Sharp inequalities via truncation

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Abstract

We show that Sobolev-Poincaré and Trudinger inequalities improve to inequalities on Lorentz-type scales provided they are stable under truncations.

1 Introduction

The classical Sobolev inequalities state that, for an open subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and all $u \in C_0^1(\Omega)$,

$$\left(\int_\Omega |u(x)|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq C_{p,n} \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}, \tag{1}$$

where $1 \leq p < n$ and $p^* = \frac{pn}{n-p}$. If Ω is connected and sufficiently nice, say has smooth boundary, then one also has the analogous Sobolev-Poincaré inequality

$$\inf_{c \in \mathbb{R}} \left(\int_\Omega |u(x) - c|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq C_{p,n}(\Omega) \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} \tag{2}$$

for all $u \in C^1(\Omega)$. In the borderline case $p = n$, $n \geq 2$, we have the Trudinger [25] inequalities (also see [20] and [26]): there exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ such that

$$\int_\Omega \exp \left(\frac{|u(x)|}{C_1 \, \|\nabla u\|_{L^n(\Omega)}} \right)^{\frac{n}{n-1}} dx \leq C_2 \, |\Omega| \tag{3}$$

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for $u \in C_0^1(\Omega)$, and when Ω is sufficiently nice with finite volume, there exist $C_1 = C_1(n, \Omega)$ and $C_2 = C_2(n)$ so that for $u \in C^1(\Omega)$

$$\inf_{c \in \mathbb{R}} \int_\Omega \exp\left(\frac{|u(x) - c|}{C_1 \|\nabla u\|_{L^n(\Omega)}}\right)^{\frac{n}{n-1}} dx \le C_2 |\Omega| . \tag{4}$$

By the density of smooth functions in the Sobolev spaces $W_0^{1,p}(\Omega)$, $W^{1,p}(\Omega)$ these inequalities extend to these spaces. The usual proof for Sobolev inequalities for the zero boundary value case is to use integral presentation formulas to estimate $|u|$ in terms of a Riesz potential of $|\nabla u|$. Except for the case $p = 1$, the inequalities can then be deduced from potential estimates; for $p = 1$ one also applies truncation techniques. Another possibility is to deduce (1) for $p = 1$ to the isoperimetric inequality and then conclude with the case $p > 1$ essentially only applying the Hölder inequality. The Trudinger inequality (3) is based on good estimates on the constant in the Sobolev inequality (1). For certain domains Ω, the inequalities (2), (4) can be reduced to (1) and (3) by extending the functions across the boundary. In many situations this is not possible and one applies various chaining techniques and relies on geometric assumptions on Ω. One is then lead to weaker inequalities where instead of, say, p^*, in (2), one only obtains an exponent $q < p^*$ and there might not be any reasonable inequalities below a fixed level p. This phenomenon also shows up in the analysis on metric spaces where the Sobolev-Poincaré-type inequalities often only hold for exponents p bounded away from 1.

The above inequalities are known to be sharp in the sense that the left-hand-sides cannot be improved on in the Orlicz scales [10] (see also [5]). However, even sharper inequalities exist in other scales. The purpose of this note is to point out that inequalities of the Sobolev-Poincaré-type always improve themselves to Lorentz-type scales if they are stable under truncation. Throughout the paper X will be a metric space with metric d and a Borel measure μ.

Theorem 1.1 *Let $\Omega \subset X$ be a domain with $\mu(\Omega) < \infty$.*

i) Fix $p, q \in (0,\infty)$. Suppose that the Sobolev-Poincaré inequality

$$\inf_{c\in\mathbb{R}} \left(\int_\Omega |u(y)-c|^q \, d\mu(y) \right)^{\frac{1}{q}} \le C_0 \left(\int_\Omega g^p(y)\, d\mu(y) \right)^{\frac{1}{p}} \tag{5}$$

is stable under truncations. Then

$$\inf_{c\in\mathbb{R}} \int_0^\infty t^{p-1} \Big[\mu\left(\{x\in\Omega : |u(x)-c| > t\}\right) \Big]^{\frac{p}{q}} dt \le C \cdot C_0^p \int_\Omega g^p(y)\, d\mu(y) \tag{6}$$

where the constant $C = C(p,q)$ depends only on p and q.

ii) Fix $s \in (1,\infty)$. Suppose that the inequality

$$\inf_{c\in\mathbb{R}} \int_\Omega \exp\left(\frac{|u(y)-c|}{C_1 \|g\|_{L^s(\Omega)}} \right)^{\frac{s}{s-1}} d\mu(y) \le C_2 \mu(\Omega) \tag{7}$$

is stable under truncations. Then there exists a constant $C = C(s, C_2)$ so that

$$\inf_{c\in\mathbb{R}} \int_0^\infty \frac{t^{s-1}}{\log^{s-1}\left(\frac{e\,\mu(\Omega)}{\mu(\{x\in\Omega:|u(x)-c|>t\})}\right)} dt \le C \cdot C_1^s \int_\Omega g^s(y)\, \mu(y). \tag{8}$$

The requirement that inequality (5) (resp. (7)) be stable under truncations means that for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\iota \in \{-1,1\}$ the pair $v_{t_1}^{t_2}$, $g_{t_1,t_2} = g\chi_{\{t_1<v\le t_2\}}$, where $v = \iota(u-b)$ and $v_{t_1}^{t_2} = \min\{\max\{0, v-t_1\}, t_2-t_1\}$, also satisfies Inequality (5) ((7)):

$$\inf_{c\in\mathbb{R}} \left(\int_\Omega \left|v_{t_1}^{t_2}(y)-c\right|^q \, d\mu(y) \right)^{\frac{1}{q}} \le C_0 \left(\int_\Omega g_{t_1,t_2}^p(y)\, d\mu(y) \right)^{\frac{1}{p}}. \tag{9}$$

The concluded inequalities in Theorem 1.1 are known to hold in the Euclidean setting when the boundary of Ω is sufficiently regular and $1 \le p < n$ and $q = p^*$ [18] and [19], [14], or $s = n$ [2], [8] and [14]. The Sobolev inequality version of *i)* is contained in [1]. Theorem 1.1 gives new information even in Euclidean spaces because no additional assumptions on the geometry of Ω are posed except for Inequality (5) or (7). Notice, however, that the conclusion in *i)* is nontrivial only when $q > p$. For sufficient condition for these inequalities see Section 4. Furthermore Theorem 1.1 when combined with results in [7] gives new inequalities on spaces that support a Poincaré inequality (see Section 3).

The idea of using truncation in connection with Sobolev-type inequalities can be traced back to the seminal paper [16] by Maz'ya. For further applications of this powerful technique see [1], [17], [7] and the references therein, and [14].

2 Proof of Theorem 1.1

We will employ the following lemma whose proof is elementary.

Lemma 2.1 *Let ν be a finite measure on a set Y. If $w \geq 0$ is a ν-measurable function such that $\nu(\{y \in Y : w(y) = 0\}) \geq \frac{\nu(Y)}{2}$, then, for every $t > 0$,*

$$\nu(\{y \in Y : w(y) > t\}) \leq 2 \inf_{c \in \mathbb{R}} \nu(\{y \in Y : |w(y) - c| > t/2\}). \tag{10}$$

We begin with the proof of the part $i)$ of Theorem 1.1:
$i)$ Choose $b \in \mathbb{R}$ such that

$$\mu(\{u \geq b\}) \geq \frac{\mu(\Omega)}{2} \text{ and } \mu(\{u \leq b\}) \geq \frac{\mu(\Omega)}{2}.$$

Here and also later we abbreviate the set $\{x \in \Omega : f(x) \geq a\}$ ($\{x \in \Omega : f(x) \leq a\}$ ...) to $\{f \geq a\}$ ($\{f \leq a\}$...). Let $v_+ = \max\{u - b, 0\}$, $v_- = -\min\{u - b, 0\}$. Then $|u - b| = v_+ + v_-$. In what follows v will denote either v_+ or v_-.

Let $0 < t_1 < t_2 < \infty$. Then the function $v_{t_1}^{t_2}$ satisfies

$$\mu\left(\{v_{t_1}^{t_2}(x) = 0\}\right) \geq \frac{\mu(\Omega)}{2}, \tag{11}$$

and we conclude using Lemma 2.1 and Inequality (5) that

$$\begin{aligned}
\left[\mu\left(\{v_{t_1}^{t_2} > t\}\right)\right]^{\frac{1}{q}} \cdot t &\leq 2^{\frac{1}{q}+1} \inf_{c \in \mathbb{R}} \left[\mu\left(\{|v_{t_1}^{t_2} - c| > t/2\}\right)\right]^{\frac{1}{q}} \cdot t/2 \\
&\leq 2^{\frac{1}{q}+1} C_0 \left(\int_\Omega g_{t_1,t_2}^p(y)\, d\mu(y)\right)^{\frac{1}{p}} \\
&= 2^{\frac{1}{q}+1} C_0 \left(\int_\Omega g^p \chi_{\{t_1 < v \leq t_2\}}\, d\mu(y)\right)^{\frac{1}{p}} \qquad (12)
\end{aligned}$$

for all $t > 0$. This yields

$$\sum_{k=-\infty}^{\infty} 2^{pk} \left[\mu(\{v_{2^k}^{2^{k+1}} > 2^k\})\right]^{\frac{p}{q}} \leq 2^{\frac{p}{q}+p} C_0^p \int_\Omega g^p(y)\, d\mu(y). \tag{13}$$

Because $\{v_{2^k}^{2^{k+1}} > 2^k\} = \{v > 2^{k+1}\}$ we have

$$\sum_{k=-\infty}^{\infty} 2^{pk} \left[\mu\left(\{v > 2^{k+1}\}\right)\right]^{\frac{p}{q}} \leq 2^{\frac{p}{q}+p} C_0^p \int_\Omega g^p(y)\, d\mu(y), \tag{14}$$

which immediately gives the estimate

$$\|v\|^q_{L^{q,p}(\Omega)} := \int_0^\infty t^{p-1} \left[\mu\left(\{v > t\}\right)\right]^{\frac{p}{q}} dt \leq C_{p,q} \cdot C_0^p \int_\Omega g^p(y)\, d\mu(y).$$

Finally

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|u-c\|^q_{L^{q,p}(\Omega)} &\leq \|u-b\|^q_{L^{q,p}(\Omega)} \\ &\leq 2^q \left(\|v_+\|^q_{L^{q,p}(\Omega)} + \|v_-\|^q_{L^{q,p}(\Omega)}\right) \\ &\leq C_{p,q} \cdot C_0^p \int_\Omega g^p(y)\, d\mu(y), \end{aligned} \tag{15}$$

as desired.

ii) As in the part *i*), choose $b \in \mathbb{R}$ such that

$$\mu\left(\{u \geq b\}\right) \geq \frac{\mu(\Omega)}{2} \text{ and } \mu\left(\{u \leq b\}\right) \geq \frac{\mu(\Omega)}{2}.$$

Let $v_+ = \max\{u-b, 0\}$, $v_- = -\min\{u-b, 0\}$. In what follows v will denote either v_+ or v_-. Fix $0 < t_1 < t_2 < \infty$. Using the expansion $\exp(t) = \sum_{k=0}^\infty \frac{t^k}{k!}$ we have that

$$\inf_{c\in\mathbb{R}} \left(\fint_\Omega \left|v_{t_1}^{t_2} - c\right|^{\frac{sk}{s-1}} d\mu\right)^{\frac{s-1}{sk}} \leq C_1 \cdot C_2^{\frac{s-1}{sk}} (k!)^{\frac{s-1}{sk}} \left(\int_\Omega g_{t_1,t_2}^s(y)\, d\mu(y)\right)^{\frac{1}{s}} \tag{16}$$

for all $k = 1, 2, 3, ...,$ and, consequently

$$\inf_{c\in\mathbb{R}} \left(\fint_\Omega \left|v_{t_1}^{t_2} - c\right|^{\frac{sm}{s-1}} d\mu\right)^{\frac{s-1}{sm}} \leq C_1 \cdot C_2^{\frac{s-1}{sm}} (m+1)^{\frac{s-1}{s}} \left(\int_\Omega g_{t_1,t_2}^s(y)\, d\mu(y)\right)^{\frac{1}{s}} \tag{17}$$

for every $m \geq 1$. Especially, we get the weak type inequality

$$\inf_{c\in\mathbb{R}} t\left[\mu\left(\{|v_{t_1}^{t_2}-c|>t\}\right)\right]^{\frac{s-1}{sm}}$$

$$\leq C_1 \cdot C_2^{\frac{s-1}{sm}} 2^{\frac{s-1}{s}}[\mu(\Omega)]^{\frac{s-1}{sm}} m^{\frac{s-1}{s}} \left(\int_\Omega g_{t_1,t_2}^s(y)\, d\mu(y)\right)^{\frac{1}{s}} \tag{18}$$

for all $m \geq 1$ and every $t > 0$. Applying Lemma 2.1 we conclude that

$$\begin{aligned} t\left[\mu\left(\{v_{t_1}^{t_2}>t\}\right)\right]^{\frac{s-1}{sm}} &\leq C_1(C_2 2/e)^{\frac{s-1}{sm}}[e\mu(\Omega)]^{\frac{s-1}{sm}} m^{\frac{s-1}{s}} \left(\int_\Omega g_{t_1,t_2}^s(y)\, d\mu(y)\right)^{\frac{1}{s}} \\ &\leq C_1 C(C_2,s)[e\mu(\Omega)]^{\frac{s-1}{sm}} m^{\frac{s-1}{s}} \left(\int_\Omega g_{t_1,t_2}^s(y)\, d\mu(y)\right)^{\frac{1}{s}} \end{aligned}$$

for all $m \geq 1$ and every $t > 0$. Next we will choose m, t_1, t_2 and t. Fix $i \in \mathbb{Z}$ and let $m = \log\left(\frac{e\mu(\Omega)}{\mu(\{v>2^{i+1}\})}\right)$. Then

$$t\left(\frac{\mu\left(\{v_{t_1}^{t_2}>t\}\right)}{e\mu(\Omega)}\right)^{\frac{s-1}{s\log\left(\frac{e\mu(\Omega)}{\mu(\{v>2^{i+1}\})}\right)}}$$

$$\leq C_1 \cdot C(C_2,s) \log^{\frac{s-1}{s}}\left(\frac{e\mu(\Omega)}{\mu\left(\{v>2^{i+1}\}\right)}\right)\left(\int_\Omega g_{t_1,t_2}^s(y)\, d\mu(y)\right)^{\frac{1}{s}} \tag{19}$$

for all $t > 0$. Choosing $t_1 = 2^i, t_2 = 2^{i+1}$ and $t = 2^i$ we arrive at

$$\frac{2^i}{\log^{\frac{s-1}{s}}\left(\frac{e\mu(\Omega)}{\mu(\{v>2^{i+1}\})}\right)}\left(\frac{\mu\left(\left\{v_{2^i}^{2^{i+1}}>2^i\right\}\right)}{e\mu(\Omega)}\right)^{\frac{s-1}{s\log\left(\frac{e\mu(\Omega)}{\mu(\{v>2^{i+1}\})}\right)}}$$

$$\leq C_1 \cdot C(C_2,s)\left(\int_\Omega g_{2^i,2^{i+1}}^s(y)\, d\mu(y)\right)^{\frac{1}{s}}. \tag{20}$$

Notice that $\{v_{2^i}^{2^{i+1}} > 2^i\} = \{v > 2^{i+1}\}$ and so

$$\left(\frac{\mu\left(\left\{v_{2^i}^{2^{i+1}}>2^i\right\}\right)}{e\mu(\Omega)}\right)^{\frac{s-1}{s\log\left(\frac{e\mu(\Omega)}{\mu(\{v>2^{i+1}\})}\right)}} = e^{\frac{1-s}{s}}. \tag{21}$$

We thus obtain

$$\frac{2^i}{\log^{\frac{s-1}{s}}\left(\frac{e\mu(\Omega)}{\mu(\{v>2^{i+1}\})}\right)} \leq C_1 \cdot C(C_2, s)\left(\int_\Omega g^s_{2^i,2^{i+1}}(y)\,d\mu(y)\right)^{\frac{1}{s}}. \tag{22}$$

We raise the estimate to the power s and sum over i. This results in

$$\sum_{i=-\infty}^{\infty} \frac{2^{si}}{\log^{s-1}\left(\frac{e\mu(\Omega)}{\mu(\{v>2^{i+1}\})}\right)} \leq C_1^s \cdot C(C_2, s)\int_\Omega g^s(y)\,d\mu(y), \tag{23}$$

and the inequality

$$\|v\|^s_{BW_s(\Omega)} := \int_0^\infty \frac{t^{s-1}}{\log^{s-1}\left(\frac{e\mu(\Omega)}{\mu(\{v>t\})}\right)}dt \leq C_1^s \cdot C(C_2, s)\int_\Omega g^s(y)\,d\mu(y) \tag{24}$$

immediately follows. Finally

$$\begin{aligned}
\inf_{c\in\mathbb{R}} \|u-c\|^s_{BW_s(\Omega)} &\leq \|u-b\|^s_{BW_s(\Omega)} \\
&\leq 2^s\left(\|v_+\|^s_{BW_s(\Omega)} + \|v_-\|^s_{BW_s(\Omega)}\right) \\
&\leq C_1^s \cdot C(C_2, s)\int_\Omega g^s(y)\,d\mu(y), \qquad (25)
\end{aligned}$$

as desired.

Remark 2.2 *Using [27, Lemma 1.8.13 and Exercice 1.7] we see that, in the case $q > p > 0$, Theorem 1.1 really gives new information. It follows that in the case $q > p > 0$, under the assumptions of Theorem 1.1, Inequality (5) is equivalent with Inequality (6), modulo the constants. Also, if*

$$\|f\|^s_{BW_s(\Omega)} := \int_0^\infty \frac{t^{s-1}}{\log^{s-1}\left(\frac{e\mu(\Omega)}{\mu(\{|f|>t\})}\right)}dt < \infty,$$

for some fixed $s > 1$, then it is easy to see that there exist constants $C_1(s)$ and $C_2(s)$ such that

$$\int_\Omega \exp\left(\frac{|f(x)|}{C_1(s)\,\|f\|_{BW_s(\Omega)}}\right)^{\frac{s}{s-1}} d\mu(x) \leq C_2(s)\mu(\Omega), \tag{26}$$

which tells that under the assumptions of Theorem 1.1 Inequality (7) is equivalent, modulo the constants, with Inequality (8). However it is easy to construct [8] a function f which satisfies

$$\int_\Omega \exp\left(C|f(x)|\right)^{\frac{s}{s-1}}\, d\mu(x) < \infty$$

with some constant C, but nevertheless $\|f\|_{BW_s(\Omega)} = \infty$.

3 Consequences of the Poincaré inequality

In this section the starting point is to assume that a pair (u, g) of locally integrable functions satisfies the $(1,p)$-Poincaré inequality

$$\inf_{c\in\mathbb{R}} ⨍_{B(x,r)} |u(y) - c|\, d\mu(y) \le K_p\, r \left(⨍_{B(x,\sigma r)} g^p(y) d\mu(y) \right)^{\frac{1}{p}} \tag{27}$$

with some $\sigma \ge 1$, $p > 0$ and $K_p > 0$, for all balls $B(x, \sigma r) \subset X$. Here $⨍_A v\, d\mu = \frac{1}{\mu(A)} \int_A v\, d\mu$. Then the function g estimates the mean oscillation of u very much the same way as $|\nabla u|$ does in the Euclidean case.

Given a function v and $0 < t_1 < t_2 < \infty$, we set

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}$$

as before. Let the pair (u, g) satisfy a $(1,p)$-Poincaré inequality in X. Assume that for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\iota \in \{-1, 1\}$, the pair $v_{t_1}^{t_2}, g\chi_{\{t_1 < v \le t_2\}}$, where $v = \iota(u - b)$, satisfies the $(1,p)$-Poincaré inequality in X (with fixed constants K_p, σ). Then we say that the pair (u, g) has a truncation property.

We assume the doubling condition: there exists a positive constant C_D such that for every $x \in X$

$$\mu(B(x, 2r)) \le C_D\, \mu(B(x, r)). \tag{28}$$

We also need a lower estimates for the measure of a ball: there exists $C_b > 0$ and $s > 1$ such that for every $x \in X$ and all $0 < r < R < \sigma \mathrm{diam}(X)$

$$\frac{\mu(B(x, r))}{\mu(B(x, R))} \ge C_b \left(\frac{r}{R}\right)^s. \tag{29}$$

Furthermore in this section we assume that X is proper and our metric is a length metric:

$$d(x, y) = \inf_{\gamma_{x,y}} \text{lenght of}\, \gamma_{x,y}, \tag{30}$$

where the infimum is taken over all curves $\gamma_{x,y}$ and $\gamma_{x,y}$ is a curve joining x to y. Proper means that closed balls are compact.

Combining Theorem 1.1 with [7, Theorem 9.7] we get the following result.

Corollary 3.1 *Let (X, d) be a metric space equipped with a metric d that satisfies (30) and a measure μ which is doubling and satisfies (29). Assume that the pair (u, g) satisfies the $(1, p)$-Poincaré inequality (27) with $p \in [1, s]$ and the pair (u, g) has the truncation property.*

i) In the case $p \in [1, s)$ we have

$$\inf_{c\in\mathbb{R}} \int_0^\infty t^{p-1} \left[\mu\left(\{|u - c| > t\}\right)\right]^{1-\frac{p}{s}} dt \le C_1 \frac{R^p}{(\mu(B))^{\frac{p}{s}}} \int_B g^p(y)\, d\mu(y).$$

ii) If in addition the space is connected and $p = s > 1$, then

$$\inf_{c\in\mathbb{R}} \int_0^\infty \frac{t^{s-1}}{\log^{s-1}\left(\frac{e\mu(B)}{\mu(\{|u-c|>t\})}\right)} dt \le C_2 R^s \fint_B g^p(y)\, d\mu(y).$$

The constants C_1 and C_2 depend on p, s, σ, K_p, C_b and C_D only.

If we have a $(1, 1)$-Poincaré inequality, then Corollary 3.1 is true for $p > 1$ even without the truncation assumption. This is proven in [15] by using sophisticated Riesz potential estimates.

There are two important examples of settings where pairs (u, g) that satisfy a $(1, p)$-Poincaré inequality and have the truncation property show up. The first is the class of spaces that support a p-Poincaré inequality for Lipschitz functions and their point-wise Lipschitz constants. See the survey [12], [7], [9] and the references in these papers. The second arises from Dirichlet spaces. Here one assumes that the Dirichlet form is local to obtain the truncation property. See e.g. [22], [23] and [24], and the references in [7, Section 10.5]. In both of these settings, the p-Poincaré inequality is only assumed for a fixed $p > 1$ and it need not necessarily hold for $p = 1$. Thus Corollary 3.1 is not covered by the "classical" Euclidean arguments.

4 The Euclidean setting

As mentioned in the introduction, the assumption (5) of Theorem 1.1 holds only for domains of certain types for fixed levels $p > 1$. Moreover, (7) can well hold when no Sobolev-Poincaré-type inequalities are true for small values of p. We confine ourselves with giving a few examples.

We say that a bounded domain $\Omega \subset \mathbb{R}^n$ is an s-John domain, $s \geq 1$, if there exists a constant $C \geq 1$, and a distinguished point $x_0 \in \Omega$ so that each point $x \in \Omega$ can be joined to x_0 (inside Ω) by a rectifiable curve (called a John curve), $\gamma : [0, l] \to \Omega$, $\gamma(0) = x$, $\gamma(l) = y$, parametrised by arc length (l depends on x), and such that distance to the boundary satisfies

$$\operatorname{dist}(\gamma(t), \partial\Omega) > C^{-1} t^s \tag{31}$$

for all $t \in [0, l]$. Note that x_0 can be replaced by any other point in Ω. The constant in (31) however, depends on the choice of x_0. In such a domain we obtain the following inequality by combining Theorem 1.1 and results in [6] and [11].

Corollary 4.1 *Let $\Omega \subset \mathbb{R}^n$ be an s-John domain, $s \geq 1$. If $1 \leq p \leq q \leq \frac{np}{(n-1)s+1-p}$, then*

$$\inf_{c \in \mathbb{R}} \int_0^\infty t^{p-1} \left| \{ |u - c| > t \} \right|^{\frac{p}{q}} dt \leq C_{p,q}(\Omega) \int_\Omega |\nabla u(x)|^p \, dx \tag{32}$$

whenever $u \in C^1(\Omega)$.

In the setting of Corollary 4.1 there are no Sobolev-Poincaré inequalities when $p < (s-1)(n-1)$, see [6]. Also notice that a simply connected plane domain ($n = 2$) of finite area that supports the inequality (32) for all $u \in C^1(\Omega)$ with some $1 \leq p < 2$ and $q = \frac{2p}{2-p}$ is necessarily 1-John by the main result in [3].

Let $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$, be a continuous, increasing and subadditive function. Then a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a $C^{0,\phi}$-domain if for every point $b \in \partial\Omega$, there exists r_0, a Cartesian system of coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with the origin at the point b, and a function $\Phi : \{x' : |x'| < r_0\} \to \mathbb{R}$ such that

i) $B(b, r_0) \cap \Omega$ has the form $\{(x', x_n) \in B(0, r_0) : x_n > \Phi(x')\}$

ii) $B(b,r_0)\cap\partial\Omega$ has the form $\{(x',x_n)\in B(0,r_0): x_n=\Phi(x')\}$

iii) $|\Phi(x')-\Phi(y')|\le\phi(|x'-y'|)$ for all $x',y'\in\mathbb{R}^{n-1}$ and $|x'|,|y'|<r_0$.

In such a domain we have the following inequality by combining Theorem 1.1 and the main result in [13] (see also [17]). Here we also used the compactness result in [7, Theorem 8.1].

Corollary 4.2 *Let $\Omega\subset\mathbb{R}^n$, $n\ge 2$, be a $C^{0,\phi}$-domain, $1\le p<q<\infty$, and assume that $\psi=\phi^{-1}$ satisfies*

$$\limsup_{t\to 0}\frac{\psi(2t)}{\psi(t)}<\infty.$$

If, for a certain $\delta>0$, ψ satisfies the condition

$$\sup_{0<t<\delta}\left(||\psi^{\frac{n-1}{q}}||_{L^q(0,t)}||\psi^{\frac{1-n}{p}}||_{L^{\frac{p}{p-1}}(t,\delta)}\right)<\infty, \tag{33}$$

then

$$\inf_{c\in\mathbb{R}}\int_0^\infty t^{p-1}\,|\{|u-c|>t\}|^{\frac{p}{q}}\,dt\le C_{p,q}(\Omega)\int_\Omega|\nabla u(x)|^p\,dx \tag{34}$$

whenever $u\in C^1(\Omega)$.

Let $\Omega\subsetneq\mathbb{R}^n$, $n\ge 2$. The quasihyperbolic distance between a pair of points $x,y\in\Omega$ is defined to be

$$k_\Omega(x,y)=\inf_{\gamma_{x,y}}\int_{\gamma_{x,y}}\frac{ds}{\operatorname{dist}(s,\partial\Omega)} \tag{35}$$

where the infimum is taken over all curves $\gamma_{x,y}$ in Ω joining x to y. Let $\beta>0$. We say that Ω satisfies a β-quasihyperbolic boundary condition if the growth condition

$$k_\Omega(x_0,x)\le\frac{1}{\beta}\log\frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{dist}(x,\partial\Omega)}+C_0 \tag{36}$$

is satisfied for all $x\in\Omega$, where x_0 is a fixed basepoint and $C_0=C_0(x_0)<\infty$. Combining Theorem 1.1 and one of the results in [21] we obtain the following corollary.

Corollary 4.3 *Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, satisfy the quasihyperbolic boundary condition (36) for some $\beta \leq 1$. Then*

$$\inf_{c\in\mathbb{R}} \int_0^\infty \frac{t^{n-1}}{\log^{n-1}\left(\frac{e|\Omega|}{|\{|u-c|>t\}|}\right)} dt \leq C_{n,\beta} \int_\Omega |\nabla u(x)|^n \, dx \tag{37}$$

for all $u \in C^1(\Omega)$.

Notice that a simply connected planar domain ($n = 2$) with finite measure that supports Inequality (37) for all $u \in C^1(\Omega)$ necessarily satisfies the quasi-hyberbolic boundary condition with some $\beta \leq 1$ by the results in [17, p.214] and [4].

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