THE COMPLEX KAKEYA MAXIMAL FUNCTION

THEMIS MITSIS

Abstract. We investigate the mapping properties of the complex version of the Kakeya maximal operator.

Let $\mathbb{CP}^n$ be the complex projective n-space equipped with the natural measure. The complex line $l_b(a)$ in $\mathbb{C}^n$ with direction $b \in \mathbb{CP}^{n-1}$ and passing through the point a is defined by

$$l_b(a) = \{z \in \mathbb{C}^n : z = a + wb, \quad w \in \mathbb{C}\},$$

where b is identified with any of its representatives. By abuse of language, we define the δ-tube $T_b^{r,\delta}(a)$ of length r, direction $b \in \mathbb{CP}^{n-1}$ and center at the point $a \in \mathbb{C}^n$ to be the δ-neighborhood of the set

$$B(a, r/2) \cap l_b(a).$$

If $r = 1$ we will use the notation $T_b^\delta(a)$. The center point a will be supressed when the context is clear.

If $f \in L^1_{loc}(\mathbb{C}^n)$, $0 < \delta < 1$, then the complex Kakeya maximal function $f_\delta^* : \mathbb{CP}^{n-1} \to \mathbb{R}$ is defined by

$$f_\delta^*(b) = \sup_{a \in \mathbb{C}^n} \frac{1}{|T_b^\delta(a)|} \int_{T_b^\delta(a)} |f|.$$

The purpose of this paper is to prove nontrivial $L^p \to L^q$ estimates for this operator.

1. Examples and conjecture

If $f = \chi_{B(0,\delta)}$, then $\|f\|_p \simeq \delta^{2n/p}$ and for each $b \in \mathbb{CP}^{n-1}$, $f_\delta^*(b) \simeq \delta^2$. Therefore the best bound one could hope for is

$$\|f_\delta^*\|_q \lesssim \delta^{2(1-n/p)} \|f\|_p. \tag{1.1}$$

For $f = \chi_{T_b^\delta(0)}$ we have $\|f\|_p \simeq \delta^{2(n-1)/p}$ and $\|f_\delta^*\|_q \gtrsim \delta^{2(n-1)/q}$. So if the above estimate holds then

$$\delta^{2(n-1)/q} \lesssim \|f_\delta^*\|_q \lesssim \delta^{2(1-n/p)} \|f\|_p \lesssim \delta^{2(1-n/p)+2(n-1)/p}.$$

Therefore we need $q \leq (n-1)p'$, where p' is the exponent conjugate to p.

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If $f=\chi_{B(0,1)}$, then $\|f\|_p \simeq 1$ and $\|f^*_\delta\|_q \simeq 1$. So in order for estimate (1.1) to hold we need $1 \lesssim \delta^{2(1-n/p)}$ which implies $p \leq n$.

These examples would reasonably lead to the following.

Conjecture. *For every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ so that*

$$\|f^*_\delta\|_q \leq C_\epsilon \left(\frac{1}{\delta}\right)^{2(n/p-1)+\epsilon} \|f\|_p$$

for $f \in L^p(\mathbb{C}^n)$, $1 \leq p \leq n$, $q \leq (n-1)p'$.

Despite the rather misleading terminology, this problem seems to be related to the problem of estimating the 2-plane transform restricted to a $2(n-1)$-dimensional subset of the (real) Grassmannian $G(2n,2)$. However, because of the geometry of the situation the combinatorial approach is applicable, and we use some of the techniques in [1] and [2] to prove the following.

Theorem 1.1. *If $f \in L^p(\mathbb{C}^2)$ then*

$$\|f^*_\delta\|_q \lesssim \left(\frac{1}{\delta}\right)^{2(2/p-1)} \left(\log\frac{1}{\delta}\right)^{1/p'} \|f\|_p$$

for $1 \leq p \leq 2$ and $q \leq p'$.

Theorem 1.2. *For all $\epsilon > 0$ there exists $C_\epsilon > 0$ so that if $f \in L^p(\mathbb{C}^n)$, $n \geq 3$, then*

$$\|f^*_\delta\|_q \leq C_\epsilon \left(\frac{1}{\delta}\right)^{2(n/p-1)+\epsilon} \|f\|_p$$

where $p = (n+2)/2$ and $q = (n-1)p'$.

Throughout this paper, the capital letter C, subscripted or otherwise, will denote various constants, not necessarily the same at each of their occurrences.We finally note the following geometric fact which will be used repeatedly. If we define a distance θ on $\mathbb{CP}^{n-1}$ by

$$\theta(b_1,b_2) = (1-|\langle b_1,b_2\rangle|^2)^{1/2}$$

where $\langle\cdot,\cdot\rangle$ is the usual Hermitian inner product, then for any $a_1,a_2 \in \mathbb{C}^n$, $b_1,b_2 \in \mathbb{CP}^{n-1}$ we have

$$\operatorname{diam}(T^\delta_{b_1}(a_1) \cap T^\delta_{b_2}(a_2)) \lesssim \frac{\delta}{\theta(b_1,b_2)}$$

and

$$|T^\delta_{b_1}(a_1) \cap T^\delta_{b_2}(a_2)| \lesssim \frac{\delta^{2n}}{\theta(b_1,b_2)^2+\delta^2}.$$

2. The case $n = 2$

In this section we use an orthogonality argument to prove Theorem 1.1. By interpolation with the trivial $L^1 \to L^\infty$ bound, it is enough to prove the following.

Proposition 2.1. *There exists a constant* $C > 0$ *such that*

$$\|f_\delta^*\|_2 \leq C \left(\log \frac{1}{\delta}\right)^{1/2} \|f\|_2$$

for all $f \in L^2(\mathbb{C}^n)$.

Proof. Let $\{b_j\}_{j=1}^M$ be a maximal δ-separated subset of $\mathbb{CP}^1$. Note that if $\theta(b, b') \leq \delta$ then for any $a \in \mathbb{C}^2$, $T_b^\delta(a)$ is contained in a fixed dilate $\widetilde{T}_{b'}^\delta(a) := T_{b'}^{C, C\delta}(a)$ of $T_{b'}^\delta(a)$. Therefore there exist $\{a_j\}_{j=1}^M \subset \mathbb{C}^2$ and $\{c_j\}_{j=1}^M \subset \mathbb{R}$ with $\|\{c_j\}_{j=1}^M\|_2 = 1$ so that

$$\begin{aligned}
\|f_\delta^*\|_2 &\leq \left(\sum_j \int_{B(b_j,\delta)} (f_\delta^*(b))^2 db\right)^{1/2} \\
&\lesssim \frac{1}{\delta}\left(\sum_j \left(\int \chi_{\widetilde{T}_{b_j}^\delta(a_j)}(y)|f(y)|dy\right)^2\right)^{1/2} \\
&= \frac{1}{\delta}\int \left(\sum_j c_j \chi_{\widetilde{T}_{b_j}^\delta(a_j)}(y)|f(y)|\right) dy \\
&\leq \frac{1}{\delta}\|f\|_2 \left(\int \left(\sum_j c_j \chi_{\widetilde{T}_{b_j}^\delta(a_j)}(y)\right)^2 dy\right)^{1/2} \\
&= \frac{1}{\delta}\|f\|_2 \left(\sum_{j,k} c_j c_k |\widetilde{T}_{b_j}^\delta(a_j) \cap \widetilde{T}_{b_k}^\delta(a_k)|\right)^{1/2} \\
&\lesssim \|f\|_2 \left(\sum_{j,k} c_j c_k \frac{\delta^2}{\theta(b_j, b_k)^2 + \delta^2}\right)^{1/2}.
\end{aligned}$$

Note that for any fixed j we have

$$\sum_k \frac{\delta^2}{\theta(b_j, b_k)^2 + \delta^2} \lesssim \int_{\mathbb{CP}^1} \frac{db}{\theta(b_j, b)^2 + \delta^2} \lesssim \log \frac{1}{\delta}.$$

The estimate now follows by Schur's test. □

3. The higher dimensional case: Preliminaries

In this section we prove two lemmas that will be needed for the main argument. For the rest of the paper we assume that $n \geq 3$.

Lemma 3.1. *Let $E \subset \mathbb{C}^n$ and $\{b_j\}_{j=1}^M$ be a β-separated subset of $\mathbb{CP}^{n-1}$. Suppose that for each j there is a tube $T_{b_j}^{r,\sigma}$ with $\sigma \leq \beta$ such that*

$$|E \cap T_{b_j}^{r,\sigma} \cap (B(a, C\sigma/\beta))^{\complement}| \geq \lambda |T_{b_j}^{r,\sigma}|$$

for all $a \in \mathbb{C}^n$. Then

$$|E| \gtrsim \lambda \sigma^{2(n-1)} M^{1/2}$$

where the implicit constant depends on n and r only.

Proof. Fix $N > 0$ and consider two cases.

(I) $\forall a \in E \quad |\{j : a \in T_{b_j}^{r,\sigma}\}| \leq N$.
(II) $\exists a \in E \quad |\{j : a \in T_{b_j}^{r,\sigma}\}| \geq N$.

In case (I) we have

$$|E| \geq \Big| \bigcup_j E \cap T_{b_j}^{r,\sigma} \Big| \geq \frac{1}{N} \sum_j |E \cap T_{b_j}^{r,\sigma}| \gtrsim \frac{M}{N} \lambda \sigma^{2(n-1)}.$$

In case (II) let $\mathcal{B}$ be a maximal $C_1\beta$-separated subset of $\{b_j : a \in T_{b_j}^{r,\sigma}\}$. Then for all $b_j, b_k \in \mathcal{B}$ with $j \neq k$ we have

$$T_{b_j}^{r,\sigma} \cap T_{b_k}^{r,\sigma} \subset B(a, C\sigma/\beta)$$

provided that C_1 has been chosen large enough. Therefore

$$\begin{aligned} |E| &\geq \Big| \bigcup_{j : b_j \in \mathcal{B}} E \cap T_{b_j}^{r,\sigma} \cap (B(a, C\sigma/\beta))^{\complement} \Big| \\ &= \sum_{j : b_j \in \mathcal{B}} |E \cap T_{b_j}^{r,\sigma} \cap (B(a, C\sigma/\beta))^{\complement}| \\ &\gtrsim N \lambda \sigma^{2(n-1)}. \end{aligned}$$

Consequently

$$|E| \gtrsim \lambda \sigma^{2(n-1)} \min\{M/N, N\} \simeq \lambda \sigma^{2(n-1)} M^{1/2}.$$

□

The next lemma is the direct higher dimensional analogue of the two-dimensional result.

Lemma 3.2. *Let $E \subset \mathbb{C}^n$ and $\{b_j\}_{j=1}^M$ be a δ-separated subset of $\mathbb{CP}^{n-1}$. Suppose that $\Pi \subset \mathbb{C}^n$ is a complex plane and that for each j there is a tube $T_{b_j}^{\delta} \subset \Pi^{C\delta}$, where $\Pi^{C\delta}$ is the $C\delta$-neighborhood of Π, such that*

$$|E \cap T_{b_j}^{\delta}| \geq \lambda |T_{b_j}^{\delta}|.$$

Then

$$|E \cap \Pi^{C\delta}| \gtrsim \lambda^2 M \delta^{2(n-1)} \left(\log \frac{1}{\delta} \right)^{-1}.$$

Proof. Note that since the tubes are contained in the $C\delta$-neighborhood of a plane, we have

$$|\{k : \sigma \leq \theta(b_k, b_j) \leq 2\sigma\}| \lesssim (\sigma/\delta)^2$$

for any j and $\sigma \geq \delta$. Therefore

$$\begin{aligned}
M\lambda\delta^{2(n-1)} &\lesssim \sum_j |E \cap \Pi^{C\delta} \cap T^\delta_{b_j}| = \int_{E\cap\Pi^{C\delta}} \sum_j \chi_{T^\delta_{b_j}} \\
&\leq |E \cap \Pi^{C\delta}|^{1/2} \Big\| \sum_j \chi_{T^\delta_{b_j}} \Big\|_2 \\
&= |E \cap \Pi^{C\delta}|^{1/2} \Big(\sum_{j,k} |T^\delta_{b_j} \cap T^\delta_{b_k}| \Big)^{1/2} \\
&\lesssim |E \cap \Pi^{C\delta}|^{1/2} \Big(M\delta^{2(n-1)} + \sum_k \sum_{j \neq k} |T^\delta_{b_j} \cap T^\delta_{b_k}| \Big)^{1/2} \\
&\lesssim |E \cap \Pi^{C\delta}|^{1/2} \Big(M\delta^{2(n-1)} \\
&+ \sum_k \sum_{l=0}^{\log(C/\delta)} \frac{\delta^{2n}}{(\delta 2^l)^2} |\{j : \delta 2^l \leq \theta(b_k, b_j) \leq \delta 2^{l+1}\}| \Big)^{1/2} \\
&\lesssim |E \cap \Pi^{C\delta}|^{1/2} \Big(M\delta^{2(n-1)} + \sum_k \sum_{l=0}^{\log(C/\delta)} \frac{\delta^{2n}}{(\delta 2^l)^2} \frac{(\delta 2^l)^2}{\delta^2} \Big)^{1/2} \\
&\lesssim |E \cap \Pi^{C\delta}|^{1/2} M^{1/2} (\log(1/\delta))^{1/2} \delta^{n-1}.
\end{aligned}$$

□

4. The main argument

In this section we prove the main lemma to the proof of the higher dimensional result.

Lemma 4.1. *Suppose $E \subset \mathbb{C}^n$, ν is a large positive number and $0 \leq \delta \leq 1/2$. Let $\{b_j\}_{j=1}^M$ be δ-separated subset of $\mathbb{CP}^{n-1}$ so that for each j there is a tube $T^\delta_{b_j}$ satisfying*

$$|T^\delta_{b_j} \cap E| \geq \lambda |T^\delta_{b_j}|$$

and

$$|T^\delta_{b_j} \cap E \cap B(a, C_0(\log(1/\delta)^{-\nu})| \leq (C_1 \log(1/\delta))^{-1} \lambda |T^\delta_{b_j}|$$

for all $a \in \mathbb{C}^n$. Then there exist positive constants κ_ν and C_ν such that

$$|E| \geq C_\nu^{-1} \lambda^2 \delta^{n-2} (\log(1/\delta))^{-\kappa_\nu} (M\delta^{2(n-1)})^{n/(2(n-1))}.$$

In the rest of this section we assume that we are working with a family of tubes satisfying the conditions above. We consider two cases of intersecting tubes. For any given N, we say that $T^\delta_{b_j}$ has property (LM) if

$$|\{z \in T^\delta_{b_j} \cap E : |\{k : z \in T^\delta_{b_k}\}| \le N\}| \ge \frac{\lambda}{2}|T^\delta_{b_j}|$$

and we say that $T^\delta_{b_j}$ has property (HM)_σ if for some $\delta \le \sigma \le 1$ we have

$$\begin{aligned}|\{z \in T^\delta_{b_j} \cap E : |\{k : z \in T^\delta_{b_k} \text{ and } \sigma \le \theta(b_k, b_j) \le 2\sigma\}| \\ \ge (C\log(1/\delta))^{-1}N\}| \ge (C\log(1/\delta))^{-1}\lambda|T^\delta_{b_j}|,\end{aligned}$$

where C is an appropriate constant. The following lemma shows that for any N, a large number of tubes must all have one property or the other.

Lemma 4.2. *Either at least $M/2$ tubes $T^\delta_{b_j}$ have property (LM), or at least $(C\log(1/\delta))^{-1}M$ tubes $T^\delta_{b_j}$ have property $(HM)_\sigma$ for some $\sigma \in [\delta, 1]$.*

Proof. Suppose less than $M/2$ tubes have property (LM). Then there are at least $M/2$ tubes $T^\delta_{b_j}$ so that

$$|\{z \in T^\delta_{b_j} \cap E : |\{k : z \in T^\delta_{b_k}\}| \ge N\}| \ge \frac{\lambda}{2}|T^\delta_{b_j}|.$$

For each point z in the above set, there are at least N tubes $T^\delta_{b_k}$ with $z \in T^\delta_{b_k}$. Therefore, there is a number $n(z)$ between 1 and $C\log(1/\delta)$ so that there are more than $(C\log(1/\delta))^{-1}N$ tubes $T^\delta_{b_k}$ containing z and satisfying $\delta 2^{n(z)} \le \theta(b_j, b_k) \le \delta 2^{n(z)+1}$. Thus for each tube $T^\delta_{b_j}$ there is an $n(j)$ such that

$$\begin{aligned}|\{z \in T^\delta_{b_j} \cap E : |\{k : z \in T^\delta_{b_k} \text{ and } \delta 2^{n(j)} \le \theta(b_k, b_j) \le \delta 2^{n(j)+1}\}| \\ \ge (C\log(1/\delta))^{-1}N\}| \ge (C\log(1/\delta))^{-1}\lambda|T^\delta_{b_j}|.\end{aligned}$$

Since this is true for more than $M/2$ tubes, there are at least $(C\log(1/\delta))^{-1}M$ tubes $T^\delta_{b_j}$ that share a common $n(j) = n_0$. Therefore each of those tubes satisfies

$$\begin{aligned}|\{z \in T^\delta_{b_j} \cap E : |\{k : z \in T^\delta_{b_k} \text{ and } \delta 2^{n_0} \le \theta(b_k, b_j) \le \delta 2^{n_0+1}\}| \\ \ge (C\log(1/\delta))^{-1}N\}| \ge (C\log(1/\delta))^{-1}\lambda|T^\delta_{b_j}|\end{aligned}$$

which proves the lemma with $\sigma = \delta 2^{n_0}$. □

The case when many of the tubes have property (LM) is easily handled.

Lemma 4.3. *If at least $M/2$ tubes have property (LM), then*

$$|E| \gtrsim \lambda \frac{M}{N}\delta^{2(n-1)}.$$

Proof. Let $\mathcal{B} = \{T_{b_j}^\delta : T_{b_j}^\delta$ has property (LM)$\}$ and put

$$S_{b_j}^\delta = \{z \in T_{b_j}^\delta \cap E : |\{k : z \in T_{b_k}^\delta\}| \leq N\}.$$

Then by the definition of property (LM)

$$|S_{b_j}^\delta| \geq \frac{\lambda}{2}|T_{b_j}^\delta|.$$

Therefore

$$|E| \geq \Big| \bigcup_{j:T_{b_j}^\delta \in \mathcal{B}} S_{b_j}^\delta \Big| \geq \frac{1}{N} \sum_{j:T_{b_j}^\delta \in \mathcal{B}} |S_{b_j}^\delta| \gtrsim \lambda \frac{M}{N} \delta^{2(n-1)}.$$

□

In the case of high multiplicity intersections we put $\widetilde{T}_{b_j}^\sigma = T_{b_j}^{100,100\sigma}$ and carry out a certain geometric cunstruction that passes a number of complex planes Π through the axes of all tubes with property $(\mathrm{HM})_\sigma$. Then the optimal estimate for tubes lying in a neighborhood $\Pi^{C\delta}$ of a plane Π is applied to each of these planes. It is shown that most of the intersection of E with the $\Pi^{C\delta}$'s lies in a small number of them giving a lower bound for $|\widetilde{T}_{b_j}^\sigma \cap E|$. Depending on whether σ is large or small we then estimate $|E|$ by $|\widetilde{T}_{b_j}^\sigma \cap E|$ if σ is large, or by using Lemma 3.1 if σ is small.

Lemma 4.4. *Fix j and assume that $T_{b_j}^\delta$ has property (HM)$_\sigma$. Then*

$$|\widetilde{T}_{b_j}^\sigma \cap E \cap (B(a, C_0(\log(1/\delta))^{-\nu}))^{\complement}| \gtrsim \lambda^3 \delta^{2(n-2)} (\log(1/\delta))^{-2(n-2)\nu-3} \sigma^2 N$$

for any $a \in \mathbb{C}^n$.

Proof. Fix a point $a \in \mathbb{C}^n$ and put

$$E' = E \setminus B(a, C_0(\log(1/\delta))^{-\nu}).$$

Let l_j be the complex line with direction b_j passing through the center of $T_{b_j}^\delta$. Define

$$\mathcal{Z} = \{z \in \mathbb{C}^n : \mathrm{dist}(z, l_j) \geq \sigma(\log(1/\delta))^{-\nu}\}$$

and

$$\mathcal{F} = \{T_{b_k}^\delta : T_{b_j}^\delta \cap T_{b_k}^\delta \neq \emptyset \text{ and } \sigma \leq \theta(b_k, b_j) \leq 2\sigma\}.$$

If $\delta \leq \sigma(\log(1/\delta))^{-\nu}$ then for all $T_{b_k}^\delta \in \mathcal{F}$ we have

$$T_{b_k}^\delta \subset \widetilde{T}_{b_j}^\sigma \quad \text{and} \quad \mathrm{diam}(T_{b_k}^\delta \cap \mathcal{Z}^{\complement}) \lesssim (\log(1/\delta))^{-\nu},$$

and so, the set $T_{b_k}^\delta \cap E' \cap \mathcal{Z}^{\complement}$ is contained in a ball $B(a', C_0(\log(1/\delta))^{-\nu})$, for some $a' \in \mathbb{C}^n$, provided that C_0 has been chosen sufficiently large. Therefore

$$|T_{b_k}^\delta \cap E' \cap \mathcal{Z}| = |T_{b_k}^\delta \cap E'| - |T_{b_k}^\delta \cap E' \cap \mathcal{Z}^{\complement}|$$

$$
\begin{aligned}
&\geq |T_{b_k}^\delta \cap E| - |T_{b_k}^\delta \cap E \cap B(a, C_0(\log(1/\delta))^{-\nu})| \\
&- |T_{b_k}^\delta \cap E \cap B(a', C_0(\log(1/\delta))^{-\nu})| \\
&\geq (1 - 2(C_1 \log(1/\delta))^{-1})\lambda |T_{b_k}^\delta| \\
&\geq \frac{\lambda}{2} |T_{b_k}^\delta|
\end{aligned}
$$

for C_1 large enough.

Now let $\{\alpha_i\}$ be a maximal δ/σ-separated subset of $\{b \in \mathbb{CP}^{n-1} : \langle b, b_j \rangle = 0\}$ and consider the complex planes Π_i spanned by b_j and α_i. These planes have the following properties which can be verified by elementary geometric arguments.

(i) Every $T_{b_k}^\delta \in \mathcal{F}$ is contained in $\Pi_i^{C\delta}$ for some i.
(ii) If $z \in \mathcal{Z}$ then $|\{i : z \in \Pi_i^{C\delta}\}| \lesssim (\log(1/\delta))^{2(n-2)\nu}$.

Therefore if we let

$$\mathcal{F}_i = \{T_{b_k}^\delta \in \mathcal{F} : T_{b_k}^\delta \subset \Pi_i^{C\delta}\},$$

then Lemma 3.2 and property (i) imply

$$|\widetilde{T}_{b_j}^\sigma \cap E' \cap \mathcal{Z} \cap \Pi_i^{C\delta}| \gtrsim \lambda^2 \delta^{2(n-1)} (\log(1/\delta))^{-1} |\mathcal{F}_i|.$$

Summing over i and using property (ii) we get

$$
\begin{aligned}
|\mathcal{F}| \leq \sum_i |\mathcal{F}_i| &\lesssim \lambda^{-2}\delta^{-2(n-1)} \log(1/\delta) \sum_i |\widetilde{T}_{b_j}^\sigma \cap E' \cap \mathcal{Z} \cap \Pi_i^{C\delta}| \\
&\lesssim \lambda^{-2}\delta^{-2(n-1)} (\log(1/\delta))^{1+2(n-2)\nu} |\widetilde{T}_{b_j}^\sigma \cap E'|. \qquad (4.1)
\end{aligned}
$$

If $\delta \geq \sigma(\log(1/\delta))^{-\nu}$, then one proves (4.1) as before, using Lemma 3.2, but now one makes the crude estimate $|\{\alpha_i\}| \lesssim (\sigma/\delta)^{2(n-2)}$ for the overlap of the sets $\{\Pi_i^{C\delta}\}$. We omit the details.

To estimate $|\mathcal{F}|$ from below, let

$$
\begin{aligned}
A_j = \{z \in T_{b_j}^\delta \cap E : |\{k : z \in T_{b_k}^\delta \text{ and } \sigma \leq \theta(b_k, b_j) \leq 2\sigma\}| \\
\gtrsim (\log(1/\delta))^{-1} N\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\mathcal{F}| &\gtrsim \sum_{k : T_{b_k}^\delta \in \mathcal{F}} |T_{b_k}^\delta \cap T_{b_j}^\delta| \frac{\sigma^2}{\delta^{2n}} \\
&= \frac{\sigma^2}{\delta^{2n}} \int_{T_{b_j}^\delta} \sum_{k : T_{b_k}^\delta \in \mathcal{F}} \chi_{T_{b_k}^\delta} \\
&\geq \frac{\sigma^2}{\delta^{2n}} \int_{A_j} \sum_{k : T_{b_k}^\delta \in \mathcal{F}} \chi_{T_{b_k}^\delta}
\end{aligned}
$$

$$\gtrsim \frac{\sigma^2}{\delta^2}(\log(1/\delta))^{-2}N\lambda \tag{4.2}$$

with the last inequality true by property $(\mathrm{HM})_\sigma$. Combining (4.1) and (4.2) we obtain the desired estimate. □

We are now in a position to prove an estimate for the case of a large number of tubes having high multiplicity intersections.

Lemma 4.5. *If for some $\sigma \in [\delta, 1]$ at least $(C\log(1/\delta))^{-1}M$ tubes $T_{b_j}^\delta$ have property $(HM)_\sigma$, then*

$$|E| \gtrsim \lambda^3\delta^{2(n-2)}N(\log(1/\delta))^{-\alpha_\nu}(M\delta^{2(n-1)})^{1/(n-1)},$$

for some $\alpha_\nu > 0$.

Proof. In what follows, $\alpha_{\nu,1}, \alpha_{\nu,2}, \alpha_{\nu,3}$ are positive constants whose exact values are irrelevant. If $\sigma \geq (\log(1/\delta))^{-\nu}(M\delta^{2(n-1)})^{1/(2(n-1))}$, then by Lemma 4.4 we have

$$\begin{aligned}|E| \geq |\widetilde{T}_{b_j}^\sigma \cap E| &\gtrsim \lambda^3\delta^{2(n-2)}(\log(1/\delta))^{-2(n-2)\nu-3}\sigma^2 N \\ &\geq \lambda^3\delta^{2(n-2)}N(\log(1/\delta))^{-\alpha_{\nu,1}}(M\delta^{2(n-1)})^{1/(n-1)}.\end{aligned}$$

If $\sigma \leq (\log(1/\delta))^{-\nu}(M\delta^{2(n-1)})^{1/(2(n-1))}$, then Lemma 4.4 implies that there are at least $(C\log(1/\delta))^{-1}M$ tubes $\widetilde{T}_{b_j}^\sigma$ with δ-separated directions b_j satisfying

$$\begin{aligned}&|\widetilde{T}_{b_j}^\sigma \cap E \cap (B(a, C(\log(1/\delta))^{-\nu}))^\complement| \\ &\qquad\gtrsim \lambda^3(\delta/\sigma)^{2(n-2)}(\log(1/\delta))^{-2(n-2)\nu-3}N|\widetilde{T}_{b_j}^\sigma|.\end{aligned}$$

Choose a maximal β-separated subset of these b_j's with $\beta = C\sigma(\log(1/\delta))^\nu$. This set has cardinality

$$\widetilde{M} \gtrsim (\log(1/\delta))^{-2\nu(n-1)-1}(\delta/\sigma)^{2(n-1)}M.$$

Applying Lemma 3.1 with $\beta = C\sigma(\log(1/\delta))^\nu$, λ replaced with

$$\lambda^3(\delta/\sigma)^{2(n-2)}(\log(1/\delta))^{-2(n-2)\nu-3}N$$

and M replaced with $\widetilde{M}$ we get

$$|E| \gtrsim \lambda^3\delta^{2(n-2)}(\log(1/\delta))^{-\alpha_{\nu,2}}N\sigma^2\left(\frac{M\delta^{2(n-1)}}{\sigma^{2(n-1)}}\right)^{1/2}.$$

The above expression is a decreasing function of σ. Therefore

$$|E| \gtrsim \lambda^3\delta^{2(n-2)}N(\log(1/\delta))^{-\alpha_{\nu,3}}(M\delta^{2(n-1)})^{1/(n-1)}.$$

Letting $\alpha_\nu = \max\{\alpha_{\nu,1}, \alpha_{\nu,3}\}$, we complete the proof. □

From here it is a relatively simple matter to prove the main lemma.

Proof of Lemma 4.1. Note that by Lemma 4.2, Lemma 4.3 and Lemma 4.5, for any N, either

$$|E| \gtrsim \lambda \frac{M}{N} \delta^{2(n-1)},$$

or

$$|E| \gtrsim \lambda^3 \delta^{2(n-2)} N (\log(1/\delta))^{-\alpha_\nu} (M\delta^{2(n-1)})^{1/(n-1)}.$$

Equating the right-hand sides of the above inequalities and solving for N we obtain

$$|E| \geq C_\nu^{-1} \lambda^2 \delta^{n-2} (\log(1/\delta))^{-\kappa_\nu} (M\delta^{2(n-1)})^{n/(2(n-1))}.$$

□

5. Completion of the proof

First, we reformulate Lemma 4.1 as follows.

Lemma 5.1. *Suppose $E \subset \mathbb{C}^n$, $A \subset \mathbb{CP}^{n-1}$, ν is a large positive number, $0 < \delta \leq 1/2$, and for each $b \in A$ a tube T_b^δ is given so that*

$$|T_b^\delta \cap E| \geq \lambda |T_b^\delta|$$

and

$$|T_b^\delta \cap E \cap B(a, C_0(\log(1/\delta))^{-\nu})| \leq (C_1 \log(1/\delta))^{-1} \lambda |T_b^\delta|$$

for all $a \in \mathbb{C}^n$. Then there exist positive constants κ_ν and C_ν so that

$$|A| \leq C_\nu (\log(1/\delta))^{\kappa_\nu} \left(\frac{|E|}{\lambda^2 \delta^{2(n-p)}} \right)^{q/p}$$

where $p = (n+2)/2$ and $q = (n-1)/p'$.

To prove Theorem 1.2, it is enough to prove the following restricted weak-type estimate at the endpoint.

Estimate W(δ). *If $E \subset \mathbb{C}^n$ and $A_\lambda = \{b \in \mathbb{CP}^{n-1} : (\chi_E)_\delta^*(b) \geq \lambda\}$, then for all $\epsilon > 0$ there exists $C_\epsilon > 0$ so that*

$$|A_\lambda| \leq C_\epsilon \delta^{-\epsilon} \left(\frac{|E|}{\lambda^p \delta^{2(n-p)}} \right)^{q/p}$$

where $p = (n+2)/2$ and $q = (n-1)/p'$.

Fix $\epsilon > 0$. We prove W(δ) by downward induction on δ. Note that W(δ) is trivial for $\delta > \delta_0$ for any given δ_0, so long as C_ϵ is large enough. Fix ν so that $\nu\epsilon > q$ and choose $\delta_0 < 1/2$ so that $\delta < \delta_0$ implies

$$(\log(1/\delta))^{q-\nu\epsilon} < C_3^{-1}$$

where C_3 is a constant to be determined later. Choose C_ϵ so that W(δ) holds if $\delta \geq \delta_0$ and so that if $\delta \leq \delta_0$ then

$$C_\nu (\log(1/\delta))^{\kappa_\nu} \leq \frac{1}{2} C_\epsilon \delta^{-\epsilon}.$$

We will show that if W($\bar{\delta}$) holds for $\bar{\delta} = \delta(\log(1/\delta))^\nu$, $\delta \le \delta_0$, then W(δ) holds. We consider two sets and estimate them separately. Let

$$A^1_\lambda = \{b \in \mathbb{CP}^{n-1} : \exists T^\delta_b \text{ with } |T^\delta_b \cap E| \ge \lambda|T^\delta_b| \text{ and}$$
$$|T^\delta_b \cap E \cap B(a, C_0(\log(1/\delta))^{-\nu})| \le (C_1 \log(1/\delta))^{-1}\lambda|T^\delta_b| \ \forall a \in \mathbb{C}^n\}$$

and

$$A^2_\lambda = \{b \in \mathbb{CP}^{n-1} : \exists T^\delta_b \text{ and } a \in \mathbb{C}^n \text{ so that}$$
$$|T^\delta_b \cap E \cap B(a, C_0(\log(1/\delta))^{-\nu})| \ge (C_1 \log(1/\delta))^{-1}\lambda|T^\delta_b|\}.$$

Then $A_\lambda \subset A^1_\lambda \cup A^2_\lambda$. Since the set A^1_λ satisfies the hypotheses of Lemma 5.1, we have

$$|A^1_\lambda| \le C_\nu(\log(1/\delta))^{\kappa_\nu}\left(\frac{|E|}{\lambda^2\delta^{2(n-p)}}\right)^{q/p} \le \frac{1}{2}C_\epsilon\delta^{-\epsilon}\left(\frac{|E|}{\lambda^p\delta^{2(n-p)}}\right)^{q/p}.$$

To estimate $|A^2_\lambda|$, note that if $b \in A^2_\lambda$, then $T^\delta_b \cap B(a, C_0(\log(1/\delta))^{-\nu})$ is contained in a tube $\widetilde{T}_b$ of dimensions, roughly, $(\log(1/\delta))^{-\nu} \times (\log(1/\delta))^{-\nu} \times \delta \times \cdots \times \delta$. Therefore

$$|\widetilde{T}_b \cap E| \gtrsim (\log(1/\delta))^{-1}\lambda|T^\delta_b| \gtrsim (\log(1/\delta))^{2\nu-1}\lambda|\widetilde{T}_b|.$$

Now let $\bar{E} = (C\log(1/\delta))^\nu E$ (the dilate of E), and $g = \chi_{\bar{E}}$. Then

$$g^*_{\bar{\delta}}(b) \ge \frac{|T^{\bar{\delta}}_b \cap \bar{E}|}{|T^{\bar{\delta}}_b|} \gtrsim \frac{|\widetilde{T}_b \cap E|}{|\widetilde{T}_b|} \ge C_2^{-1}(\log(1/\delta))^{2\nu-1}\lambda$$

for C_2 large enough. So, by the inductive hypothesis,

$$|A^2_\lambda| \le C_\epsilon\bar{\delta}^{-\epsilon}\left(\frac{|\bar{E}|}{(C_2^{-1}\lambda(\log(1/\delta))^{2\nu-1})^p\bar{\delta}^{2(n-p)}}\right)^{q/p}.$$

Since $|\bar{E}| = (C\log(1/\delta))^{2n\nu}|E|$ we get

$$|A^2_\lambda| \le \frac{1}{2}C_\epsilon\delta^{-\epsilon}\left(\frac{|E|}{\lambda^p\delta^{2(n-p)}}\right)^{q/p},$$

provided that C_3 has been chosen large enough. Therefore

$$|A_\lambda| \le |A^1_\lambda| + |A^2_\lambda| \le C_\epsilon\delta^{-\epsilon}\left(\frac{|E|}{\lambda^p\delta^{2(n-p)}}\right)^{q/p}$$

proving the estimate.

References

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Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland
E-mail address: mitsis@maths.jyu.fi