# $(n,2)$-SETS HAVE FULL HAUSDORFF DIMENSION

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Abstract. We prove that a set containing translates of every 2-plane must have full Hausdorff dimension.

## 1. Introduction

This is a continuation of [4] where a partial result on the problem under investigation was obtained. Since that paper is unpublished work, we will reproduce certain parts of it for the sake of completeness.

An $(n,2)$-set in $\mathbb{R}^n$ is a subset $E \subset \mathbb{R}^n$ containing a translate of every 2-dimensional plane.

The natural question that arises is whether $E$ must have positive Lebesgue measure. This turns out to be true in low dimensions. Marstrand [3] proved that $(3,2)$-sets have positive measure. Bourgain [1] showed the same for $(4,2)$-sets and made a connection with the Kakeya conjecture. In higher dimensions the question is open. However, it has been known for some time that if $n > 4$ then $\dim_H(E) \geq (2n+2)/3$, where $\dim_H$ denotes Hausdorff dimension. This follows from the estimates for the 2-plane transform due to Christ [2]. Recent work by the author [4] has led to the mild improvement $\dim_H(E) \geq (2n+3)/3$. In the present paper we modify the argument in [4], which in turn is based on geometric-combinatorial ideas inspired by Wolff [6], to obtain full dimension. Namely we prove the following.

**Theorem 1.1.** *Suppose $n > 4$ and let $E \subset \mathbb{R}^n$ be an $(n,2)$-set. Then $dim_H(E) = n$.*

## 2. Terminology and notation

$S^{n-1} \subset \mathbb{R}^n$ is the $(n-1)$-dimensional unit sphere.

$B(a,r)$ is the closed ball of radius $r$ centered at the point $a$.

For $X \subset \mathbb{R}^n$, $X^\perp$ denotes its orthogonal complement.

If $e \in S^{n-1}$, $a \in \mathbb{R}^n$ then $L_e(a) = \{a + te : t \in \mathbb{R}\}$ is the line in the $e$-direction passing through the point $a$.

If $e \in S^{n-1}$, $a \in \mathbb{R}^n$, $\beta > 0$ then $T_e^\beta(a) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, L_e(a)) \leq \beta\}$ is the infinite tube with axis $L_e(a)$ and cross-section radius $\beta$.

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$\mathcal{L}^k$ denotes $k$-dimensional Lebesgue measure and $\mathcal{L}^0$ counting measure. When the context is clear we will use the notation $|\cdot|$ for all these measures.

Let $\mathcal{G}_n$ be the Grassmannian manifold of all 2-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the unique probability measure $\gamma_{n,2}$ which is invariant under the action of the orthogonal group. The elements of $\mathcal{G}_n$ will be refered to as *direction planes*.

If $P_1, P_2 \in \mathcal{G}_n$, then their distance is defined by

$$d(P_1, P_2) = \|\mathrm{proj}_{P_1} - \mathrm{proj}_{P_2}\|$$

where $\mathrm{proj}_P : \mathbb{R}^n \to P$ is the orthogonal projection onto $P$.

A set of points or direction planes is called *$\rho$-separated* if the distance between any two of its elements is at least $\rho$.

If $P \in \mathcal{G}_n,\ 1 \leq l \leq 4,\ \delta > 0$ then $P^{l,\delta}$ is a rectangle of dimensions $l \times l \times \underbrace{\delta \times \cdots \times \delta}_{n-2}$, that is, the image of $[0,l] \times [0,l] \times [0,\delta] \times \cdots \times [0,\delta]$ under a rotation and a translation, such that its faces with dimensions $l \times l$ are parallel to $P$. Such a set will be refered to as a *$\delta$-plate* or simply as a *plate*. When $l = 1$ the superscript $l$ will be supressed.

If $P_1^{l,\delta} \cap P_2^{l,\delta} \neq \emptyset$ and $d(P_1, P_2) = \theta$ we will say that the plates *intersect at angle $\theta$*.

The letter $C$ will denote various constants whose values may change from line to line. Similarly, $C_\epsilon$ will denote constants depending on $\epsilon$. If we need to keep track of the value of a constant through a calculation we will use subscripted letters $C_1, C_2, \ldots$ or the notation $\widetilde{C}$. $x \lesssim y$ means $x \leq Cx$ and $x \simeq y$ means $(x \lesssim y \ \&\ y \lesssim x)$.

Finally, note that

$$\gamma_{n,2}(\{P \in \mathcal{G}_n : d(P, P_0) \leq \delta\}) \simeq \delta^{2(n-2)} \text{ for all } P_0 \in \mathcal{G}_n,\ \delta \leq 1.$$

So if $\mathcal{A} \subset \mathcal{G}_n$ and $\mathcal{B}$ is a maximal $\delta$-separated subset of $\mathcal{A}$ then

$$\gamma_{n,2}(\mathcal{A}) \lesssim |\mathcal{B}|\delta^{2(n-2)}.$$

Further, if $\mathcal{A} \subset \mathcal{G}_n$ is $\delta$-separated and $\mathcal{B}$ is a maximal $\eta$-separated subset of $\mathcal{A}$ with $\eta \geq \delta$ then

$$|\mathcal{B}| \gtrsim |\mathcal{A}|(\delta/\eta)^{2(n-2)}.$$

## 3. Auxiliary Lemmas

The following technical lemma allows us to control the intersection of two plates.

**Lemma 3.1.** *Let $P_1^{l,\eta}, P_2^{l,\eta}$ be two plates such that $d(P_1, P_2) \leq 1/2$. Then there exists a tube $T_e^{\beta}(a)$ with $\beta = C\eta/d(P_1, P_2)$ such that*

$$P_1^{l,\eta} \cap P_2^{l,\eta} \subset T_e^{\beta}(a).$$

*In particular*

$$|P_1^{l,\eta} \cap P_2^{l,\eta}| \lesssim \frac{\eta^{n-1}}{d(P_1, P_2)}.$$

*Proof.* Choose $a, b \in P_1^{l,\eta} \cap P_2^{l,\eta}$ so that $|a-b| = \operatorname{diam}(P_1^{l,\eta} \cap P_2^{l,\eta})$ and let $r = |a-b|$, $\theta = d(P_1, P_2)$, $e = (a-b)/|a-b|$. If $r \leq C_1\eta/\theta$ then $B(a, 2r) \subset T_e^{\beta}(a)$, so we may assume that $r \geq C_1\eta/\theta$. We can also assume that $P_2$ is the $x_1x_2$-plane, $a = \mathbf{0}$, $b = (b_1, 0, \overline{b})$, $\overline{b} \in \mathbb{R}^{n-2}$. Since $\theta \leq 1/2$, by simple linear algebra, we can write $P_1 = \{(s, t, s\overline{u} + t\overline{v}) : s, t \in \mathbb{R}\}$, where $\overline{u}, \overline{v} \in \mathbb{R}^{n-2}$, $|\overline{u}|, |\overline{v}| \lesssim 1$. Now, if $x = (x_i) \in S^{n-1}$ then $\operatorname{proj}_{P_1}(x) = (s_0, t_0, s_0\overline{u} + t_0\overline{v})$, where

$$s_0 = \frac{(x_1 + \langle u, x\rangle)(1 + |\overline{v}|^2) - (x_2 + \langle v, x\rangle)\langle \overline{u}, \overline{v}\rangle}{1 + |\overline{u}|^2 + |\overline{v}|^2 + |\overline{u}|^2|\overline{v}|^2 - \langle \overline{u}, \overline{v}\rangle^2},$$

$$t_0 = \frac{(x_2 + \langle v, x\rangle)(1 + |\overline{u}|^2) - (x_1 + \langle u, x\rangle)\langle \overline{u}, \overline{v}\rangle}{1 + |\overline{u}|^2 + |\overline{v}|^2 + |\overline{u}|^2|\overline{v}|^2 - \langle \overline{u}, \overline{v}\rangle^2},$$

$$u = (0, 0, \overline{u}), \ v = (0, 0, \overline{v}).$$

Therefore

$$\theta = \sup_{|x|=1} |\operatorname{proj}_{P_1}(x) - \operatorname{proj}_{P_2}(x)| \lesssim |\overline{u}| + |\overline{v}|.$$

Since $b \in P_1^{l,\eta} \cap P_2^{l,\eta}$ there exists $(s, t, s\overline{u} + t\overline{v}) \in P_1$ such that

$$|b_1 - s| \lesssim \eta, \ |t| \lesssim \eta, \ |s\overline{u} + t\overline{v} - \overline{b}| \lesssim \eta, \ |\overline{b}| \lesssim \eta.$$

Then

$$|s| \geq |b| - |\overline{b}| - |b_1 - s| \geq r - C\eta \geq (1 - CC_1^{-1})r \gtrsim r$$

for $C_1$ sufficiently large. Hence

$$|\overline{u}| \leq \frac{|s\overline{u} + t\overline{v} - \overline{b}| + |t\overline{v}| + |\overline{b}|}{|s|} \lesssim \frac{\eta}{r}.$$

Consequently

$$|\overline{v}| = |\overline{v}| + |\overline{u}| - |\overline{u}| \geq C^{-1}\theta - C\eta/r \geq (C^{-1} - CC_1^{-1})\theta \gtrsim \theta$$

for $C_1$ large enough.

Now let $y = (y_i) \in P_1^{l,\eta} \cap P_2^{l,\eta}$. Then there exist $z_1 = (s_1, t_1, s_1\overline{u} + t_1\overline{v}) \in P_1$, $z_2 = (s_2, t_2, \overline{\mathbf{0}}) \in P_2$ such that $|y - z_1| \lesssim \eta$, $|y - z_2| \lesssim \eta$. Therefore

$$|s_1\overline{u} + t_1\overline{v}| \lesssim \eta, \ |s_1| \leq |z_1| \leq |z_1 - y| + |y| \lesssim \eta + r \lesssim r.$$

It follows that

$$|t_1| \le \frac{|s_1\overline{u} + t_1\overline{v}| + |s_1\overline{u}|}{|\overline{v}|} \lesssim \frac{\eta + r(\eta/r)}{\theta} \lesssim \frac{\eta}{\theta}.$$

Hence

$$\begin{aligned} \mathrm{dist}(y, x_1 - \mathrm{axis}) &\le |y - (s_1, 0, \overline{\mathbf{0}})| \le |y - z_1| + |t_1| + |s_1\overline{u} + t_1\overline{v}| \\ &\lesssim \eta + \eta/\theta + \eta \lesssim \eta/\theta. \end{aligned}$$

We conclude that

$$\begin{aligned} \mathrm{dist}(y, L_e(\mathbf{0})) &\le |y - y_1 b_1^{-1} b| \le \mathrm{dist}(y, x_1 - \mathrm{axis}) + |y_1||b_1|^{-1}|\overline{b}| \\ &\lesssim \eta/\theta + \eta \lesssim \eta/\theta \end{aligned}$$

proving the first assertion of the lemma. To prove the second assertion, note that

$$|P_1^{l,\eta} \cap P_2^{l,\eta}| \le |P_1^{l,\eta} \cap T_e^{\beta}(a)| \lesssim \frac{\eta^{n-1}}{d(P_1, P_2)}.$$

□

The proof of Theorem 1.1 will be, essentially, a reduction to the 3-dimensional case via the Radon transform. We give the definitions.

For a function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying the appropriate integrability conditions, the Radon transform

$$\mathcal{R}f : S^2 \times \mathbb{R} \to \mathbb{R}$$

is defined by

$$\mathcal{R}f(e,t) = \int\limits_{\langle e,x\rangle = t} f(x)\, d\mathcal{L}^2(x).$$

It is proved in Oberlin and Stein [5] that for any measurable set $E \subset \mathbb{R}^3$ one has the following estimate.

$$\|\mathcal{R}\chi_E\|_{3,\infty} \lesssim \|\chi_E\|_{3/2}$$

where

$$\|\mathcal{R}\chi_E\|_{3,\infty} = \left( \int\limits_{S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \right)^{1/3}$$

and $d\sigma$ is surface measure.

We can discretize this result as follows.

**Lemma 3.2.** *Suppose $E$ is a set in $\mathbb{R}^3$, $\lambda \le 1$ and let $\{P_k\}_{k=1}^M$ be a $\delta$-separated set in $\mathcal{G}_3$ such that for each $k$ there is plate $P_k^{l,C\delta}$ satisfying*

$$|P_k^{l,C\delta} \cap E| \ge \lambda |P_k^{l,C\delta}|.$$

*Then*

$$|E| \gtrsim \lambda^{3/2} M^{1/2} \delta.$$

*Proof.* For each $e \in S^2$ let $P_e$ be the plane with normal $e$ passing through the origin. Then there is a $\delta$-separated set $\{e_k\}_{k=1}^M$ on $S^2$ such that $P_k = P_{e_k}$. Note that since $1 \le l \le 4$, for each $e \in B(e_k, \delta/2) \cap S^2$ we have

$$\lambda\delta \le |P_k^{l,C\delta} \cap E| \le \int_{I_e} \mathcal{L}^2((P_e + x) \cap E) d\mathcal{L}^1(x)$$

where $I_e$ is an interval on $P_e^\perp$ with $\mathcal{L}^1(I_e) \lesssim \delta$. Therefore there exists $x_e \in I_e$ such that

$$\lambda \lesssim \mathcal{L}^2((P_e + x_e) \cap E).$$

Hence

$$\lambda \lesssim \sup_t \mathcal{R}\chi_E(e,t).$$

We conclude that

$$\begin{aligned} \lambda^3\delta^2 M &\lesssim \sum_k \int_{B(e_k,\delta/2)\cap S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \\ &\le \int_{S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \\ &= \|\mathcal{R}\chi_E\|_{3,\infty}^3 \lesssim \|\chi_E\|_{3/2}^3 = |E|^2. \end{aligned}$$

□

This, in turn, gives rise to the following higher dimensional analogue.

**Lemma 3.3.** *Suppose $E$ is a set in $\mathbb{R}^n$, $\lambda \le 1$, $\Pi \subset \mathbb{R}^n$ is a 3-plane and $\{P_k\}_{k=1}^M$ is a $\delta$-separated set in $\mathcal{G}_n$ such that for each $k$ there exists a plate $P_k^\delta$ satisfying*

$$P_k^\delta \subset \Pi^{\widetilde{C}\delta} \text{ and } |P_k^\delta \cap E| \ge \lambda |P_k^\delta|$$

*where $\Pi^{\widetilde{C}\delta} = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Pi) \le \widetilde{C}\delta\}$ is the $\widetilde{C}\delta$-neighborhood of $\Pi$. Then*

$$|E \cap \Pi^{\widetilde{C}\delta}| \gtrsim \lambda^3 M^{1/2} \delta^{n-2}.$$

*Proof.* Whithout loss of generality we may asssume that $\Pi$ is the $x_1x_2x_3$-plane. Since $P_k^\delta \subset \Pi^{\widetilde{C}\delta}$ there is a direction plane $Q_k \subset \Pi$ such that $d(P_k, Q_k) \lesssim \delta$. Therefore we can find a plate $Q_k^{2,C_1\delta}$ with $P_k^\delta \subset Q_k^{2,C_1\delta}$. It follows that

$$|Q_k^{2,C_1\delta} \cap E \cap \Pi^{\widetilde{C}\delta}| \ge \lambda\delta^{n-2}.$$

Let $\mathcal{B}$ be a maximal $C_2\delta$-separated subset of $\{P_k\}_{k=1}^M$ and put $\mathcal{B}' = \{Q_k : P_k \in \mathcal{B}\}$. Then for $Q_j, Q_k \in \mathcal{B}'$, $j \ne k$, we have

$$d(Q_j, Q_k) \ge d(P_j, P_k) - d(P_j, Q_j) - d(P_k, Q_k) \ge (C_2 - C)\delta \ge \delta$$

for $C_2$ sufficiently large.

Now for each $Q_k \in \mathcal{B}'$ let

$$L_k = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \leq \frac{\lambda\delta}{C_3} \right\},$$

$$H_k = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \geq \frac{\lambda\delta}{C_3} \right\}.$$

Note that

$$\mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \lesssim \delta, \text{ for all } x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp.$$

Hence

$$\begin{aligned}
\lambda\delta^{n-2} &\leq |Q_k^{2,C_1\delta} \cap E \cap \Pi^{\widetilde{C}\delta}| \\
&= \int\limits_{B(0,\widetilde{C}\delta)\cap\Pi^\perp} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\
&= \int\limits_{L_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\
&+ \int\limits_{H_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\
&\leq \frac{\lambda\delta}{C_3} C\delta^{n-3} + C\delta\mathcal{L}^{n-3}(H_k).
\end{aligned}$$

Therefore, $\mathcal{L}^{n-3}(H_k) \gtrsim \lambda\delta^{n-3}$ for $C_3$ sufficiently large.

Next, let $M' = |\mathcal{B}'|$ and define

$$L = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : |\{k : x \in H_k\}| < \frac{\lambda M'}{C_4} \right\},$$

$$H = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : |\{k : x \in H_k\}| \geq \frac{\lambda M'}{C_4} \right\}.$$

Then

$$\begin{aligned}
\lambda\delta^{n-3}M' &\lesssim \sum_k \int \chi_{H_k} = \int\limits_H \sum_k \chi_{H_k} + \int\limits_L \sum_k \chi_{H_k} \\
&\leq M'\mathcal{L}^{n-3}(H) + \frac{\lambda M'}{C_4}\mathcal{L}^{n-3}(L) \\
&\leq M'\mathcal{L}^{n-3}(H) + \frac{\lambda M'}{C_4} C\delta^{n-3}.
\end{aligned}$$

Therefore $\mathcal{L}^{n-3}(H) \gtrsim \lambda\delta^{n-3}$ for $C_4$ sufficiently large.

Note that for each $x \in H$ there are at least $\lambda M'/C_4$ plates in $\Pi + x$, that is, plates in a copy of $\mathbb{R}^3$, with $\delta$-separated direction planes and such that the 3-dimensional measure of their intersection with $E \cap (\Pi + x)$ is at least $C^{-1}\lambda\delta$. Hence, by Lemma 3.2

$$\mathcal{L}^3(E \cap (\Pi + x)) \gtrsim \lambda^{3/2}(\lambda M')^{1/2}\delta.$$

We conclude that

$$\begin{aligned}|E \cap \Pi^{\widetilde{C}\delta}| &\geq \int_H \mathcal{L}^3(E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x)\\ &\gtrsim \lambda\delta^{n-3}\lambda^{3/2}(\lambda M')^{1/2}\delta\\ &\simeq \lambda^3 M^{1/2}\delta^{n-2}.\end{aligned}$$

□

## 4. The main argument

Theorem 1.1 will be a consequence of the following.

**Proposition 4.1.** *Suppose $E$ is a set in $\mathbb{R}^n$, $\lambda \leq 1$ and $\{P_j\}_{j=1}^M$ is a $\delta$-separated set in $\mathcal{G}_n$ with* $\operatorname{diam}(\{P_j\}_{j=1}^M) \leq 1/2$, *such that for each $j$ there is plate $P_j^\delta$ satisfying*

$$|P_j^\delta \cap E| \geq \lambda |P_j^\delta|.$$

*Then*

$$|E| \geq C_\epsilon^{-1}\delta^\epsilon \lambda^{(n+2)/2} M^{1/2}\delta^{n-2}$$

*Proof.* The idea of the proof is to find a single plate with high multiplicity and then apply repeatedly Lemma 3.3 to the plates intersecting it. To this end fix a number $\mu$ and let

$$\mathcal{C}'_\mu = \left\{ P_j^\delta : |\{x \in P_j^\delta \cap E : |\{k : x \in P_k^\delta\}| \leq \mu\}| \geq \frac{\lambda}{2}\delta^{n-2} \right\},$$

$$\mathcal{C}''_\mu = \left\{ P_j^\delta : |\{x \in P_j^\delta \cap E : |\{k : x \in P_k^\delta\}| \geq \mu\}| \geq \frac{\lambda}{2}\delta^{n-2} \right\}.$$

Then

$$\left\{P_j^\delta\right\}_{j=1}^M = \mathcal{C}'_\mu \cup \mathcal{C}''_\mu.$$

We consider two cases.

CASE I. (Low multiplicity) $\mathcal{C}''_\mu = \emptyset$.
CASE II. (High multiplicity) $\mathcal{C}''_\mu \neq \emptyset$.

In case I, for each $1 \leq j \leq M$ let

$$A'_j = \left\{x \in P_j^\delta \cap E : |\{k : x \in P_k^\delta\}| \leq \mu\right\}.$$

Then

$$|E| \geq |\bigcup_{j=1}^M A'_j| \geq \frac{1}{\mu}\sum_{j=1}^M |A'_j| \geq \frac{1}{2}\frac{M}{\mu}\lambda\delta^{n-2}.$$

Therefore

$$\mu \geq \frac{1}{2}\frac{M}{|E|}\lambda\delta^{n-2}.$$

So letting

$$\mu_0 = \frac{1}{4}\frac{M}{|E|}\lambda\delta^{n-2} \tag{1}$$

we see that we cannot have case I for $\mu_0$. Consequently, there is a plate $P^\delta := P^\delta_{j_0}$ such that

$$|\{x \in P^\delta \cap E : |\{k : x \in P^\delta_k\}| \geq \mu_0\}| \geq \frac{\lambda}{2}\delta^{n-2}.$$

Note that for each $x \in P^\delta \cap E$ with $|\{k : x \in P^\delta_k\}| \geq \mu_0$ we have

$$\{k : x \in P^\delta_k\} = \bigcup_{i=1}^{\log(C/\delta)} \{k : x \in P^\delta_k \text{ and } \delta 2^{i-1} \leq d(P_k, P) < \delta 2^i\}.$$

Therefore, by the pigeonhole principle, there is an integer $i(x)$ with $1 \leq i(x) \leq \log(C/\delta)$ such that

$$|\{k : x \in P^\delta_k \text{ and } \delta 2^{i(x)-1} \leq d(P_k, P) < \delta 2^{i(x)}\}| \geq (\log(C/\delta))^{-1}\mu_0.$$

And so,

$$\begin{aligned}&\{x \in P^\delta \cap E : |\{k : x \in P^\delta_k\}| \geq \mu_0\} \\ &= \bigcup_{i=1}^{\log(C/\delta)} \{x \in P^\delta \cap E : |\{k : x \in P^\delta_k \text{ and } \delta 2^{i-1} \leq d(P_k, P) < \delta 2^i\}| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \geq (\log(C/\delta))^{-1}\mu_0\}.\end{aligned}$$

Applying the pigeonhole principle again, we see that there exists a number $\rho := \delta 2^{i_0 - 1}$ and a set $A \subset P^\delta \cap E$ of measure

$$|A| \gtrsim (\log(C/\delta))^{-1}\lambda\delta^{n-2} \tag{2}$$

such that for every $x \in A$

$$|\{k : x \in P^\delta_k \text{ and } \rho \leq d(P_k, P) < 2\rho\}| \geq (\log(C/\delta))^{-1}\mu_0. \tag{3}$$

Heuristically, (2) and (3) tell us that a large number of plates intersect $P^\delta$ at approximately the same angle. We are going to estimate this number using the bound for the measure of their pairwise intersections. To do this, define

$$\mathcal{D} = \{P^\delta_k : P^\delta_k \cap P^\delta \neq \emptyset \text{ and } \rho \leq d(P_k, P) < 2\rho\}.$$

Then, by Lemma 3.1, we have

$$\begin{aligned}|\mathcal{D}| &\gtrsim \sum_{k : P^\delta_k \in \mathcal{D}} |P^\delta_k \cap P^\delta| \frac{\rho}{\delta^{n-1}} \\ &= \frac{\rho}{\delta^{n-1}} \int_{P^\delta} \sum_{k : P^\delta_k \in \mathcal{D}} \chi_{P^\delta_k} \\ &\geq \frac{\rho}{\delta^{n-1}} \int_{A} \sum_{k : P^\delta_k \in \mathcal{D}} \chi_{P^\delta_k}\end{aligned}$$

$$\geq \frac{\rho}{\delta^{n-1}}|A|(\log(C/\delta))^{-1}\mu_0$$

$$\gtrsim (\log(C/\delta))^{-2}\lambda^2\frac{\rho}{\delta}\frac{M}{|E|}\delta^{n-2}. \tag{4}$$

Where the last inequality follows from (1) and (2) and the one before last from (3).

We are now in a position to carry out a geometric construction, in the spirit of [6], which will allow us to use Lemma 3.3.

Let $\{e_i\}_i$ be a maximal $\delta/\rho$-separated set of points on the $(n-3)$-dimensional unit sphere $S^{n-1} \cap P^\perp$ and let

$$\Pi_i = c + \Pi_i'$$

where $c$ is the center of $P^\delta$ and $\Pi_i'$ is the 3-plane spanned by $e_i$ and $P$. Then for each $P_k^\delta \in \mathcal{D}$ there exists an $i$ such that $P_k^\delta \subset \Pi_i^{\widetilde{C}\delta}$, where $\Pi_i^{\widetilde{C}\delta}$ is the $\widetilde{C}\delta$-neighborhood of $\Pi_i$. To see this, let $y \in P_k^\delta$, $w \in P_k^\delta \cap P^\delta$. Then there exists $z \in P_k$ with $|z| \lesssim 1$, $|y - w - z| \lesssim \delta$. Write

$$z = z_1 + z_2 \in P \oplus P^\perp,$$

$$c - w = w_1 + w_2 \in P \oplus P^\perp.$$

Then $|z_2| \lesssim \rho$, $|w_2| \lesssim \delta$. Choose $e_i$ so that $|z_2 - |z_2|e_i| \lesssim \delta$. Hence

$$y = ((y - w - z) + (z_2 - |z_2|e_i) - w_2) + (z_1 - w_1 + |z_2|e_i) + c \in \Pi_i^{\widetilde{C}\delta}.$$

Therefore, if we let

$$\mathcal{D}_i = \left\{ P_k^\delta \in \mathcal{D} : P_k^\delta \subset \Pi_i^{\widetilde{C}\delta} \right\}$$

then

$$\mathcal{D} = \bigcup_i \mathcal{D}_i.$$

Now let $\gamma = \lambda(\log(1/\delta))^{-1}$ and consider two cases.

CASE I. $\delta \leq \gamma\rho$.
CASE II. $\delta \geq \gamma\rho$.

In case I let

$$\mathcal{X} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, c + P) \leq \gamma\rho\}.$$

Then for each $P_k^\delta \in \mathcal{D}$

$$P_k^\delta \cap \mathcal{X} \subset P_k^{2\gamma\rho} \cap \mathcal{X}.$$

Hence, by Lemma 3.1, $P_k^\delta \cap \mathcal{X}$ is contained in a tube of cross-section radius $C\gamma$. Therefore

$$\begin{aligned} |P_k^\delta \cap (E \cap \mathcal{X}^\complement)| &= |P_k^\delta \cap E| - |P_k^\delta \cap E \cap \mathcal{X}| \\ &\geq |P_k^\delta \cap E| - |P_k^\delta \cap \mathcal{X}| \end{aligned}$$

$$\geq \lambda\delta^{n-2} - C\lambda(\log(1/\delta))^{-1}\delta^{n-2}$$
$$\geq \frac{\lambda}{2}\delta^{n-2}$$

for $\delta$ sufficiently small.

Now, if $\operatorname{dist}(x, c+P) \geq \gamma\rho$ then $x$ belongs to at most $C\gamma^{-(n-3)}$ sets $\Pi_i^{\widetilde{C}\delta}$. To see this suppose $c = \mathbf{0}$, $x \in \Pi_i^{\widetilde{C}\delta}$ and write $x = u+w \in P \oplus P^\perp$. Then $|w| \geq \gamma\rho$, $|w - \langle w, e_i\rangle e_i| \lesssim \delta$. Therefore, either $|w - |w|e_i| \lesssim \delta$, or $|w + |w|e_i| \lesssim \delta$. We conclude that the set of all possible $e_i$ is contained in

$$B(w/|w|, C\delta/(\gamma\rho)) \cup B(-w/|w|, C\delta/(\gamma\rho))$$

and therefore has cardinality at most $C\gamma^{-(n-3)}$. Consequently

$$\begin{aligned}
|E| &\geq |\bigcup_i E \cap \mathfrak{X}^{\mathsf{C}} \cap \Pi_i^{\widetilde{C}\delta}| \\
&\gtrsim \gamma^{n-3} \sum_i |(E \cap \mathfrak{X}^{\mathsf{C}}) \cap \Pi_i^{\widetilde{C}\delta}| \\
&\gtrsim \gamma^{n-3}\lambda^3\delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2}
\end{aligned}$$

where the last inequality follows from Lemma 3.3 applied to the set $E \cap \mathfrak{X}^{\mathsf{C}}$, the families of plates $\{\mathcal{D}_i\}_i$ and the 3-planes $\{\Pi_i\}_i$.

In case II, since $|\{\Pi_i\}_i| \lesssim (\rho/\delta)^{n-3}$, we have

$$\begin{aligned}
|E| &\geq |\bigcup_i E \cap \Pi_i^{\widetilde{C}\delta}| \\
&\gtrsim (\delta/\rho)^{n-3} \sum_i |E \cap \Pi_i^{\widetilde{C}\delta}| \\
&\geq \gamma^{n-3} \sum_i |E \cap \Pi_i^{\widetilde{C}\delta}| \\
&\gtrsim \gamma^{n-3}\lambda^3\delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2}
\end{aligned}$$

with the last inequality true by Lemma 3.3 applied to the set $E$, the families of plates $\{\mathcal{D}_i\}_i$ and the 3-planes $\{\Pi_i\}_i$.

We conclude that in either case

$$|E| \gtrsim \gamma^{n-3}\lambda^3\delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2}. \tag{5}$$

To estimate the sum above, note that $\Pi_i^{\widetilde{C}\delta}$, being the $\widetilde{C}\delta$-neighborhood of a copy of $\mathbb{R}^3$, can contain at most $C(\rho/\delta)^2$ plates whose direction planes are $\delta$-separated and at distance approximately $\rho$ from $P$. Therefore

$$|\mathcal{D}| \leq \sum_i |\mathcal{D}_i| \lesssim \frac{\rho}{\delta} \sum_i |\mathcal{D}_i|^{1/2}. \tag{6}$$

Combining (4), (5) and (6) we obtain

$$|E| \geq C_\epsilon^{-1}\delta^{2\epsilon}\lambda^{n+2}\frac{M}{|E|}\delta^{2(n-2)}$$

where the logarithmic factors have been absorbed into $C_\epsilon^{-1}\delta^{2\epsilon}$. Consequently

$$|E| \geq C_\epsilon^{-1}\delta^{\epsilon}\lambda^{(n+2)/2}M^{1/2}\delta^{n-2}$$

proving the proposition. □

## 5. Proof of Theorem 1.1

Let $F$ be an $(n,2)$-set. Using measure theory, we can find a compact set $E \subset F$ and a set $\mathcal{A} \subset \mathcal{G}_n$ of positive measure so that $\operatorname{diam}(\mathcal{A}) \leq 1/2$ and for every $P \in \mathcal{A}$ there is a square $S_P$ of unit area such that $S_P$ is parallel to $P$ and

$$\mathcal{L}^2(S_P \cap E) \geq 1/2.$$

Fix a covering $\{B(x_i, r_i)\}$ of $E$ and let

$$I_k = \left\{i : 2^{-k} \leq r_i \leq 2^{-(k-1)}\right\}, \quad \nu_k = |I_k|,$$

$$E_k = E \cap \bigcup_{i \in I_k} B(x_i, r_i), \quad \widetilde{E}_k = \bigcup_{i \in I_k} B(x_i, 2r_i).$$

Then, by the pigeonhole principle, one can find a $k$ and a set $\mathcal{B} \subset \mathcal{A}$ of measure at least $C^{-1}k^{-2}$ so that

$$\mathcal{L}^2(S_P \cap E_k) \gtrsim k^{-2}, \text{ for all } P \in \mathcal{B}.$$

Let $\{P_j\}_{j=1}^M$ be a maximal $2^{-k}$-separated set in $\mathcal{B}$. Then

$$M \gtrsim k^{-2}2^{2k(n-2)}$$

and for each $P_j$ there is a plate $P_j^{2^{-k}}$ such that

$$|P_j^{2^{-k}} \cap \widetilde{E}_k| \gtrsim k^{-2}|P_j^{2^{-k}}|.$$

So, by Proposition 4.1

$$|\widetilde{E}_k| \geq C_\epsilon^{-1}k^{-\alpha}2^{-k\epsilon}$$

where $\alpha = n+3$. On the other hand

$$|\widetilde{E}_k| \lesssim \nu_k 2^{-kn}.$$

Therefore

$$\nu_k \geq C_\epsilon^{-1}k^{-\alpha}2^{k(n-\epsilon)}.$$

Consequently

$$\sum_i r_i^{n-2\epsilon} \gtrsim \nu_k 2^{-k(n-2\epsilon)} \geq C_\epsilon^{-1}k^{-\alpha}2^{k\epsilon} \geq \widetilde{C}_\epsilon^{-1}.$$

We conclude that $\dim_H(F) = n$.

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