MILTON'S CONJECTURE ON THE REGULARITY OF SOLUTIONS TO ISOTROPIC EQUATIONS

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Abstract. We present examples showing that the threshold for the integrability of the gradient of solutions to planar isotropic equations is $\frac{2K}{K-1}$. The main tools are p-laminates and Beltrami Operators.

1. Introduction

In this paper we investigate the regularity of solutions to the isotropic equation

$$\operatorname{div}\big(\rho(z)\nabla u(z)\big) = 0, \qquad z \in Q \tag{1.1}$$

where Q is a cube in the plane $\mathbb{R}^2$, $u \in W^{1,2}(Q,\mathbb{R})$ and $\rho \in L^\infty(Q,[\frac{1}{K},K])$ is real valued. Through the whole paper K is an arbitrary constant greater than one. In [PS] Piccinini and Spagnolo proved that the solutions of (1.1) are locally Hölder continuous with exponent $\frac{4}{\pi}Arctan(\frac{1}{K})$. This shows that in this regard isotropic equations differ drastically from general linear elliptic equations,

$$\operatorname{div}\big(\sigma(z)\nabla u(z)\big) = 0, \qquad z \in Q, \tag{1.2}$$

where $\sigma(z) \in \mathbf{M}^{2\times 2}$ with $\sigma(z) = \sigma(z)^t$ and $\frac{1}{K}|\xi|^2 \leq \langle \xi, \sigma(z)\xi\rangle \leq K|\xi|^2$ for every $\xi \in \mathbb{R}^2$ and a.e.z in Q. As in the isotropic case we require $u \in W^{1,2}(Q,\mathbb{R})$.

It goes back to Morrey [Mo] that the threshold for the Hölder regularity of the solution in the anisotropic case is only $\frac{1}{K}$. Thus, in terms of Hölder continuity, solutions to the isotropic equation belong to a more regular class of functions than in the general case.

The regularity of solutions to a P.D.E is also studied in terms of the integrability of the gradient. By the Sobolev Embedding Theorem if the gradient is integrable with exponent $p > 2$, the function is Hölder continuous with exponent $1-\frac{2}{p}$. A crucial phenomenon in elliptic PDE is that weak solutions which are a priory only in $W^{1,2}_{loc}(Q)$ automatically belong to $W^{1,p}_{loc}(Q)$ for some $p > 2$ (see [Bo] for $n = 2$ and [Me] for arbitrary n). More precisely the gradients satisfy the so-called reverse

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Hölder inequalities, that is, for every ball $B(a,r)$ such that $B(a,2r)$ is compactly included in Q there exists a constant $C(K,p)$ such that

$$(1.3) \qquad \int_{B(a,r)} |\nabla u|^p dz \leq C(K,p)\Big(\int_{B(a,2r)} |\nabla u(z)|^2 dz\Big)^{\frac{p}{2}}.$$

In understanding the properties of a given subclass of elliptic PDE is important to find the supremum of those exponents p for which gradients of weak solutions satisfy reverse Hölder inequalities. This supremum is called the threshold exponent of that class. The value of the threshold exponent is relevant in applications, because it measures "the highest possible concentration of the field" (See, for example, [Mi] [LN] and the references therein for the relation of the threshold to several questions in physics.)

The threshold for anisotropic equations like (1.2) in the plane was established by Astala, Leonetti and Nesi to be equal to $\frac{2K}{K-1}$. The result was obtained by Leonetti and Nesi in [LN] as a consequence of the higher integrability results for gradients of quasiregular mappings due to Astala [As1]. In the proofs in [As1] the complex structure of $\mathbb{R}^2$ is essential and hence, the higher dimensional case remains as a challenging open problem. We see that in the anisotropic case the threshold for the integrability of the gradient $\frac{2K}{K-1}$ and the Sobolev embedding yield the "right" Hölder regularity, $\frac{1}{K}$ (up to the end point). In fact, the example showing the sharpness of both results is the same: the real part of the radial stretching $f(z) = z|z|^{\frac{1}{K}-1}$.

The search for the threshold for the integrability of the gradient in the isotropic case has also drawn the attention of the researchers, see for example [AN] and [LN]. A natural question is if also here the bounds for the Hölder continuity and the integrability of the gradient are related by the Sobolev embedding Theorem.

The situation in the anisotropic case and the results in [PS] indicate that the higher integrability threshold for the isotropic equations might be larger than $\frac{2K}{K-1}$. However, the intuition coming from physics, led Graeme Milton to conjecture the opposite. The underlying physical problem relies on the fact that the matrix valued function σ in (1.2) can be thought of as to express the electric conductivity properties of certain material. In [Mi] Milton suggested conductivity matrices $\rho_j I$ where the concentration of the related fields ∇u_j should be high enough to prevent any uniform integrability better than $\frac{2K}{K-1}$. However since his remarkable work appeared in 1986, a mathematical proof of this fact was lacking to the best of our knowledge. In this work we rigorously prove Milton assertion showing that his physical intuition led him to the right answer. We present the result in the following form.

Theorem 1.1. *Let $K > 1$. There exist sequences of functions $\{\rho_j\} \in L^\infty(Q, \{K, \frac{1}{K}\})$ and $\{u_j\} \in W^{1,2}(Q, \mathbb{R})$ with $\|u_j\|_{W^{1,2}} \leq 1$, such that*

$$\mathit{div}\big(\rho_j(z)\nabla u_j(z)\big) = 0, \qquad a.e\ z \in Q, \tag{1.4}$$

and for every compact set R of positive measure contained in Q

$$\lim_{j\to\infty} \int_R |\nabla u_j(z)|^{\frac{2K}{K-1}} dz = \infty.$$

In fact our construction gives that the sequence $\{u_j\}$ is uniformly bounded in $W^{1,p}(Q, \mathbb{R})$ with $1 \leq p < \frac{2K}{K-1}$. This must be the case, since the result of Astala-Leonetti-Nesi states that the bounds on the $W^{1,2}$ norm imply bounds on the $W^{1,p}$ norm for the above range of p.

Weak reverse Hölder inequalities imply also sharp regularity results for the Dirichlet problem

$$\text{div }\big(\sigma(z)\nabla u(z)\big) = \text{div } F \qquad z \in Q, \tag{1.5}$$

where σ is as in (1.2) and $F \in L^p$ (see [IS]). In this regard the theorem 1.1 is easily seen to imply the following corollary.

Corollary 1.2. *There exist functions $\rho \in L^\infty(Q, \{K, \frac{1}{K}\})$, $u \in W^{1,2}(Q, \mathbb{R})$ and a vector field $F \in L^\infty(Q, \mathbb{R}^2)$ such that*

$$\mathit{div}\ (\rho(z)\nabla u(z)) = \mathit{div}\ F, \tag{1.6}$$

and

$$\int_Q |\nabla u(z)|^{\frac{2K}{K-1}} dz = \infty.$$

Our approach to study the equation (1.1) is based upon considering the flow $\rho(z)\nabla u(z)$ as a rotated potential. Most of the notation used below is standard and explained in section 2. However we need to introduce immediately the following sets. Let us associate to every positive number ρ a 2-dimensional subspace E_ρ of the space of 2×2 matrices $\mathbf{M}^{2\times 2}$ as follows:

$$E_\rho = \{\begin{pmatrix} X \\ J\rho X \end{pmatrix} \text{ where } X \in \mathbb{R}^2 \text{ and } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}. \tag{1.7}$$

Denote

$$E = E_K \cup E_{K^{-1}}. \tag{1.8}$$

Then it can be seen that u is a solution to a linear isotropic equation (1.1) with $\rho(z) = \{K, K^{-1}\}$ almost everywhere if and only if u is the real part of a function $f \in W^{1,2}(Q, \mathbb{R}^2)$ such that

$$Df(z) \in E$$

for almost every $z \in Q$. Moreover, set $k = \frac{K-1}{K+1}$. Then $Df(z) \in E$ if and only if there exists $\mu \in L^\infty(Q, \{-k, k\})$ such that

$$\overline{\partial} f - \mu \overline{\partial f} = 0. \tag{1.9}$$

The function μ is called the second complex dilatation of the mapping f. It gives information about how the linear map $Df(z)$ distorts discs into ellipses. The modulus of μ expresses the maximal stretching of the ellipses, and its argument the direction of maximal stretching of these ellipses.

For the sake of completeness the relation between the equations (1.1) and (1.9) is discussed in section 6.

Next we observe that the integrability properties of Df are completely encoded in its distributional measure $Df_\sharp\mathcal{L}^n_Q$ (c.f section 2) since,

$$\frac{1}{|Q|}\int_Q |Df(z)|^p dz = \int_{\mathbf{M^{2\times 2}}} |\lambda|^p dDf_\sharp(\mathcal{L}^n_Q)(\lambda).$$

Thus our strategy will be the following: Firstly we construct a probability measure $\nu \in \mathcal{M}(E)$ such that

$$\int_{\mathbf{M^{2\times 2}}} |\lambda|^{\frac{2K}{K-1}} d\nu(\lambda) = \infty.$$

Since ν has support in E, if it was the distribution of the gradient of some Sobolev function the problem would be concluded. This need not be the case, but using the theory of laminates (See section 3) we can at least show the existence of a sequence $\{f_j\}$ uniformly bounded in each $W^{1,p}(Q,\mathbb{R}^m)$, $2 \le p < \frac{2K}{K-1}$ such that

$$Df_{j\sharp}(\mathcal{L}^n_Q) \overset{\star}{\rightharpoonup} \nu \text{ in } \mathcal{M}(\mathbf{M^{2\times 2}}). \tag{1.10}$$

Whenever (1.10) holds we say that the sequence $\{Df_j\}$ *generates* the measure ν. The last difficulty is that a priory the sequence $\{Df_j\}$ does not stay in E almost everywhere. This can be handled by several means. One option is based in adapting the recent new methods for solving partial differential inclusions (see [DM],[K],[MS]) and in particular Proposition 4.42 in [K]) to our situation. However, the proof would be more technical and specific. We have chosen to follow a somehow more direct (familiar) and general route based on the so-called Beltrami Operators. Using them we can find another sequence $\{g_j\}$ such that $Dg_j(z) \in E$ almost every $z \in Q$ and it also *generates* ν. The latter argument is related to those of [AF], where the Beltrami Operators were applied to analyze the so-called Quasiregular Gradient Young measures. These operators have turned out to be an efficient tool in clarifying a wealth of questions concerning the study of the best exponents in planar P.D.E and related topics [As2],[F],[AF]. Their invertibility properties and other issues are described in the recent work of Astala, Iwaniec and Saksman [AIS].

Once a sequence $\{g_j\}$ as above is obtained, an easy argument shows that the real parts of g_j and the conductivity coefficients $\{\rho_j\}$ associated to the second complex dilatations of g_j prove the Theorem 1.1 to be true.

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2. Notation

Let Q denote the n dimensional unit square $Q = \{z \in \mathbb{R}^n : |z_i| \leq 1\}$, $B(a,r) = \{z \in \mathbb{R}^n : |z-a| \leq r\}$ and $R \Subset Q$ means that R is a compact subset of Q. Concerning matrices, $\mathbf{M^{m\times n}}$ is the space of $m\times n$ matrices. The tensor product $a \otimes d$ with $a \in \mathbb{R}^m$ and $d \in \mathbb{R}^n$ denotes the rank-one matrix $(a_i d_j)$. It maps $v \in \mathbb{R}^n$ to $\langle v, d\rangle a$. Here, $\langle \cdot, \cdot \rangle$ represents the Euclidean scalar product. Unless otherwise indicated for a matrix A, $|A|$ represents the euclidean norm of A. We denote closed balls in the space of matrices $\mathbf{M^{m\times n}}$ by $B(r)$, i.e $B(r) = \{A \in \mathbf{M^{m\times n}} : |A| \leq r\}$. Similarly, $B_\infty(r) = \{A \in \mathbf{M^{m\times n}} : |A| \geq r\}$. The plane of diagonal matrices in $\mathbf{M^{2\times 2}}$ is denoted by D. We will use the notation

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = (d_1, d_2). \tag{2.1}$$

For a matrix $A \in \mathbf{M^{2\times 2}}$ we will also use complex coordinates $A = (A_z, A_{\overline{z}})$. Here, $A_z \in \mathbb{C}$ and $A_{\overline{z}} \in \mathbb{C}$ satisfy the following relation: Let us identify a vector $w = (x, y) \in \mathbb{R}^2$ with the complex number $w = x + iy$. Then it holds that for every vector $w \in \mathbb{R}^2$

$$Aw = A_z \cdot w + A_{\overline{z}} \cdot \overline{w},$$

where $\overline{w}$ denotes the complex conjugate of w. Using this notation

$$\begin{aligned} E_K &= \{A = (A_z, A_{\overline{z}}) \in \mathbf{M^{2\times 2}} : A_{\overline{z}} = k\overline{A_z}\} \text{ and} \\ E_{\frac{1}{K}} &= \{A = (A_z, A_{\overline{z}}) \in \mathbf{M^{2\times 2}} : A_{\overline{z}} = -k\overline{A_z}\}, \end{aligned} \tag{2.2}$$

where the sets E_ρ were introduced in (1.7) and $k = \frac{K-1}{K+1}$. We use also complex coordinates for the differential of a mapping $f \in W^{1,p}(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$;

$$Df(z) = (\partial f(z), \overline{\partial} f(z)).$$

Concerning measures $\mathcal{M}(\mathbf{M^{2\times 2}})$ stands for the set of Radon measures in $\mathbf{M^{2\times 2}}$, δ_A is a Dirac delta at A, spt ν stands for the support of ν and $\stackrel{*}{\rightharpoonup}$ means convergence in the weak star topology. For a set E, $|E|$ denotes its Lebesgue measure. Let Ω be a bounded measurable set. Then $\mathcal{L}^n_\Omega$ stands for the normalized Lebesgue measure restricted Ω so that $\mathcal{L}^n_\Omega(\Omega) = 1$. Let f be a measurable function $f : \Omega \to \mathbb{R}^m$ and N a Borel set in $\mathbb{R}^m$. Then the push-forward $\mathcal{L}^n_\Omega$ under f is given by

$$f_\sharp(\mathcal{L}^n_\Omega)(N) = \mathcal{L}^n_\Omega(f^{-1}(N)).$$

We call $f_\sharp(\mathcal{L}^n_\Omega)$ the distribution measure of f. Finally the threshold $\frac{2K}{K-1}$ is denoted by p_K.

3. Laminates

In this section we describe a process to build probability measures which arise as weak star limits of distribution measures of gradients of Sobolev functions. The class of probability measures obtained by this process, named as laminates, were introduced in [P1] to provide examples of the so-called Homogeneous Gradient Young measures. In the setting of homogenization, lamination of materials has been present from the very beginning, since it provides one of the few situations where the relation Microstructure-Macrostructure is relatively well understood. We recall the basics of Laminates, referring to [P], [K] for further details. The reader familiar with Gradient Young measures will recognize features of this theory in the discussion below.

Let us start with a matrix $A \in \mathbf{M^{m\times n}}$. Suppose that there exist matrices $B, C \in \mathbf{M^{m\times n}}$, a real parameter $\lambda \in [0,1]$ and vectors $a \in \mathbb{R}^m, d \in \mathbb{R}^n$ such that

$$A = \lambda B + (1-\lambda)C,$$

$$B - C = a \otimes d; \tag{3.1}$$

Whenever (3.1) is satisfied we say that B and C are *rank-one connected* and that $[B, C]$ is a *rank-one segment.* Therefore using this jargon, A is supposed to belong to certain rank-one segment.

Let h be the saw-tooth function on the real line, obtained as the periodic extension of

$$h(x) = \begin{cases} \lambda x & \text{if } 0 \leq x \leq 1-\lambda \\ -(1-\lambda)x + (1-\lambda) & \text{if } 1-\lambda \leq x \leq 1. \end{cases}$$

We define

$$f(z) = Az - ah(\langle z, d\rangle)$$

for z in the unit cube Q. Clearly $f \in W^{1,\infty}(Q, \mathbb{R}^m)$ and $Df_\sharp(\mathcal{L}^n_Q)$ is equal to the measure $\nu = \lambda\delta_B + (1-\lambda)\delta_C$. Unfortunately, to iterate this process in order to obtain less trivial laminates we would need that f had affine boundary values. Hence to avoid the problem we consider the sequence $\{f_j\}_{j=1}^\infty$, $f_j(z) = \frac{1}{j}f(jz)$. For each j the distribution measure $Df_{j_\sharp}(\mathcal{L}^n_Q)$ is still equal to ν . Moreover, we have obtained the following advantages. Firstly, if $R \Subset Q$ has positive measure we get that

$$Df_{j_\sharp}(\mathcal{L}^n_R) \stackrel{\star}{\rightharpoonup} \nu.$$

Secondly, the above property still holds when instead of $\{f_j\}$ we put another sequence $\{\tilde{f}_j\}$ such that $\lim_{j\to\infty} |z \in Q : Df_j \neq D\tilde{f}_j| = 0$. This modification can be made so that the boundary values of every $\tilde{f}_j$ are equal to A and $|z \in Q : Df_j(z) \neq D\tilde{f}_j(z)| \leq \frac{1}{j}$ (see the picture 1 and for example lemma 3.2 in [K] for a proof).

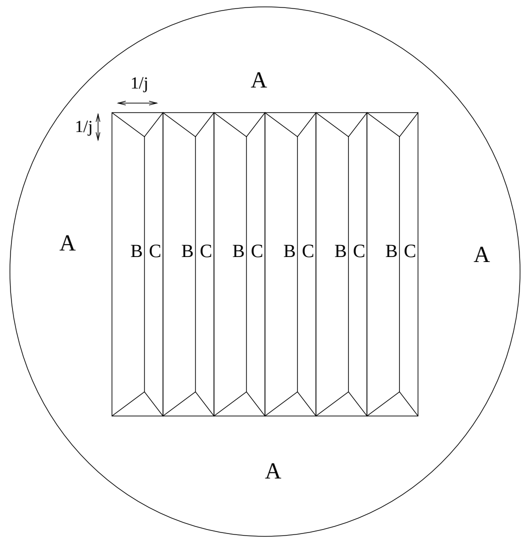

FIGURE 1. If a piecewise affine mapping f has gradient equal to A in a certain region and A belongs to a rank-one segment $[B, C]$, we can replace A by layers where the gradient is equal to B and C and an interface region. The size of the interface is controlled by the number of layers we take.

Lastly, it is clear that arguing by means of similarities, we can replace the unit cube Q in the above argument by an arbitrary cube $\tilde{Q}$.

Therefore, this procedure associates to every cube $\tilde{Q} \subset \mathbb{R}^n$ a sequence $\{\tilde{f}_j\}$ bounded in $W^{1,\infty}(\tilde{Q}, \mathbb{R}^m)$ such that for every $\tilde{R} \Subset \tilde{Q}$ with positive measure

$$D\tilde{f}_{j\sharp}(\mathcal{L}^n_{\tilde{R}}) \xrightarrow{\star} \nu \tag{3.2}$$

and $\tilde{f}_j(z) - Az \in W_0^{1,\infty}(\tilde{Q}, \mathbb{R}^m)$. We say that we have splitted the measure δ_A as $\lambda\delta_B + (1-\lambda)\delta_C$.

Next, suppose that B belongs to a rank-one segment $[D, E]$, $B = \lambda_2 D + (1-\lambda_2)E$, $0 < \lambda_2 < 1$. The idea is to substitute every layer where the gradient of f_j was equal to B by finer layers of D and E. It is important that the scale of the layers of D and E must be small enough in relation with the scale of the layers of B and C. Formally, denote $\Omega_j = \{z \in Q : Df_j(z) = B\}$ and take a finite collection of dyadic cubes $\{Q_j^i\}_{i=1}^{N_j} \subset \Omega_j$ such that $|\Omega_j \setminus \cup_i^{N_j} Q_j^i| \le \frac{1}{j}$.

Inside of each of the cubes Q_j^i we construct the sequence $\{f_{j,i}^k\}_{k=1}^\infty \in W^{1,\infty}(Q_j^i, \mathbb{R}^m)$ obtained as in (3.2) replacing A by B, ν by $\lambda_2\delta_D + (1-\lambda_2)\delta_E$ and $\tilde{Q}$ by Q_j^i. We choose $k(j) = jN_j$ to have that $\cup_{i=1}^{N_j}|z \in Q_j^i : Df_{j,i}^{k(j)} \neq \{D, E\}| \le \frac{1}{j}$.

Due to the affine boundary values of each $f_{j,i}^k$ we can weld them together and define

$$f_j^2(z) = \begin{cases} f_{j,i}^{k(j)}(z) & \text{if } z \in Q_j^i, \\ f_j(z) & \text{otherwise.} \end{cases}$$

A direct computation shows that for every $R \Subset Q$ with positive measure, the weak star limit of the sequence of measures $\{Df_{j\,\sharp}^2(\mathcal{L}_R^n)\}$ is $\nu^2 = \lambda(\lambda_2\delta_D + (1-\lambda_2)\delta_E) + (1-\lambda)\delta_C$. Clearly, we can iterate this construction as long as we have enough relations in term of rank-one connections. The obtained measures will be generated by gradients in the sense of (1.10). We arrive to the class of prelaminates.

Definition 3.1. The family of prelaminates $\mathcal{PL}$ is the smallest family of probability measures on $\mathbf{M}^{\mathbf{m}\times\mathbf{n}}$ such that

(1) $\mathcal{PL}$ contains all Dirac masses in $\mathbf{M}^{\mathbf{m}\times\mathbf{n}}$.
(2) Let $\nu = \sum_{i=1}^k \lambda_i\delta_{A_i} \in \mathcal{PL}$ and let $A_1 = \lambda B + (1-\lambda)C$ where $\lambda \in [0,1]$ and $[B,C]$ is a rank-one segment. Then the probability measure $\sum_{i=2}^k \lambda_i\delta_{A_i} + \lambda_1(\lambda\delta_B + (1-\lambda)\delta_C) \in \mathcal{PL}$.

Theorem 3.2. *Let ν be a prelaminate supported in the ball $B(r) \subset \mathbf{M}^{\mathbf{m}\times\mathbf{n}}$. Then there exists a sequence $\{f_j\} \in W^{1,\infty}(Q,\mathbb{R}^m)$ such that for every $R \Subset Q$ with positive measure*

(1) $\|f_j\|_{1,\infty} \leq Cr$,
(2) $Df_{j_\sharp}(\mathcal{L}_R^n) \xrightarrow{\star} \nu$.

Proof:
This theorem can be found in many places in the literature since it follows from the fact that laminates are homogeneous Gradient Young measures, [P1]. The interested reader can complete a proof using the above scheme and an induction argument. □

Finally laminates are defined as weak($\star$) limits of prelaminates in $\mathcal{M}(\mathbf{M}^{\mathbf{m}\times\mathbf{n}})$. A laminate which is not a prelaminate is called an infinite-rank laminate.

Definition 3.3. Let ν be a probability measure on $\mathbf{M}^{\mathbf{m}\times\mathbf{n}}$ and $1 \leq p < \infty$. Then ν is said to be a $p-$*laminate* if there exists a sequence of prelaminates ν_j such that

a) $\sup_j \int_{\mathbf{M}^{\mathbf{m}\times\mathbf{n}}} |\lambda|^p d\nu_j(\lambda) < \infty$,
b) $\nu_j \xrightarrow{\star} \nu$ in $\mathcal{M}(\mathbf{M}^{\mathbf{m}\times\mathbf{n}})$.

Theorem 3.4. *Let ν a $p-$ laminate. Then there exists a sequence $\{f_j\}$ uniformly bounded in $W^{1,p}(Q,\mathbf{R^2})$ such that*

$$Df_{j_\sharp}(\mathcal{L}_R^n) \xrightarrow{\star} \nu$$

for every compact subset R of Q with positive measure.

Proof:

In the case of compactly supported laminates this theorem is proved in the literature (See [P], chapter 9). The proof follows from Theorem 3.2 and a diagonalization argument. For the case of finite p let us apply Theorem 3.2 to each prelaminate ν_j. For each j we obtain a sequence $\{f_j^i\}_{i=1}^{\infty}$ uniformly bounded in $W^{1,\infty}(Q,\mathbb{R}^m)$ such that $Df^i_{j_\sharp}(\mathcal{L}^n_R) \overset{\star}{\rightharpoonup} \nu_j$ as i tends to ∞. The uniform bound on the $W^{1,\infty}$ norms of the f_j^i gives that

$$\lim_{i\to\infty}\int_Q |Df_j^i(x)|^p dx = \int_{\mathbf{M^{m\times n}}} |\lambda|^p d\nu_j(\lambda).$$

Putting this together with the assumption a) in the definition of p laminate gives the uniform bounds for the p-norms of the $\{f_j^i\}_{i,j=1}^{\infty}$ and thus for the generating subsequence. □

The following remark, on the particular nature of the laminate we are going to deal with, will simplify the proofs in section 5.

Remark 3.5. If the structure of an infinite-rank laminate is sufficiently simple it behaves essentially as a prelaminate in the following sense. Let ν be a prelaminate and f_j the generating sequence described in Theorem 3.2. Set $\Omega_j = \{z \in Q : Df_j(z) \notin \operatorname{spt}(\nu)\}$. Then it follows from the construction that

$$\lim_{j\to\infty} |\Omega_j| = 0. \tag{3.3}$$

It is easy to see that if a p-laminate ν is purely atomic and $\lim_{j\to\infty}\nu(\operatorname{spt}(\nu)\setminus \operatorname{spt}(\nu_j)) = 0$, ν_j as in Definition 3.3 b), the sequence $\{f_j\}$ obtained from Theorem 3.4 satisfies property (3.3) as well.

4. The staircase-laminate

This section will be devoted to constructing a laminate ν supported in the set E presented in formula (1.8), and satisfying

$$\int_{\mathbf{M^{2\times 2}}} |\lambda|^{p_K} d\nu(\lambda) = \infty \text{ and } \int_{\mathbf{M^{2\times 2}}} |\lambda|^p d\nu < \infty \tag{4.1}$$

for every $p < p_K$. In fact, we will not need the whole set E since the laminate ν will live on the intersection of E with the plane of all diagonal matrices $D = (d_1, d_2)$; c.f.(2.1). Recall that in this plane the only rank-one directions are horizontal and vertical lines. Moreover using notation (2.1) we have that

$$E_K \cap D = \{(a, Ka) : a \in \mathbb{R}\} \text{ and } E_{K^{-1}} \cap D = \{(a, K^{-1}a) : a \in \mathbb{R}\}.$$

The condition (4.1) can only be achieved if the support of ν is unbounded, thus ν must be an infinite-rank laminate. Hence we will construct a sequence of prelaminates ν^n, such that

$$\nu^n \overset{\star}{\rightharpoonup} \nu \text{ in } \mathcal{M}(\mathbf{M^{2\times 2}}).$$

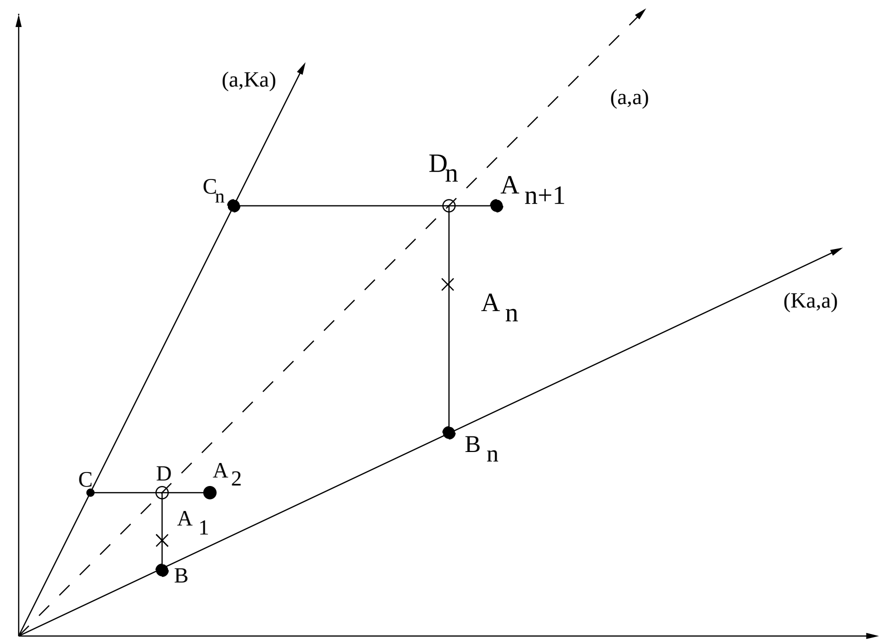

FIGURE 2. *"Steps" 1 and n. Black dots denote the support of the measure, the cross is the centre of mass, and white dots are auxiliary matrices.*

For ν to be a probability measure we need that $\lim_{n\to\infty} \nu^n(B_\infty(R)) = 0$ for every positive R. In fact the integrability of the measure ν depends on the rate at which $\nu(B_\infty(R))$ goes to 0 when R tends to ∞. The opening of the cone $\mathcal{Q}$,

$$\mathcal{Q} \equiv \{(x,y) \in D : K^{-1} \leq \frac{y}{x} \leq K\},$$

determines if it is possible to find a laminate ν supported in Q such that $\nu(B_\infty(R))$ converges to 0 slowly enough for (4.1) to hold.

We will firstly describe how certain sequences of matrices in $\mathcal{Q}$ give naturally rise to infinite rank-laminates and after that we will choose an appropriate sequence to create the measure ν.

Our construction will resemble an staircase (see figure 3). Thus, we start by describing how to build its steps. Take two diagonal matrices $A_1 = (A_1^1, A_1^2), A_2 = (A_2^1, A_2^2) \in \mathcal{Q}$. We will use the partial ordering,

$$A_1 \leq A_2 \iff A_1^1 \leq A_2^1 \text{ and } A_1^2 \leq A_2^2. \tag{4.2}$$

Given such a pair of matrices, the matrices $B = (A_1^1, \frac{1}{K}A_1^1) \in E_{K^{-1}}$, $D = (A_1^1, A_2^2)$ and $C = (\frac{A_2^2}{K}, A_2^2) \in E_K$ satisfy that $A_1 \in [B, D]$, $D \in [C, A_2]$. In addition, $[B, D]$ is a vertical segment and $[C, A_2]$ is an horizontal segment i.e they are rank-one segments. (For a quick illustration see figure 2. Observe that although in the figure $A_1^1 = A_2^2$, this is not required in the general construction). Let λ_1 and $\lambda_2 \in [0,1]$ be parameters such that

$$A_1 = \lambda_1 B + (1-\lambda_1)D,$$

and

$$D = \lambda_2 C + (1-\lambda_2)A_2.$$

Plugging the latter expression into the former we obtain that

$$A_1 = \lambda_1 B + (1-\lambda_1)\big(\lambda_2 C + (1-\lambda_2)A_2\big). \tag{4.3}$$

In the language of measures, (4.3) means

$$A_1 = \int_{\mathbf{M}^{2\times 2}} t d\nu_1(t),$$

where ν_1 is the measure

$$\nu_1 = \lambda_1 \delta_B + (1-\lambda_1)\big(\lambda_2 \delta_C + (1-\lambda_2)\delta_{A_2}\big). \tag{4.4}$$

The figure 2 shows how this construction looks like if the matrices are near the origin, (A_1 as the center of mass of a measure supported on B, C and A_2), or if they are relatively far away (A_n as the center of mass of a measure supported on B_n, C_n and A_{n+1}).

Now let us suppose that we are given a sequence of matrices $\{A_n\}_{n=1}^{\infty} \in \mathcal{Q}$ ordered as in (4.2), $A_n \leq A_{n+1}$ for every n. We can repeat the explained construction with $A_1 = A_n$ and $A_2 = A_{n+1}$. This yields a sequence of step measures $\{\nu_n\}_{n=1}^{\infty}$. We would like to paste the measures ν_n together to obtain a new measure ν. Let us sketch the idea that will be made rigorous below for a concrete choice of the matrices A_n. Consider the measure ν_1 as in (4.4). Replace δ_{A_2} in the definition of ν_1 by the measure ν_2. This defines a new measure

$$\nu^2 = \lambda_1\delta_B+(1-\lambda_1)\left(\lambda_2\delta_C+(1-\lambda_2)\left(\lambda_3\delta_{B_2}+(1-\lambda_3)\big(\lambda_4\delta_{C_2}+(1-\lambda_4)\delta_{A_3}\big)\right)\right).$$

Here the new parameters and matrices come from the definition of the step measure ν_2. Since ν^2 has an atom δ_{A_3}, we proceed by replacing it by the step measure ν_3 to obtain a new measure ν^3 with an atom at A_4. We continue iteratively obtaining a sequence of probability measures $\{\nu^n\}$. Finally the measure ν is defined as the weak star limit of this sequence. Besides the condition on the ordering (4.2), the only restriction on the sequence $\{A_n\}$ is that ν should have finite p^{th}-moment for some $1 < p < \infty$. We further observe that we have only used rank-one segments at every step of the construction, so it follows that if ν has finite pth moment for some $p > 1$, ν is a laminate. The figure 3 helps to understand the process just loosely explained, which will be completely understood by analogy with the concrete example explained below these lines.

Let us concentrate now in obtaining the measure ν such that (4.1) holds. We consider the sequence $A_n = \{(n+1, n)\}_{n=n_0}^{\infty}, n_0 \geq \frac{1}{K-1}$.

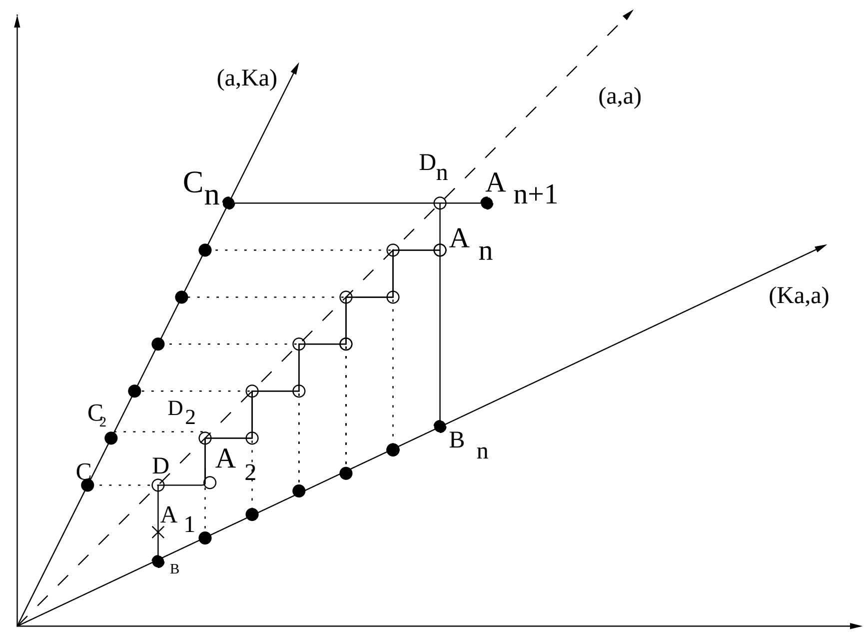

FIGURE 3. The staircase at level n.

Since $\{A_n\}$ is well ordered and contained in $\mathcal{Q}$ we can use the scheme indicated above to construct prelaminates ν^n with centre of mass A_{n_0}, and supported on the set $E \cup \{A_{n+1}\}$. To avoid keeping track of n_0 everywhere we assume without loss of generality that $n_0 = 1$.

Let start with the measure δ_{A_1}. Clearly the following relations hold,

$$(2,1) = \tfrac{K}{2(K-1)}(2, \tfrac{2}{K}) + (1 - \tfrac{K}{2(K-1)})(2,2), \tag{4.5}$$

$$(2,2) = \tfrac{K}{2(K-1)+K}(\tfrac{2}{K}, 2) + (1 - \tfrac{K}{2(K-1)+K})(3,2). \tag{4.6}$$

Thus, in the above notation, $B = (2, \frac{2}{K})$, $D = (2,2)$ and $C = (\frac{2}{K}, 2)$. Hence A_1 is the centre of mass of the probability measure ν_1 defined by

$$\nu_1 = \tfrac{K}{2(K-1)}\delta_{(2,\frac{2}{K})} + (1 - \tfrac{K}{2(K-1)})\Big(\tfrac{K}{2(K-1)+K}\delta_{(\frac{2}{K},2)} + (1 - \tfrac{K}{2(K-1)+K})\delta_{(3,2)}\Big),$$

which is a prelaminate. Furthermore it can be expressed as $\nu_1 = \mu_1 + \lambda_1\delta_{A_2}$, where μ_1 is a new measure supported in the set $E \cap B(2C_K)$. The constant C_K is equal to $|B| = |C|$, explicitly $C_K = |(1, \frac{1}{K})| = \frac{\sqrt{1+K^2}}{K}$. It will appear often below since it is a natural parameter in our construction. We say that we are one "step" up in the staircase.

The construction gives that $\nu_1(B_\infty(2C_K)) = \nu_1(A_2) = \lambda_1$ where

$$\lambda_1 = (1 - \frac{K}{2(K-1)+K})(1 - \frac{K}{2(K-1)}).$$

We can repeat the same operation at level n since,

$$(n+1,n) = \tfrac{K}{(n+1)(K-1)}(n+1,\tfrac{n+1}{K}) + (1-\tfrac{K}{(n+1)(K-1)})(n+1,n+1)$$
$$(n+1,n+1) = \tfrac{K}{(n+1)(K-1)+K}(\tfrac{n+1}{K},n+1) + (1-\tfrac{K}{(n+1)(K-1)+K})(n+2,n+1).$$

The structure relations in terms of rank-one connections are the same as in (4.5) and (4.6). Hence A_n can be expressed as the center of mass of the laminate ν_n defined by

$$\nu_n = \tfrac{K}{(n+1)(K-1)}\delta_{(n+1,\frac{n+1}{K})} + (1-\tfrac{K}{(n+1)(K-1)})\Big(\tfrac{K}{(n+1)(K-1)+K}\delta_{(\frac{n+1}{K},n+1)} + (1-\tfrac{K}{(n+1)(K-1)+K})\delta_{(n+2,n+1)}\Big).$$

As before there exists a measure μ_n supported in $E\cap B((n+1)C_K)$ such that ν_n splits as $\nu_n = \mu_n + \lambda_n\delta_{A_{n+1}}$ and $\nu_n(B_\infty((n+1)C_K)) = \lambda_n$. The value of λ_n will be important:

$$\lambda_n = (1-\frac{K}{(n+1)(K-1)+K})(1-\frac{K}{(n+1)(K-1)}). \tag{4.7}$$

Next, we paste the steps together to obtain a truncated staircase. The formal procedure is done by induction. Let us start with $n=1$. Remember that

$$\nu_1 = \mu_1 + \lambda_1\delta_{A_2}.$$

We define

$$\nu^2 = \mu_1 + \lambda_1\nu_2.$$

Declare $\mu^2 = \nu^2_{|B(3C_K)}$. Then ν^2 splits in the form $\nu^2 = \mu^2 + \lambda_1\lambda_2\delta_{A_3}$ and μ^2 is supported in $E\cap B(3C_k)$. Since A_3 is the center of mass of ν_3, we defined $\nu^3 = \mu^2 + \lambda_1\lambda_2\nu_3$. Now we can find μ^3 as before and continue inductively. The previous procedure gives the definitions:

$$\nu^n = \mu^{n-1} + (\Pi_{i=1}^{n-1}\lambda_i)\nu_n,$$

and

$$\mu^n = \nu^n_{|B((n+1)C_K)}.$$

This defines the truncated staircases. Observe that it follows from the construction that

$$\nu^m(B_\infty((n+1)C_K) = \nu^n(B_\infty((n+1)C_K) = \nu^n(A_{n+1}) = \Pi_{i=1}^n\lambda_i \tag{4.8}$$

for every $m\geq n$. Finally, we let the staircase grow infinitely and obtain:

Definition 4.1. (The staircase-laminate) Let ν^n be as above. Then the staircase-laminate ν is defined by:

$$\nu = \lim_{n\to\infty}\nu^n \qquad \text{in the weak star topology of } \mathcal{M}(\mathbf{M^{2\times 2}}). \tag{4.9}$$

Next thing we need to guarantee is that ν is a probability measure and that it has the appropriated growth.

First we observe that by the Cavalieri principle,

$$\int_{\mathbf{M^{2\times 2}}} |\lambda|^p d\nu(\lambda) = p\int_0^\infty t^{p-1}\nu(B_\infty(t))dt.$$

Dividing $\mathbf{M^{2\times 2}}$ into rings we obtain:

$$\frac{1}{p}\int_{\mathbf{M^{2\times2}}}|\lambda|^p d\nu(\lambda)=\sum_{n=1}^{\infty}\int_{nC_K}^{(n+1)C_K} t^{p-1}\nu(B_\infty(t))dt. \tag{4.10}$$

By (4.8), (4.10) is less than

$$\sum_{n=1}^\infty \int_{nC_K}^{(n+1)C_K}((n+1)C_K)^{p-1}\nu(B_\infty(nC_K))dt = \sum_{n=1}^\infty((n+1)C_K)^{p-1}\nu^{n-1}(A_n),$$

and bigger than

$$\sum_{n=1}^\infty \int_{nC_K}^{(n+1)C_K}(nC_K)^{p-1}\nu(B_\infty((n+1)C_K))dt = \sum_{n=1}^\infty(nC_K)^{p-1}\nu^{n}(A_{n+1}).$$

Therefore we have obtained that for every p, there are constants, C_1, C_2, depending on K, such that

$$C_2\sum_{n=1}^{\infty} n^{p-1}\nu^n(A_{n+1})\le \int_{\mathbf{M^{2\times2}}}|\lambda|^p d\nu(\lambda)\le C_1\sum_{n=1}^{\infty}(n+1)^{p-1}\nu^{n-1}(A_n).$$

We compare the above sums with $\sum_{n=1}^\infty \frac{1}{n}$. Thus, if

$$\begin{aligned}&\liminf_{n\to\infty} n^{p_K}\nu^n(A_{n+1})>0\\ &\limsup_{n\to\infty} n^{p_K}\nu^n(A_{n+1})<\infty\end{aligned} \tag{4.11}$$

it follows that $\int_{\mathbf{M^{2\times2}}}|\lambda|^{p_K}d\nu(\lambda) = \infty$ and for every $p<p_K$, $\int_{\mathbf{M^{2\times2}}}|\lambda|^p d\nu(\lambda)<\infty$.

After plugging the value of $\nu^n(A_{n+1})$ into (4.11) we are led to study the behavior of the sequence, $a_n=\Pi_{i=1}^n\lambda_i$. The following basic manipulation show that it behaves like n^{-p_K}. Firstly we handle the product by using (4.7) and taking logarithms. This gives

$$|\log(a_n)+\sum_{i=1}^{n}\frac{2K}{(i+1)(K-1)}|\le c(n).$$

where $\sup_n c(n)\le C\le\infty$. Since $p_K=\frac{2K}{K-1}$ and $|\log(n)-\sum_{i=1}^n\frac{1}{i+1}|\le c_0$, we arrive to

$$|\log(a_n)+p_K\log(n)|\le c_1<\infty$$

for every $n\in\mathbb{N}$.

Therefore, (4.11) is satisfied and the staircase laminate ν verifies (4.1).

Remark 4.2. It is the fact that the auxiliary values $\{A_n\}$ are asymptotically closed to the set $E_1\cap D=\{(a,a): a\in\mathbb{R}\}$ that characterizes the integrability of the measure ν. Generally let $\{\tilde{A_n}\}\subset\mathcal{Q}$ be a sequence of auxiliary matrices ordered as in (4.2) and such that

$$\lim_{n\to\infty}\frac{\tilde{A_n}^2}{\tilde{A_n}^1}=t.$$

Then it can be shown that if we perform the above scheme to obtain a laminate $\tilde{\nu}$, the threshold for the integrability of $\tilde{\nu}$ is equal to

$$\frac{K}{(K-t)} + \frac{K}{(K-\frac{1}{t})}.$$

5. Correcting sequences via Beltrami Operators

Consider the staircase-laminate ν. By the construction it is a p-laminate in the sense of definition 3.3 for every $1 < p < p_K$. Hence by Theorem 3.4 there exists a sequence $\{f_j\} \in W^{1,p}(Q, \mathbb{R}^2)$ such that

$$Df_{j\sharp}(\mathcal{L}^n_R) \overset{\star}{\rightharpoonup} \nu \tag{5.1}$$

for every $R \Subset Q$ with positive measure and $\{f_j\}$ is uniformly bounded in $W^{1,p}(Q, \mathbb{R}^2)$ for every $1 < p < p_K$. In addition, the measure ν satisfies the requirements of the Remark 3.5 and is supported on the set $E \setminus B(0, \varepsilon)$ for some $\varepsilon > 0$. Thus,

$$\lim_{j\to\infty} |\{z \in Q : Df_j(z) \notin E \text{ or } Df_j(z) = 0\}| = 0. \tag{5.2}$$

The fact that E is related to an elliptic equation allows one to "project" the sequence $\{f_j\}$ to a sequence $\{g_j\}$ such that $\{Dg_j\}$ take values in E and converges to Df_j in L^p.

Proposition 5.1. *Let ν be the staircase laminate and $\{f_j\}$ the generating sequence with the properties 5.1 and 5.2. Let $k = \frac{K-1}{K+1}$. Then there exists a sequence of Beltrami coefficients $\mu_j \in L^\infty(Q, \{k, -k\})$ and a sequence $\{g_j\} \in W^{1,p}(Q, \mathbb{R}^m)$ for all $2 \le p < p_K$ such that:*

$$\overline{\partial} g_j(z) - \mu_j(z)\overline{\partial g_j}(z) = 0 \tag{5.3}$$

for a.e. z in Q, and

$$\lim_{j\to\infty} \|Df_j - Dg_j\|_{L^p(Q)} \to 0. \tag{5.4}$$

Proof:
Declare $\Omega_j = \{z \in Q : Df_j(z) \in E \text{ and } Df_j(z) \neq 0\}$. Then property (5.2) reads as

$$\lim_{j\to\infty} |Q \setminus \Omega_j| = 0. \tag{5.5}$$

Define

$$\mu_j(z) = \begin{cases} \frac{\overline{\partial} f_j(z)}{\overline{\partial f_j}(z)} & \text{if } z \in \Omega_j, \\ k & \text{if } z \in Q \setminus \Omega_j. \end{cases}$$

It is easy to see from the expression of the set E in complex coordinates (2.2) that $\mu_j \in L^\infty(Q, \{k, -k\})$. The key point in the proof is that (5.5) implies that each $\{f_j\}$ satisfies a non homogeneous Beltrami equation with right hand side going to zero in L^p for every $p < p_K$. The argument is the following: By the definition of μ_j

$$\overline{\partial} f_j(z) - \mu_j(z)\overline{\partial f_j}(z) = (\overline{\partial} f_j(z) - k\overline{\partial f_j(z)})\chi_{Q\setminus\Omega_j}(z) \tag{5.6}$$

in Q. Let $h_j(z) = (\overline{\partial} f_j(z) - k\overline{\partial f_j}(z))\chi_{Q\setminus\Omega_j}(z)$ and consider exponents $2 < p < p' < p_K$. Then Hölder's inequality with exponents $\frac{p'}{p}, \frac{p'}{p'-p}$ implies that $\int_{\mathbb{C}} |h_j|^p dz \leq C\|Df_j\|_{L^{p'}(Q)}|\Omega_j|^{\frac{p'-p}{p'}}$. By the definition of ν, $\|Df_j\|_{p'}$ is uniformly bounded and hence

$$\lim_{j\to\infty} \int_{\mathbb{C}} |h_j|^p dz \to 0. \tag{5.7}$$

for every $p < p_K$. Here after, the argument is similar to those in [AF]. We sketch the proof, that goes in the same way that the proof of Theorem 1.2 in [AF]. First we extend μ_j to the whole complex plane as;

$$\tilde{\mu_j}(z) = \begin{cases} \mu_j(z) & \text{if } z \in Q \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Now we use the two integral operators naturally related to the theory of quasiconformal mappings; The Cauchy transform P,

$$Pf(z) = \frac{-1}{2\pi}\int_{\mathbb{C}} \frac{f(w)}{z-w}dw$$

and the Beurling-Ahlfors transform S

$$(Sf)(z) = -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}dw.$$

For definitions and proofs of their properties see [Al]. These operators satisfy that for smooth compactly supported h,

$$\overline{\partial} P(h) = h, \quad \partial P(h) = S(h). \tag{5.9}$$

Moreover both operators are continuous from $L^p(\mathbb{C})$ into itself thus continuous from $L^p(\mathbb{C})$ into itself if $1 < p < \infty$. Therefore the above equalities (5.9) extend to $L^p, p > 1$ in the distributional sense.

Then if we consider the sequence $\{F_j\}$

$$F_j = P((I - \tilde{\mu_j}\overline{S})^{-1}(h_j))$$

it is easy to see that $g_j = (F_j - f_j)\chi_Q$ satisfies the equation

$$\overline{\partial} g_j - \mu_j\overline{\partial g_j} = 0 \text{ in } Q. \tag{5.10}$$

Furthermore by (5.7), $\{h_j\}$ tends to zero in L^p. Recall that through the whole proof we are assuming that $2 \leq p < p_K$. Hence by Theorem 3 in [AIS] $(I - \tilde{\mu_j}\overline{S})^{-1}$ is a bounded operator from L^p into itself ([AIS]). It follows that $\{(I - \tilde{\mu_j}\overline{S})^{-1}h_j\}$ tends to zero in $L^p(\mathbb{C})$ as well. This fact together with (5.9) and the boundness of S imply that $\{DF_j\}$ also converges to zero in $L^p(\mathbb{C})$. We have proved that $\{g_j\}$ satisfy (5.3) and (5.4). □

We will need that the convergence in L^p implies that the limit of the distributional measures are the same, i.e (5.4) implies that for every $R \Subset Q$ with positive measure

$$\lim_{j\to\infty} Df_{j\sharp}(\mathcal{L}^n_R) = \lim_{j\to\infty} Dg_{j\sharp}(\mathcal{L}^n_R) \tag{5.11}$$

in the weak star topology of $\mathcal{M}(\mathbf{M^{2\times 2}})$. Actually convergence in measure is enough for (5.11) to hold.

Remark 5.2. The above Proposition holds for every $W^{1,p}$-GYM supported in E with $1+k<p$. In this case we do not know if property 5.2 holds since a priory the generating sequence only converges to the set in measure. Therefore, a more subtle argument is needed to choose the Beltrami coefficients. One option is using the so-called Measurable Selection Lemma to find a projection of the generating sequence to the set. Other is noticing that $E = F^{-1}(0)$ with $F(A) = \min\{|A_{\overline{z}} - k\overline{A_z}|^p, |A_{\overline{z}} + k\overline{A_z}|^p\}$. Then the fact that $\int_{\mathbb{C}} F(\lambda)d\nu(\lambda) = 0$ gives a choice of appropriate Beltrami coefficients.

6. Proof of the theorem 1.1

We start with the staircase-laminate ν constructed in section 3. Firstly, by Theorem 3.4 applied to ν we obtain a generating sequence $\{f_j\}$. Then Proposition 5.1 provides a generating sequence $\{g_j\}$ and a sequence of Beltrami coefficients $\{\mu_j\} \in L^\infty(Q, \{k, -k\})$ such that

$$\overline{\partial} g_j(z) - \mu_j(z)\overline{\partial g_j(z)} = 0 \tag{6.1}$$

a.e.z in Q. Next, let us recall that if $g_j(z) = u_j(z) + iv_j(z)$ for u_j and $v_j \in W^{1,2}_{loc}(Q)$, it is an algebraic computation to show that

$$\operatorname{div}(\rho_j(z)\nabla u_j(z)) = 0 \tag{6.2}$$

in Q, where $\rho_j(z) = \frac{1-\mu_j(z)}{1+\mu_j(z)}$. For the sake of completeness we show the calculation. First we observe that

$$2\overline{\partial} g_j(z) = \nabla u_j(z) + J\nabla v_j(z)$$

$$2\overline{\partial g_j(z)} = \nabla u_j(z) - J\nabla v_j(z).$$

Thus, (6.1) becomes

$$\nabla u_j(z) + J\nabla v_j(z) = \mu_j(z)(\nabla u_j(z) - J\nabla v_j(z)).$$

a.e.z in Q. After rearranging this equation we obtain

$$\frac{1-\mu_j(z)}{1+\mu_j(z)}\nabla u_j(z) = -J\nabla v_j(z). \tag{6.3}$$

We obtain (6.2) by recalling that J is the Hodge $\star$ operator in dimension 2, this is it sends curl free vector fields to divergence free vector fields. Now, when $\mu_j(z)=k$, $\rho_j(z)=\frac{1-k}{1+k}=\frac{1}{K}$ and when $\mu_j(z)=-k$ $\rho_j(z)=\frac{1+k}{1-k}=K$ so (6.2) is an elliptic isotropic equation like (1.1). Moreover, the bounds on the ρ_j and (6.3) imply that for every $1\le p<\infty$

$$\int_R |\nabla u_j(z)|^p dz \le \int_R |Dg_j(z)|^p dz \le C(K)\int_R |\nabla u_j(z)|^p dz. \tag{6.4}$$

To conclude we use the following basic consequence of monotone convergence theorem.

Lemma 6.1. *(*[BDF] *Proposition 2.15) Let $\nu_j \overset{\star}{\rightharpoonup} \nu$ in $\mathcal{M}(\mathbf{M^{2\times 2}})$ and f be a continuous function on $\mathbf{M^{2\times 2}}$. Then,*

$$\int_{\mathbf{M^{2\times 2}}} f(\lambda)d\nu(\lambda) \le \liminf_{j\to\infty}\int_{\mathbf{M^{2\times 2}}} f(\lambda)d\nu_j(\lambda).$$

Take ν equal to the staircase laminate, $\nu_j = Dg_{j\sharp}(\mathcal{L}^n_R)$ and $f(\lambda)=|\lambda|^{p_K}$ in the lemma. The left hand side is equal to ∞ by (4.1) which yields that

$$\liminf_{j\to\infty}\frac{1}{|R|}\int_R |Dg_j|^{p_K}=\infty, \tag{6.5}$$

and by (6.4) that

$$\liminf_{j\to\infty}\int_R |\nabla u_j|^{p_K}=\infty.$$

Moreover since $\{Dg_j\}$ is uniformly bounded in $L^p(Q)$ for every $p<p_K$ (6.4) implies, after normalization, that

$$\|u_j(z)\|_{W^{1,2}(Q,\mathbb{R})}\le 1.$$

Therefore, $\rho_j(z)=\frac{1-\mu_j(z)}{1+\mu_j(z)}$ and $u_j(z)$ equal to the real part of $g_j(z)$ prove Theorem 1.1. □

Proof of the Corollary 1.2

Consider a union of disjoint balls $\{B_i(a_i,r_i)\}_{i=1}^{\infty}\Subset Q$. Let $\eta\in C_0^\infty(B(0,1))$ be a cut-off function such that $\eta(z)=1$ if $|z|\le\frac{1}{2}$. For every i let T_i, $T_i(z)=\frac{z-a_i}{r_i}$, be a similarity satisfying $T_i(B_i)=B(0,1)$. Define then, $\eta_i=r_i\eta(T_i(z))\in C_0^\infty(B_i)$. Use now (6.5) with $R=B(a_i,\frac{r_i}{2})$ to select $j(i)$ such that

$$\int_{B(a_i,\frac{r_i}{2})}|Dg_{j(i)}(z)|^{p_K}dz\ge\frac{1}{r_i}. \tag{6.6}$$

Setting $g_{j(i)}=g_i$ gives sequences $\{g_i\}_{i=1}^\infty,\{\mu_i\}_{i=1}^\infty$. Let

$$\mu=\sum_{i=1}^\infty \mu_i\chi_{B_i}+k\chi_{Q\setminus\cup_{i=1}^\infty B_i} \tag{6.7}$$

and

$$g = \sum_{i=1}^{\infty} g_i \eta_i. \tag{6.8}$$

Then we have that

$$\overline{\partial} g(z) - \mu \overline{\partial g}(z) = F \tag{6.9}$$

in Q. Explicitly the vector field F is given by

$$F = \sum_{i=1}^{\infty} (\overline{\partial} \eta_i(z) - \mu_i \overline{\partial \eta_i}(z)) g_i.$$

Thus, we can use the Sobolev embedding for the g_i and that $\|\nabla \eta_i\|_\infty = \|\nabla \eta\|_\infty$ to see that $F \in L^\infty(Q)$. Regarding the integrability of $|Dg|^{p_K}$ we have that

$$\int_Q |Dg|^{p_K} dz \geq \sum_{i=1}^{\infty} r_i \int_{B(a_i, \frac{r_i}{2})} |Dg_i|^{p_K},$$

which, after plugging (6.6), implies that

$$\int_Q |Dg(z)|^{p_K} dz = \infty. \tag{6.10}$$

Finally as in the homogeneous case we look at the equation satisfied by the real part of g. Namely if $g = u + iv$, (6.9) read as

$$\nabla u(z) + J \nabla v(z) = \mu(z)(\nabla u(z) - J \nabla v(z)) + 2F.$$

and after rearrangement

$$\frac{1 - \mu(z)}{1 + \mu(z)} \nabla u(z) = -J \nabla v(z) + \frac{1}{1 + \mu(z)} 2F.$$

Therefore we conclude that if $\rho(z) = \frac{1-\mu(z)}{1+\mu(z)}$

$$\operatorname{div}(\rho \nabla u) = \operatorname{div} \frac{1}{1 + \mu(z)} 2F.$$

This together with (6.10) proves the corollary. □

Remark 6.2. From the viewpoint of physics, the exponent p_K being low means that there areas where the concentration of the electric field is quantitatively high. In [Mi] is purposed to investigate whether there are areas where the electric field is especially feeble. For this question we search for the largest exponent q_K such that for every $q < q_K$

$$\int_R |\nabla u(z)|^{-q} dz < C(R, p).$$

where u is a solution to (1.1). To avoid technical problems with the singular set we assume that the quasiregular mapping f such that $Df =$

$\begin{pmatrix} \nabla u \\ J\rho\nabla u \end{pmatrix}$ is a a local homeomorphism everywhere. This for example is guaranteed if we assume affine boundary values for u (see [LN]).

Essentially the same example shows that $q_K = \frac{2}{K-1}$. We consider the same laminate ν and a generating sequence $\{f_j\}$. We define the same sequence of Beltrami coefficients $\{\tilde{\mu_j}\}$ as in (5.8). The difference is that this time we correct the sequence f_j with the Beltrami operator $(I - \tilde{\mu_j}S)^{-1}$ to obtain a sequence $\{g_j\}$ of solutions of

$$\overline{\partial} g_j(z) - \tilde{\mu_j}(z)\partial g_j(z) = 0 \tag{6.11}$$

in Q. Observe that we are not taking the conjugate of $\partial g_j(z)$ so $\tilde{\mu_j}$ are standard complex dilatations ([Al]). Assume for a moment that the functions g_j are injective. Then, the composition rule for Beltrami coefficients ([Al]) shows that the functions g_j^{-1} satisfy the equations

$$\overline{\partial} g_j^{-1}(z) + \tilde{\mu_j}(g_j^{-1}(z))\overline{\partial g_j^{-1}(z)} = 0 \tag{6.12}$$

on $g_j(Q)$.

Therefore, by the discussion in section 6, the real parts of g_j^{-1} satisfy an isotropic equation like (1.1). Now, a change of variables gives that $\int_{g_j(R)} J_{g_j^{-1}}(z)^{\frac{-1}{K-1}} dz = \int_R J_{g_j}(w)^{\frac{K}{K-1}} dw$ for every R compactly contained in Q. Notice that at this point one have to be careful because the domains $g_j(Q)$ are not the same. However, by the quasisymmetry of g_j we can find a domain $\tilde{Q}$, such that $\{g_j(Q)\}$ converges in the Hausdorff metric to $\tilde{Q}$. In turn this observation yields the result. If the g_j are not injective (and they need no to be) we have to use another result from [AF], Theorem 1.5. In our setting that result implies that there exists another sequence $\{F_j\}$ of injective solutions of (6.11) which generate the staircase laminate ν and hence, whose p_K-norm blows up. It follows that the sequence $\{ReF_j^{-1}{}_{|\tilde{Q}}\}$ prove that $q_K \leq \frac{2}{K-1}$. The other inequality follows from [As1], [LN].

Remark 6.3. In the sequence we have obtained we have no control in the boundary values. This can be fixed in the following way. Firstly, it is clear that the generating sequence $\{f_j\}$ for the Laminate can be assumed to have affine boundary values. Then once we have our choice of Beltrami coefficients $\{\mu_j\}$ we can associate to them the right scalars $\{\rho_j\}$ as above. Then we look at the following boundary value problem

$$\begin{aligned} &\text{div } (\rho_j \nabla u_j) = 0 \\ &u_j - Re(f_j) \in W_0^{1,2}(Q) \end{aligned} \tag{6.13}$$

Then it can be shown (but it is a lengthier computation that the one we have presented) that since $\{Re(f_j)\}$ solves an isotropic equation with right hand side going to zero $\|\nabla u_j - \nabla(Re(f_j))\|_{L^2} \to 0$. It is not hard to see that this it is enough to guarantee that

$\lim_{j\to\infty}\int_R |\nabla u_j|^{p_K} dz = \infty$ where R is a compact set with positive measure.

References

[Al] Ahlfors, L.V. *Lectures on quasiconformal mappings*, Van Nostrand, Princenton 1966.

[As1] Astala, K. Area distortion of quasiconformal mappings, Acta math., 173, (1994), 37-60.

[As2] Astala, K. In preparation.

[AIS] Astala, K; Iwaniec, T; Saksman, E; Beltrami operators in the plane. Duke Math. J. 107 (2001), no. 1, 27–56.

[AF] Astala, K; Faraco, D. Quasiregular mappings and Young measures. Proc. Roy. Soc .Edinburgh Sect A. (To appear)

[AN] Astala,K; Nesi, V. Composites and quasiconformal mappings:New optimal bounds in two dimensions. Calc. Var. Partial Differential Equations (To appear)

[BDF] Braides, Andrea; Defranceschi, Anneliese Homogenization of multiple integrals. Oxford Lecture Series in Mathematics and its Applications, 12. The Clarendon Press, Oxford University Press, New York, 1998.

[Bo] Boyarski , B. V. Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients. (Russian) Mat. Sb. N.S. 43(85) 1957 451–503.

[DM] Dacorogna, B; Marcellini, P *Implicit partial differential equations.* Progress in Nonlinear Differential Equations and their Applications, 37. Birkhuser 1999

[F] Faraco, D. Tartar Conjecture and Beltrami Operators. In preparation

[G] Gehring, F. W. The L^p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. 130 (1973), 265–277.

[IS] Iwaniec, T; Sbordone, C. Quasiharmonic fields. Ann. Inst. H. Poincar Anal. Non Linaire 18, (2001) no. 5,519–572.

[K] Kirheim, B. Geometry and rigidity of microstructures. Habilitation Thesis, Leipzig 2001.

[LN] Leonetti, F; Nesi, V. Quasiconformal solutions to certain first order systems and the proof of a conjecture of G. W. Milton, J. Math. Pures. Appl (9) 76 (1997), 109-124.

[Me] Meyers, Norman G. An L^pe-estimate for the gradient of solutions of second order elliptic divergence equations. Ann. Scuola Norm. Sup. Pisa (3) 17 1963 189–206.

[Mi] Milton, G. Modelling the properties of composites by laminates, in *Homogenization and effective moduli of materials and media*, Ericksen, Kiderleher, Kohn, Lions, eds., IMA volumes in mathematics and its applications 1, Springer-Verlag, New York, 1986.

[Mo] Morrey C. B. On the solution of quasi-linear elliptic partial differential equations. Trans AM. Math. Soc. 43, 1938,126-166

[MS] Müller, S; Šverák, V. Unexpected solutions of first and second order partial differential equations, Doc Math. J. DMV, Extra Vol. ICM 98, pp 691-702

[P1] Pedregal, P. Laminates and microstructure, Europ. J. Appl. Math. 4 1993, 121-149

[P] Pedregal, P. *Parametrized Measures and Variational principles.* l Birkhauser , (1997).

[PS] Piccinini L.C, Spagnolo, S. On the Hölder continuity of solutions of second order elliptic equations in two variables. Ann. Scuola Norm. Sup. Pisa (3) 26 (1972), 391–402.

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