Mappings of finite distortion: The sharp modulus of continuity

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Abstract

We establish an essentially sharp modulus of continuity for mappings of subexponentially integrable distortion.

1   Introduction

We consider mappings $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ is a domain. Such a mapping is said to have finite distortion if the following three conditions are satisfied:

1. $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n)$.

2. The Jacobian determinant $J(x, f)$ of f is locally integrable.

3. There is a measurable function $K_O = K_O(x) \geq 1$, finite almost everywhere, such that f satisfies the distortion inequality

$$|Df(x)|^n \leq K_O(x)J(x, f) \qquad \text{a.e.} \tag{1}$$

Notice that when $K_O \in L^\infty$, we recover the class of mappings of bounded distortion, also called quasiregular mappings. In this case, f is locally Hölder continuous with exponent K_O^{-1} and this exponent is sharp. By this we mean that f has such a representative; this comment also applies to what follows. Recently, (non-constant) mappings of finite distortion with subexponentially integrable distortion K_O have been shown to share many of the nice properties of mappings of bounded distortion such as being open and discrete [5], [12], [14]. Here the subexponential integrability of K_O requires that

$$\exp(\mathcal{A}(K_O)) \in L^1_{loc}(\Omega)$$

with $\mathcal{A}$ smooth and strictly increasing, with

(i) $\displaystyle\int_1^\infty \frac{\mathcal{A}'(t)}{t}\, dt = \infty$, and so that

(ii) there exists $t_0 \in (0, \infty)$ such that $\mathcal{A}'(t)t$ increases to infinity for $t \geq t_0$.

In particular, under this assumption, a mapping f of finite distortion is continuous. The integrability assumption on K_O is sharp as regards continuity (and openness and discreteness) in the sense that mappings of finite distortion with $\exp(\mathcal{B}(K_O)) \in L^1_{loc}(\Omega)$ and without a continuous representative can be constructed whenever $\int_1^\infty \frac{\mathcal{B}'(t)}{t}\, dt < \infty$, see [14].

Let us consider the case $\exp\left(\lambda K_O\right) \in L^1_{loc}(\Omega)$. By the results in [7], f then has (locally) a modulus of continuity of the type

$$|f(x) - f(y)| \leq C/\log\log(R/|x-y|).$$

It is also known that better bounds can be obtained when the the constant λ is sufficiently large: given $s > 0$, the (local) modulus of continuity

$$|f(x) - f(y)| \leq C(\log(R/|x-y|))^{-s}$$

holds provided $\lambda \geq \lambda(s, n)$. This conclusion is established in [6] as a corollary to a surprising regularity property of mappings with exponentially integrable distortion that does not hold for small λ. In this paper we establish an essentially sharp modulus of continuity by a different technique.

Theorem 1.1 *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion whose distortion function K_O satisfies the integrability condition*

$$K := \int_B exp(\lambda K_O(x))\, dx < \infty \tag{2}$$

where $\lambda > 0$ and $B = B(x_0, R) \subset\subset \Omega$. Then for every small $\epsilon > 0$ and all $x, y \in B\left(x_0, \left(\frac{R}{8}\right)^e \left[\frac{n}{\omega_{n-1}} K\right]^{\frac{1-e}{n}}\right)$ we have the estimate

$$|f(x) - f(y)| \leq C_{K,R,n,\lambda}(\epsilon) \frac{\left(\int_B J(z, f)\, dz\right)^{\frac{1}{n}}}{\log^{\frac{\lambda}{n} - \epsilon}\left(\frac{nK}{\omega_{n-1}|x-y|^n}\right)}. \tag{3}$$

Here ω_{n-1} is the surface measure of $\partial B(0, 1)$.

The following example shows that our modulus of continuity is essentially sharp.

Example 1.2 *Let $\lambda > 0$. Then there exists a continuous mapping $f : B(0, 1) \to \mathbb{R}^n$ of finite distortion such that*

$$\int_{B(0,1)} exp(\lambda K_O(x, f))\, dx < \infty \tag{4}$$

and points $x_j, y_j \in B(0, 1)$ so that $x_j \to 0$, $y_j \to 0$ and

$$|f(x_j) - f(y_j)| \log^{\frac{\lambda}{n} + \epsilon}\left(\frac{1}{|x_j - y_j|}\right) \to \infty \tag{5}$$

for every $\epsilon > 0$.

We believe that the claim of Theorem 1.1 holds with the exponent λ/n, but we have not been able to verify this.

Theorem 1.1 is new even for homeomorphic solutions to the Beltrami equation in the plane. By the results in [1] and [9], one obtains a logarithmic modulus of continuity of the type of given in Theorem 1.1 for solutions to the Beltrami equation but with a worse exponent.

By the discussion above and Theorem 1.1, one would expect for an essentially sharp modulus of continuity even under the weaker assumption that $exp\, \mathcal{A}(K_O) \in L^1_{loc}(\Omega)$ with $\mathcal{A}$ satisfying (i) and (ii). This is indeed the case and such a result will be given in Section 4.

Let us close this introduction by briefly sketching the proof of Theorem 1.1. We mimic the lines of reasoning by Morrey [16] and Reshetnyak [17] for the case of mappings of bounded distortion as follows. We prove a decay estimate on the integrals of the Jacobian of f over balls by establishing a differential inequality for these integrals. This is done by employing a suitable isoperimetric inequality established in [8]; the crucial point behind the isoperimetric inequality is that, under our assumptions, the pointwise Jacobian of f coincides with the distributional Jacobian. In the (classical) case of mappings of bounded distortion, the decay estimate on the Jacobian guarantees that f belongs to a Morrey class and is thus Hölder continuous [16], [17], [10]. In our setting, the decay order is too weak to imply even continuity for a general mapping. To bypass this difficulty, we insert the fundamental ideas from our initial proof of continuity in [7] to show that, for our mappings, the decay order suffices to yield the desired modulus of continuity.

2   Orlicz spaces

Theorem 1.1 will be obtained as a corollary to a more general result. Let us replace the assumption $\exp(\lambda K_O) \in L^1_{loc}(\Omega)$ with $\exp(\mathcal{A}(K_O)) \in L^1_{loc}(\Omega)$, where $\mathcal{A}$ is an Orlicz function. We call a infinitely differentiable and strictly increasing function $\mathcal{A} : [0, \infty) \to [0, \infty)$ with $\mathcal{A}(0) = 0$ and $\lim_{t \to \infty} \mathcal{A}(t) = \infty$ an Orlicz function. We will assume for all $\Omega' \subset\subset \Omega$ that

$$\int_{\Omega'} \exp \mathcal{A}(K_O(x)) \, dx < \infty, \tag{6}$$

where

$$\int_1^\infty \frac{\mathcal{A}'(s)}{s} \, ds = \frac{1}{\beta} \int_0^{\sqrt[\beta]{\exp\frac{C}{(\mathcal{A}(1))}}} \frac{dt}{t \mathcal{A}^{-1} \left(\log \frac{C}{t^\beta} \right)} = \infty, \tag{7}$$

for all $C, \beta > 0$. This is condition (i) from the introduction. We wish to warn the reader that conditions (6) and (7) do not require K_O even to be locally integrable and thus an additional technical assumption on $\mathcal{A}$ has to be posed. To fill up this gap we assume that $\mathcal{A}$ satisfies condition (ii): $t\mathcal{A}'(t)$ increases for large t to infinity.

Proposition 2.1 *Assume that $\mathcal{A}$ is an Orlicz function satisfying (i) and (ii). Then we have the point-wise inequality*

$$P(KJ) \leq J + exp(\mathcal{A}(K)) - 1 \tag{8}$$

for all $K, J \geq 0$, where the Orlicz function P satisfies the integrability condition

$$\int_1^\infty \frac{P(s)}{s^2} = \infty \tag{9}$$

and also the technical condition that for every $\epsilon > 0$ we have

$$(t^{-1}P(t))' \leq 0 \leq (t^{\epsilon-1}P(t))' \tag{10}$$

for all $t \geq t_{2.1}(\epsilon, \mathcal{A})$.

We refer to [14, Lemma 2.1 and Lemma 2.3] for the proof of Proposition 2.1.

Lemma 2.2 *Assume that $\mathcal{A}$ is an Orlicz function satisfying (ii) and let $p \in [1, \infty)$. Then there exists $t_{2.2} = t_{2.2}(p, \mathcal{A}) \in (0, \infty)$ such that the function*

$$t \to t^{-p} exp(\mathcal{A}(t))$$

is increasing on $(t_{2.2}, \infty)$.

Proof. The claim follows from the identity

$$\frac{d}{dt} t^{-p} \exp(\mathcal{A}(t)) = t^{-p-1} \exp(\mathcal{A}(t)) \left[\mathcal{A}'(t)t - p \right].$$

Lemma 2.3 *Assume that $\mathcal{A}$ is an Orlicz function satisfying (ii) and let $p \in [1, \infty)$. Then there exists $t_{2.3} = t_{2.3}(p, \mathcal{A}) \in (0, \infty)$ such that the function*

$$t \to \frac{t}{\mathcal{A}^{-1}(\log t^p)}$$

is increasing on $(t_{2.3}, \infty)$.

Proof. Because

$$\frac{d}{dt} \exp\left(\frac{\mathcal{A}(t)}{p}\right) t^{-1} = \exp\left(\frac{\mathcal{A}(t)}{p}\right) t^{-2} \left(\frac{1}{p}\mathcal{A}'(t)t - 1\right)$$

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we find a number $\tilde{t}_0(p, \mathcal{A})$ such that $\exp\left(\frac{\mathcal{A}(t)}{p}\right) t^{-1}$ is increasing for $t > \tilde{t}_0$.

We conclude that $\frac{t}{\mathcal{A}^{-1}(\log t^p)} = \frac{\exp\left(\frac{\mathcal{A}(\mathcal{A}^{-1}(p \log t))}{p}\right)}{\mathcal{A}^{-1}(p \log t)}$ is increasing on $(t_{2.3}, \infty)$,

where $t_{2.3} = \exp\left(\frac{\mathcal{A}(\tilde{t}_0)}{p}\right)$.

Lemma 2.4 *Assume that $\mathcal{A}$ is an Orlicz function satisfying (ii) and let $p \in [1, \infty)$. Then there exists $t_{2.4} = t_{2.4}(p, \mathcal{A}) \in (0, \infty)$ such that the function*

$$t \to exp(\mathcal{A}(t^{\frac{1}{p}}))$$

is convex on $(t_{2.4}, \infty)$.

Proof. Because

$$\frac{d}{dt} \exp(\mathcal{A}(t^{\frac{1}{p}})) = \frac{1}{p} \exp(\mathcal{A}(t^{\frac{1}{p}})) \mathcal{A}'(t^{\frac{1}{p}}) t^{\frac{1}{p}-1},$$

it suffices to show that the function

$$t \to \frac{\exp(\mathcal{A}(t^{\frac{1}{p}}))}{t}$$

is increasing for large values of t. This holds by Lemma 2.2 and so the claim follows.

3 Isoperimetric-type inequality

A crucial tool in establishing the correct modulus of continuity in our case is the following integral-type isoperimetric inequality for mappings in the space $\{f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n) : P(|Df|^n) \in L_{loc}^1(\Omega) \text{ and } J(x, f) \geq 0 \text{ a.e. in } \Omega\}$.

Lemma 3.1 *Assume that an Orlicz function P satisfies the divergence condition (9) and the technical condition (10). Let $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n)$, $n \geq 2$, satisfy $P(|Df|^n) \in L_{loc}^1(\Omega)$ with $J(x, f) \geq 0$ almost everywhere in Ω. Then for each $B(x_0, R) \subset\subset \Omega$, we have*

$$\int_{B(x_0, r)} J(z, f) \, dz \leq (n \sqrt[n-1]{\omega_{n-1}})^{-1} \left(\int_{\partial B(x_0, r)} |Df(z)|^{n-1} \, dz \right)^{\frac{n}{n-1}} \tag{11}$$

for almost every $0 < r < R$.

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We refer to [8, Proposition 5.1] for the proof of Lemma 3.1. The proof of Lemma 3.1 is based on the fact that we can integrate by parts against the Jacobian under the assumptions of Lemma 3.1. This means that

$$\int_\Omega \varphi(x) J(x, f) \, dx = - \int_\Omega f_i J(x, f_1, ..., f_{i-1}, \varphi, f_{i+1}, ..., f_n) \, dx \qquad (12)$$

for all test functions $\varphi \in C_0^\infty(\Omega)$ and each index $i = 1, ..., n$.

We close this section by giving a simple proof for a weaker version of the isoperimetric-type inequality. This is weaker in two senses. The power of $|Df|$ will be larger than $n-1$, and the constant somewhat larger. This weaker version is shown below to yield a version of Theorem 1.1 with $\frac{\lambda}{n}$ replaced by $\frac{\lambda}{n^2 A_p(n)}$, where $p \in (n-1, n)$.

Lemma 3.2 *Under the assumptions of Lemma 3.1, for all $p \in (n-1, n)$, we have that*

$$\int_{B(x_0, r)} J(z, f) \, dz \leq \frac{A_p(n)}{\sqrt[n-1]{\omega_{n-1}}} |\partial B(x_0, r)|^{\frac{n}{n-1}} \left(\fint_{\partial B(x_0, r)} |Df(z)|^p \, dz \right)^{\frac{n}{p}} \qquad (13)$$

for almost every $0 < r < R$. The constant $A_p(n)$ is given by the formula

$$A_p(n) = \left(\frac{\omega_{n-1}}{\omega_{n-2}} \right)^{\frac{1}{p}} \left(\int_0^\pi \sin^{\frac{1-n}{p-1}}(\theta) \, d\theta \right)^{\frac{p-1}{p}}. \qquad (14)$$

Because

$$\pi \leq A_p(n) \leq \pi \frac{p-1}{p-n+1},$$

see [10, Lemma 3.10.1], (13) is weaker than (11).

Proof. Let $j \in \{1, 2, 3...\}$. Choose a mollifier $\eta \in C_0^\infty(B(0,1))$ and let $\eta_j(x) = j^n \eta(jx)$ and $\varphi_j = \eta_j * \chi_{B(x_0, r-1/j)}$. Then $\varphi_j \in C_0^\infty(\Omega)$ when j is sufficiently large and $\sup\{|\nabla \varphi_j(x)| : x \in \Omega\} \leq j$.

Because $p > n-1$, the mapping f is continuous on $\partial B(x_0, r)$ for almost every $r \in (0, R)$. Fix such an r and pick $a \in \partial B(x_0, r)$. By [3] or [2, Theorem

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1.1], we know that we can integrate by parts against the Jacobian and so

$$
\begin{aligned}
\left| \int_\Omega \varphi_j(x) J(x,f)\, dx \right| &= \left| - \int_\Omega (f_1(x) - f_1(a)) J(x, \varphi_j, f_2, ..., f_n)\, dx \right| \\
&\leq \int_\Omega |f_1 - f_1(a)||\nabla\varphi_j||Df|^{n-1} \\
&\leq \operatorname{osc}(f_1, \partial B(x_0, r)) j \int_{r-\frac{1}{j}}^{r} \int_{\partial B(x_0, s)} |Df|^{n-1} \quad (15)
\end{aligned}
$$

for almost every $r \in (0, R)$. Here we used the standard notation

$$
\operatorname{osc}(f_1, \partial B(x_0, r)) = \sup_{\partial B(x_0, r)} f_1(x) - \inf_{\partial B(x_0, r)} f_1(x).
$$

Letting $j \to \infty$ and using the monotone convergence theorem and Lebesgue differentiation theorem, we conclude that

$$
\int_{B(x_0, r)} J(x, f)\, dx \leq \operatorname{osc}(f_1, \partial B(x_0, r)) \int_{\partial B(x_0, r)} |Df(x)|^{n-1}\, dx \quad (16)
$$

for almost every $r \in (0, R)$. The oscillation lemma ([10, Lemma 3.10.1]) tells us that

$$
\operatorname{osc}(f_1, \partial B(x_0, r)) \leq A_p(n) r \left(\fint_{\partial B(x_0, r)} |\nabla f_1|^p \right)^{\frac{1}{p}}, \quad (17)
$$

where the constant $A_p(n)$ is given by the formula (13). Combining the inequality (16) with the oscillation lemma and using Hölder's inequality, we conclude that

$$
\int_{B(x_0, r)} J(z, f)\, dz \leq \frac{A_p(n)}{\sqrt[n-1]{\omega_{n-1}}} |\partial B(x_0, r)|^{\frac{n}{n-1}} \left(\fint_{\partial B(x_0, r)} |Df(z)|^p\, dz \right)^{\frac{n}{p}} \quad (18)
$$

for almost every $0 < r < R$, as desired.

4   Proof of Theorem 1.1

The elements in the Sobolev space $W^{1,1}_{loc}(\Omega)$ are equivalence classes of functions which agree almost everywhere in Ω. In order to study the fine properties of a function $u \in W^{1,1}_{loc}(\Omega)$, it is convenient to use the representative $\tilde{u}$, defined by the formula

$$
\tilde{u}(x) = \limsup_{r \to 0} \fint_{B(x, r)} u(z)\, dz.
$$

If $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$, then we set $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n)$. It is well known that if $v : \mathbb{R}^n \to \mathbb{R}$ is locally integrable, then

$$\lim_{r \to 0} \fint_{B(x,r)} |v(x) - v(z)| \, dz = 0$$

for almost all $x \in \mathbb{R}^n$, and thus $\tilde{f} = f$ almost everywhere in Ω.

Let $\mathcal{A}$ be an Orlicz function satisfying the integrability condition (7), $n \in \{2, 3, 4...\}$, $K > 0$ and $\beta > 0$. We introduce the strictly increasing function $\alpha(r) = \alpha_{\mathcal{A}, K, n, \beta}(r)$ defined for $0 < r^n < \frac{nK}{\omega_{n-1}}$ by the formula

$$\alpha_{\mathcal{A}, K, n, \beta}(r) = \sup \left\{ t \in \left(0, \frac{r}{2}\right) : \int_t^{r/2} \frac{ds}{s \mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1} s^n}\right)} \, ds \geq \beta \right\}. \quad (19)$$

Now we can formulate our main theorem. After stating the result we show that this technical version easily yields Theorem 1.1; for a slightly simpler version see Remark 4.4 below.

Theorem 4.1 *Assume that an Orlicz function $\mathcal{A}$ satisfies (i) and (ii). Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion whose distortion function satisfies the integrability condition*

$$K = \int_B exp(\mathcal{A}(K_O(x))) \, dx < \infty, \quad (20)$$

where $B = B(x_0, R) \subset\subset \Omega$. Then, for all small $\epsilon > 0$, we have

$$\left| \tilde{f}(x) - \tilde{f}(y) \right| \leq C_{\mathcal{A}, K}(n, \beta) exp \left((\epsilon - 1) \int_{2\alpha^{-1}(|x-y|)}^R \frac{dt}{t \mathcal{A}^{-1}\left(\log C_{\mathcal{A}, n}(\epsilon) \frac{nK}{\omega_{n-1} t^n}\right)} \right)$$
$$\left(\int_B J(z, f) \, dz \right)^{\frac{1}{n}} \quad (21)$$

whenever $x, y \in B(x_0, \alpha(R/4))$.

We will split the proof of Theorem 4.1 into two parts, Lemma 4.2 and Lemma 4.3. Before we go to the proof of Theorem 4.1, let us show how Theorem 1.1 follows from Theorem 4.1: Let $\mathcal{A}(t) = \lambda t$, $\lambda > 0$ and $s(r) = \left(\frac{r}{2}\right)^e \left[\frac{nK}{\omega_{n-1}}\right]^{\frac{1-e}{n}}$. First we observe that

$$\int_{s(r)}^{r/2} \frac{dt}{t \mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1} t^n}\right)} = \frac{\lambda}{n}$$

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and so $\alpha_{\mathcal{A},K,n,\lambda/n}(r) \geq s(r)$ for all $r < \sqrt[n]{\frac{nK}{\omega_{n-1}}}$. Furthermore

$$\exp\left((\epsilon-1)\int_{2s^{-1}(|x-y|)}^{R} \frac{dt}{t\mathcal{A}^{-1}\left(\log C_{\mathcal{A},n}(\epsilon)\frac{nK}{\omega_{n-1}t^n}\right)}\right) =$$

$$\left(\frac{\log\left(C_{\mathcal{A},n}(\epsilon)\frac{nK}{\omega_{n-1}R^n}\right)}{\log\left(\frac{C_{\mathcal{A},n}(\epsilon)}{2^{n+\frac{n}{e}}}\left[\frac{nK}{\omega_{n-1}|x-y|^n}\right]^{\frac{1}{e}}\right)}\right)^{\frac{\lambda}{n}(1-\epsilon)}$$

and so Theorem 1.1 follows.

Lemma 4.2 *Under the hypothesis of Theorem 4.1, we have*

$$\left|\tilde{f}(x) - \tilde{f}(y)\right|^n \int_r^{R/2} \frac{dt}{t\mathcal{A}^{-1}\left(\log\frac{nK}{\omega_{n-1}t^n}\right)} \leq C_{\mathcal{A},K}(n)\int_{B(x_0,R)} J(z,f)\,dz \quad (22)$$

whenever $x, y \in B(x_0, r) \subset B(x_0, R/2)$.

Proof. Combining the distortion inequality $|Df(x)|^n \leq K_O(x)J(x,f)$ with Proposition 2.1, we obtain that $P(|Df|^n) \in L^1_{loc}(\Omega)$, where $\int_1^\infty \frac{P(s)}{s^2} = \infty$ and for every $\epsilon > 0$ the function $t \to t^{\epsilon-1}P(t)$ is increasing for all $t \geq t_{2,1}(\mathcal{A},\epsilon)$. Using Proposition 2.6 in [14] and Proposition 2.4 in [14], we conclude that the coordinate functions of f are weakly monotone (see [7, Definition 1.5]). This is based on the fact that $J(\cdot,f)$ coincides with the distributional Jacobian i.e. (12) holds. Let $p = n - \frac{1}{2}$, $r < R/2$ and $x, y \in B(x_0, r)$. Then $|Df| \in L^p_{loc}(\Omega)$. Using Lemma 7.2 in [7] that holds for mappings whose coordinate functions are weakly monotone we have the estimate

$$\frac{|\tilde{f}(x) - \tilde{f}(y)|}{C(n,p)t} \leq \left(\fint_{\partial B(x_0,t)} |Df|^p\right)^{\frac{1}{p}} \quad (23)$$

for almost every $t \in [r, R]$. Write $B_s = B(x_0, s)$ and $A_i = B_{2^i r} \setminus B_{2^{i-1}r}$, for all $i \in \{1, 2, ...\}$. Define

$$G_i = \left\{t \in [2^{i-1}r, 2^i r] : \int_{\partial B_t} \exp(\mathcal{A}(K_O(x)))\,dx \leq \frac{3}{2^{i-1}r}\int_{A_i} \exp(\mathcal{A}(K_O(x)))\,dx\right\}$$

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for all $i \in \{1, 2, ...\} \cap [1, \log_2 \frac{R}{r}] = I$. Because $2r \leq R$, we see that $I \neq \emptyset$. Using Fubini's theorem, we conclude that

$$|G_i| \geq \frac{2^{i-1}}{2} r,$$

for all $i \in I$. Combining the distortion inequality and Hölder's inequality with the inequality (23), we have that

$$\frac{|\tilde{f}(x) - \tilde{f}(y)|^n}{C(n,p)t^n} \leq \left(\fint_{\partial B_t} |K_O(x)|^{\frac{p}{n-p}} \right)^{\frac{n-p}{p}} \fint_{\partial B_t} J(x, f) \, dx \qquad (24)$$

for almost every $t \in [r, R]$. By Lemma 2.2, we find a number $t_{2.2}(\frac{p}{n-p}, \mathcal{A})$ such that the function $t \to t^{-\frac{p}{n-p}} \exp \mathcal{A}(t)$ is increasing on $(t_{2.2}, \infty)$. Let $t \in [r, R]$ be such that $\fint_{\partial B_t} \exp(\mathcal{A}(K_O(x))) \, dx < \infty$, and pick τ so that

$$\exp \mathcal{A}(\tau) = \fint_{\partial B_t} \exp(\mathcal{A}(K_O(x))) \, dx.$$

Write $\lambda = \max\{\tau, t_{2.2}\}$. Then we estimate

$$
\begin{aligned}
\fint_{\partial B_t} |K_O(x)|^{\frac{p}{n-p}} \, dx &\leq \frac{1}{|\partial B_t|} \int_{\partial B_t \cap \{|K_O| > \lambda\}} |K_O(x)|^{\frac{p}{n-p}} \, dx \\
&\quad + \int_{\partial B_t \cap \{|K_O| \leq \lambda\}} |K_O(x)|^{\frac{p}{n-p}} \, dx \\
&\leq \frac{\lambda^{\frac{p}{n-p}}}{\mathcal{A}(\lambda)} \fint_{\partial B_t} \exp(\mathcal{A}(K_O(x))) \, dx + \lambda^{\frac{p}{n-p}} \leq 2\lambda^{\frac{p}{n-p}}.
\end{aligned}
$$

It follows that

$$\frac{|\tilde{f}(x) - \tilde{f}(y)|^n}{t} \leq C(p, n)\lambda \int_{\partial B_t} J(x, f) \, dx \qquad (25)$$

and thus

$$
\begin{aligned}
\frac{|\tilde{f}(x) - \tilde{f}(y)|^n}{t} &\leq C(p, n, \mathcal{A})\mathcal{A}^{-1} \left(\log \left(\fint_{\partial B_t} \exp(\mathcal{A}(K_O(x))) \, dx \right) \right) \\
&\quad \int_{\partial B_t} J(x, f) \, dx \qquad (26)
\end{aligned}
$$

for almost every $t \in [r, R]$. Fix $i \in I$. For almost every $t \in G_i$, we have that

$$\frac{|\tilde{f}(x) - \tilde{f}(y)|^n}{t} \leq C(p, n, \mathcal{A})\mathcal{A}^{-1} \left(\log \left(\frac{6K}{\omega_{n-1} t^{n-1} 2^i r} \right) \right) \int_{\partial B_t} J(x, f) \, dx. \qquad (27)$$

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Integrating this estimate over the set G_i with respect to t, we arrive at

$$|\tilde{f}(x) - \tilde{f}(y)|^n \int_{G_i} \frac{dt}{t\mathcal{A}^{-1}\left(\log \frac{6nK}{\omega_{n-1}t^n}\right)} \leq C(p, n, \mathcal{A}) \int_{A_i} J(x, f)\, dx. \qquad (28)$$

By Lemma 2.3, we fix $t_{2.3} = t_{2.3}(n, \mathcal{A}) \geq 1$ so that the function $h : t \to \left[t\mathcal{A}^{-1}\left(\log\left(\frac{1}{t^n}\right)\right)\right]^{-1}$ is decreasing on $(0, \frac{1}{t_{2.3}})$. Then the function $t \to h\left(\frac{t}{t_{2.3}} \sqrt[n]{\frac{\omega_{n-1}}{6nK}}\right)$ is decreasing on $(0, \sqrt[n]{\frac{6nK}{\omega_{n-1}}}) \supset (0, R)$. Combining this with the estimate $|G_i| \geq \frac{2^{i-1}}{2}r$, we conclude that

$$|\tilde{f}(x) - \tilde{f}(y)|^n \int_{2^{i-2}3r}^{2^i r} \frac{dt}{t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}t^n}\right)} \leq C(p, n, \mathcal{A}) \int_{A_i} J(x, f)\, dx. \qquad (29)$$

Because

$$\int_{2^{i-2}3r}^{2^i r} \frac{dt}{t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}t^n}\right)} = \int_{2^{i-1}r}^{2^{i-2}3r} \frac{ds}{(s+2^{i-2}r)\mathcal{A}^{-1}\left(\log\left(\frac{t_{2.3}^n nK}{\omega_{n-1}(s+2^{i-2}r)^n}\right)\right)}$$

$$\geq \int_{2^{i-1}r}^{2^{i-2}3r} \frac{1}{3} \frac{ds}{s\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}s^n}\right)}, \qquad (30)$$

we obtain the estimate

$$|\tilde{f}(x) - \tilde{f}(y)|^n \int_{2^{i-1}r}^{2^i r} \frac{dt}{t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}t^n}\right)} \leq C(p, n, \mathcal{A}) \int_{A_i} J(x, f)\, dx. \qquad (31)$$

Summing over the set I, we arrive at

$$|\tilde{f}(x) - \tilde{f}(y)|^n \int_r^{R/2} \frac{dt}{t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}t^n}\right)} \leq C(p, n, \mathcal{A}) \int_{B_R} J(x, f)\, dx. \qquad (32)$$

In the case $t \in (0, \sqrt[n]{\frac{nK}{\omega_{n-1}t_{2.3}}})$, we have that

$$\frac{1}{t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n K}{t^n}\right)} = \frac{1}{t_{2.3}} \frac{t_{2.3}}{t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}t^n}\right)}$$

$$\geq \frac{1}{t_{2.3}} \frac{1}{t\mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1}t^n}\right)}. \qquad (33)$$

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Here we used the fact that the function $s \to \frac{s}{\mathcal{A}^{-1}(\log s^n)}$ is increasing on $(t_{2.3}, \infty)$. Furthermore

$$\sup \left\{ \frac{t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}t^n}\right)}{t\mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1}t^n}\right)} : \frac{nK}{\omega_{n-1}t_{2.3}} \leq t^n \leq R^n \left(\leq \frac{nK}{\omega_{n-1}}\right) \right\} \leq C_{\mathcal{A},K}(n) \tag{34}$$

and so also in the case $R^n \geq t^n \geq \frac{nK}{\omega_{n-1}t_{2.3}}$, we have the inequality

$$t\mathcal{A}^{-1}\left(\log \frac{nK}{\omega_{n-1}t^n}\right) \geq C_{\mathcal{A},K}(n)t\mathcal{A}^{-1}\left(\log \frac{t_{2.3}^n nK}{\omega_{n-1}t^n}\right). \tag{35}$$

Combining the inequality (32) with the estimates (33) and (35) we complete the proof of Lemma 4.2.

Inequality (22) together with the following lemma give us the desired modulus of continuity.

Lemma 4.3 *Under the hypothesis of Theorem 4.1, for every small $\epsilon > 0$, we have*

$$\int_{B(x_0,r)} J(x,f)\,dx \leq 2\exp\left((\epsilon - n)\int_{2r}^R \frac{dt}{t\mathcal{A}^{-1}\left(\log C_{\mathcal{A},n}(\epsilon)\frac{nK}{\omega_{n-1}t^n}\right)}\right)$$
$$\int_{B(x_0,R)} J(x,f)\,dx \tag{36}$$

whenever $r \in (0, R/2)$.

Proof. Using Proposition 2.1 we see that the assumptions of Lemma 3.1 are fulfilled, and so

$$\int_{B(x_0,s)} J(x,f)\,dx \leq (n^{\,n-1}\!\sqrt{\omega_{n-1}})^{-1}|\partial B(x_0,r)|^{\frac{n}{n-1}}\left(\fint_{\partial B(x,s)} |Df|^p\right)^{\frac{n}{p}} \tag{37}$$

for almost every $0 < s < R$ and all $p \geq n - 1$. To get the sharp estimate, we will later take $p = n-1$; the general value of p allows us to show that already Lemma 3.2 is sufficient to establish a logarithmic modulus of continuity. Write $B_r = B(x_0, r)$. Fix $\epsilon \in (0,1)$, $p \in [n-1, n)$, and $i \in \{1, 2, 3, ...\}$.

We denote the interval $(R2^{-\epsilon i}, R2^{-\epsilon(i-1)})$ by $\triangle_{i,\epsilon}$ and the annulus $B_{R2^{-\epsilon(i-1)}} \setminus B_{R2^{-\epsilon i}}$ by $A_{i,\epsilon}$. Using Fubini's theorem we observe that for every $\delta > 0$ the set

$$E_i = \left\{ t \in \triangle_{i,\epsilon} : \int_{\partial B_t} |Df|^p \leq \frac{(1+\delta)}{|\triangle_{i,\epsilon}|} \int_{A_{i,\epsilon}} |Df(x)|^p \, dx \right\} \qquad (38)$$

has a positive measure.

Choosing $r \in \triangle_{i,\epsilon}$ so that r lies in the set E_i and so that the inequality (37) holds, we obtain the estimate

$$
\begin{aligned}
\int_{B_{R2^{-\epsilon i}}} J(x,f) \, dx &\leq \int_{B_r} J(x,f) \, dx \leq (n \sqrt[n-1]{\omega_{n-1}})^{-1} |\partial B_r|^{\frac{n}{n-1}} \left(\int_{\partial B_r} |Df|^p \right)^{\frac{n}{p}} \\
&\leq (n \sqrt[n-1]{\omega_{n-1}})^{-1} \left(\frac{1+\delta}{|\triangle_{i,\epsilon}|} \right)^{\frac{n}{p}} |\partial B_r|^{\frac{n}{n-1} - \frac{n}{p}} \\
&\qquad \left(\int_{A_{i,\epsilon}} |Df(x)|^p \, dx \right)^{\frac{n}{n-1}}
\end{aligned}
$$

By Lemma 2.4 we can fix $t_{2.4} = t_{2.4}(p/(n-p), \mathcal{A}) \in (0, \infty)$ so that the function $t \to \exp \mathcal{A}\left(t^{\frac{n-p}{p}} \right)$ is convex on $(t_{2.4}, \infty)$. We set

$$\tilde{K}_O(x) = \begin{cases} K_O(x), & K_O(x) > t_{2.4} \\ t_{2.4}, & K_O(x) \leq t_{2.4}. \end{cases} \qquad (39)$$

Combining the distortion inequality $|Df(x)|^n \leq \tilde{K}_O(x) J(x,f)$ with Hölder's inequality, we find that

$$
\begin{aligned}
\int_{B_{R2^{-\epsilon i}}} J(x,f) \, dx &\leq (n \sqrt[n-1]{\omega_{n-1}})^{-1} \left(\frac{1+\delta}{|\triangle_{i,\epsilon}|} \right)^{\frac{n}{p}} |\partial B_r|^{\frac{n}{n-1} - \frac{n}{p}} \\
&\qquad \left(\int_{A_{i,\epsilon}} |\tilde{K}_O|^{\frac{p}{n-p}} \right)^{\frac{n-p}{p}} \int_{A_{i,\epsilon}} J(x,f) \, dx. \qquad (40)
\end{aligned}
$$

Jensen's inequality applied to the convex function $(t_{2.4}, \infty) \to (0, \infty) : \tau \to \exp \mathcal{A}\left(\tau^{\frac{n-p}{p}} \right)$ yields

$$
\begin{aligned}
\int_{B_{R2^{-\epsilon i}}} J(x,f) \, dx &\leq (n \sqrt[n-1]{\omega_{n-1}})^{-1} \left(\frac{1+\delta}{|\triangle_{i,\epsilon}|} \right)^{\frac{n}{p}} |A_{i,\epsilon}|^{\frac{n-p}{p}} \\
&\qquad |\partial B_r|^{\frac{n}{n-1} - \frac{n}{p}} \mathcal{A}^{-1}\left(\log \fint_{A_{i,\epsilon}} \exp \mathcal{A}(\tilde{K}_O) \right) \int_{A_{i,\epsilon}} J(x,f) \, dx
\end{aligned}
$$

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and computations show that

$$
\int_{B_{R2^{-\epsilon i}}} J(x,f)\, dx \;\leq\; \left(\frac{1+\delta}{n}\right)^{\frac{n}{p}} \left(\frac{2^{\epsilon n}-1}{2^{\epsilon}-1}\right)^{\frac{n-p}{p}} R2^{-\epsilon i}\, |\triangle_{i,\epsilon}|^{-1}
$$
$$
\mathcal{A}^{-1}\left(\log\left(\frac{\exp\mathcal{A}(t_{2.4})nK2^{\epsilon n}}{\omega_{n-1}R^n 2^{-\epsilon n(i-1)}(2^{\epsilon n}-1)}\right)\right)
$$
$$
\int_{A_{i,\epsilon}} J(x,f)\, dx \tag{41}
$$

Letting $\delta \to 0$, using the elementary inequalities $\epsilon a \log 2 \leq 2^{a\epsilon}-1 \leq \epsilon a 2^{a\epsilon}\log 2$ for all $\epsilon, a \geq 0$ and the fact that $\epsilon \leq 1$, we conclude that

$$
\int_{B_{R2^{-\epsilon i}}} J(x,f)\, dx \;\leq\; \frac{2^{\frac{n\epsilon}{p}}}{n} R2^{-\epsilon i}\mathcal{A}^{-1}\left(\log\frac{\tilde{C}nK}{\omega_{n-1}R^n 2^{-\epsilon n(i-1)}\epsilon}\right)
$$
$$
\int_{A_{i,\epsilon}} J(x,f)\, dx\, |\triangle_{i,\epsilon}|^{-1}. \tag{42}
$$

Here and also in what follows we denote the constant $\exp\mathcal{A}(t_{2.4}(p/(n-p),\mathcal{A}))$ by $\tilde{C}$. At this point we would like to warn the reader that the constant $\tilde{C}$ tends to infinity if we let $p \to n$.

Next we introduce two auxiliary functions. We set

$$
u_{i,\epsilon}(x) = \int_{B_{R2^{-\epsilon i}}} J(x,f)\, dx + \frac{x-R2^{-\epsilon i}}{|\triangle_{i,\epsilon}|}\int_{A_{i,\epsilon}} J(x,f)\, dx
$$

and

$$
u_\epsilon = \sum_{i=1}^{\infty} u_{i,\epsilon}(x)\chi_{[R2^{-\epsilon i},R2^{-\epsilon(i-1)})}(x)
$$

for every $x \in \mathbb{R}$. Using the inequality (42), we see that

$$
u_{i,\epsilon}(x) \leq \left[\frac{2^{\frac{n\epsilon}{p}}}{n} R2^{-\epsilon i}\mathcal{A}^{-1}\left(\log\frac{\tilde{C}nK}{\omega_{n-1}R^n 2^{-\epsilon n(i-1)}\epsilon}\right) + x - R2^{-\epsilon i}\right]u'_{i,\epsilon}(x) \tag{43}
$$

for all $x \in \triangle_{i,\epsilon}$. Combining the inequality $R2^{-\epsilon(i-1)} - R2^{-\epsilon i} \leq R2^{-\epsilon i}2\epsilon$ with the fact that $\left[\mathcal{A}^{-1}(\log\frac{\tilde{C}nK}{\omega_{n-1}R^n 2^{-\epsilon n(i-1)}\epsilon})\right]^{-1} \leq \left[\mathcal{A}^{-1}(\log\frac{2^{\epsilon n}\tilde{C}nK}{\epsilon\omega_{n-1}R^n})\right]^{-1} = B_\epsilon$ (notice that $B_\epsilon \to 0$, as $\epsilon \to 0$), we conclude with

$$
u_{i,\epsilon}(x) \leq \left(\frac{2^{\frac{n\epsilon}{p}}}{n} + 2B_\epsilon\epsilon\right)x\mathcal{A}^{-1}\left(\log\frac{\tilde{C}nK}{\omega_{n-1}x^n\epsilon}\right)u'_{i,\epsilon}(x) \tag{44}
$$

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for all $x \in \triangle_{i,\epsilon}$.

Fix $\delta \in (0, R/10^6)$ and define

$$v_{\epsilon,\delta}(x) = \exp\left(-\int_\delta^x \left[\left(\frac{2^{\frac{n\epsilon}{p}}}{n} + 2B_\epsilon\epsilon\right) t \mathcal{A}^{-1}\left(\log\frac{\tilde{C}nK}{\omega_{n-1}t^n\epsilon}\right)\right]^{-1} dt\right) \quad (45)$$

for all $x \in (\delta, R)$. Using the inequality (44), we see that

$$
\frac{d}{dx}(v_{\epsilon,\delta}u_\epsilon)(x) = v_{\epsilon,\delta}(x)\left[-\left(\left(\frac{2^{\frac{n\epsilon}{p}}}{n} + 2B_\epsilon\epsilon\right) x \mathcal{A}^{-1}\left(\log\frac{\tilde{C}nK}{\omega_{n-1}x^n\epsilon}\right)\right)^{-1}\right.
$$
$$
\left. u_\epsilon(x) + u_\epsilon'(x)\right] \geq 0 \quad (46)
$$

for all $x \in (\delta, R) \cap \cup_{i=1}^\infty \triangle_{i,\epsilon}$ and so the continuous function $v_{\epsilon,\delta}u_\epsilon$ is non-decreasing on (δ, R). Let $2r \in (\delta, R - \delta)$. Choosing $i_r \in \mathbb{N}$ so that $r \in [R2^{-\epsilon i_r}, R2^{-\epsilon(i_r-1)})$, we find that

$$v_{\epsilon,\delta}(r)\int_{B_{R2^{-\epsilon i_r}}} J(x,f)\,dx \leq 2v_{\epsilon,\delta}(R-\delta)\int_{B_{R-\delta}} J(x,f)\,dx. \quad (47)$$

Since the Jacobian of f is non-negative almost everywhere this yields the estimate

$$
\int_{B_{2^{-\epsilon}2r}} J(x,f)\,dx \leq 2\exp\left[-\int_{2r}^{R-\delta}\left[\left(\frac{2^{\frac{n\epsilon}{p}}}{n} + 2B_\epsilon\epsilon\right) t\right.\right.
$$
$$
\left.\left.\mathcal{A}^{-1}\left(\log\frac{\tilde{C}nK}{\omega_{n-1}t^n\epsilon}\right)\right]^{-1} dt\right]\int_{B_{R-\delta}} J(x,f)\,dx. \quad (48)
$$

Letting $\delta \to 0$ and using the estimate $\epsilon \leq 1$, we find that

$$
\int_{B_r} J(x,f)\,dx \leq 2\exp\left[-\int_{2r}^{R}\left[\left(\frac{2^{\frac{n\epsilon}{p}}}{n} + 2B_\epsilon\epsilon\right) t\right.\right.
$$
$$
\left.\left.\mathcal{A}^{-1}\left(\log\frac{\tilde{C}nK}{\omega_{n-1}t^n\epsilon}\right)\right]^{-1} dt\right]\int_{B_R} J(x,f)\,dx \quad (49)
$$

for all $0 < r < R/2$. Choosing $p = n - 1$, the constant $\tilde{C}$ depends only on $\mathcal{A}$ and n. Furthermore, using the fact that $\frac{2^{\frac{n\epsilon}{n-1}}}{n} + 2B_\epsilon\epsilon \to \frac{1}{n}$, as $\epsilon \to 0$, we obtain the desired inequality (36).

Now we complete the proof of the entire theorem as follows.
Given $x \in B(x_0, \alpha\left(\frac{R}{4}\right))$ we consider the ball $B(x_0, r)$, $r = \alpha_{\mathcal{A}, n, \beta}^{-1}(|x - x_0|)$.
By Lemma 4.2 we have the estimate

$$|\tilde{f}(x) - \tilde{f}(x_0)| \leq C_{\mathcal{A}, K}(n, \beta) \left(\int_{B(x_0, r)} J(z, f) \, dz \right)^{\frac{1}{n}}. \tag{50}$$

Using Lemma 4.3 we further obtain the estimate

$$\int_{B(x_0, r)} J(z, f) \, dz \leq 2\exp\left((\epsilon - n) \int_{2r}^{R} \frac{dt}{t\mathcal{A}^{-1}\left(\log C_{\mathcal{A}, n}(\epsilon)\frac{nK}{\omega_{n-1}t^n} \right)} \right)$$
$$\int_{B(x_0, R)} J(x, f) \, dx. \tag{51}$$

Combining the inequality (50) with the estimate (51) we finally conclude the desired modulus of continuity (21).

Remark 4.4 *The integral in (21) of Theorem 4.1 can be taken from $|x - y|$ to R when $|x - y|$ is sufficiently small. To see this, notice that the ratio of this integral, taken from $|x - y|$ to $\alpha^{-1}(|x - y|)$, with the corresponding integral from $|x - y|$ to R, tends to zero when $|x - y|$ tends to zero. Thus the small error made in changing the limits can be imbedded in the estimate by changing the ϵ in front of the integral in (21) to 2ϵ. We leave the details to the reader.*

Remark 4.5 *If we use the inequality (13) instead of the isoperimetric inequality (11) in the proof of Lemma 4.3, then under the assumptions of Lemma 4.3, for every small $\epsilon > 0$, we have*

$$\int_{B(x_0, r)} J(x, f) \, dx \leq 2exp\left(\left(\epsilon - \frac{1}{A_{n-\epsilon}(n)} \right) \int_{2r}^{R} \frac{dt}{t\mathcal{A}^{-1}\left(\log C_{\mathcal{A}, n}(\epsilon)\frac{nK}{\omega_{n-1}t^n} \right)} \right)$$
$$\int_{B(x_0, R)} J(x, f) \, dx \tag{52}$$

whenever $r \in (0, R/2)$. The constant $A_{n-\epsilon}(n)$ is given by the formula (14). This results in the claim of Theorem 4.1 with $\epsilon - 1$ replaced by $\epsilon - \frac{1}{n A_{n-\epsilon}(n)}$, which is weaker than our claim. However in the setting of Theorem 1.1, this weaker estimate still gives us a logarithmic modulus of continuity, this time with the exponent $\frac{\lambda}{n^2 A_{n-\epsilon}(n)} - \epsilon$.

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5   Construction of Example 1.2

We close this paper by showing that the modulus of continuity we gave above in Section 4 cannot be substantially improved on. To see that Example 5.1 qualifies for that purpose, make a change of variables to reduce the integrand to the form given in Theorem 4.1, and notice that the role of α^{-1} is not significant when $|x - y|$ is small; see Remark 4.4. Example 1.2 is contained as a special case in Example 5.1.

Example 5.1 *Assume that an Orlicz function $\mathcal{A}$ satisfies (i). Then there exists a continuous mapping of finite distortion $f : B(0,1) \to \mathbb{R}^n$ such that*

$$\int_{B(0,1)} exp(\mathcal{A}(K_O(x, f)))\, dx < \infty \tag{53}$$

and points $x_j, y_j \in B(0,1)$ so that $x_j \to 0$, $y_j \to 0$, and

$$|f(x_j) - f(y_j)|\, exp\left(\int_{|x_j - y_j|}^{1/2} \frac{ds}{s\mathcal{A}^{-1}\left(\frac{n}{1+\epsilon} \log \frac{1}{s}\right)} \right) \to \infty \tag{54}$$

for every $\epsilon > 0$.

Let $r \in (0,1)$. Given a ball $B = \{x \in \mathbb{R}^n : |x - a| < r\}$ and $\epsilon > 0$ we consider the radial stretching

$$g(x) = a + r\frac{x - a}{|x - a|}\rho_r(|x - a|), \tag{55}$$

where

$$\rho_r(t) = \exp\left(-\int_t^r \frac{ds}{s\mathcal{A}^{-1}\left(\frac{n}{1+\epsilon} \log \frac{1}{s}\right)} \right)$$

for all $t \in (0, r)$. We may calculate Jacobian of g by using the familiar formula

$$J(x, g) = r^n \rho_r'(|x - a|)\left(\frac{\rho_r(|x - a|)}{|x - a|} \right)^{n-1} \tag{56}$$

(see the equations (5.43 and (5.44) in [10])). The distortion function is

$$K_O(x, g) = \frac{|Dg(x)|^n}{J(x, g)} = \frac{\rho_r(|x - a|)}{|x - a|\rho_r'(|x - a|)} \tag{57}$$

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see the formula (5.49) in [10]. Here we used the assumption $|x - a| < 1$. Using polar coordinates we compute the integral of the Jacobian as

$$\int_B J(x, g) \, dx = \frac{\omega_{n-1}}{n} r^n \int_0^r \rho_r^{n-1}(t) \rho_r'(t) \, dt = \frac{\omega_{n-1}}{n} r^n \rho_r^n(t) \Big|_0^r = |B|. \qquad (58)$$

Combining the equation (57) with the fact that

$$\frac{t \rho_r'(t)}{\rho_r(t)} = \frac{1}{\mathcal{A}^{-1} \left(\frac{n}{1+\epsilon} \log \frac{1}{t} \right)}$$

we see that

$$\int_B \exp(\mathcal{A}(K_O(x, g))) \, dx = \int_B \left(\frac{1}{|x|} \right)^{\frac{n}{1+\epsilon}} dx. \qquad (59)$$

We will now glue together several versions of g with variable ϵ, r, and a. Let $\epsilon_i = \frac{1}{i}$, $i = 2, 3, \dots$. For all $i = 2, 3, 4 \dots$ we choose $r_i \in (0, 2^{-i}]$ so that $\int_{B(0, r_i)} |x|^{-\frac{n}{1+\epsilon_i}} \, dx \leq 2^{-i}$. Let $a_i = (2^{-i+2}, 0, \dots, 0)$, for $i \geq 4$, set $B_i = B(a_i, r_i)$, and define $G = \cup_{i \geq 4} B_i$.

Define the mapping $f : B(0, 1) \to \mathbb{R}^n$ by setting

$$f(x) = \begin{cases} a_i + r_i \frac{x - a_i}{|x - a_i|} \rho_{r_i}(|x - a_i|) & \text{for } x \in B_i, \\ x & \text{for } x \in B(0, 1) \setminus G. \end{cases} \qquad (60)$$

On account of the formula (59) we find that

$$\int_G \exp(\mathcal{A}(K_O(x, f))) \leq \sum_{i=2}^{\infty} 2^{-i} < \infty. \qquad (61)$$

We also have the trivial estimate

$$\int_{B(0,1) \setminus G} \exp(\mathcal{A}(K_O(x, f))) \leq \exp(\mathcal{A}(1)) |B(0, 1)|. \qquad (62)$$

Thus (53) holds.

To verify that $J(\cdot, f) \in L^1(B(0, 1))$ we use the equation (58),

$$\int_G J(x, f) \, dx \leq \sum_{i=1}^{\infty} |B_i| < \infty. \qquad (63)$$

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and the trivial estimate

$$\int_{B(0,1)\backslash G} J(x,f) = |B(0,1)\backslash G|. \tag{64}$$

Let us now explain how to choose the points x_i, y_i. Set $x_i = a_i$. If $y_i \in B_i$, then

$$|f(x_i) - f(y_i)| = r_i \rho_{r_i}(|x_i - y_i|)$$

$$= r_i \exp\left(-\int_{|x_i-y_i|}^{r_i} \frac{ds}{s\mathcal{A}^{-1}\left(\frac{n}{1+\epsilon_i}\log\frac{1}{s}\right)}\right).$$

Using the changes of variable $\mathcal{A}(t) = \frac{n}{1+k\epsilon_i}\log\frac{1}{s}$, $k = 1, 2$, we see that

$$r_i \exp\left(-\int_{|x_i-y_i|}^{r_i} \frac{ds}{s\mathcal{A}^{-1}\left(\frac{n}{1+\epsilon_i}\log\frac{1}{s}\right)}\right) \geq i \exp\left(-\int_{|x_i-y_i|}^{1/2} \frac{ds}{s\mathcal{A}^{-1}\left(\frac{n}{1+2\epsilon_i}\log\frac{1}{s}\right)}\right)$$

when y_i chosen to be sufficiently close to x_i. The claim follows by noting that

$$\mathcal{A}^{-1}\left(\frac{n}{1+\epsilon}\log\frac{1}{s}\right) \leq \mathcal{A}^{-1}\left(\frac{n}{1+2\epsilon_i}\log\frac{1}{s}\right)$$

when $2\epsilon_i \leq \epsilon$.

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