Metric Definition of μ-homeomorphisms

S. Kallunki* and P. Koskela*

Dedicated to Fred and Lois Gehring

Abstract

We give sufficient metric conditions for a homeomorphism to belong to a Sobolev class. We give an improvement on the result of Gehring's on the metric definition of quasiconformality where "limsup" is replaced with "liminf".

1 Introduction

The analytic definition of quasiconformality declares that a homeomorphism f between domains Ω and Ω' in $\mathbf{R}^n$ is quasiconformal if $f \in W^{1,n}_{loc}(\Omega, \Omega')$ and there exists a constant K so that

$$|Df(x)|^n \leq KJ_f(x) \qquad \text{a.e. in } \Omega.$$

Because the Jacobian of any homeomorphism $f \in W^{1,1}_{loc}(\Omega, \Omega')$ is locally integrable, the regularity assumption on f in this definition can naturally be relaxed to $f \in W^{1,1}_{loc}(\Omega, \Omega')$. There has been considerable interest recently in so-called μ-homeomorphisms that form a natural generalization of the concept of a quasiconformal mapping in dimension two. To be more precise, we consider homeomorphisms $f \in W^{1,1}_{loc}(\Omega, \Omega')$ so that

$$|Df(x)|^n \leq K(x)J_f(x) \qquad \text{a.e. in } \Omega \tag{1}$$

[0] *2000 Mathematics Subject Classification:* 30C62, 30C65

*Both authors partially supported by the Academy of Finland, project 41933, and S.K. by the foundation Vilho, Yrjö ja Kalle Väisälän rahasto. This research was completed when P.K. was visiting at the University of Michigan as the Fred and Lois Gehring professor.

with $K(x) \geq 1$ and $\exp(\lambda K) \in L^1_{loc}(\Omega)$ for some $\lambda > 0$. A class of mappings equivalent to this was introduced by David in [1] and further studied in [17], [16]. He considered the Beltrami equation

$$\overline{\partial} f(z) = \mu(z) \partial f(z)$$

and essentially showed that a homeomorphic solution $f \in W^{1,1}_{loc}(\Omega, \Omega')$ exists (in the planar case) when $|\mu(z)| \leq 1$ almost everywhere and

$$\exp\left(C \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \right) \in L^1_{loc}(\Omega);$$

for this generality see [17]. These mappings in fact belong to $\cap_{p<2} W^{1,p}_{loc}(\Omega, \Omega')$, they are differentiable a.e. and preserve the null sets for the two dimensional Lebesgue measure. These conclusions hold with 2 replaced by n in any dimension for mappings with an exponentially integrable distortion in the sense of (1); see [13], [14].

Quasiconformal mappings can alternatively be defined using metric quantities: Let Ω and Ω' be domains in $\mathbf{R}^n$ and $f \colon \Omega \to \Omega'$ a homeomorphism. Recall that f is then either sense-preserving or sense-reversing; we will throughout the paper assume that all the homeomorphisms we deal with are sense-preserving. Then the distortion of f at a point $x \in \Omega$ is

$$H_f(x) = \limsup_{r \to 0} H_f(x, r) \tag{2}$$

or

$$h_f(x) = \liminf_{r \to 0} H_f(x, r), \tag{3}$$

where

$$H_f(x, r) = \frac{L_f(x, r)}{l_f(x, r)},$$

and

$$L_f(x, r) = \sup\{|f(x) - f(y)| : |x - y| \leq r\},$$
$$l_f(x, r) = \inf\{|f(x) - f(y)| : |x - y| \geq r\}.$$

By $|x - y|$ we denote the euclidean distance between x and y. Now f is quasiconformal if and only if the distortion H_f is uniformly bounded, i.e.

$$H_f(x) \leq H < \infty \text{ for all } x \in \Omega. \tag{4}$$

According to a result by Gehring [3, Thm 8], the uniform boundedness of H_f can be relaxed to the requirement that $H_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-dimensional measure and $H_f \leq H$ a.e. Quite recently it was observed that, first of all, H_f in (4) can be replaced with h_f. For this result that quickly found applications in complex dynamics see the paper [7] by Heinonen and Koskela. Secondly, we established in [11] a version of the result of Gehring's by showing that it suffices to assume that $h_f(x) \leq H$ outside a set of σ-finite $(n-1)$-measure. This result was partially motivated by the need for tools of this type in complex dynamics (see [4]).
In the case of μ-homeomorphisms or more generally homeomorphisms $f \in W^{1,1}(\Omega, \Omega')$ that satisfy (1) with some suitably well integrable K, there is no real hope in obtaining a metric definition that would yield the same class of mappings: the σ-finiteness of the exceptional set is crucial even in the quasiconformal setting and under integrability conditions on K, H_f can well be infinite in a set of dimension larger than $n-1$. Thus the best one can hope for is a sufficient metric condition. Again, the quest for such a condition is partially motivated by complex dynamics; μ-homeomorphism appear naturally in conjugation problems [5], [6]. According to a result by Kallunki and Martio [12], in the planar case, a homeomorphism $f\colon \Omega \to \Omega'$ for which $H_f \in L^p_{loc}(\Omega)$ for some $p > 2$ and $H_f(x) < \infty$ outside a set of σ-finite length indeed belongs to $W^{1,1}_{loc}(\Omega, \Omega')$ and is differentiable a.e. Also see [17] for a related result. For simplicity and for the relevance for complex dynamics we here and from this on mostly concentrate on the planar case. In this paper we establish new results in terms of h_f. They rely on our first theorem that gives control on the distortion of shapes by means of integrals of h_f.

Theorem 1.1 *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^2$ such that $h_f(x) < \infty$ outside a set E of σ-finite length and $h_f \in L^2_{loc}(\Omega)$. Then*

$$L_f(x,r) \leq l_f(x,r) \exp\Big(C \fint_{B(x,2r)} h_f^2(y)\, dy\Big) \tag{5}$$

for each $x \in \Omega$ and every $r > 0$ such that $B(x,2r) \subset \Omega$. The constant C is an absolute constant. In particular, f is differentiable almost everywhere.

By the σ-finite length of a set we mean that the set has a countable cover by sets of finite length.

As an immediate consequence of this theorem and its higher dimensional analog given in Section 4 we obtain the following corollary that gives a full extension of the result of Gehring's discussed above; also see Theorem 1.3 below. It is an improvement on our main result in [11].

Corollary 1.2 *Let $\Omega, \Omega' \subset \mathbf{R}^n$ be domains and suppose that $f : \Omega \to \Omega'$ is a homeomorphism. Suppose that there is a set E of σ-finite $(n-1)$-measure and a constant H such that $h_f(x) < \infty$ outside E in Ω and*

$$h_f(x) \leq H$$

almost everywhere in Ω. Then f is quasiconformal.

We close this introduction with two regularity results. The first of them gives a sufficient metric condition for a mapping to be a μ-homeomorphism.

Theorem 1.3 *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^2$ such that $h_f(x) < \infty$ outside a set E of σ-finite length. There is a constant C', independent of E and f, such that*

$$\exp(C' h_f^2) \in L^1_{loc}(\Omega)$$

implies that $f \in W^{1,2}_{loc}(\Omega, \Omega')$ and that (1) holds with $\exp(C'K^2) \in L^1_{loc}(\Omega)$.

Notice that we obtain a stronger conclusion than simply the exponential integrability of the distortion and that the asserted regularity of the mapping is stronger than one would expect. The regularity will be deduced from [8], [9]. It would be interesting to know if already the exponential integrability of h_f could guarantee that the mapping belongs to $W^{1,1}_{loc}(\Omega, \Omega')$.
If we assume that h_f is sufficiently regular, then we do obtain that $W^{1,1}_{loc}(\Omega, \Omega')$ under a substantially weaker integrability assumption on h_f. This is the content of Theorem 1.4 that is based on Theorem 1.1 and results in [12]. It extends a related conclusion in [12].

Theorem 1.4 *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^2$ such that $h_f(x) < \infty$ and*

$$\limsup_{r \to 0} \fint_{B(x,r)} h_f^2(x)\, dx < \infty \qquad (6)$$

outside a set E of σ-finite length, and $h_f \in L^{2+\epsilon}_{loc}(\Omega)$ with $\epsilon > 0$. Then $f \in W^{1,1}_{loc}(\Omega)$.

The paper is organized as follows. In Section 2 we prove Theorem 1.1. Section 3 is devoted to the proofs of Theorems 1.3 and 1.4. The last section contains the formulation and the outline of the proof of Theorem 1.1 in $\mathbf{R}^n$.

2 The local quasisymmetry condition

In this section we prove Theorem 1.1.
Proof of Theorem 1.1. If inequality (5) holds, then

$$H_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} < \infty \text{ for a.e. } x \in \Omega.$$

This guarantees the differentiability of f almost everywhere due to the Rademacher-Stepanov theorem (see e.g [12]).
The proof of inequality (5) is somewhat technical. The argument is an improvement on the techniques in [7] and [11]; for the convenience of the reader we will repeat even the part of the original reasoning from [7] that need not be altered.
First fix $x_0 \in \Omega$ and $r > 0$ with $B(x_0, 2r) \subset \Omega$. We can assume that

$$L_f(x_0, r) > 3l_f(x_0, r).$$

Let $1 \le p < 2$ and $\epsilon > 0$. Define

$$A = \overline{B}(f(x_0), L) \setminus B(f(x_0), l),$$

where $L = L_f(x_0, r)$ and $l = l_f(x_0, r)$. For each $k = 0, 1, 2, \ldots$ write

$$A_k = \{y \in f^{-1}(A) \cap B(x_0, 2r) : 2^k \le h_f(y) < 2^{k+1}\}.$$

The set A_k is a Borel set, $f^{-1}(A) \cap B(x_0, 2r) \setminus E = \cup_k A_k$ and for every k there exists open U_k such that $A_k \subset U_k$ and

$$|U_k| \le |A_k| + \frac{\epsilon}{2^k (2^{\frac{2p}{2-p}})^k}.$$

Fix k. Now for every $y \in A_k$ there is $r_y > 0$ such that

(i) $0 < r_y < \frac{1}{10}\min\{d(f^{-1}(\overline{B}(f(x_0), l)), f^{-1}(\mathbf{R}^2 \setminus B(f(x_0), L))),$
$d(y, \partial B(x_0, 2r))\}$,

(ii) $\operatorname{diam}(fB_y) < 2^{-j_0-3}L$,

(iii) $H_f(y, r_y) < 2^{k+1}$, and

(iv) $B_y \subset U_k$.

Here $B_y = B(y, r_y)$ and j_0 is the least positive integer with $2^{-j_0}L < l$.
We have obtained a family of balls B_y such that they satisfy the conditions (i) and (ii) and if $y \in A_k$ then B_y satisfies the condition (iii) for k. By the Besicovitch covering theorem we find balls $\overline{B}_1, \overline{B}_2, \ldots$ from balls $\overline{B}(y, r_y)$ so that

$$f^{-1}(A) \cap B(x_0, 2r) \setminus E \subset \bigcup_j \overline{B}_j \subset B(x_0, 2r)$$

and $\sum_j \chi_{\overline{B}_j}(x) \leq C(2)$ for every $x \in \mathbf{R}^2$. Here and in what follows notation like $C(2)$ indicates that this constant will depend on the dimension when the argument is extended to cover the higher dimensional setting. For these balls we know that

$$|f\overline{B}_j| \leq C(2)\operatorname{diam}(f\overline{B}_j)^2$$

and when $y_j \in A_k$ (here y_j is the center of $\overline{B}_j$)

$$|f\overline{B}_j| \geq \frac{C(2)\operatorname{diam}(f\overline{B}_j)^2}{2^2(2^{k+1})^2}.$$

Let us define

$$\rho(x) = (\log \frac{L}{l})^{-1} \sum_j \frac{\operatorname{diam}(f\overline{B}_j)}{d(fB_j, f(x_0))} \frac{1}{\operatorname{diam}(B_j)} \chi_{2B_j}(x).$$

The function ρ is measurable, because it is a countable sum of simple functions.
By a general estimate on L^p-norms of weighted sums of characteristic functions, the L^p-norms, $1 \leq p < \infty$, of ρ are comparable to the corresponding norms of the function where the characteristic functions χ_{2B_j} are replaced

with χ_{B_j} (c.f. [2]). Thus, knowing that $\sum \chi_{B_j} \leq C(2)$, we arrive at the estimate

$$\int_{B(x_0,2r)} \rho(x)^p\, dx \leq C(2,p)\left(\log\frac{L}{l}\right)^{-p} \sum_j \left(\frac{\operatorname{diam}(f\overline{B}_j)}{d(fB_j, f(x_0))}\frac{1}{\operatorname{diam}(B_j)}\right)^p |B_j|.$$

Using the fact that $\operatorname{diam} fB_j^2 \leq C(2)|fB_j|(2^{k+1})^2$ when $y_j \in A_k$ and Hölder's inequality, we thus obtain

$$\begin{aligned}\int_{B(x_0,2r)} \rho(x)^p\, dx \ &\leq\ C(2,p)(\log\frac{L}{l})^{-p}\left(\sum_j \frac{|fB_j|}{d(fB_j, f(x_0))^2}\right)^{\frac{p}{2}} \\ &\qquad \left(\sum_k \sum_{y_j\in A_k} (2^k)^{2p/(2-p)}\operatorname{diam}(B_j)^2\right)^{\frac{2-p}{2}}.\end{aligned}$$

It is easy to see by regrouping the balls depending on their distance from $f(x_0)$ and by using the estimate $\sum \chi_{B_i} \leq C(2)$ that

$$\sum_j \frac{|fB_j|}{d(fB_j, f(x_0))^2} \leq C(2)\log\frac{L}{l}.$$

The approximation of the second term is a little bit trickier. First $\operatorname{diam} B_j^2 \leq C(2)|B_j| = C(2)(|B_j \cap A_k| + |B_j \setminus A_k|)$. The double sum over the $|B_j \cap A_k|$-terms can be estimated by the integral of $h_f^{2p/(2-p)}$ and the double sum over the $|B_j \setminus A_k|$-terms turns out to be no more than a constant times ϵ, because $\cup_{y_j\in A_k} B_j \subset U_k$ and $|U_k| \leq |A_k| + \frac{\epsilon}{2^k(2^{\frac{2p}{2-p}})^k}$. Therefore

$$\sum_k \sum_{y_j\in A_k} (2^k)^{2p/(2-p)}\operatorname{diam}(B_j)^2 \leq C(2)\left(\int_{B(x_0,2r)} h_f(x)^{2p/(2-p)}\, dx + \epsilon\right).$$

Because ϵ was arbitrary we conclude that

$$\int_{B(x_0,2r)} \rho(x)^p\, dx \leq C(2,p)(\log\frac{L}{l})^{-p/2}\left(\int_{B(x_0,2r)} h_f(x)^{2p/(2-p)}\, dx\right)^{(2-p)/2}.$$

In the following we will actually choose $p = 1$.

Our next goal is to find a lower bound on the integral of ρ. For this, define $F_1 = f^{-1}(\overline{B}(f(x_0), l))$ and $F_2 = f^{-1}(\mathbf{R}^2 \setminus B(f(x_0), L)) \cap B(x_0, 2r)$. Take a point $y \in F_1 \cap S(x_0, r)$. By applying an auxiliary rotation we may assume that $y = x_0 + (r, 0)$. Consider the line segments J_t parallel to the imaginary axis through the points $x_0 + (t, 0)$, $0 \le t \le r$, which join two points of $S(x_0, 2r)$. Assume first that $\int_{J_t} \rho \ge \frac{1}{2000}$ for each t in a set $A \subset [0, r]$ with $m(A) > r/2$. Here m refers to the Lebesgue measure on the line. Then it follows from the Fubini theorem that

$$\int_{B(x_0,2r)} \rho \ge \frac{r}{4000}.$$

Suppose then that $\int_{J_t} \rho \le \frac{1}{2000}$ for each t in a set A with $m(A) > r/2$. Now $m(\{0 \le t \le r : E \cap J_t \text{ is uncountable }\}) = 0$, and $m(\{r \le s \le 2r : E \cap S(x_0, s)$ is uncountable $\}) = 0$ because E has σ-finite length, compare [18, 30.16].
Take a radius $r < s < 2r$ and a number $t \in A$ such that both $E \cap S(x_0, s)$ and $E \cap J_t$ are countable.
Pick the balls $V_1, V_2, \ldots$ from balls the $\overline{B}_1, \overline{B}_2, \ldots$ for which $V_i \cap S(x_0, s) \ne \emptyset$ or $V_i \cap J_t \ne \emptyset$. Write $\gamma = J_t \cup S(x_0, s)$. Then the connected set γ intersects both F_1 and F_2 and thus $f(\gamma)$ is a connected set that intersects both $\overline{B}(f(x_0), l)$ and $\mathbf{R}^2 \setminus B(f(x_0), L)$. Moreover, the sets $f(V_i)$ cover $f(\gamma)$ up to a countable set.
Now

$$\int_\gamma \rho \ge \frac{1}{2}(\log \frac{L}{l})^{-1} \sum_i \frac{\operatorname{diam} f V_i}{d(fV_i, f(x_0))}.$$

If $f(V_i)$ touches the annulus $A_j = B(f(x_0), 2^{j+1}l) \setminus B(f(x_0), 2^j l)$, where $j = 0, \ldots, j_0 - 1$, then $d(fV_i, f(x_0)) \le 2^{j+1}l$, and because the connected sets $f(V_i)$ cover $f(\gamma)$ up to a countable set,

$$\int_\gamma \rho \ge (\log \frac{L}{l})^{-1} \sum_{j=0}^{j_0-1} \frac{1}{4} \ge \frac{1}{1000}.$$

Because $t \in A$, we have that

$$\int_{J_t} \rho \le \frac{1}{2000}$$

and it follows that

$$\int_{S(x_0,s)} \rho \ge \frac{1}{2000}.$$

From this estimate we obtain by using the Fubini theorem that

$$\int_{B(x_0,2r)} \rho(x) \geq \int_r^{2r} (\int_{S(y,s)} \rho) \, ds \geq \frac{1}{2000} r.$$

Thus we in both cases have the estimate

$$\int_{B(x_0,2r)} \rho(x) \, dx \geq Cr.$$

Combining now the lower bound with the upper bound we finally have with the choice $p = 1$ that

$$C(2)r \leq C(2)(\log \frac{L}{l})^{-1/2} (\int_{B(x_0,2r)} h_f(x)^2 \, dx)^{1/2}.$$

This gives the claim.

3 Metric conditions for μ-homeomorphisms

In this section we will prove Theorems 1.4 and 1.3.

Proof of Theorem 1.4. Taking the $\limsup_{r\to 0}$ in inequality (5) we see that

$$H_f(x) = \limsup_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)} < \infty$$

outside a set E of σ-finite length. Moreover, by Theorem 1.1, f is differentiable almost everywhere. By an elementary argument one sees that $H_f(x) = h_f(x)$ everywhere in the set where both f is differentiable and $\min_{|e|=1} |Df(x)e| > 0$. To employ Theorem 1.2 in [12] (according to which the conditions $H_f(x) < \infty$ σ-a.e. and $H_f \in L^{2+\epsilon}_{loc}(\Omega)$ guarantee that $f \in W^{1,1}_{loc}(\Omega, \Omega'))$ we would like to have $H_f = h_f$ almost everywhere in Ω. The exceptional set where f is differentiable, $h_f(x) < \infty$, and $\min_{|e|=1} |Df(x)e| = 0$ thus causes a potential danger. On the other hand, in this set, $|Df(x)| = 0$, and it is easy to check that the image of the intersection of this exceptional set with any line has zero length. Combining this fact with the proof of Theorem 1.2 in [12] we conclude that f belongs to the Sobolev class $W^{1,1}_{loc}(\Omega, \Omega')$.

For the proof of 1.3 we need two lemmas, the first of which is a version of the standard absolute continuity result for quasicormal mappings, tailored for our setting.

Lemma 3.1 *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^2$ such that*

$$L_f(x,r) \le l_f(x,r)\varphi(x) \qquad \text{whenever } B(x,2r) \subset \Omega, \tag{7}$$

where $\varphi \in L^2_{loc}(\Omega)$. Then f is absolutely continuous on almost all lines parallel to the coordinate axes.

Proof. Let $Q \subset\subset \Omega$ be an open 2-interval and suppose $Q = I \times J$ where $I =]a,b[\in \mathbf{R}^1$ and $J =]c,d[\subset \mathbf{R}^1$. For each Borel set $E \subset I$ we set $\eta(E) = |f(E \times J)|$. Then η is a finite Borel measure in I and hence it has a finite derivative $\eta'(y)$ for almost every $y \in I$ by the Radon-Nikodym theorem. Choose $y \in I$ such that (i) $\eta'(y)$ exists and (ii) $\varphi \in L^2(\{y\} \times J)$. The latter is possible due to the Fubini theorem. We will prove that f is absolutely continuous on the segment $\{y\} \times J$ which will prove the theorem.
Define $J' = \{y\} \times J$. Now let $F \subset J'$ be compact. We wish to estimate $\mathcal{H}^1(fF)$. Choose $0 < \epsilon < \operatorname{dist}(F, \partial J)/2$ and $t > 0$. Let $0 < \delta_1 \le 1$ be the number given by Lemma 31.1 in [18] for the set F. We will soon state what this lemma gives us. Choose δ_2 such that, if $0 < r < \delta_2$, then $|f(x) - f(z)| < t$ whenever $x, z \in Q$ and $|x - z| \le 2r$. Denote $\delta = \min\{\delta_1, \delta_2, \epsilon\}$. Choose $0 < r < \delta$. Now Lemma 31.1 in [18] gives a covering $\triangle_1, \dots, \triangle_p$ of F with intervals in J' so that (i) $\operatorname{diam}(\triangle_i) = r$ for $1 \le i \le p$, (ii) each point of J' belongs to at most two different $\triangle_i$, and (iii) each $\overline{\triangle}_i$ is contained in the ϵ-neighborhood of F in J'.
Now, because $\varphi \in L^2(J')$, there are points $x_i \in \overline{\triangle}_i$ such that

$$\varphi(x_i) \le 2 \inf_{x \in \overline{\triangle}_i} \varphi(x) < \infty. \tag{8}$$

Set $A_i = B^2(x_i, r)$. Now $\triangle_i \subset A_i$ and $A_i \subset \overline{B}^1(y,r) \times J$. Because $\operatorname{diam}(fA_i) < t$ we have that $\mathcal{H}^1_t(fF) \le \sum \operatorname{diam}(fA_i) \le 2 \sum L_i$, where $L_i = L_f(x_i, r)$. Denote similarly $l_i = l_f(x_i, r)$. Using (7) we obtain the estimate

$$\begin{aligned} \mathcal{H}^1_t(fF)^2 &\le 2^2 \left(\sum_i L_i \right)^2 \le 2^2 \left(\sum_i l_i \varphi(x_i) \right)^2 \\ &= \frac{2^2}{r} \left(\sum_i l_i r^{1/2} \varphi(x_i) \right)^2 ; \end{aligned}$$

notice that $B(x_i, 2r) \subset \Omega$. By Hölder's inequality we further conclude that

$$\mathcal{H}^1_t(fF)^2 \leq \frac{2^2}{\Omega_2 r} \sum_i |f(A_i)| \sum_i r\varphi^2(x_i),$$

where $\Omega_2 = |B^2(0,1)|$. Because no point belongs to more than two of the sets A_i, $\sum |fA_i| \leq 10\,\eta(\overline{B^1}(y,r))$. Since the points x_i satisfy (8) we arrive at

$$\mathcal{H}^1_t(fF)^2 \leq \frac{10 \cdot 2^2}{\Omega_2} \frac{\eta(\overline{B^1}(y,r))}{r} \left(\int_{F+\epsilon} \varphi^2(x)\, dx \right).$$

Here $F + \epsilon$ is the ϵ-neighborhood of F in J'. Letting first $r \to 0$ and then $\delta_1 \to 0$ and finally $\epsilon \to 0$ and $t \to 0$ we deduce that

$$\mathcal{H}^1(fF)^2 \leq C(2)\eta'(y) \left(\int_F \varphi^2(x)\, dx \right).$$

The absolute continuity of f on J' follows from this estimate, see Lemma 30.9 in [18].

Lemma 3.2 *Let u be a non-negative function so that*

$$\exp(C'u) \in L^1_{loc}(\Omega) \tag{9}$$

and let $p > 1$. Then, for each compact set $F \subset \Omega$

$$\exp(\epsilon C' M(\chi_F u)) \in L^p_{loc}(\Omega) \tag{10}$$

where ϵ depends only on p, and M is the usual Hardy-Littlewood maximal operator.

The proof of this lemma is a simple computation which is based on the fact that the Hardy-Littlewood maximal operator is bounded from L^q to L^q when $1 < q < \infty$ and on the expansion of the exponential function as a power series.

The proof of Theorem 1.3. From Theorem 1.1 we have that f is differentiable almost everywhere. Next we would like to show that f is absolutely continuous on lines. For this it is enough to show that f is absolutely continuous on lines in every square $Q \subset\subset \Omega$. Fix such a square Q. Now inequality (5) gives if $r < \frac{1}{2}d(Q, \partial\Omega)$ and $x \in Q$ that

$$L_f(x,r) \leq l_f(x,r) \exp(CM(\chi_F h_f^2)(x)),$$

where $F = \{y \in \Omega : d(y,Q) \leq \frac{1}{2}d(Q,\partial\Omega)\}$. Lemma 3.2 shows that the function $\exp(CM(\chi_F h_f^2)(x)) \in L^2_{loc}(\Omega)$ when the constant C' is chosen correctly. By Lemma 3.1 we see that f is absolutely continuous on almost all lines parallel to the coordinate axes.

The next goal in our proof is to check that $f \in W^{1,1}_{loc}(\Omega, \Omega')$. Because f is absolutely continuous on almost all lines parallel to the coordinate axes, it suffices to show that $|Df| \in L^1_{loc}(\Omega)$. Here Df is the matrix obtained from the partial derivatives of the coordinate mappings. It exists at a.e. point of Ω. Now, for a.e. $x \in \Omega$, f is differentiable, $Df(x)$ exists and $h_f(x) < \infty$. Fix such a point x. Because f was assumed to be sense-preserving, the Jacobian determinant, $J_f(x)$ has to be non-negative. If $J_f(x) > 0$, then clearly

$$|Df(x)|^2 \leq h_f(x) J_f(x). \tag{11}$$

On the other hand, if $J_f(x) = 0$, then elementary linear algebra shows that $\min_{|e|=1} |Df(x)e| = 0$. From the assumption $h_f(x) < \infty$ it then follows that $Df(x)$ has to be the zero matrix. Thus (11) holds for a.e. $x \in \Omega$. Because the Jacobian of each a.e. differentiable homeomorphism is locally integrable (c.f. [15, p. 360]) we thus conclude from (11) and our integrability assumption on h_f that $|Df|$ is locally integrable, in fact locally p-integrable for any $p < 2$.

We are left to show that $f \in W^{1,2}_{loc}(\Omega, \Omega')$. By the previous paragraph, $f \in W^{1,1}_{loc}(\Omega, \Omega')$, $J_f \in L^1_{loc}(\Omega)$, and

$$|Df(x)|^2 \leq K(x) J_f(x)$$

a.e. where $K = h_f$ satisfies $\exp(C'K^2) \in L^1_{loc}(\Omega)$. Then $\exp(\lambda K) \in L^1_{loc}(\Omega)$ for every $\lambda > 0$ and [8] or the main theorem in [9] gives us the desired regularity.

4 The higher dimensional setting

We have similar results in $\mathbf{R}^n$, but the proofs are more technical. For simplicity we only cover the extension of Theorem 1.1 to higher dimensions and refer to reader to [10] for the analogs for Theorem 1.3 and Theorem 1.4 in higher dimensions.

Theorem 4.1 *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^n$ such that $h_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-measure and $h_f \in L^{p^*}_{loc}(\Omega)$, with some $n-1 < p < n$. Here $p^* = \frac{pn}{n-p}$. Then*

$$L_f(x,r) \le l_f(x,r) \exp\left((C(n,p) ⨍_{B(x,2r)} h_f^{p*}(y)\, dy)^{\frac{1}{p^*}\frac{n}{n-1}} \right) \tag{12}$$

for every $x \in \Omega$ and each $r > 0$ such that $B(x,2r) \subset \Omega$. In particular, f is differentiable almost everywhere.

Proof. The first part of the proof is the same as the beginning of the proof of Theorem 1.1. We simply replace the number 2 there with n. We then end up with the estimate

$$\int_{\mathbf{R}^n} \rho^p(y)\, dy \le C(n,p)(\log \frac{L}{l})^{\frac{p}{n}(1-n)} \left(\int_{B(x_0,2r)} h_f(x)^{np/(n-p)}\, dx \right)^{(n-p)/n}.$$

The claim follows from this inequality provided we can find a suitable lower bound on the p-integral of ρ. Such an estimate is obtained by a simple modification of the reasoning in [11] (Lemma 2.1 there holds with n replaced by p when $p > n-1$, and in Lemma 2.3 there the set E_b is not needed). Explicitly we have the lower bound

$$\int_{\mathbf{R}^n} \rho^p(y)\, dy \ge C(n) r^{n-p}.$$

For the details see [10]. By combining these two facts the claim follows.

References

[1] David, G., *Solutions de l'equation de Beltrami avec $\|\mu\|_\infty = 1$*, Ann. Acad. Sci. Fenn. Ser. AI Math. **13** (1988), 25–70.

[2] Bojarski, B., Remarks on Sobolev imbedding inequalities, Complex analysis, Joensuu 1987, 52–68, Lecture Notes in Mathematics 1351, Springer-Verlag, 1988.

[3] Gehring, F.W., *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[4] Graczyk, J., and Smirnov, S., *Non-uniform hyperbolicity in complex dynamics I,II*, preprint.

[5] Haïssinsky, P., *Chirurgie parabolique*, C. R. Acad. Sci. Paris Sér. 1 Math. **327** (1998), 195–198.

[6] Haïssinsky, P., *Chirurgie croisée*, Bull. Soc. Math. France **128** (2000), 599–654.

[7] Heinonen, J. and Koskela, P., *Definitions of quasiconformality*, Invent. Math. **120** (1995), 61–79.

[8] Iwaniec, T., Koskela, P. and Martin, G., *Mappings of BMO-distortion and Beltrami type operators*, J. Anal. Math., to appear.

[9] Iwaniec, T., Koskela, P., Martin, G. and Sbordone, C., *Mappings of finite distortion: $L^n logL$-integrability*, preprint.

[10] Kallunki, S., *Mappings of finite distortion: The metric definition*, in preparation.

[11] Kallunki, S. and Koskela, P., *Exceptional sets for the definition of quasiconformality*, Amer. J. Math. **122** (2000), 735–743.

[12] Kallunki, S. and Martio, O., *ACL homeomorphisms and linear dilatation*, to appear in Proc. Amer. Math. Soc.

[13] Kauhanen, J., Koskela, P. and Malý, J., *Mappings of finite distortion: condition N*, Mich. Math. J. **49** (2001), 169–181.

[14] Koskela, P. and Malý, J., *Mappings of finite distortion: The zero set of the Jacobian*, preprint.

[15] Rado, T. and Reichelderfer, P.V., *Continuous transformations in analysis*, Die Grundlehren der matematischen Wissenschaften 75, Springer-Verlag, 1955.

[16] Ryazanov, V., Srebro, U. and Yakubov, E., *BMO-quasiconformal mappings*, J. Anal. Math. **83** (2001), 1–20.

[17] Tukia, P., *Compactness properties of μ-homeomorphisms*, Ann. Acad. Sci. Fenn. Ser. AI Math. **16** (1991), 47 – 69.

[18] Väisälä, J., *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics 229. Springer-Verlag, 1971.

University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35, Fin-40351 Jyväskylä
Finland
sakallun@maths.jyu.fi pkoskela@maths.jyu.fi

and (P.K.)
University of Michigan
Department of Mathematics
Ann Arbor, MI 48109
U.S.A