\alpha > 0$, $t < T$),

$$\begin{aligned}
0 &\leq J(t) \leq \frac{1}{2t} \int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) \\
&+ C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))}.
\end{aligned} \tag{29}$$

Moreover, note that

$$\begin{aligned}
&\frac{d}{dt}J(t) \\
= \ & -\frac{1}{t}J(t) + \frac{1}{t}\int_X |D(u\phi)(x)|^2\, p(t,x_0,x)\, d\mu(x) \\
\geq \ & \frac{-1}{2t^2}\int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) - C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))} \\
+ \ & \frac{1}{t}\int_X |Dw(x)|^2\, p(t,x_0,x)\, d\mu(x) \\
\geq \ & \frac{1}{t}\left(\int_X |Dw(x)|^2\, p(t,x_0,x)\, d\mu(x) - \frac{1}{2t}\int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x)\right) \\
- \ & C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))}.
\end{aligned}$$

Now, for fixed $t$, either

$$\int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) \leq 2t\int_X |Dw(x)|^2\, p(t,x_0,x)\, d\mu(x)$$

or

$$\int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) > 2t\int_X |Dw(x)|^2\, p(t,x_0,x)\, d\mu(x).$$

In the first case,

$$\frac{d}{dt}J(t) \geq -C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))}. \tag{30}$$

In the second case, we use (21) and (27) in order to obtain the estimate

$$\begin{aligned}
&\int_X |Dw(x)|^2\, p(t,x_0,x)\, d\mu(x) \leq \frac{1}{2t}\int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) \\
\leq \ & \frac{1}{2t}\int_{B(x_0,t^{\frac{1}{3}})} w^2(t,x)\, p(t,x_0,x)\, d\mu(x) + \frac{1}{2t}\int_{X\setminus B(x_0,t^{\frac{1}{3}})} w^2(t,x)\, p(t,x_0,x)\, d\mu(x) \\
\leq \ & C_r t^{\frac{2}{3}\alpha-1} + \frac{1}{t}C_r ||u||^2_{L^\infty(B(x_0,8r))}\int_{X\setminus B(x_0,t^{\frac{1}{3}})} t^{\frac{-Q}{2}} e^{\frac{-d(x_0,x)^2}{2C_1 t}} e^{\frac{-d(x_0,x)^2}{C_2(lt)}}\, d\mu(x) \\
\leq \ & C_r t^{\frac{2}{3}\alpha-1} + C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))},
\end{aligned}$$

where $l = 2C_1/C_2$ and $\alpha > 0$. Note that

$$\begin{aligned}
\int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x) \ &= \ \int_X \big(u(x)\phi(x)\big)^2\, p(t,x_0,x)\, d\mu(x) \\
&- \ \left(\int_X u(x)\phi(x)\, p(t,x_0,x)\, d\mu(x)\right)^2.
\end{aligned}$$

Thus, using inequality (5), we have

$$\begin{aligned}
\frac{d}{dt}J(t) &\geq \frac{1}{t}\left(\int_X |Dw(x)|^2\, p(t,x_0,x)\, d\mu(x) - \frac{1}{2t}\int_X w^2(t,x)\, p(t,x_0,x)\, d\mu(x)\right) \\
(31)\quad &- C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))} \geq -C\int_X |Dw(x)|^2\, p(t,x_0,x)\, d\mu(x) \\
&- C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))} \geq -C_r e^{-Ct^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))} \\
&- C_r t^{\frac{2}{3}\alpha-1} - C_r e^{-Ct^{\frac{-1}{3}}} ||u||_{L^2(B(x_0,8r))}
\end{aligned}$$

in the second case. Since (30) implies (31), inequality (31) holds in either of the two cases.

By the fundamental theorem of calculus,

$$\begin{aligned}
J(0) &:= \lim_{t\to 0^+} \frac{1}{t}\int_0^t \int_X |D(u\phi)(x)|^2\, p(s,x_0,x)\, d\mu(x)\, ds \\
&= \lim_{s\to 0^+} \int_X |D(u\phi)(x)|^2\, p(s,x_0,x)\, d\mu(x) = \lim_{s\to 0^+} T_s|D(u\phi)(x_0)|^2 \\
&= |D(u\phi)(x_0)|^2 = |Du(x_0)|^2
\end{aligned}$$

(for almost every $x_0 \in X$). Because $J(T) - J(0) = J(T) - |Du(x_0)|^2 = \int_0^T \frac{d}{dt}J(t)\, dt$, combining inequalities (29) and (31) gives

$$\begin{aligned}
|Du(x_0)|^2 &= J(T) - \int_0^T J'(t)\, dt \leq \frac{1}{2T}\int_X w^2(T,x)\, p(T,x_0,x)\, d\mu(x) \\
&+ C_r e^{-CT^{\frac{-1}{3}}} ||u||^2_{L^2(B(x_0,8r))} + C_r\int_0^T ||u||^2_{L^2(B(x_0,8r))} e^{-Ct^{\frac{-1}{3}}} \\
&+ t^{\frac{2}{3}\alpha-1}\, dt \leq C_{T,r}||u||^2_{L^2(B(x_0,8r))}.
\end{aligned}$$

Thus we have an a priori local bound on the Cheeger derivative of $u$;

$$|Du(x_0)| \leq C_{T,r}||u||^2_{L^2(B(x_0,8r))}.$$

Now by a local chaining argument using chains of balls converging to $x, y \in \Omega$ and using the local 2-Poincaré inequality, we see that

$$|u(x) - u(y)| \leq C_{T,r}||u||_{L^2(B(x_0,10r))} d(x,y)$$

whenever $x, y \in \Omega$ satisfy $B(x, 10d(x,y)) \subset\subset \Omega$. Actually, this estimate is first only obtained a.e., but it then extends to hold for all points. For this reasoning we need to know that the metric space $X$ is quasiconvex, i.e. that there exists $C > 0$ such that for every $x, y \in X$ there exists a closed curve joining $x$ and $y$ whose length does not exceed $Cd(x,y)$. By Semmes' Theorem we know that this is true in our case, see for instance [6]. Thus $u$ is locally Lipschitz continuous on $\Omega$, proving the theorem. $\square$

# 5 Curvature conditions

Let the assumptions of Theorem 1.1 be valid, except for the Sobolev inequality (5). Then we can define the "square of the length of the gradient" (opérateur carré du champ)

$$\Gamma(u,v)(x) = \frac{1}{2}\big(A(uv)(x) - u(x)Av(x) - v(x)Au(x)\big)$$

pointwise whenever $u$, $v$ and $uv \in D(A)$. By Lemma 3.1, $\Gamma(u,v)(x)$ coincides with $Du(x) \cdot Dv(x)$ in our situation. Let us assume that we have a dense subspace $\mathcal{S} \subset N^{1,2}(X)$ such that we can also define, for all $u, v \in \mathcal{S}$, the operator

$$\Gamma_2(u,v)(x) = \frac{1}{2}\Big(A\big(\Gamma(u,v)\big)(x) - \Gamma(u,Av)(x) - \Gamma(v,Au)(x)\Big)$$

pointwise. Then, following [3], we say that the diffusion semigroup has curvature greater or equal to some $\kappa \in \mathbb{R}$, if for every $u \in \mathcal{S}$ and $x \in X$,

$$\Gamma_2(u,u) \geq \kappa\Gamma(u,u). \tag{32}$$

Now we have the following result which is part of Proposition 2.1 of [2].

**Proposition 5.1.** *Assume that the subspace $\mathcal{S}$ is as above, and that the diffusion semigroup has curvature greater or equal to some $\kappa \in \mathbb{R}$. Then, for every $u \in N^{1,2}(X)$, each $t > 0$, and for almost every $x_0 \in X$,*

$$\int_X \big(u(x) - T_t u(x_0)\big)^2 p(t,x_0,x)\,d\mu(x) \leq \frac{1-e^{-2\kappa t}}{\kappa} \int_X |Du(x)|^2\, p(t,x_0,x)\,d\mu(x). \tag{33}$$

*When $\kappa = 0$, $\frac{1-e^{-2\kappa t}}{\kappa}$ is replaced by $2t$.*

It is easy to check, using an argument as in (22), that inequality (33) implies the Sobolev inequality (5) when $t < t_0$ for some $t_0$ and when the constant $C$ is sufficiently large. Consequently, our Sobolev inequality (5) holds when the curvature of the diffusion semigroup is bounded from below, in particular for Riemannian manifolds with Ricci curvature bounded from below. Here the generator $A$ is the Laplace-Beltrami operator. In this case, the lower curvature bound of the diffusion semigroup is equivalent to the lower Ricci curvature bound, see [3], [2] for a discussion. Here $\mathcal{S} = C^\infty \cap L^2(X)$.

Let us briefly comment on Euclidean spaces with smooth Ahlfors-regular weights. When the weight $w \in C^2(\mathbb{R}^n)$, we can calculate $\Gamma(u,u)$ and $\Gamma_2(u,u)$ for every $u \in C_0^\infty(\mathbb{R}^n)$, and we have

$$\Gamma_2(u,u)(x) = A(|\nabla u|^2)(x) - 2\nabla u(x) \cdot \nabla(Au)(x) \geq \kappa |\nabla u(x)|^2 = \kappa \Gamma(u,u)(x)$$

if $\frac{1}{w^2}\big(|\nabla w(x)|^2 - w(x)\Delta w(x)\big) \geq \kappa$. Thus the curvature is locally bounded from below whenever $w \in C^2(\mathbb{R}^n)$ is a locally Ahlfors-regular weight. One could also consider weights in those Riemannian manifolds mentioned above.

Let us return to one of the examples discussed in the introduction. Define $X_\alpha \subset \mathbb{R}^2$ by

$$X_\alpha = \left\{(x,y) = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \,:\, \varphi \in [0,\alpha],\, r \geq 0\right\}.$$

Restrict the Euclidean metric and the Lebesgue measure to $X_\alpha$. Recall from the introduction that, when $\pi < \alpha < 2\pi$, Cheeger-harmonic functions need not be locally Lipschitz. When $\alpha = \pi$, $X_\alpha$ is the closed upper half-plane. We wish to show that the assumptions of Theorem 1.1 are satisfied in this case, as stated in the introduction. For this, it suffices to show that those $L^2$-integrable $C^\infty$-functions $u$, for which $\partial_2 u$ vanishes in some $\epsilon$-neighborhood of $\{y = 0\}$, are dense in $N^{1,2}(\overline{\mathbb{R}_2^+})$. This is sufficient because, for them, the generator $A$ coincides with the Laplacian $\Delta$, and hence we have the lower curvature bound zero. Here the Newtonian functions are the extensions of functions in $N^{1,2}(\mathbb{R}_2^+) = W^{1,2}(\mathbb{R}_2^+)$ to the closure $\{y = 0\}$.

So, let $u \in W^{1,2}(\mathbb{R}_2^+)$. Because $\int_{\mathbb{R}} |\partial_1 u(x,\cdot)|^2\, dx$ is integrable in $\mathbb{R}^+$, we can use the Fubini Theorem to find a decreasing sequence $(\epsilon_i)$ of positive real numbers converging to zero such that

$$\epsilon_i \int_{\mathbb{R}} |\partial_1 u(x,\epsilon_i)|^2\, dx \to 0$$

as $i \to \infty$. Now, for each $i$, define $u_i$ by setting

$$u_i(x,y) = \begin{cases} u(x,y), & y \geq \epsilon_i \\ u(x,\epsilon_i), & y < \epsilon_i. \end{cases}$$

Then $u_i \in W^{1,2}(\mathbb{R}_2^+)$. Next take a convolution approximation $u_{\epsilon_i}$ to $u_i$ in $y \geq \frac{1}{2}\epsilon_i$ so that the smoothing kernel has support in $B(0, \frac{1}{4}\epsilon_i)$, and extend $u_{\epsilon_i}$ to the rest of $\mathbb{R}_2^+$ in the obvious manner. Then the sequence $(u_{\epsilon_i})$ converges to $u$ in the $W^{1,2}(\mathbb{R}_2^+)$-norm, $u_{\epsilon_i} \in C^\infty(\mathbb{R}_2^+) \cap W^{1,2}(\mathbb{R}_2^+)$, and $\partial_2 u_{\epsilon_i}$ vanishes in an $\frac{\epsilon_i}{2}$-neighborhood of the $x$-axis. Thus we have the desired dense subspace of $W^{1,2}(\mathbb{R}_2^+)$, and we can apply Proposition 5.1. Lipschitz continuity also holds in $X_\alpha$ when $\alpha < \pi$.

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