PATHWISE CONNECTIVITY OF A CONFORMAL BOUNDARY

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Abstract. We show that, in dimensions $n \geq 3$, the metric boundary of a conformal deformation of the unit ball is pathwise connected, and even of bounded turning, provided the conformal scaling factor satisfies a Harnack inequality and the volume growth of the deformed space is at most euclidean.

Following [1] we consider conformal deformations of the unit ball of $\mathbb{R}^n$, $n \geq 2$, of the following type. Let $\rho : \mathbb{B}^n \to (0, \infty)$ be a continuous function satisfying for some constants $A \geq 1$ and $B > 0$,

$$HI(A) \qquad 1/A \leq \frac{\rho(x)}{\rho(y)} \leq A \quad \text{whenever } x, y \in B_z = B(z, \frac{1}{2}(1 - |z|)) \text{ for any } z \in \mathbb{B}^n,$$

and

$$VG(B) \qquad \mu_\rho(B_\rho(x, r)) \leq Br^n \quad \text{for all } x \in \mathbb{B}^n, \text{ and } r > 0.$$

Here $\mu_\rho(E) = \int_E \rho^n dm_n$ and $B_\rho(x, r)$ is a ball center centered at x with radius r with respect to the metric d_ρ, defined for $x, y \in \mathbb{B}^n$ by the formula

$$d_\rho(x, y) = \inf_\gamma \int_\gamma \rho ds,$$

where the infimum is taken over all curves in $\mathbb{B}^n$ with endpoints x and y. We will call a function ρ satisfying both $HI(A)$ and $VG(B)$ for some constants A and B *a conformal density*. Notice that $HI(A)$ requires that the density ρ satisfies a Harnack inequality and $VG(B)$ that the volume growth is at most euclidean. The *ρ-boundary of the unit*

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ball, $\partial_\rho\mathbb{B}^n$, is defined as $\overline{(\mathbb{B}^n, d_\rho)} \setminus (\mathbb{B}^n, d_\rho)$. Here we take the abstract closure of the metric space $(\mathbb{B}^n, d_\rho)$. The metric d_ρ then extends in a natural way to this boundary. Thanks to [1], $\partial_\rho\mathbb{B}^n$ can be identified with the set of those points $\varsigma \in \partial\mathbb{B}^n$ for which the integral of ρ along the ray $[0, \varsigma)$ is finite. It was shown in [1] that many of the properties of quasiconformal images of balls are shared by the metric space $(\mathbb{B}^n, d_\rho)$, and the size of $\partial_\rho\mathbb{B}^n$ in the metric d_ρ was further examined in [2].

The study of the connectivity properties of $\partial_\rho\mathbb{B}^n$ was initiated in [7]. In particular, it was shown that, in dimensions $n \geq 3$, for all $\varsigma, \vartheta \in \partial_\rho\mathbb{B}^n$ outside a small exceptional set, there exists a path, i.e. a continuous mapping, $\gamma : ([0,1], |.|) \to (\partial_\rho\mathbb{B}^n, d_\rho)$ that connects ς and ϑ. In fact, γ can be taken to be Hölder continuous. It remained however open if $(\partial_\rho\mathbb{B}^n, d_\rho)$ is pathwise connected when $n \geq 3$; it is easy to give examples in dimension two where this fails.

In this note, we show that $(\partial_\rho\mathbb{B}^n, d_\rho)$ is indeed pathwise connected for $n \geq 3$ by establishing the following theorem.

Theorem A. *Let $n \geq 3$, and assume that ρ is a conformal density on $\mathbb{B}^n$. Then $(\partial_\rho\mathbb{B}^n, d_\rho)$ is pathwise connected. Moreover, any pair ς, ϑ of points in $\partial_\rho\mathbb{B}^n$ can be joined by a path γ in $\partial_\rho\mathbb{B}^n$ with $diam_\rho(\gamma) \leq C(A,B)d_\rho(\varsigma, \vartheta)$.*

Here $\mathrm{diam}_\rho(\gamma)$ refers to the diameter of the image of γ in the ρ-metric. It is then natural to inquire if we could replace the diameter by length in Theorem A. This turns out not to be the case, as is easily seen by taking, for example, $\rho(t) = (1-t)^{-1/2}$. Then there are no non-trivial rectifiable paths on $\partial_\rho\mathbb{B}^n$. See [1] for the fact that this ρ satisfies our assumptions; examples of this type can even be realized as quasiconformal mappings [3]. Thus our metric space is, in the lingo of quasiconformal mappings, of bounded turning but not quasiconvex. When ρ arises from a quasiconformal mapping, the bounded turning condition translates to a property of the boundary of image domain in the internal metric. The results by Gehring [4] show that the complement of the image domain is bounded turning in the euclidean metric, but our conclusion appears to be new even in this setting.

It would be interesting to know if, in dimensions $n \geq 4$, one has even stronger connectivity properties, say, if the boundary is then simply connected. Furthermore, it is not clear to us to what extent the underlying metric space $(\mathbb{B}^n, |.|)$ can be replaced with a more general space; for an extension of a large part of [1] to a more general setting see [5].

1. Preliminaries

We denote by $C(\alpha, \beta, ...)$ a positive constant whose value can be chosen in a way that is dependent only on quantities $\alpha, \beta, ...$ and nothing else. By writing $[x, y)$ we mean the set $\{ty + (1-t)x : \ 0 \leq t < 1\}$. Given $\varsigma \in \partial_\rho \mathbb{B}^n$, $\lambda \in (0,1)$, and $r \in (0,1]$ we set

$$\mathrm{Cone}(\varsigma, \lambda, r) = \bigcup \{B\big(t\varsigma, \lambda(1-t)\big) : 1 - r \leq t < 1\}.$$

We need the following version of a density estimate originally proven in [7]. In the original version, the radius of the euclidean ball is r, and instead of $\frac{4}{5}$, the constant in the definition of δ is $\frac{5}{9}$. However, by $HI(A)$ and Lemma 1.3 in [7], the argument in the proof of Theorem 3.1 in [7] applies with these modifications. The version stated below can also be obtained by combining 2.1 and Lemma 3.2 in [1] with Proposition 4.3 in [6] and using an auxiliary Möbius transformation.

We formulate the estimate in terms of *the Hausdorff content* H^∞_α (of a set E) by which we mean the number

$$H^\infty_\alpha(E) = \inf\{\sum_{k=1}^{\infty} r_k^\alpha : \ E \subset \bigcup_{k=1}^{\infty} B(x_k, r_k)\}.$$

Lemma 1.1. *Let $\varsigma \in \partial_\rho \mathbb{B}^n$ and $0 < r \leq 1$. Set $\delta = diam_\rho(Cone(\varsigma, \frac{4}{5}, r))$. Then there exists a constant $C = C(B, n) > 0$ such that*

$$H^\infty_1(\big(B(\varsigma, \frac{3}{2}r) \cap \partial\mathbb{B}^n\big) \backslash B_\rho(\varsigma, \hat{r})) \leq \frac{Cr}{\Big(\log(1 + \frac{\hat{r}}{2\delta})\Big)^{n-1}}$$

for $\hat{r} \geq 2\delta$.

We will also repeatedly use the Gehring-Hayman theorem from [1].

Theorem 1.2. (Gehring-Hayman Theorem) *If γ is hyperbolic geodesic in $\mathbb{B}^n$, then*

$$\int_\gamma \rho ds \leq C d_\rho(x,y),$$

where $C = C(A,B,n)$.

2. Pathwise connectivity of the boundary

In the thesis [7, Theorem 3.1], the following result was established.

Theorem 2.1. *Let $0 < \alpha \leq 1$ and $n \geq 3$. There is a set $E \subset \partial\mathbb{B}^n$ such that $H^\infty_\alpha(E) = 0$ with the following property. For every pair of points $\varsigma, \omega \in \partial_\rho\mathbb{B}^n \backslash E$, there exists a euclidean path $\gamma \subset \partial_\rho\mathbb{B}^n \backslash E$ that connects ς and ω, and and so that the identity mapping $id : (\gamma, |.|) \to (\gamma, d_\rho)$ is Hölder continuous with exponent α/n along γ.*

Examples show that one cannot take $E = \emptyset$. In order to prove Theorem A, we must thus give up Hölder continuity.

We split the proof into lemmas. *To simplify our notation, from now on, the balls we are dealing with in both metrics are supposed to be contained as sets in $\partial\mathbb{B}^n$, if not stated otherwise.* Recall here that $\partial_\rho\mathbb{B}^n$ can be identified as a subset of $\partial\mathbb{B}^n$.

Lemma 2.2. *Let $\varsigma \in \partial_\rho\mathbb{B}^n$ and $0 < r \leq 1$. Then there exists a constant $C = C(A,B,n) > 0$ and $j \geq 1$ so that*

$$H^\infty_1(B(\varsigma, \frac{3}{2}r) \setminus E_j) < \frac{r}{100},$$

where

$$E_j = \{\omega \in \partial_\rho\mathbb{B}^n \cap B_\rho(\varsigma, C\hat{r}) : \ \rho(tw) \leq (1-t)^{-1+1/n} \text{ for all } t \in [1-1/j, 1)\}$$

and $\hat{r} = diam_\rho(Cone(\varsigma, \frac{4}{5}, r))$.

Proof. It follows from Lemma 1.1 that there is a constant $C = C(B, n) > 0$ such that

$$H_1^\infty(B(\varsigma, \frac{3}{2}r)\backslash B_\rho(\varsigma, C\hat{r})) < \frac{r}{200}.$$

Thus it is enough to show that we have an appropriate growth estimate for $\rho(t\omega)$ on sufficiently many radii.

Define

$$D_j = \{\omega \in B_\rho(\varsigma, C\hat{r}) : \ \rho(t\omega) > (1-t)^{\frac{1}{n}-1} \text{ for some } t \in [1 - \frac{1}{j}, 1)\}.$$

It suffices to show that there exists $j_0 \in \mathbb{N}$ such that

$$H_1^\infty(B(\varsigma, \frac{3}{2}r) \cap D_{j_0}) < \frac{r}{200}.$$

For that purpose, fix j and consider points $\omega \in B(\varsigma, \frac{3}{2}r) \cap D_j$ such that

$$\rho(t_\omega \omega) > (1-t_\omega)^{\frac{1}{n}-1} \quad \text{for some } t_\omega \in [1 - \frac{1}{j}, 1). \tag{2.3}$$

From $HI(A)$ it follows that there exists a constant $C = C(A, n)$ such that

$$\begin{aligned} \int_{B_{t_\omega \omega}} \rho^n dm_n &\geq C\rho(t_\omega \omega)^n (1-t_\omega)^n \\ &\geq C(1-t_\omega)^{1-n}(1-t_\omega)^n = C(1-t_\omega). \end{aligned}$$

Write also

$$F = \bigcup_{\varphi \in B(\varsigma, \frac{3}{2}r) \cap B_\rho(\varsigma, C\hat{r})} \mathrm{Cone}(\varphi, \frac{1}{2}, 1).$$

Then F is open and, by the Gehring-Hayman theorem, there is a finite l such that $\mathrm{length}_\rho[0, \varphi) \leq l$ for every φ. Thus by $HI(A)$ (cf. Lemma 1.3 in [7]), $\mathrm{diam}_\rho(F) < \infty$. Moreover, from $VG(B)$ it follows that

$$\mu_\rho(F) = \int_F \rho^n dm_n < \infty.$$

Define $u(x) = \rho(x)^n$ for $x \in F$ and $u(x) = 0$ elsewhere. Then $u \in L^1(\mathbb{R}^n)$.

For each $\omega \in B(\varsigma, \frac{3}{2}r) \cap D_j$, pick a closed ball $\bar{B}_\omega = \bar{B}(\omega, \frac{3}{2}(1-t_\omega)) \subset \mathbb{R}^n$ so that the inequality in (2.3) holds. By the Besicovitch covering theorem, there is a constant $C' = C'(n)$, and a cover of $B(\varsigma, \frac{3}{2}r) \cap D_j$ by countably many of these balls, call them $\bar{B}_i$, which cover also the Whitney balls $\hat{B}_i = B(t_{\omega_i}\omega_i, \frac{1}{2}(1-t_{\omega_i}))$, such that

$$\begin{aligned} H_1^\infty(B(\varsigma, \frac{3}{2}r) \cap D_j) &\leq \sum_i (\frac{3}{2}(1-t_{\omega_i})) \\ &\leq \frac{(3/2)}{C} \sum_i \int_{\hat{B}_i} \rho^n dm_n \\ &\leq \frac{(3/2)C'}{C} \int_{\cup_i B_i} u dm_n \\ &= C'' \int_{\cup_i B_i} u \, dm_n. \end{aligned}$$

Since $u \in L^1(\mathbb{R}^n)$ and $t_{\omega_i} \in [1-\frac{1}{j}, 1)$, the last integral above can be made, by enlarging j, as small as we wish. $\square$

Given a 2-sphere $S \subset \partial\mathbb{B}^n$, $\varsigma \in \partial\mathbb{B}^n$, and $r > 0$ we write

$$S^1(\varsigma, r) = \{\omega \in \partial\mathbb{B}^n \cap S : \ |\varsigma - \omega| = r\}.$$

Our next lemma shows that the sets E_j from Lemma 2.3 contain suitable circles and circular arcs.

Lemma 2.4. *Let $\varsigma \in \partial_\rho\mathbb{B}^n$, $0 < r \leq 1$. Suppose that $S \subset \partial\mathbb{B}^n$ is a 2-sphere with $\varsigma \in S$. Then there is $j \geq 1$, a radius $\sqrt{2}r < r' < \frac{3}{2}r$, and a spherical arc $J \subset S$ that joins $S^1(\varsigma, \frac{r}{2})$ to $S^1(\varsigma, \frac{3}{2}r)$ so that $J \cup S^1(\varsigma, r') \subset E_j$.*

Proof. By Lemma 2.2,

$$H_1^\infty(B(\varsigma, \frac{3}{2}r) \setminus E_j) < \frac{r}{100},$$

when j is sufficiently large. Let P be the projection from $S \cap (B(\varsigma, \frac{3}{2}r) \setminus B(\varsigma, \frac{r}{2}))$ to $S^1(\varsigma, r)$. Then P is Lipschitz continuous with constant 2. Thus

$$H_1^\infty(P(S \cap (B(\varsigma, \frac{3}{2}r) \setminus B(\varsigma, \frac{r}{2})) \setminus E_j)) \leq 2H_1^\infty(B(\varsigma, \frac{3}{2}r) \setminus E_j).$$

Because $H_1^\infty(S^1(\varsigma, r)) \geq r/\sqrt{2}$, we conclude that there is a circular arc $J \subset S \cap E_j$, contained in $\overline{B}(\varsigma, \frac{3}{2}r) \setminus B(\varsigma, \frac{r}{2})$, which joins $S^1(\varsigma, \frac{r}{2})$ to $S^1(\varsigma, \frac{3}{2}r)$. The existence of a desired radius $\sqrt{2}r < r' < \frac{3}{2}r$ follows by similar reasoning as above. Observe that $S \cap (B(\varsigma, \frac{3}{2}r) \setminus B(\varsigma, \sqrt{2}r))$ can be projected into J using a Lipschitz mapping with constant 1. Moreover, $H_1^\infty(J \cap (B(\varsigma, \frac{3}{2}r) \setminus B(\varsigma, \sqrt{2}r))) \geq (3/2 - \sqrt{2})r/4$. $\square$

Lemma 2.5. *Let E_j be as above, and let $\gamma : ([0,1], |.|) \to (E_j, |.|)$ be continuous. Then $id \circ \gamma : ([0,1], |.|) \to (E_j, d_\rho)$ is continuous and*

$$diam_\rho(\gamma) + d_\rho(\gamma, \varsigma) \leq 3C\, diam_\rho(Cone(\varsigma, \frac{4}{5}, r)),$$

where C is the constant in Lemma 2.2.

Proof. Let $0 < \epsilon < 1/j$ and $t_0 \in [0,1]$. Write $\varphi = \gamma(t_0)$. Choose $\delta > 0$ so small that for every $t \in [0,1]$ with $|t - t_0| < \delta$,

$$|\gamma(t) - \gamma(t_0)| \leq \epsilon.$$

Write $\varphi = \gamma(t_0)$ and $\phi = \gamma(t)$.

Now, since $\varphi, \phi \in E_j$ and $\epsilon < 1/j$,

$$\begin{aligned} \text{length}_\rho[(1-\epsilon)\varphi, \varphi) &= \int_{1-\epsilon}^{1} \rho(t\varphi)dt \\ &\leq \int_{1-\epsilon}^{1} (1-t)^{\frac{1}{n}-1} dt = n\epsilon^{1/n}. \end{aligned} \tag{2.6}$$

By (2.6) and $HI(A)$ (cf. Lemma 1.3 in [7]), there is a constant $C = C(A) > 0$ such that

$$\text{diam}_\rho(\text{Cone}(\varphi, \frac{1}{2}, \epsilon)) \leq Cn\epsilon^{1/n}.$$

This estimate also holds for ϕ. Since the cones at φ and ϕ intersect, φ and ϕ can be joined in $\mathbb{B}^n$ with a path whose ρ-length is at most $2Cn\epsilon^{1/n}$. In other words,

$$d_\rho(\varphi, \phi) \leq 2Cn\epsilon^{1/n}.$$

The continuity of $id \circ \gamma$ follows. The last assertion of the lemma follows from the definition of E_j and the triangle inequality. $\square$

Proof of Theorem A. Given $\varsigma, \vartheta \in \partial_\rho \mathbb{B}^n$, pick a 2-sphere $S \subset \partial \mathbb{B}^n$ so that $\varsigma, \vartheta \in S$. Let $r = |\varsigma - \vartheta|/2$. Using Lemma 2.4 and Lemma 2.5 we find circles $S_{\varsigma,1}$ and $S_{\vartheta,1}$ centered at ς and at ϑ of radii $\sqrt{2}r < r^1_\varsigma < \frac{3}{2}r$ and $\sqrt{2}r < r^1_\vartheta < \frac{3}{2}r$ and corresponding circular arcs J^1_ς, J^1_ϑ so that all these sets are the images of ρ-paths and contained in $B_\rho(\varsigma, r_1) \cup B_\rho(\vartheta, r_1)$, where

$$r_1 = C \max\{\operatorname{diam}_\rho(\operatorname{Cone}(\varsigma, \frac{4}{5}, r)), \operatorname{diam}_\rho(\operatorname{Cone}(\vartheta, \frac{4}{5}, r))\},$$

in which C is the constant of Lemma 2.2. By elementary geometry, $(S_{\varsigma,1} \cup J^1_\varsigma) \cap (S_{\vartheta,1} \cup J^1_\vartheta) \neq \emptyset$. Therefore, $S_{\varsigma,1} \cup S_{\vartheta,1} \cup J^1_\varsigma \cup J^1_\vartheta$ is connected.

We continue by repeating the use of Lemma 2.4 and Lemma 2.5 for $\frac{r}{2}$. We obtain analogous sets $S_{\varsigma,2}$, $S_{\vartheta,2}$, J^2_ς, J^2_ϑ, contained in $B_\rho(\varsigma, r_2) \cup B_\rho(\vartheta, r_2)$, where

$$r_2 = C \max\{\operatorname{diam}_\rho(\operatorname{Cone}(\varsigma, \frac{4}{5}, \frac{r}{2})), \operatorname{diam}_\rho(\operatorname{Cone}(\vartheta, \frac{4}{5}, \frac{r}{2}))\}.$$

The union of these new sets together with the sets from the first step is again connected. Continue in this manner inductively. The desired ρ-path γ is now obtained by gluing together suitable pieces of the arcs and circles; notice that $\operatorname{diam}_\rho(\operatorname{Cone}(\omega, \frac{4}{5}, 2^{-i}r))$ tends to zero when $i \to \infty$ and $\omega \in \partial_\rho \mathbb{R}^n$.

To see that $\operatorname{diam}_\rho(\gamma) \leq C(A, B) d_\rho(\varsigma, \vartheta)$, notice first that the image of γ is contained in $B_\rho(\varsigma, r_1) \cup B_\rho(\vartheta, r_1)$, and

$$\operatorname{Cone}(\varsigma, \frac{4}{5}, r) \cap \operatorname{Cone}(\vartheta, \frac{4}{5}, r) \neq \emptyset.$$

On the other hand, it follows from $HI(A)$ and the Gehring-Hayman theorem that $r_1 \leq C d_\rho(\varsigma, \vartheta)$, where $C = C(A, B, n)$ (cf. Lemma 1.7 in [7]). The claim follows. $\square$

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