# AN IMPROVED BOUND FOR THE HAUSDORFF DIMENSION OF $(n, 2)$-SETS

THEMIS MITSIS

ABSTRACT. We improve on the bound for the Hausdorff dimension of sets containing a translate of every 2-plane.

## 1. Introduction

An $(n, 2)$-set is a subset of $\mathbb{R}^n$ containing a translate of every 2-plane. It is natural to ask whether such a set must be, in some sense, large. In low dimensions the answer is affirmative. Marstrand [3] proved that $(3, 2)$-sets have positive Lebesgue measure and Bourgain [1] showed that the same is true for $(4, 2)$-sets. In higher dimensions it is an open question whether an $(n, 2)$-set must have positive measure. We refer the reader to [1] for a discussion on the relation of this problem to the Kakeya conjecture.

On the other hand, the estimates for the 2-plane transform due to Christ [2] imply that for $n > 4$, an $(n, 2)$-set has Hausdorff dimension at least $(2n + 2)/3$.

In this paper we use the endpoint estimate for the Radon transform in $\mathbb{R}^3$ together with geometric-combinatorial ideas in the spirit of Wolff [5] to improve on this bound. Namely, we prove the following.

**Theorem 1.1.** *If $n > 4$ then the Hausdorff dimension of an $(n, 2)$-set is at least $(2n + 3)/3$.*

## 2. Preliminaries

We set out the notation and the terminology we will be using.

$S^{n-1} \subset \mathbb{R}^n$ is the $(n-1)$-dimensional unit sphere.

$B(a, r)$ is the closed ball of radius $r$ centered at the point $a$.

For $X \subset \mathbb{R}^n$, $X^\perp$ stands for its orthogonal complement.

If $e \in S^{n-1}$, $a \in \mathbb{R}^n$ then $L_e(a) = \{a + te : t \in \mathbb{R}\}$ is the line in the $e$-direction passing through the point $a$.

If $e \in S^{n-1}$, $a \in \mathbb{R}^n$, $\beta > 0$ then $T_e^\beta = \{x \in \mathbb{R}^n : \text{dist}(x, L_e(a)) \leq \beta\}$ is the infinite tube with axis $L_e(a)$ and cross-section radius $\beta$.

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$\mathcal{L}^k$ denotes $k$-dimensional Lebesgue measure and $\mathcal{L}^0$ counting measure. When the context is clear we will use the notation $|\cdot|$ for all these measures.

Let $\mathcal{G}_n$ be the Grassmannian manifold of all 2-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the unique probability measure which is invariant under the action of the orthogonal group. The elements of $\mathcal{G}_n$ will be refered to as *direction planes*.

If $P_1, P_2 \in \mathcal{G}_n$, then their distance is defined by

$$d(P_1, P_2) = \|\mathrm{proj}_{P_1} - \mathrm{proj}_{P_2}\|$$

where $\mathrm{proj}_P : \mathbb{R}^n \to P$ is the orthogonal projection onto $P$.

A set of points or direction planes is called *$\rho$-separated* if the distance between any two of its elements is at least $\rho$.

If $P \in \mathcal{G}_n,\ 1 \leq l \leq 4,\ \delta > 0$ then $P^{l,\delta}$ is a rectangle of dimensions $l \times l \times \underbrace{\delta \times \cdots \times \delta}_{n-2}$, that is, the image of $[0,l] \times [0,l] \times [0,\delta] \times \cdots \times [0,\delta]$ under a rotation and a translation, such that its faces with dimensions $l \times l$ are parallel to $P$. Such a set will be refered to as a *$\delta$-plate* or simply as a *plate*. When $l = 1$ the superscript $l$ will be supressed.

If $P_1^{l,\delta} \cap P_2^{l,\delta} \neq \emptyset$ and $d(P_1, P_2) = \theta$ we will say that the plates *intersect at angle $\theta$*.

The letter $C$ will denote various constants whose values may change from line to line. Similarly, $C_\epsilon$ will denote constants depending on $\epsilon$. If we need to keep track of the value of a constant through a calculation we will use subscripted letters $C_1, C_2, \ldots$ or the notation $\widetilde{C}$.

Finally, $x \lesssim y$ means $x \leq Cx$ and $x \simeq y$ means $(x \lesssim y \ \&\ y \lesssim x)$.

We close this section with a geometric lemma which allows us to control the intersection of two plates containing a line segment of given length. It can be proved by coordinate geometry.

**Lemma 2.1.** *Let $P_1^{l,\eta}, P_2^{l,\eta}$ be two plates such that $d(P_1, P_2) \leq 1/2$. Then for any $a, b \in P_1^{l,\eta} \cap P_2^{l,\eta}$, $r > 0$ with $r \leq |a-b| \leq 2r$ we have*

$$P_1^{l,\eta} \cap P_2^{l,\eta} \cap B(a, 2r) \subset T_e^{\beta}(a),$$

*where $e = (a-b)/|a-b|$ and $\beta = C\eta/d(P_1, P_2)$.*

Note that the lemma implies that for any two plates $P_1^{l,\eta}, P_2^{l,\eta}$ with $d(P_1, P_2) \leq 1/2$, there exist $e \in S^{n-1},\ a \in \mathbb{R}^n$ so that

$$P_1^{l,\eta} \cap P_2^{l,\eta} \subset T_e^{\beta}(a),$$

where $\beta = C\eta/d(P_1, P_2)$. Therefore

$$|P_1^{l,\eta} \cap P_2^{l,\eta}| \leq |P_1^{l,\eta} \cap T_e^{\beta}(a)| \lesssim \frac{\eta^{n-1}}{d(P_1, P_2)}.$$

## 3. An auxiliary estimate

In this section we prove a lemma which gives a lower bound for the measure of a set having large intersection with a family of plates with well-separated direction planes. The proof of Theorem 1.1 will be based on setting up a suitable configuration of plates and applying this lemma for appropriately chosen values of the parameters involved.

**Lemma 3.1.** *Suppose $E$ is a set in $\mathbb{R}^n$, $\beta, \kappa \le 1$ and $\mathcal{B} = \{P_j\}_{j=1}^M$ is an $\eta$-separated subset of $\mathcal{G}_n$ with* $\operatorname{diam}(\mathcal{B}) \le 1/2$*, such that for each $j$ there is a plate $P_j^{l,\eta}$ satisfying*

$$|P_j^{l,\eta} \cap E \cap \left(T_e^{\beta}(a)\right)^{\complement}| \ge \kappa |P_j^{l,\eta}|$$

*for all $e \in S^{n-1}$, $a \in \mathbb{R}^n$. Then*

$$|E| \gtrsim (\log(C/\beta))^{-5/3} \beta^{2(n-2)/3} \kappa |\mathcal{B}|^{1/3} \eta^{n-2}.$$

*Proof.* For each $a \in P_j^{l,\eta} \cap E$ let

$$q(j,a,i) = \inf_e |P_j^{l,\eta} \cap E \cap B(a, \beta 2^i) \cap \left(T_e^{\beta}(a)\right)^{\complement}|.$$

Then

$$q(j,a,0) = 0 \text{ and } q\left(j,a,\log(C/\beta)\right) \ge \kappa |P_j^{l,\eta}|.$$

Therefore there is a smallest $i(j,a)$ such that

$$q(j,a,i(j,a)) \ge \frac{\kappa}{2} |P_j^{l,\eta}|.$$

Since there are at most $\log(C/\beta)$ possible $i(j,a)$, there is an $i(j)$ and a set $A_j' \subset P_j^{l,\eta} \cap E$ of measure

$$|A_j'| \gtrsim (\log(C/\beta))^{-1} \kappa |P_j^{l,\eta}|$$

such that for each $a \in A_j'$

$$q(j,a,i(j)) \ge \frac{\kappa}{2} |P_j^{l,\eta}| \text{ and } q(j,a,i(j)-1) \le \frac{\kappa}{2} |P_j^{l,\eta}|.$$

Since there are $M$ plates and at most $\log(C/\beta)$ possible $i(j)$ there is an $i_0$ and a set $\mathcal{C}' \subset \{P_j^{l,\eta}\}$ such that

$$|\mathcal{C}'| \gtrsim (\log(C/\beta))^{-1} M$$

and for each $P_j^{l,\eta} \in \mathcal{C}'$ and each $a \in A_j'$

$$|P_j^{l,\eta} \cap E \cap B(a, \beta 2^{i_0}) \cap \left(T_e^{\beta}(a)\right)^{\complement}| \ge \frac{\kappa}{2} |P_j^{l,\eta}|$$

for all $e \in S^{n-1}$ and

$$|P_j^{l,\eta} \cap E \cap B(a, \beta 2^{i_0 - 1}) \cap \left(T_{e(j,a)}^{\beta}(a)\right)^{\complement}| \le \frac{\kappa}{2} |P_j^{l,\eta}|$$

for some $e(j,a) \in S^{n-1}$. It follows that

$$\begin{aligned}
\frac{\kappa}{2}|P_j^{l,\eta}| &\leq |P_j^{l,\eta} \cap E \cap \left(T^{\beta}_{e(j,a)}(a)\right)^{\complement}| \\
&- |P_j^{l,\eta} \cap E \cap B(a, \beta 2^{i_0-1}) \cap \left(T^{\beta}_{e(j,a)}(a)\right)^{\complement}| \\
&= |P_j^{l,\eta} \cap E \cap \left(B(a, \beta 2^{i_0-1})\right)^{\complement} \cap \left(T^{\beta}_{e(j,a)}(a)\right)^{\complement}| \\
&\leq |P_j^{l,\eta} \cap E \cap \left(B(a, \beta 2^{i_0-1})\right)^{\complement}| \\
&= \sum_{k=0}^{\log(C/\beta)} |P_j^{l,\eta} \cap E \cap \left(B(a, \beta 2^{i_0+k}) \setminus B(a, \beta 2^{i_0+k-1})\right)|.
\end{aligned}$$

Therefore there is a $k(j,a)$ such that

$$\begin{aligned}
&|P_j^{l,\eta} \cap E \cap \left(B(a, \beta 2^{i_0+k(j,a)}) \setminus B(a, \beta 2^{i_0+k(j,a)-1})\right)| \\
&\qquad\qquad \gtrsim \left(\log(C/\beta)\right)^{-1} \kappa |P_j^{l,\eta}|.
\end{aligned}$$

By the same argument as before, we conclude that there is a number $r = \beta 2^{i_0+k_0-1}$ and a set $\mathcal{C} \subset \mathcal{C}'$ with

$$|\mathcal{C}| \gtrsim \left(\log(C/\beta)\right)^{-2} M \tag{1}$$

so that for each $P_j^{l,\eta} \in \mathcal{C}$ there is a subset $A_j \subset A_j'$ of measure

$$|A_j| \gtrsim \left(\log(C/\beta)\right)^{-2} \kappa |P_j^{l,\eta}| \tag{2}$$

such that for each $a \in A_j$

$$|P_j^{l,\eta} \cap E \cap B(a, 2r) \cap \left(T^{\beta}_{e}(a)\right)^{\complement}| \gtrsim \kappa |P_j^{l,\eta}| \tag{3}$$

for all $e \in S^{n-1}$ and

$$|P_j^{l,\eta} \cap E \cap (B(a,2r) \setminus B(a,r))| \gtrsim \left(\log(C/\beta)\right)^{-1} \kappa |P_j^{l,\eta}|. \tag{4}$$

We will make different estimates for $|E|$ depending on the overlap of the sets $A_j$. We fix a number $N$ and consider two cases.

CASE I. $\forall a \in \mathbb{R}^n \quad |\{j : a \in A_j\}| \leq N$.
CASE II. $\exists a \in \mathbb{R}^n \quad |\{j : a \in A_j\}| \geq N$.

In case I we have

$$\begin{aligned}
|E| &\geq |\bigcup_{j: P_j^{l,\eta} \in \mathcal{C}} A_j| \geq \frac{1}{N} \sum_{j: P_j^{l,\eta} \in \mathcal{C}} |A_j| \\
&\gtrsim \frac{1}{N} |\mathcal{C}| \left(\log(C/\beta)\right)^{-2} \kappa \eta^{n-2} \\
&\gtrsim \frac{M}{N} \left(\log(C/\beta)\right)^{-4} \kappa \eta^{n-2}
\end{aligned} \tag{5}$$

where we have used (1) and (2).

In case II, we fix a number $\mu$ and consider two subcases.

(II)$_1$. $\forall b \in B(a,2r) \setminus B(a,r) \quad |\{j : a \in A_j, b \in P_j^{l,\eta}\}| \le \mu$.
(II)$_2$. $\exists b \in B(a,2r) \setminus B(a,r) \quad |\{j : a \in A_j, b \in P_j^{l,\eta}\}| \ge \mu$.

In subcase (II)$_1$ we have

$$\begin{aligned} |E| &\ge \Big| \bigcup_{j:a\in A_j} P_j^{l,\eta} \cap E \cap (B(a,2r) \setminus B(a,r)) \Big| \\ &\ge \frac{1}{\mu} \sum_{j:a\in A_j} |P_j^{l,\eta} \cap E \cap (B(a,2r) \setminus B(a,r))| \\ &\gtrsim \frac{N}{\mu} (\log(C/\beta))^{-1} \kappa \eta^{n-2} \end{aligned} \tag{6}$$

where the last inequality follows from (4).

In subcase (II)$_2$ let $\mathcal{B}'$ be a maximal $C_1\eta/\beta$-separated subset of $\{P_j : a \in A_j, b \in P_j^{l,\eta}\}$. Then $|\mathcal{B}'| \gtrsim \mu\beta^{2(n-2)}$. Note that if $P_j, P_k \in \mathcal{B}'$ then by Lemma 2.1

$$P_j^{l,\eta} \cap P_k^{l,\eta} \cap B(a,2r) \subset T_e^{C\beta/C_1}(a) \subset T_e^{\beta}(a)$$

where $e = (a-b)/|a-b|$, provided that $C_1$ has been chosen large enough. Therefore the family

$$\left\{ P_j^{l,\eta} \cap E \cap B(a,2r) \cap \left(T_e^{\beta}(a)\right)^{\complement} : P_j \in \mathcal{B}' \right\}$$

is disjoint. Consequently

$$\begin{aligned} |E| &\ge \Big| \bigcup_{j:P_j\in\mathcal{B}'} P_j^{l,\eta} \cap E \cap B(a,2r) \cap \left(T_e^{\beta}(a)\right)^{\complement} \Big| \\ &= \sum_{j:P_j\in\mathcal{B}'} |P_j^{l,\eta} \cap E \cap B(a,2r) \cap \left(T_e^{\beta}(a)\right)^{\complement}| \\ &\gtrsim |\mathcal{B}'|\kappa\eta^{n-2} \\ &\gtrsim \beta^{2(n-2)}\mu\kappa\eta^{n-2} \end{aligned} \tag{7}$$

where we have used (3).

So, in case II we see that choosing

$$\mu = N^{1/2} (\log(C/\beta))^{-1/2} \beta^{-(n-2)}$$

(6) and (7) imply that

$$|E| \gtrsim (\log(C/\beta))^{-1/2} \beta^{n-2} \kappa N^{1/2} \eta^{n-2}. \tag{8}$$

Choosing

$$N = M^{2/3} (\log(C/\beta))^{-7/3} \beta^{-2(n-2)/3}$$

(5) and (8) yield

$$|E| \gtrsim (\log(C/\beta))^{-5/3} \beta^{2(n-2)/3} \kappa M^{1/3} \eta^{n-2}$$

proving the lemma. $\square$

## 4. The 3-plane estimate

In this section we use the mapping properties of the Radon transform in $\mathbb{R}^3$ to derive an estimate for the measure of a set intersecting a family of plates which are contained in a neighborhood of a 3-plane. In the proof of Theorem 1.1 we will end up summing such estimates.

For a function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying the appropriate integrability conditions, the Radon transform

$$\mathcal{R}f : S^2 \times \mathbb{R} \to \mathbb{R}$$

is defined by

$$\mathcal{R}f(e,t) = \int\limits_{\langle e,x\rangle = t} f(x) d\mathcal{L}^2(x).$$

It is proved in Oberlin and Stein [4] that for any measurable set $E \subset \mathbb{R}^3$ one has the following estimate.

$$\|\mathcal{R}\chi_E\|_{3,\infty} \lesssim \|\chi_E\|_{3/2}$$

where

$$\|\mathcal{R}\chi_E\|_{3,\infty} = \left( \int\limits_{S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \right)^{1/3}$$

and $d\sigma$ is surface measure.

We can discretize this result as follows.

**Lemma 4.1.** *Suppose $E$ is a set in $\mathbb{R}^3$, $\lambda \leq 1$ and let $\{P_k\}_{k=1}^M$ be a $\delta$-separated set in $\mathcal{G}_3$ such that for each $k$ there is plate $P_k^{l,C\delta}$ satisfying*

$$|P_k^{l,C\delta} \cap E| \geq \lambda |P_k^{l,C\delta}|.$$

*Then*

$$|E| \gtrsim \lambda^{3/2} M^{1/2} \delta.$$

*Proof.* For each $e \in S^2$ let $P_e$ be the plane with normal $e$ passing through the origin. Then there is a $\delta$-separated set $\{e_k\}_{k=1}^M$ on $S^2$ such that $P_k = P_{e_k}$. Note that since $1 \leq l \leq 4$, for each $e \in B(e_k, \delta/2) \cap S^2$ we have

$$\lambda\delta \leq |P_k^{l,C\delta} \cap E| \leq \int\limits_{I_e} \mathcal{L}^2((P_e + x) \cap E) d\mathcal{L}^1(x)$$

where $I_e$ is an interval on $P_e^{\perp}$ with $\mathcal{L}^1(I_e) \lesssim \delta$. Therefore there exists $x_e \in I_e$ such that

$$\lambda \lesssim \mathcal{L}^2((P_e + x_e) \cap E).$$

Hence

$$\lambda \lesssim \sup_t \mathcal{R}\chi_E(e,t).$$

We conclude that

$$\begin{aligned} \lambda^3\delta^2 M &\lesssim \sum_k \int\limits_{B(e_k,\delta/2)\cap S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \\ &\leq \int\limits_{S^2} (\sup_t \mathcal{R}\chi_E(e,t))^3 d\sigma(e) \\ &= \|\mathcal{R}\chi_E\|_{3,\infty}^3 \lesssim \|\chi_E\|_{3/2}^3 = |E|^2. \end{aligned}$$

□

This, in turn, gives rise to the following higher dimensional analogue.

**Lemma 4.2.** *Suppose $E$ is a set in $\mathbb{R}^n$, $\lambda \leq 1$, $\Pi \subset \mathbb{R}^n$ is a 3-plane and $\{P_k\}_{k=1}^M$ is a $\delta$-separated set in $\mathcal{G}_n$ such that for each $k$ there exists a plate $P_k^\delta$ satisfying*

$$P_k^\delta \subset \Pi^{\widetilde{C}\delta} \text{ and } |P_k^\delta \cap E| \geq \lambda |P_k^\delta|$$

*where $\Pi^{\widetilde{C}\delta} = \{x \in \mathbb{R}^n : \mathrm{dist}(x,\Pi) \leq \widetilde{C}\delta\}$ is the $\widetilde{C}\delta$-neighborhood of $\Pi$. Then*

$$|E \cap \Pi^{\widetilde{C}\delta}| \gtrsim \lambda^3 M^{1/2} \delta^{n-2}.$$

*Proof.* Whithout loss of generality we may asssume that $\Pi$ is the $x_1x_2x_3$-plane. Since $P_k^\delta \subset \Pi^{\widetilde{C}\delta}$ there is a direction plane $Q_k \subset \Pi$ such that $d(P_k, Q_k) \lesssim \delta$. Therefore we can find a plate $Q_k^{2,C_1\delta}$ with $P_k^\delta \subset Q_k^{2,C_1\delta}$. It follows that

$$|Q_k^{2,C_1\delta} \cap E \cap \Pi^{\widetilde{C}\delta}| \geq \lambda\delta^{n-2}.$$

Let $\mathcal{B}$ be a maximal $C_2\delta$-separated subset of $\{P_k\}_{k=1}^M$ and put $\mathcal{B}' = \{Q_k : P_k \in \mathcal{B}\}$. Then for $Q_j, Q_k \in \mathcal{B}'$, $j \neq k$, we have

$$d(Q_j,Q_k) \geq d(P_j,P_k) - d(P_j,Q_j) - d(P_k,Q_k) \geq (C_2 - C)\delta \geq \delta$$

for $C_2$ sufficiently large.

Now for each $Q_k \in \mathcal{B}'$ let

$$L_k = \left\{ x \in B(0,\widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \leq \frac{\lambda\delta}{C_3} \right\},$$

$$H_k = \left\{ x \in B(0,\widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \geq \frac{\lambda\delta}{C_3} \right\}.$$

Note that

$$\mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \lesssim \delta, \text{ for all } x \in B(0, \widetilde{C}\delta) \cap \Pi^{\perp}.$$

Hence

$$\begin{aligned}
\lambda\delta^{n-2} &\leq |Q_k^{2,C_1\delta} \cap E \cap \Pi^{\widetilde{C}\delta}| \\
&= \int\limits_{B(0,\widetilde{C}\delta)\cap\Pi^{\perp}} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\
&= \int\limits_{L_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\
&+ \int\limits_{H_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\
&\leq \frac{\lambda\delta}{C_3} C\delta^{n-3} + C\delta\mathcal{L}^{n-3}(H_k).
\end{aligned}$$

Therefore, $\mathcal{L}^{n-3}(H_k) \gtrsim \lambda\delta^{n-3}$ for $C_3$ sufficiently large.

Next, let $M' = |\mathcal{B}'|$ and define

$$L = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^{\perp} : |\{k : x \in H_k\}| < \frac{\lambda M'}{C_4} \right\},$$

$$H = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^{\perp} : |\{k : x \in H_k\}| \geq \frac{\lambda M'}{C_4} \right\}.$$

Then

$$\begin{aligned}
\lambda\delta^{n-3}M' &\lesssim \sum_k \int \chi_{H_k} = \int\limits_H \sum_k \chi_{H_k} + \int\limits_L \sum_k \chi_{H_k} \\
&\leq M'\mathcal{L}^{n-3}(H) + \frac{\lambda M'}{C_4}\mathcal{L}^{n-3}(L) \\
&\leq M'\mathcal{L}^{n-3}(H) + \frac{\lambda M'}{C_4} C\delta^{n-3}.
\end{aligned}$$

Therefore $\mathcal{L}^{n-3}(H) \gtrsim \lambda\delta^{n-3}$ for $C_4$ sufficiently large.

Note that for each $x \in H$ there are at least $\lambda M'/C_4$ plates in $\Pi + x$, that is, plates in a copy of $\mathbb{R}^3$, with $\delta$-separated direction planes and such that the 3-dimensional measure of their intersection with $E \cap (\Pi + x)$ is at least $C^{-1}\lambda\delta$. Hence, by Lemma 4.1

$$\mathcal{L}^3(E \cap (\Pi + x)) \gtrsim \lambda^{3/2}(\lambda M')^{1/2}\delta.$$

We conclude that

$$\begin{aligned}
|E \cap \Pi^{\widetilde{C}\delta}| &\geq \int\limits_H \mathcal{L}^3(E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\
&\gtrsim \lambda\delta^{n-3}\lambda^{3/2}(\lambda M')^{1/2}\delta
\end{aligned}$$

$$\simeq \lambda^3 M^{1/2}\delta^{n-2}.$$

□

## 5. The Hausdorff dimension bound

Theorem 1.1 will be a consequence of the following.

**Proposition 5.1.** *Suppose $E$ is a set in $\mathbb{R}^n$, $\lambda \leq 1$ and $\{P_j\}_{j=1}^M$ is a $\delta$-separated set in $\mathcal{G}_n$ with* $\operatorname{diam}(\{P_j\}_{j=1}^M) \leq 1/2$, *such that for each $j$ there is plate $P_j^\delta$ satisfying*

$$|P_j^\delta \cap E| \geq \lambda |P_j^\delta|.$$

*Then*

$$|E| \geq C_\epsilon^{-1}\delta^\epsilon \lambda^\alpha M^{(2n-3)/(6(n-2))}\delta^{n-2}$$

*where $\alpha$ is a positive constant depending on $n$.*

*Proof.* Our first objective is to find a large family of plates intersecting many other plates at approximately the same angle. To this end, fix a number $\mu$ and for each $x \in P_j^\delta \cap E$ let

$$S(j,x,\mu) = \left\{b \in P_j^\delta \cap E : |\{k : x, b \in P_k^\delta\}| \leq \mu\right\}.$$

We say that a plate $P_j^\delta$ has property (LM)-$\mu$ if

$$\left|\left\{x \in P_j^\delta \cap E : |S(j,x,\mu)| \geq \frac{\lambda}{2}\delta^{n-2}\right\}\right| \geq \frac{\lambda}{2}\delta^{n-2}$$

and it has property (HM)-$\mu$ if

$$\left|\left\{x \in P_j^\delta \cap E : |S(j,x,\mu)| \leq \frac{\lambda}{2}\delta^{n-2}\right\}\right| \geq \frac{\lambda}{2}\delta^{n-2}.$$

Then each plate falls into at least one of these categories. We consider two cases.

CASE I. There is a set $\mathcal{C}'$ of at least $M/2$ plates with property (LM)-$\mu$.

CASE II. There is a set $\mathcal{C}''$ of at least $M/2$ plates with property (HM)-$\mu$.

In case I let

$$A_j' = \left\{x \in P_j^\delta \cap E : |S(j,x,\mu)| \geq \frac{\lambda}{2}\delta^{n-2}\right\}.$$

We fix a number $N$ and consider two subcases

(I)$_1$. $\forall a \in \mathbb{R}^n \quad |\{j : a \in A_j'\}| \leq N$.
(I)$_2$. $\exists a \in \mathbb{R}^n \quad |\{j : a \in A_j'\}| \geq N$.

In subcase $(\mathrm{I})_1$

$$|E| \geq |\bigcup_{j:P_j^\delta \in \mathcal{C}'} A'_j| \geq \frac{1}{N} \sum_{j:P_j^\delta \in \mathcal{C}'} |A'_j| \geq \frac{M}{4N}\lambda\delta^{n-2}. \tag{9}$$

In subcase $(\mathrm{I})_2$

$$|E| \geq |\bigcup_{j:a \in A'_j} S(j,a,\mu)| \geq \frac{1}{\mu} \sum_{j:a\in A'_j} |S(j,a,\mu)| \geq \frac{N}{2\mu}\lambda\delta^{n-2}. \tag{10}$$

Choosing $N = 2^{-1/2}M^{1/2}\mu^{1/2}$, (9) and (10) imply that

$$|E| \geq \frac{1}{2^{3/2}}\frac{M^{1/2}}{\mu^{1/2}}\lambda\delta^{n-2}.$$

So letting

$$\mu_0 = \frac{1}{16}\frac{M}{|E|^2}\lambda^2\delta^{2(n-2)} \tag{11}$$

we see that we cannot have case I for $\mu_0$. Consequently, there is a set $\mathcal{C}''$ of at least $M/2$ plates so that for each $P_j^\delta \in \mathcal{C}''$ there is a set $A''_j \subset P_j^\delta \cap E$ of measure

$$|A''_j| \geq \frac{\lambda}{2}\delta^{n-2}$$

such that for each $a \in A''_j$

$$|\left\{b \in P_j^\delta \cap E : |\{k : a, b \in P_k^\delta\}| \geq \mu_0\right\}| \geq \frac{\lambda}{2}\delta^{n-2}.$$

Fix $a \in A''_j$. Then, by the pigeonhole principle, for each $b \in P_j^\delta \cap E$ with $|\{k : a, b \in P_k^\delta\}| \geq \mu_0$, there is an $i(j,a,b)$, $1 \leq i(j,a,b) \leq \log(C/\delta)$, such that

$$\begin{aligned}|\{k : a, b \in P_k^\delta \text{ and } \delta 2^{i(j,a,b)-1} \leq d(P_j, P_k) \leq \delta 2^{i(j,a,b)}\}|& \\ \geq (\log(C/\delta))^{-1}\,\mu_0&.\end{aligned}$$

Since there are at most $\log(C/\delta)$ possible $i(j,a,b)$, then for a given $a \in A''_j$ there is an $i(j,a)$ such that

$$\begin{aligned}|\{b \in P_j^\delta \cap E : |\{k : a, b \in P_k^\delta \text{ and } \delta 2^{i(j,a)-1} \leq d(P_j, P_k) \leq \delta 2^{i(j,a)}\}|& \\ \geq (\log(C/\delta))^{-1}\,\mu_0\}| \gtrsim (\log(C/\delta))^{-1}\lambda\delta^{n-2}&.\end{aligned}$$

By the same argument, for a given $P_j^\delta \in \mathcal{C}''$ there is a set $A_j \subset A''_j$ of measure

$$|A_j| \gtrsim (\log(C/\delta))^{-1}\lambda\delta^{n-2} \tag{12}$$

and an $i(j)$ such that for each $a \in A_j$

$$\begin{aligned}|\{b \in P_j^\delta \cap E : |\{k : a, b \in P_k^\delta \text{ and } \delta 2^{i(j)-1} \leq d(P_j, P_k) \leq \delta 2^{i(j)}\}|& \\ \geq (\log(C/\delta))^{-1}\,\mu_0\}| \gtrsim (\log(C/\delta))^{-1}\lambda\delta^{n-2}&.\end{aligned}$$

Since there are at least $M/2$ plates in $\mathcal{C}''$ we conclude that there is a number $\rho = \delta 2^{i_0 - 1}$ and a subset $\mathcal{C} \subset \mathcal{C}''$ with

$$|\mathcal{C}| \gtrsim (\log(C/\delta))^{-1} M \tag{13}$$

so that for each $P_j^\delta \in \mathcal{C}$ and each $a \in A_j$

$$\begin{aligned} &|\{b \in P_j^\delta \cap E : |\{k : a, b \in P_k^\delta \text{ and } \rho \le d(P_j, P_k) \le 2\rho\}| \\ &\qquad \ge (\log(C/\delta))^{-1}\, \mu_0\}| \gtrsim (\log(C/\delta))^{-1} \lambda \delta^{n-2}. \end{aligned} \tag{14}$$

Heuristically, (12) and (14) tell us that each member of $\mathcal{C}$ intersects a large number of plates at angle approximately $\rho$. We are going to estimate this number by exploiting the bound for the measure of their intersection.

For each $P_j^\delta \in \mathcal{C}$ and each $a \in A_j$ let

$$\mathcal{A}_j(a) = \left\{P_k^\delta : a \in P_k^\delta \text{ and } \rho \le d(P_j, P_k) \le 2\rho\right\},$$

$$\begin{aligned} A_j(a) = \{b \in P_j^\delta \cap E : &|\{k : a, b \in P_k^\delta \text{ and } \rho \le d(P_j, P_k) \le 2\rho\}| \\ &\ge (\log(C/\delta))^{-1}\, \mu_0\}. \end{aligned}$$

Then, by the remark following Lemma 2.1, we have

$$\begin{aligned} |\mathcal{A}_j(a)| &\gtrsim \sum_{k: P_k^\delta \in \mathcal{A}_j(a)} |P_j^\delta \cap P_k^\delta| \frac{\rho}{\delta^{n-1}} \\ &= \frac{\rho}{\delta^{n-1}} \int_{P_j^\delta} \sum_{k: P_k^\delta \in \mathcal{A}_j(a)} \chi_{P_k^\delta} \\ &\ge \frac{\rho}{\delta^{n-1}} \int_{A_j(a)} \sum_{k: P_k^\delta \in \mathcal{A}_j(a)} \chi_{P_k^\delta} \\ &\ge \frac{\rho}{\delta^{n-1}} |A_j(a)| (\log(C/\delta))^{-1} \mu_0 \\ &\gtrsim (\log(C/\delta))^{-2} \lambda (\rho/\delta) \mu_0 \end{aligned} \tag{15}$$

where the last inequality follows from (14). Therefore, if we let

$$\mathcal{D}_j = \left\{P_k^\delta : P_j^\delta \cap P_k^\delta \ne \emptyset \text{ and } \rho \le d(P_j, P_k) \le 2\rho\right\}$$

then

$$\begin{aligned} |\mathcal{D}_j| &\gtrsim \sum_{k: P_k^\delta \in \mathcal{D}_j} |P_j^\delta \cap P_k^\delta| \frac{\rho}{\delta^{n-1}} \\ &= \frac{\rho}{\delta^{n-1}} \int_{P_j^\delta} \sum_{k: P_k^\delta \in \mathcal{D}_j} \chi_{P_k^\delta} \\ &\ge \frac{\rho}{\delta^{n-1}} \int_{A_j} \sum_{k: P_k^\delta \in \mathcal{D}_j} \chi_{P_k^\delta} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho}{\delta^{n-1}} \int\limits_{A_j} |\mathcal{A}_j(a)| da \\
&\gtrsim (\log(C/\delta))^{-3}\lambda^2(\rho/\delta)^2\mu_0 \qquad (16)
\end{aligned}
$$

where the last inequality follows from (12) and (15).

Note that the preceding counting argument enabled us to gain an extra $\rho/\delta$ factor. This will play an important role in what follows.

We are now in a position to carry out a construction which will allow us to use the 3-plane estimate of Section 4.

For each $P_j^\delta \in \mathcal{C}$, suppose $\{e_{ji}\}_i$ is a maximal $\delta/\rho$-separated set of points on the $(n-3)$-dimensional unit sphere $S^{n-1} \cap P_j^\perp$ and let

$$\Pi_{ji} = c_j + \Pi'_{ji}$$

where $c_j$ is the center of $P_j^\delta$ and $\Pi'_{ji}$ is the 3-plane spanned by $e_{ji}$ and $P_j$. Using geometry, one can show that for each $P_k^\delta \in \mathcal{D}_j$ there exists an $i$ such that $P_k^\delta \subset \Pi_{ji}^{\widetilde{C}\delta}$, where $\Pi_{ji}^{\widetilde{C}\delta}$ is the $\widetilde{C}\delta$-neighborhood of $\Pi_{ji}$. Therefore, if we let

$$\mathcal{D}_{ji} = \left\{ P_k^\delta \in \mathcal{D}_j : P_k^\delta \subset \Pi_{ji}^{\widetilde{C}\delta} \right\}$$

then

$$\mathcal{D}_j = \bigcup_i \mathcal{D}_{ji}.$$

Now for each $P_j^\delta \in \mathcal{C}$ let $\widetilde{P}_j^\rho = P_j^{4,C\rho}$ be a plate with direction plane $P_j$, the same center as $P_j^\delta$ and the indicated dimensions. Note that if $C$ is large enough then for each $P_k^\delta \in \mathcal{D}_j$ we have $P_k^\delta \subset \widetilde{P}_j^\rho$. We will show that these dilated plates have large intersection with $E$. More precisely, we claim that for all $e \in S^{n-1}$, $a \in \mathbb{R}^n$

$$|\widetilde{P}_j^\rho \cap E \cap (T_e^\gamma(a))^{\complement}| \geq C_\epsilon^{-1}\delta^\epsilon\lambda^{n+4}\frac{\rho M}{\delta|E|^2}\delta^{3(n-2)} \qquad (17)$$

where $\gamma = \lambda(\log(1/\delta))^{-1}$. To see this, fix a plate $\widetilde{P}_j^\rho$, $e \in S^{n-1}$, $a \in \mathbb{R}^n$ and let

$$E' = E \cap (T_e^\gamma(a))^{\complement}.$$

Note that for every $P_k^\delta \in \mathcal{D}_j$ we have

$$
\begin{aligned}
|P_k^\delta \cap E'| &\geq |P_k^\delta \cap E| - |P_k^\delta \cap T_e^\gamma(a)| \\
&\geq \lambda\delta^{n-2} - C\lambda(\log(1/\delta))^{-1}\delta^{n-2} \\
&\geq \frac{3}{4}\lambda\delta^{n-2}
\end{aligned}
$$

for $\delta$ sufficiently small.

We consider two cases.

CASE I. $\delta \leq \gamma\rho$.

CASE II. $\delta \geq \gamma\rho$.

In case I let

$$\mathcal{X}_j = \{x \in \mathbb{R}^n : \operatorname{dist}(x, c_j + P_j) \leq \gamma\rho\}.$$

Then for each $P_k^\delta \in \mathcal{D}_j$

$$P_k^\delta \cap \mathcal{X}_j \subset P_k^{2\gamma\rho} \cap \mathcal{X}_j.$$

Hence, by Lemma 2.1, $P_k^\delta \cap \mathcal{X}_j$ is contained in a tube of cross-section radius $C\gamma$. Therefore

$$\begin{aligned}
|P_k^\delta \cap (\widetilde{P}_j^\rho \cap E' \cap \mathcal{X}_j^{\complement})| &= |P_k^\delta \cap E' \cap \mathcal{X}_j^{\complement}| \\
&= |P_k^\delta \cap E'| - |P_k^\delta \cap E' \cap \mathcal{X}_j| \\
&\geq |P_k^\delta \cap E'| - |P_k^\delta \cap \mathcal{X}_j| \\
&\geq \frac{\lambda}{2}\delta^{n-2}
\end{aligned}$$

for $\delta$ sufficiently small.

Now, if $\operatorname{dist}(x, c_j + P_j) \geq \gamma\rho$ it is easy to see that $x$ belongs to at most $C\gamma^{-(n-3)}$ sets $\Pi_{ji}^{\widetilde{C}\delta}$. Consequently

$$\begin{aligned}
|\widetilde{P}_j^\rho \cap E'| &\geq |\bigcup_i \widetilde{P}_j^\rho \cap E' \cap \mathcal{X}_j^{\complement} \cap \Pi_{ji}^{\widetilde{C}\delta}| \\
&\gtrsim \gamma^{n-3} \sum_i |(\widetilde{P}_j^\rho \cap E' \cap \mathcal{X}_j^{\complement}) \cap \Pi_{ji}^{\widetilde{C}\delta}| \\
&\gtrsim \gamma^{n-3}\lambda^3\delta^{n-2} \sum_i |\mathcal{D}_{ji}|^{1/2}
\end{aligned}$$

where the last inequality follows from Lemma 4.2 applied to the set $\widetilde{P}_j^\rho \cap E' \cap \mathcal{X}_j^{\complement}$, the families of plates $\{\mathcal{D}_{ji}\}_i$ and the 3-planes $\{\Pi_{ji}\}_i$.

In case II, since $|\{\Pi_{ji}\}_i| \lesssim (\rho/\delta)^{n-3}$, we have

$$\begin{aligned}
|\widetilde{P}_j^\rho \cap E'| &\geq |\bigcup_i \widetilde{P}_j^\rho \cap E' \cap \Pi_{ji}^{\widetilde{C}\delta}| \\
&\gtrsim (\delta/\rho)^{n-3} \sum_i |(\widetilde{P}_j^\rho \cap E') \cap \Pi_{ji}^{\widetilde{C}\delta}| \\
&\geq \gamma^{n-3} \sum_i |(\widetilde{P}_j^\rho \cap E') \cap \Pi_{ji}^{\widetilde{C}\delta}| \\
&\gtrsim \gamma^{n-3}\lambda^3\delta^{n-2} \sum_i |\mathcal{D}_{ji}|^{1/2}
\end{aligned}$$

with the last inequality true by Lemma 4.2 applied to the set $\widetilde{P}_j^\rho \cap E'$, the families of plates $\{\mathcal{D}_{ji}\}_i$ and the 3-planes $\{\Pi_{ji}\}_i$.

We conclude that in either case

$$|\widetilde{P}_j^\rho \cap E'| \gtrsim \gamma^{n-3}\lambda^3\delta^{n-2} \sum_i |\mathcal{D}_{ji}|^{1/2}. \tag{18}$$

To estimate the sum above, note that $\Pi_{ji}^{\widetilde{C}\delta}$, being the $\widetilde{C}\delta$-neighborhood of a copy of $\mathbb{R}^3$, can contain at most $C(\rho/\delta)^2$ plates whose direction planes are $\delta$-separated and at distance approximately $\rho$ from $P_j$. Therefore

$$|\mathcal{D}_j| \leq \sum_i |\mathcal{D}_{ji}| \lesssim \frac{\rho}{\delta} \sum_i |D_{ji}|^{1/2}. \tag{19}$$

Combining (11), (16), (18) and (19) we obtain

$$|\widetilde{P}_j^\rho \cap E'| \geq C_\epsilon^{-1} \delta^\epsilon \lambda^{n+4} \frac{\rho M}{\delta |E|^2} \delta^{3(n-2)}$$

as claimed.

Note that (17) trivially implies that

$$|E| \geq C_\epsilon^{-1} \delta^{\epsilon/3} \lambda^{(n+4)/3} (\rho/\delta)^{1/3} M^{1/3} \delta^{n-2}. \tag{20}$$

Now let $\mathcal{B}$ be a maximal $C\rho$-separated subset of $\{P_j : P_j^\delta \in \mathcal{C}\}$. Then

$$|\mathcal{B}| \gtrsim (\delta/\rho)^{2(n-2)} |\mathcal{C}| \gtrsim (\log(C/\delta))^{-1} (\delta/\rho)^{2(n-2)} M.$$

Rewriting (17) as

$$|\widetilde{P}_j^\rho \cap E \cap (T_e^\gamma(a))^{\complement}| \geq C_\epsilon^{-1} \delta^\epsilon \lambda^{n+4} \frac{\rho M}{\delta |E|^2 \rho^{n-2}} \delta^{3(n-2)} |\widetilde{P}_j^\rho|$$

we see that the family $\{\widetilde{P}_j^\rho : P_j \in \mathcal{B}\}$ satisfies the conditions of Lemma 3.1 with $l = 4$, $\eta = C\rho$, $\beta = \gamma = \lambda(\log(1/\delta))^{-1}$ and

$$\kappa = C_\epsilon^{-1} \delta^\epsilon \lambda^{n+4} \frac{\rho M}{\delta |E|^2 \rho^{n-2}} \delta^{3(n-2)}.$$

Hence, after some algebra

$$|E| \geq C_\epsilon^{-1} \delta^\epsilon \lambda^{(5n+13)/9} (\delta/\rho)^{(2n-7)/9} M^{4/9} \delta^{n-2}. \tag{21}$$

If $\rho \geq \delta M^{1/(2(n-2))}$ then (20) yields

$$|E| \geq C_\epsilon^{-1} \delta^\epsilon \lambda^{(n+4)/3} M^{(2n-3)/(6(n-2))} \delta^{n-2}. \tag{22}$$

If $\rho \leq \delta M^{1/(2(n-2))}$ then (21) gives

$$|E| \geq C_\epsilon^{-1} \delta^\epsilon \lambda^{(5n+13)/9} M^{(2n-3)/(6(n-2))} \delta^{n-2}. \tag{23}$$

Letting $\alpha = (5n+13)/9$ and combining (22) and (23) we get

$$|E| \geq C_\epsilon^{-1} \delta^\epsilon \lambda^\alpha M^{(2n-3)/(6(n-2))} \delta^{n-2}$$

proving the proposition. $\square$

The proof of Theorem 1.1 is now, essentially, Lemma 2.15 in [1]. We give a sketch for the convenience of the reader.

Let $F$ be an $(n, 2)$-set. Using measure theory, we can find a compact set $E \subset F$ and a set $\mathcal{A} \subset \mathcal{G}_n$ of positive measure so that $\operatorname{diam}(\mathcal{A}) \leq 1/2$

and for every $P \in \mathcal{A}$ there is a square $S_P$ of unit area such that $S_P$ is parallel to $P$ and

$$\mathcal{L}^2(S_P \cap E) \geq 1/2.$$

Fix a covering $\{B(x_i, r_i)\}$ of $E$ and let

$$I_k = \left\{i : 2^{-k} \leq r_i \leq 2^{-(k-1)}\right\}, \quad \nu_k = |I_k|,$$

$$E_k = E \cap \bigcup_{i \in I_k} B(x_i, r_i), \quad \widetilde{E}_k = \bigcup_{i \in I_k} B(x_i, 2r_i).$$

Then, by the pigeonhole principle, one can find a $k$ and a set $\mathcal{B} \subset \mathcal{A}$ of measure at least $C^{-1}k^{-2}$ so that

$$\mathcal{L}^2(S_P \cap E_k) \gtrsim k^{-2}, \text{ for all } P \in \mathcal{B}.$$

Let $\{P_j\}_{j=1}^M$ be a maximal $2^{-k}$-separated set in $\mathcal{B}$. Then

$$M \gtrsim k^{-2}2^{2k(n-2)}$$

and for each $P_j$ there is a plate $P_j^{2^{-k}}$ such that

$$|P_j^{2^{-k}} \cap \widetilde{E}_k| \gtrsim k^{-2}|P_j^{2^{-k}}|.$$

So, by Proposition 5.1

$$|\widetilde{E}_k| \geq C_\epsilon^{-1}k^{-\xi}2^{-k(n-s+\epsilon)}$$

where $\xi = 2\alpha + (2n-3)/(3(n-2))$, $s = (2n+3)/3$. On the other hand

$$|\widetilde{E}_k| \lesssim \nu_k 2^{-kn}.$$

Therefore

$$\nu_k \geq C_\epsilon^{-1}k^{-\xi}2^{k(s-\epsilon)}.$$

Consequently

$$\sum_i r_i^{s-2\epsilon} \gtrsim \nu_k 2^{-k(s-2\epsilon)} \geq C_\epsilon^{-1}k^{-\xi}2^{k\epsilon} \geq \widetilde{C}_\epsilon^{-1}.$$

We conclude that the Hausdorff dimension of $F$ is at least $s$.

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Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland

*E-mail address*: mitsis@math.jyu.fi