Mappings of finite distortion:

$L^n \log^\alpha L$-integrability

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1. Introduction

Recently systematic studies of mappings of finite distortion have emerged in the geometric function theory. The connection with deformations of elastic bodies in the theory of nonlinear elasticity is perhaps a primary motivation for such studies. The regularity properties of these mappings and solutions of the associated PDEs are the basic theme in [1], [2], [4], [10], [11], [13], [15], [18], [20], [21], [25], [26], [27], [29]. Those papers proved rather clearly that the class of mappings with exponentially integrable distortion function is optimal in many respects. Other related works on mappings with finite distortion are [12], [23], [24], [9], [5], [6], [14], [19], [22], [28]. In [10] the authors established, in *even* dimensions, improved regularity under the a priori assumption that the mappings in question have integrable Jacobian determinants. Although

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they did suspect that the results obtained in [10] and [1] remain valid in all dimensions, there were quite significant obstacles to overcome. In this paper we continue this theme, refining and extending this earlier work to all dimensions.

Let Ω be an open subset of $I\!R^n$. We shall consider mappings $f : \Omega \to I\!R^n$ in the Sobolev class $W^{1,1}_{\rm loc}(\Omega, I\!R^n)$. Thus the differential matrix $Df(x) \in I\!R^{n\times n}$ is defined at almost every point $x \in \Omega$.

Definition 1. A mapping $f \in W^{1,1}_{\rm loc}(\Omega, I\!R^n)$ is said to have finite distortion if:

i) The Jacobian determinant $J(x, f) = detDf(x)$ is locally integrable.

ii) There is a measurable function $K = K(x) \geq 1$, finite almost everywhere, such that

$$||Df(x)||^n \leq K(x)J(x, f). \tag{1}$$

Note that the operator norm of a matrix M is being used here, $||M|| = \max\{|M\xi|; |\xi| = 1\}$. If f has finite distortion, then (1) holds for the function K that is defined to take the value 1 when either $J(x, f) = 0$ or $Df(x)$ does not exist and the value $||Df(x)||^n/J(x, f)$ otherwise. We call this function the distortion function of f.

Every mapping of finite distortion solves a nonlinear system of first order PDEs, the so-called Beltrami system. This in turn gives rise to a degenerate elliptic equation of the second order; an analogue of the familiar n-harmonic equation. Our main idea is to approximate these second order equations by more regular ones. The solutions of the regular equations approximate our mapping f. We then exploit a result of [15] which provides a priori estimates for the solutions of the regular equations. As a final step we show that f coincides with the weak limit of the regular solutions. More importantly, the estimates are preserved in passing to the limit.

Theorem 1. *For each dimension $n \geq 2$ and $\alpha \geq 0$ there exists $\lambda_\alpha(n) \geq 1$ such that if $f : \Omega \to I\!R^n$ has finite distortion and the distortion function $K = K(x)$*

of f satisfies

$$\int_\Omega e^{\lambda K(x)} dx < \infty$$

for some $\lambda \geq \lambda_\alpha(n)$, then

$$\int_B |Df(x)|^n \log^\alpha \left(1 + \frac{|Df(x)|}{|Df|_B}\right) dx \leq C_\alpha(n) \int_{2B} J(x,f) dx$$

for all concentric balls $B \subset 2B \subset \Omega$. Here $|Df|_B$ stands for the integral average of $|Df|$ over the ball B.

The assumption in Theorem 1 on the size of λ is necessary as is easily seen by considering the mapping $f(x) = \frac{x}{|x| \log^s(1/|x|)}$, $s > 0$, for $x \in B(0, 1/2)$, that has finite and exponentially integrable distortion.

As a consequence of Theorem 1 and our earlier work in [1], [10] we now obtain the following modulus of continuity and volume distortion estimates for mappings of exponentially integrable distortion. These estimates extend the corresponding results by G. David [4] in the complex plane.

Corollary 1. *For each dimension $n \geq 2$ and $s \geq 0$ there exists $\lambda_s(n) \geq 1$ such that if the distortion function $K = K(x)$ of a mapping $f : \Omega \to \mathbf{R}^n$ of finite distortion satisfies*

$$\int_\Omega e^{\lambda K(x)}\, dx < \infty$$

for some $\lambda \geq \lambda_s(n)$, then given $B(a, 2R) \subset \Omega$ there is a constant C so that

$$|f(x) - f(y)| \leq \frac{C}{\log^s(\frac{2R}{|x-y|})} \tag{2}$$

whenever $x, y \in B(a, R)$ and

$$|f(E)| \leq \frac{C}{\log^s(2 + \frac{1}{|E|})} \tag{3}$$

whenever $E \subset B(a, R)$ is compact.

We close the introduction with a result that extends a removable singularity theorem from [10], originally only given in even dimensions, to all dimensions. For the concept of the capacity referred to in the statement see Section 9.

Theorem 2. *There exists a number $\lambda(n) > 0$ with the following properties. Let $E \subset \mathbf{R}^n$ be a closed set of zero $L^n \log^{n-1} L$–capacity and let*

$$f : \Omega \setminus E \to \mathbf{R}^n \tag{4}$$

be a bounded mapping of finite distortion and assume that the distortion function K satisfies

$$\int_{\Omega\setminus E} e^{\lambda K(x)}\, dx < \infty$$

for some $\lambda \geq \lambda(n)$. Then f extends to a mapping of finite distortion in Ω with exponentially integrable distortion function.

2. Algebraic Preliminaries

Here we recall some commonly used formulas.
Recall that the adjoint Jacobian matrix $Adj\, Df(x) \in I\!R^{n\times n}$ satisfies the rule

$$D^t f(x)\, Adj\, Df(x) = J(x,f)\mathbf{I}\,, \tag{2.1}$$

where $D^t f(x)$ denotes the transpose to $Df(x)$: the entries of $Adj\, Df(x)$ are the cofactors of $Df(x)$. In particular

$$\|Adj\, Df(x)\| \leq \|Df(x)\|^{n-1}\,. \tag{2.2}$$

It is well known that the adjoint differential matrix is divergence free

$$\text{div}\,[Adj\, Df(x)] = 0\,.$$

Precisely, this means that

$$\int_\Omega \langle Adj\, Df(x),\, D\varphi_j(x)\rangle dx = 0 \tag{2.3}$$

whenever $f \in W^{1,n-1}_{\text{loc}}(\Omega, I\!R^n)$ and $\varphi_j \in C_0^\infty(\Omega, I\!R^n)$. Here and in the sequel we use the inner product of matrices defined by

$$\langle X, Y\rangle = \text{Trace}\ X^t Y\,.$$

This inner product gives rise to another norm in $\mathbb{R}^{n\times n}$, called Hilbert–Schmidt norm and denoted by $|M|^2 = \text{Trace } M^t M$.
To every mapping f of finite distortion there corresponds the Beltrami equation

$$D^t f(x)\, Df(x) = J(x,f)^{\frac{2}{n}} G(x)\,, \tag{2.4}$$

where $G(x)$ is the distortion tensor of f: a symmetric, positive definite matrix of determinant 1. Here G is determined by (2.4) when $J(x,f) > 0$ and we set $G = \mathbf{I}$ otherwise.
Associated with $G(x)$ is the energy integral

$$\mathcal{E}[h] = \int_\Omega E(x,\, Dh) dx\,,$$

where $E(x,M) = \langle MG^{-1}(x), M\rangle^{\frac{n}{2}}$. It is the essence of the analytic theory of mappings with finite distortion that these mappings minimize the energy functional, subject to a Dirichlet boundary condition. The Euler–Lagrange equation takes the form

$$\text{div } A(x,\, Df) = 0\,,$$

where $A : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is given by

$$A(x,M) = \langle MG^{-1}(x),\, M\rangle^{\frac{n-2}{2}} MG^{-1}(x)\,.$$

The energy integrand takes the form

$$E(x,M) = \langle A(x,M), M\rangle\,.$$

Notice that

$$E(x,M) \geq n^{\frac{n}{2}} \det(M)$$

for any M and that for $M = Df(x)$ and f satisfying the Beltrami equation we have

$$E(x,Df) = n^{\frac{n}{2}} J(x,f)\,.$$

In particular, our mapping f has finite energy, since $J(x,f)$ is assumed to be integrable. We shall make use of the following elementary, though quite involved inequality:

$$\frac{1}{n}|M|^n + \frac{n-1}{n}|A(x,M)|^{\frac{n}{n-1}} \le K(x)E(x,M) \tag{2.5}$$

for $M \in \mathbb{R}^{n\times n}$.

Using the terminology of the next section, this inequality tells us that a mapping of distortion $K = K(x)$ gives rise to an n-harmonic couple $\Phi = [A(x, Df),\, Df]$.

3. p-Harmonic couples

A pair $\Phi = [A, M]$ of matrix fields $A(x)$ and $M(x)$ such that

$$\text{div } A(x) = 0$$

$$M(x) = DF \qquad , \qquad F \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$$

is said to be p-harmonic, $1 < p < \infty$ of distortion $K = K(x) \ge 1$, if

$$\frac{1}{p}|M(x)|^p + \frac{p-1}{p}|A(x)|^{\frac{p}{p-1}} \le K(x)\langle A(x), M(x)\rangle \ . \tag{3.1}$$

The energy of Φ is defined by

$$\mathcal{E}[\Phi] = \int_\Omega \langle A(x), M(x)\rangle dx \ .$$

In what follows we shall assume that this integral converges. We will be interested in the distortion K of the exponential class $\mathrm{EXP}(\Omega)$. Precisely we shall assume that

$$\int_\Omega e^{\lambda K(x)} dx < \infty \tag{3.2}$$

for a certain $\lambda > 0$.

Under this condition, the distortion inequality (3.1) implies

$$M \in L^p \log^{-1} L(\Omega) \quad \text{ and } \quad A \in L^{\frac{p}{p-1}} \log^{-1} L(\Omega)\,,$$

by Hölder's inequality in Orlicz spaces.

Recall from the previous section that, for a given mapping $f : \Omega \to I\!R^n$ of exponentially integrable distortion, the pair

$$\Phi = [A(x, Df),\ Df(x)]$$

is an n-harmonic couple of the same finite distortion $K = K(x)$ as in (1.1). This paper relies on the following a priori estimate concerning integrability properties of p-harmonic couples.

Theorem 2. *Let $[A, M]$ be a p-harmonic couple. For every integer $m \geq 0$ there exists $\lambda_p(m, n) \geq 1$ such that if (3.2) holds with some $\lambda \geq \lambda_p(m, n)$ and $\Omega' \subset \Omega$ is relatively compact and $M \in L^p \log^m L(\Omega')$, then*

$$\|M\|^p_{L^p \log^m L(\Omega')} \leq C_p(\Omega') \int_\Omega \langle A(x), M(x)\rangle dx\,. \tag{3.3}$$

Here the dependence on Ω' is only in terms of $d(\Omega', \partial\Omega)$.

Let us emphasize explicitly here that this result requires the left hand side to be finite. The proof of Estimate (3.3) can be found in [15]. As a matter of fact we are going to use it for the exponent p close to n. More specifically, we always assume that

$$n - \frac{1}{2} \leq p \leq n\,. \tag{3.4}$$

Remark 3.1

It is important to realize that in this range of exponents both $\lambda_p(m, n)$ and $C_p(\Omega')$ can be made independent of p.

4. The Approximation

First we approximate the distortion tensor $G = G(x)$ by uniformly elliptic ones

$$G_\epsilon(x) = \frac{G(x) + \epsilon \mathbf{I}}{\mathbf{I} + \epsilon G(x)}, \qquad 0 \leq \epsilon \leq 1\ . \tag{4.1}$$

Here the quotient notation simply means that we multiply the matrix $G + \epsilon \mathbf{I}$ by the inverse of $\mathbf{I} + \epsilon G$. We believe that in what follows the analogous symbols are self-explanatory. In particular we have

$$G_\epsilon^{-1}(x) = \frac{G^{-1}(x) + \epsilon \mathbf{I}}{\mathbf{I} + \epsilon G^{-1}(x)} \ . \tag{4.2}$$

To ensure convergence of the integrals we will need to work with exponents slightly below the dimension. Then we set $p = n - \epsilon$, where $0 < \epsilon \leq \frac{1}{2}$. Accordingly, we introduce the perturbation

$$E_\epsilon(x, M) = \langle A_\epsilon(x, M), M \rangle = \langle M G_\epsilon^{-1}(x), M \rangle^{\frac{n-\epsilon}{2}} \tag{4.3}$$

of the energy integrand, where

$$A_\epsilon(x, M) := \langle M G_\epsilon^{-1}(x), M \rangle^{\frac{p-2}{2}} M G_\epsilon^{-1}(x) \tag{4.4}$$

for $x \in \Omega$ and $M \in I\!\!R^{n \times n}$.

It is a matter of elementary analysis of the eigenvalues of $G_\epsilon(x)$ that we have

$$\frac{1}{p}|M|^p + \frac{p-1}{p} \, |A_\epsilon(x, M)|^{\frac{p}{p-1}} \leq K(x) E_\epsilon(x, M) \tag{4.5}$$

with the same distortion function $K = K(x)$ as in (3.1).

Next we fix a ball B compactly contained in Ω and consider the minimizer $h = h_\epsilon \, : \, B \to I\!\!R^n$ of the p-harmonic variational integral

$$\mathcal{E}_p[h] = \int_B E_\epsilon(x, Dh) dx \, . \tag{4.6}$$

This minimizer is taken under to the Dirichlet condition

$$h \in f + W_0^{1, n-\epsilon}(B, I\!\!R^n) \, .$$

This makes sense because $f \in W^{1, n-\epsilon}(B, I\!\!R^n)$ for each $\epsilon > 0$, see the discussion in the following section. The existence and uniqueness of h_ϵ are easily established by the usual convexity arguments. The Euler–Lagrange equation for h_ϵ reads as

$$\operatorname{div} \, A_\epsilon(x, Dh_\epsilon) = 0 \, .$$

Because the equation is uniformly elliptic, the minimizer h_ϵ in fact belongs to $W^{1,q}_{loc}(B, I\!R^n)$ for some $q > n - e$, see e.g. [8]. We deduce from here that the pair $\Phi_\epsilon = [A_\epsilon(x, Dh_\epsilon),\ Dh_\epsilon)]$ is a p-harmonic couple of distortion K, that is

$$\frac{1}{p}|Dh_\epsilon(x)|^p + \frac{p-1}{p}|A_\epsilon(x, Dh_\epsilon)|^{\frac{p}{p-1}} \leq K(x)E_\epsilon(x, Dh_\epsilon) \tag{4.7}$$

for almost every $x \in B$.
As K satisfies the condition (3.2) with $\lambda = \lambda(m, n)$ sufficiently large, Theorem 2 applies to each h_ϵ:

$$\|Dh_\epsilon\|^{n-\epsilon}_{L^{n-\epsilon}\log^m L(B')} \leq C_{m,n}(B') \int_B E_\epsilon(x, Dh_\epsilon)dx \tag{4.8}$$

for every relatively compact subset $B' \subset B$. Note that the constant $C_{m,n}(B')$ is independent of ϵ, see Remark 3.1.

Our nearest goal is to show that the mappings $h_\epsilon : B \to I\!R^n$ converge to some $h : B \to I\!R^n$ weakly in every space $W^{1,s}(B', I\!R^n)$, where $1 \leq s < n$ and B' is an arbitrary compact subdomain of B. We prove this fact in a few steps. Let us begin with some integrability properties of the Dirichlet data.

5. $|Df|$ belongs to $L^n \log^{-1} L(\Omega)$

By the definition of mappings of finite distortion, the Jacobian determinant $J(x, f)$ is (locally) integrable on Ω; regarding Theorem 1 we may assume that the integrability is global. On the other hand, the distortion function K lies in $\mathrm{EXP}(\Omega)$. We find, by using Hölder's inequality in Orlicz spaces, that

$$K(x)J(x, f) \in L \log^{-1} L(\Omega)\ .$$

This, in view of the distortion inequality (2.5), implies

$$|Df| \in L^n \log^{-1} L(\Omega)\ , \tag{5.1}$$

as claimed.

In particular, we extract two elementary consequences from this fact. First

$$f \in W^{1,n-\epsilon}(\Omega, I\!R^n) \qquad \text{for every} \quad 0 < \epsilon \leq 1 \,.$$

Second, it is known [16] that (5.1) implies that f belongs to the grand Sobolev space $W^{1,n)}(\Omega, I\!R^n)$, that is

$$\lim_{\epsilon \to 0} \epsilon \int_\Omega |Df(x)|^{n-\epsilon}\, dx = 0 \,. \tag{5.2}$$

We also observe the following upper semicontinuity formula

$$\begin{aligned} &\limsup_{\epsilon\to 0} \int_\Omega E_\epsilon(x, Df)\, dx \leq \\ &\leq \int_\Omega E(x, Df)dx = n^{\frac{n}{2}} \int_\Omega J(x, f)dx \,. \end{aligned} \tag{5.3}$$

This is immediate from the elementary pointwise inequality

$$\begin{aligned} E_\epsilon(x, M) &\leq \left[\langle MG^{-1}(x), M\rangle + \epsilon |M|^2\right]^{\frac{n-\epsilon}{2}} \leq \\ &\leq (1+\delta)\langle MG^{-1}(x), M\rangle^{\frac{n-\epsilon}{2}} + \epsilon C_\delta |M|^{n-\epsilon} \,, \end{aligned} \tag{5.4}$$

where δ can be any positive parameter whereas the constant C_δ can be quite large. Applying this inequality to $M = Df$, and integrating we obtain

$$\begin{aligned} \limsup_{\epsilon\to 0} \int_\Omega E_\epsilon(x, Df)dx &\leq (1+\delta) \int_\Omega E(x, Df)dx \\ &\quad + C_\delta \lim_{\epsilon\to 0} \epsilon \int_\Omega |Df|^{n-\epsilon} dx \;. \end{aligned}$$

This last limit equals zero and we conclude with inequality (5.3).

6. A Weakly Converging Sequence $\{h_j\}$

We show here first that there exist positive numbers ϵ_j converging to zero such that the sequence of mappings $h_j = h_{\epsilon_j}(x)$ converges to a mapping $h : B \to I\!R^n$, weakly in every space $W^{1,s}(B')$ with $1 \leq s < n$ and relatively compact subdomains $B' \subset B$.

Indeed, if we fix $B' \subset\subset B$, then by the estimate (4.8) we obtain

$$\int_{B'} |Dh_\epsilon|^{n-\epsilon} dx \leq \|Dh_\epsilon\|^{n-\epsilon}_{L^{n-\epsilon}\log^m(B')} \leq \tag{6.1}$$

$$\leq C_{m,n}(B') \int_B E_\epsilon(x, Dh_\epsilon) dx \tag{6.2}$$

$$\leq C_{m,n}(B') \int_B E_\epsilon(x, Df) dx \leq \tag{6.3}$$

$$\leq 2C_{m,n}(B') \int_\Omega E(x, Df)\, dx$$

provided ϵ is sufficiently small, depending of course on B', by (5.3).

Finally, a routine diagonal selection method gives the sequence $\{h_j\}$. We next show that the limit mapping $h : B \to I\!R^n$ actually belongs to $W^{1,p}_{\text{loc}}(B, I\!R^n)$, with $P(t) = t^n \log^m(e+t)$ and satisfies

$$\|Dh\|^n_{L^n \log^m L(B')} \leq C_{m,n}(B') \int_\Omega E(x, Df)\, dx\ . \tag{6.4}$$

This estimate follows from the calculation above. Indeed, for sufficiently large j, with fixed B', we can write

$$\|Dh_j\|^{n-\epsilon_j}_{L^{n-\epsilon_j} \log^m L(B')} \leq 2C_{m,n}(B') \int_\Omega E(x, Df) dx\ .$$

Inequality (6.4) follows by simply applying the lower semicontinuity of the norms $\| \,.\, \|_{L^{n-\delta}\log^m L}$ with δ arbitrarily small.

To establish Theorem 1 we need only show that

$$h(x) = f(x) \qquad \text{for a.e. } x \in \Omega\ .$$

This fact still requires some work due to insufficient regularity of both h and f. We proceed by first showing that:

7. The Limit Mapping Has Finite Energy

In fact we shall see that h and f have the same energy on the ball B :

$$\begin{aligned} \int_B E(x, Dh) dx &= \int_B E(x, Df) dx = \\ &= n^{\frac{n}{2}} \int_B J(x, f) dx. \end{aligned} \tag{7.1}$$

As h_ϵ is a minimizer of the functional (4.6), we have

$$\int_B E_\epsilon(x, Dh_\epsilon)dx \leq \int_B E_\epsilon(x, Df)dx.$$

Fix an arbitrary compact set $B^{'} \subset B$ and numbers $\delta > 0$ and $1 \leq s < n$. We use the algebraic inequality

$$\langle M(\delta I + G)^{-1}, M\rangle \leq \langle MG_\epsilon^{-1}, M\rangle \tag{7.2}$$

for $0 \leq \epsilon \leq \delta$, whose proof will be given at the end of this section. Thus

$$\begin{aligned}
\int_{B'} \langle Dh(\delta I + G)^{-1}, Dh\rangle^{\frac{s}{2}} dx &\leq \lim_{j\to\infty} \int_{B'} \langle Dh_j(\delta I + G)^{-1}, Dh_j\rangle^{\frac{s}{2}} dx \\
&\leq \lim_{j\to\infty} \int_{B} \langle Dh_j G_{\epsilon_j}^{-1}, Dh_j\rangle^{\frac{s}{2}} dx \\
&\leq \lim_{j\to\infty} |B|^{\frac{n-\epsilon_j - s}{n-\epsilon_j}} \left[\int_B \langle Dh_j G_{\epsilon_j}^{-1}, Dh_j\rangle^{\frac{n-\epsilon_j}{2}} dx\right]^{\frac{s}{n-\epsilon_j}} \\
&= |B|^{\frac{n-s}{n}} \lim_{j\to\infty} \left[\int_B E_{\epsilon_j}(x, Dh_j)dx\right]^{\frac{s}{n-\epsilon_j}} \\
&\leq |B|^{\frac{n-s}{n}} \lim_{j\to\infty} \left[\int_B E_{\epsilon_j}(x, Df)dx\right]^{\frac{s}{n-\epsilon_j}} \\
&= |B|^{\frac{n-s}{n}} \left[\int_B E(x, Df)dx\right]^{\frac{s}{n}}.
\end{aligned}$$

Letting s tend to n yields

$$\int_{B'} \langle Dh(\delta I + G)^{-1}, Dh\rangle^{\frac{n}{2}} \leq \int_B E(x, Df)dx.$$

Next we may pass to the limit as $\delta \to 0$. Since $B^{'}$ was an arbitrary compact set in B, we conclude with the inequality

$$\int_B E(x, Dh)dx \leq \int_B E(x, Df)dx. \tag{7.3}$$

The distortion inequality together with the hypothesis that K lies in the exponential class give

$$|Dh| \in L^n \log^{-1} L(B). \tag{7.4}$$

The proof of the reverse inequality substantially relies on the fact that f solves the Beltrami equation (2.4). This equation allows us to replace the integrand

$E(x, Df)$ by the Jacobian determinant (null Lagrangian). Precisely, we can write

$$
\begin{aligned}
\int_B E(x, Df)dx &= n^{\frac{n}{2}} \int_B J(x, f)dx = \\
&= n^{\frac{n}{2}} \lim_{\epsilon \to 0} \int_B |Df(x)|^{-\epsilon} J(x, f)dx = \\
&= n^{\frac{n}{2}} \lim_{\epsilon \to 0} \int_B \left[\frac{J(x, f)}{|Df(x)|^{\epsilon}} - \frac{J(x, h)}{|Dh(x)|^{\epsilon}} \right] dx \\
&+ n^{\frac{n}{2}} \limsup_{\epsilon \to 0} \int_B \frac{J(x, h)}{|Dh(x)|^{\epsilon}} dx
\end{aligned} \tag{7.5}
$$

At this point we make use of a deep result concerning Jacobians [7], namely

$$
\left| \int_B \left[\frac{J(x, f)}{|Df(x)|^{\epsilon}} - \frac{J(x, h)}{|Dh(x)|^{\epsilon}} \right] dx \right| \leq C\epsilon \int_B \left[|Df(x)|^{n-\epsilon} + |Dh(x)|^{n-\epsilon} \right] dx \tag{7.6}
$$

The estimate holds for arbitrary mappings f, h in $W^{1,n-\epsilon}(B, \mathbb{R}^n)$ provided these mappings coincide on ∂B. Also recall that for the functions $|Df(x)|$ and $|Dh(x)|$ in the Orlicz space $L^n \log^{-1} L(B)$, the right hand side tends to zero as $\epsilon \to 0$. Hence we have

$$
\begin{aligned}
\int_B E(x, Df)dx &= n^{\frac{n}{2}} \lim_{\epsilon \to 0} \int_B \frac{J(x, h)}{|Dh(x)|^{\epsilon}} dx \leq \\
&\leq \lim_{\epsilon \to 0} \int_B \frac{E(x, Dh)}{|Dh(x)|^{\epsilon}} dx = \int_B E(x, Dh)dx,
\end{aligned} \tag{7.7}
$$

by Monotone Convergence Theorem. It follows from the previous estimate that both f and h have the same energy.

Consequently we must have equality in the middle step of this calculations, which happens only if

$$
n^{\frac{n}{2}} J(x, h) = E(x, Dh) = \langle DhG^{-1}(x), Dh \rangle^{\frac{n}{2}}. \tag{7.8}
$$

We recall that $detG(x) \equiv 1$. An analysis of Hadamard's inequality for determinants reveals that in order to have equality at (7.8), the Jacobian matrix $Dh(x)$ must satisfy the same Beltrami equation as Df does:

$$
D^t h(x) Dh(x) = J(x, h)^{\frac{2}{n}} G(x). \tag{7.9}
$$

In this way we have arrived at two solutions of the same first order PDE. These solutions coincide on ∂B. Our last step will be to prove that $h = f$.

Proof of (7.2).

We begin by diagonalizing G :

$$G = O^t \Gamma O$$

where $\Gamma = diag(\gamma_1, \cdots, \gamma_n)$. We see that

$$(\delta I + G)^{-1} = O^t(\delta I + \Gamma)^{-1}$$

and

$$G_\epsilon^{-1} = O^t(\frac{\Gamma + \epsilon I}{I + \epsilon\Gamma})^{-1}.$$

Inequality (7.2) thus reduces to showing that

$$\langle M'(\delta I + G)^{-1}, M'\rangle \leq \langle M'(\frac{\Gamma + \epsilon I}{I + \epsilon\Gamma})^{-1}, M'\rangle$$

for $M' = [m_{ij}] = MO^t$. This inequality reads as

$$\Sigma_{i,j=1}^n m_{ij}^2(\delta = \gamma_j)^{-1} \leq \Sigma_{i,j=1}^2 m_{ij}^2(\frac{\gamma_j + \epsilon}{1 + \epsilon\gamma_j})^{-1},$$

which is immediate from

$$(\delta = \gamma_j)^{-1} \leq (\frac{\gamma_j + \epsilon}{1 + \epsilon\gamma_j})^{-1}.$$

8. The Uniqueness, $h = f$

Denote $g = f - h \in W_0^{1,P}(B, I\!R^n)$, $P(t) = t^n \log^{-1}(e + t)$. With the notation of section 2, we have

(8.1) $$A(x, Df) = n^{\frac{n-2}{2}} Adj Df, \quad |A(x, Df)| \leq C_n |Df|^{n-1}$$

(8.2) $$A(x, Dh) = n^{\frac{n-2}{2}} Adj Dh, \quad |A(x, Dh)| \leq C_n |Dh|^{n-1}$$

The function $M \to A(x,M)$ is monotone in $I\!R^{n\times n}$, i.e.

$$\langle A(x,M_1)-A(x,M_2), M_1-M_2\rangle$$

and thus for every $0<\epsilon\leq 1$, we can write

$$\begin{aligned} 0 &\leq \int_B \langle A(x,Df)-A(x,Dh), |Df-Dh|^{-\epsilon}(Df-Dh)\rangle = \\ &= n^{\frac{n-2}{2}}\int_B \langle Adj\,Df - Adj\,Dh, |Df-Dh|^{-\epsilon}(Df-Dh)\rangle \end{aligned} \tag{8.3}$$

Next we decompose the matrix field $|Dg|^{-\epsilon}Dg \in L^{n+\epsilon}(B, I\!R^{n\times n})$ according to the well known Hodge decomposition, see [IS1]. Precisely we can write

$$|Dg|^{-\epsilon}Dg = D\chi + F \tag{8.4}$$

where $\chi\in W_0^{1,n+\epsilon}(B,I\!R^n)$ and F is a divergence free matrix field in $L^{n+\epsilon}(B,I\!R^{n\times n})$, since $n+\epsilon < \frac{n}{1-\epsilon}$.

The essence in this procedure lies in the estimate

$$\|F\|_{\frac{n-\epsilon}{1-\epsilon}} \leq C_n\epsilon\|Dg\|_{n-\epsilon}^{1-\epsilon}. \tag{8.5}$$

We then split the integral at (8.3) accordingly. As the Adjoint differential is divergence free, the first term vanishes:

$$\int_B \langle Adj\,Df - Adj\,Dh, D\chi\rangle dx = 0. \tag{8.6}$$

Indeed this certainly holds if $\chi\in C_0^\infty(B,I\!R^n)$, see the identity (2.3). The case when $\chi\in W_0^{1,n+\epsilon}(B,I\!R^n)$ follows by an approximation argument. For this we take into account that

$$|Adj\,Df - Adj\,Dh| \leq C_n(|Df|^{n-1}+|Dh|^{n-1}) \in L^{\frac{n+\epsilon}{n+\epsilon-1}}(B) \tag{8.7}$$

Hence the first term vanishes.

The second term can be estimated as follows

$$\begin{aligned} \int_B \langle Adj\,Df - Adj\,Dh, F\rangle &\leq c_n\int_B (|Df|^{n-1}+|Dh|^{n-1})|F| \leq \\ &\leq c_n\left[\int_B (|Df|^{n-\epsilon}+|Dh|^{n-\epsilon})dx\right]^{\frac{n-1}{n-\epsilon}}\left[\int_B |F|^{\frac{n-\epsilon}{1-\epsilon}}\right]^{\frac{1-\epsilon}{n-\epsilon}} \\ &\leq c_n\epsilon\int_B (|Df|^{n-\epsilon}+|Dh|^{n-\epsilon})dx \end{aligned}$$

Letting ϵ go to zero, we conclude

$$\int_B \langle A(x, Df) - A(x, Dh), Df - Dh \rangle = 0.$$

Hence $Df = Dh$ and $f = h$ completing the proof of Theorem 1.

9. Proof of the removability theorem

A compact set $E \subset \mathbf{R}^n$ is said to have zero $L^n \log L^{n-1}$–capacity if there is a bounded open set Ω containing E such that

$$\inf \int_\Omega |\nabla \phi|^n \log^{n-1}(3 + |\nabla \phi|) = 0 \tag{9.1}$$

where the infimum is taken with respect to all functions $\phi \in C_0^\infty(\Omega)$ which are equal to 1 on some neighborhood of the set E.

There are many elementary properties of this capacity which follow as in the classical setting for the more usual n–capacity. We will use the fact that a set of zero $L^n \log L^{n-1}$–capacity has vanishing $(n-1)$-dimensional measure; in fact such a set is of Hausdorff dimension zero. It is easy to construct Cantor sets that have capacity zero in our sense. A closed set has zero $L^n \log L^{n-1}$–capacity if and only if it can be written as the countable union of compact sets of zero $L^n \log L^{n-1}$–capacity.

Proof. Since E has vanishing $(n-1)$-dimensional measure, it suffices to show that

$$|Df| \in L^n \log^{-1} L_{loc}(\Omega). \tag{9.2}$$

Indeed, it then follows that $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$ and because $J(x, f) \geq 0$ a.e. we further deduce from (9.2) and [16] that $J(x, f)$ is locally integrable in Ω. Let $\eta \in C_0^\infty(\Omega)$ be an arbitrary test function. We denote by E' the intersection of E with the support of η. There exists a sequence of functions $\{\phi_j\}_{j=1}^\infty$ such that for each j we have

1. $\phi_j \in C_0^\infty(\Omega)$,

2. $0 \leq \phi_j \leq 1$,

3. $\phi_j = 1$ on some neighborhood U_j of E',

4. $\lim_{j\to\infty} \phi_j(x) = 0$ for almost all $x \in \mathbf{R}^n$,

5. $\lim_{j\to\infty} \int_\Omega |\nabla\phi_j|^n \log^{n-1}(3 + |\nabla\phi_j|) = 0$

We set

$$\varphi_j = (1 - \phi_j)\eta \in C_0^\infty(\Omega \setminus E') \tag{9.3}$$

The distortion inequality $|Df|^n \leq KJ$ and the inequality

$$ab \leq a\log(1 + a) + e^b - 1,$$

for non-negative real numbers, imply

$$\begin{aligned} \frac{|\varphi_j Df|^n}{\log(3 + |\varphi_j Df|^n)} &\leq \frac{K|\varphi_j|^n J}{\log(3 + |\varphi_j|^n J)} \\ &\leq \frac{1}{\lambda}\left[|\varphi_j|^n J + \exp(\lambda K)\right] \end{aligned} \tag{9.4}$$

We now integrate both sides of this inequality over Ω and use integration by parts on the term containing the Jacobian determinant. Here we rely on a result from [16] according to which

$$\int_G \phi df_1 \wedge ... \wedge df_n \; dx = -\int_G f_1 d\phi \wedge ... \wedge df_n \; dx$$

whenever $\phi \in C_0^\infty(G)$ and $f \in W^{1,1}_{loc}(G, \mathbf{R}^n)$ is a Sobolev mapping with $|Df| \in L^n \log^{-1} L_{loc}(G)$ and $J(x, f) \geq 0$ a.e. Notice that the support of φ_j does not intersect E. We so obtain

$$\begin{aligned} \lambda \int_\Omega \frac{|\varphi_j Df|^n}{\log(3 + |\varphi_j Df|^n)} &\leq n\int_\Omega |\varphi_j|^{n-1}|\nabla\varphi_j||f||Df|^{n-1} \\ &\quad + \int_\Omega e^{\lambda K(x)} dx \end{aligned} \tag{9.5}$$

Next we use the inequality

$$ab^{n-1} \leq C(n)\left[\frac{b^n}{\log(3 + b^n)} + a^n \log^{n-1}(3 + a)\right] \tag{9.6}$$

for non–negative numbers a and b. This yields the inequality

$$\begin{aligned}
&\lambda \int_\Omega \frac{|\varphi_j Df|^n}{\log(3+|\varphi_j Df|^n)} \\
\leq\ & C(n) \int_\Omega \frac{|\varphi_j Df|^n}{\log(3+|\varphi_j Df|^n)} \\
+\ & C(n) \int_\Omega |\nabla\varphi_j|^n |f|^n \log^{n-1}(3+|f||\nabla\varphi_j|) + \int_\Omega e^{\lambda K(x)}.
\end{aligned}$$

If λ is sufficiently large, then the first part of the right hand side is absorbed in the left and we have

$$\begin{aligned}
&\lambda \int_\Omega \frac{|\varphi_j Df|^n}{\log(3+|\varphi_j Df|^n)} \\
\leq\ & C(n) \int_\Omega |\nabla\varphi_j|^n |f|^n \log^{n-1}(3+|f||\nabla\varphi_j|) + \int_\Omega e^{\lambda K(x)}.
\end{aligned} \tag{9.7}$$

Recall that $|f|$ is assumed bounded in Ω and also that $\nabla\varphi_j = (1-\phi_j)\nabla\eta - \eta\nabla\phi_j$. It follows from the conditions defining ϕ_j that we can pass to the limit (as $j \to \infty$) in equation (9.7) to obtain

$$\lambda \int_\Omega \frac{|\eta Df|^n}{\log(3+|\eta Df|^n)} \leq C(n) \int_\Omega |\nabla\eta|^n |f| \log(3+|f||\nabla\eta|) + \int_\Omega e^{\lambda K(x)},$$

so that $|Df| \in L^n \log^{-1} L(G)$ for G any compact subset of Ω on which η does not vanish. Since $\eta \in C_0^\infty(\Omega)$ was arbitrary, we conclude that $|Df| \in L^n \log^{-1} L(G)$ for any compact subset of Ω. This then proves the removability theorem.

10. The distortion K in $exp(\Omega)$

Let us denote with $exp(\Omega)$ the closure of $L^\infty(\Omega)$ in the space $EXP(\Omega)$. In [3] it is proven that, for $g \in EXP(\Omega)$

$$dist_{EXP}(g, L^\infty) = \inf\left\{\mu > o : \fint_\Omega e^{\frac{|g(x)|}{\mu}} dx < \infty\right\}$$

and this implies immediately that $g \in exp(\Omega)$ iff

$$\fint_\Omega e^{\lambda g(x)} dx < \infty \qquad \text{for any} \lambda > 0.$$

Combining with our Theorem 1 we conclude that if $K \in exp(\Omega)$ then locally $|Df| \in L^n \log^\alpha L$ for every α.

Bibliography

[1] K. Astala, T. Iwaniec, P. Koskela and G. Martin, Mappings of BMO-bounded distortion, Math. Ann. 317, (2000), 703–726.

[2] M. A. Brakalova and J. A. Jenkins, On solutions of the Beltrami equation, J. Anal. Math. 76, (1998), 67-92.

[3] M. Carozza and C. Sbordone, The distance to L^∞ in some function spaces and applications, Differential Integral Equations, 10, (4) (1997), 599-607.

[4] G. David, Solutions de l'equation de Beltrami avec $\|\mu\|_\infty$, Ann. Acad. Sci. Fenn. Ser. A I Math., 13 (1998), 25-70.

[5] F. W. Gehring and T. Iwaniec, The limit of mappings with finite distortion, Ann. Acad. Sci. Fenn. Math. 24 (1999), 253-264.

[6] V. M. Goldstein and S. K. Vodop'yanov, Quasiconformal mappings and space of functions with generalized first derivatives, Sibirsk. Mat. Z. 17 (1976), 515-531.

[7] L. Greco, T. Iwaniec, C. Sbordone and B. Stroffolini, Degree formulas for maps with nonintegrable Jacobians, Topol. Methods Nonlinear Anal. 6 (1995), 81-95.

[8] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory*, Oxford University Press, 1993.

[9] J. Heinonen and P. Koskela, Sobolev mappings with integrable dilatation, Arch. Rational Mech. Anal. 125 (1993), 81-97.

[10] T. Iwaniec, P. Koskela and G. Martin, Mappings of BMO-distortion and Beltrami type operators, preprint (1999).

[11] T. Iwaniec, P. Koskela and J. Onninen, Mappings of finite distortion: Monotonicity and continuity, Invent. Math. 144 (2001) 3, 507–531.

[12] T. Iwaniec and G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford Univ. Press (to appear).

[13] T. Iwaniec and G. Martin, *The Beltrami Equation*, Memoirs of Amer. Math. Soc. (to appear).

[14] T. Iwaniec and G. Martin, Squeezing the Sierpinski sponge, to appear.

[15] T. Iwaniec, L. Migliaccio, G. Moscariello and A. Passarelli di Napoli, A priori estimates for nonlinear elliptic complexes, preprint (2000).

[16] T. Iwaniec and C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, Arch. Rational Mech. Anal. 119 (1992), 129-143.

[17] T. Iwaniec and C. Sbordone, Div-curl fields of finite distortion, C. R. Acad. Sci. Paris, t. 327, Ser. I (1998), 729-734.

[18] T. Iwaniec and C. Sbordone, Quasiharmonic fields, Ann. Inst. Poincarè, Analyse Nonlineaire (to appear).

[19] T. Iwaniec and V. Sverak, On mappings with integrable dilatation, Proc. Amer. Math. Soc. 118 (1993), 181-188.

[20] J. Kauhanen, P. Koskela and J. Maly, Mappings of finite distortion: Discreteness and Openness, Arch. Rational Mech. Anal. (to appear).

[21] J. Kauhanen, P. Koskela and J. Maly, Mappings of finite distortion: Condition N, Michigan Math. J. 49 (2001), 169-181.

[22] O. Lehto, *Univalent functions and Teichmuller spaces,* G.T.M. 109, Springer-Verlag (1987).

[23] J. Manfredi and E. Villamor, An extension of Reshetnyak's Theorem, Indiana Univ. Math. J. 47 (1998), 1131-1145.

[24] J. Manfredi and E. Villamor, Mappings with integrable dilatation in higher dimensions, Bull. Amer. Math. Soc. (N.S.) 32, n.2 (1995), 235-240.

[25] L. Migliaccio and G. Moscariello, Mappings with unbounded dilatation, preprint n. 3, (1997), Università degli Studi di Salerno.

[26] V. Potemkin and V. Ryazanov, On the noncompactness of David classes, Ann. Acad. Sci. Fenn. Math. 23, (1998), 191-204.

[27] V. Ryazanov, U. Srebro and E. Yakubov, BMO-Quasiconformal mappings, J. Anal. Math. J. Anal. Math. 83 (2001), 1-20..

[28] U. Srebro and E. Yakubov, Branched folded maps and alternating Beltrami equations, J. Anal. Math. 70 (1996), 65-90.

[29] P. Tukia, Compactness properties of μ-homeomorphisms, Ann. Acad. Sci. Fenn. Ser. A I Math. 16 (1991), 47-69.

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